Categorical Logic
Exercise Sheet 2

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1. **Subobjects.** Consider an object X in a category. A subobject of X is an object S together with a monomorphism $m\colon S \rightarrowtail X$. Two subobjects (S, m) and (S', m') are isomorphic if there is an isomorphism $i\colon S \to S'$ such that $m = m'i$. We write $\mathrm{Sub}(X)$ for the class of all subobjects of X, up-to isomorphism.

   Given two subobjects (S, m) and (S', m'), we write $(S, m) \leq (S', m')$ if there is a morphism $m''\colon S \to S'$ such that $m = m'm''$. This defines a partial order on $\mathrm{Sub}(X)$.

   (a) Consider a set X in the category of sets. Describe an isomorphism between the partial order $(\mathrm{Sub}(X), \leq)$ and the usual partial order of subsets, $(\mathcal{P}(X), \subseteq)$.

   (b) Describe what a monomorphism is in the category of monoids and in the category of arrows. For both categories, give a characterization of $\mathrm{Sub}(X)$, similar to the subsets in question (a).

   (c) We write 1 for a terminal object. What is $\mathrm{Sub}(1)$ in the following categories? (i) the category of sets; (ii) the category of monoids; (iii) the category of arrows.

2. **Kernel pairs and equivalence relations.** The kernel pair of a morphism $f\colon X \to Y$ is the pullback of f along itself.

   (a) What is the kernel pair of a function, in the category of sets?

   (b) Consider a category with finite limits. Show that the kernel pair of a morphism $f\colon X \to Y$ can be understood as the meaning of the formula $(x, x'.\, f(x) = f(x'))$.

   (c) There is a characterization of models of the theory of equivalence relations on the next page. Show that it is correct. If you are short of time, just check one or two of the axioms.

   (d) Show that a kernel pair is always an equivalence relation.

   (e) A category is sometimes said to be *exact* if every equivalence relation arises as a kernel pair. Show that the category of sets is exact.

   (f) Describe the models of the theory of equivalence relations in the category of monoids (as if for someone who likes monoids but doesn't like category theory).

   (g) *[Optional.]* Let G be an Abelian group. Describe a bijective correspondence between

      i. equivalence relations on G

      ii. subgroups of G.

Notes on the theory of equivalence relations. An equivalence relation on an object A in a category with finite limits is a subobject $R \rightarrowtail A \times A$ that satisfies the following conditions. We will consider the composite projection morphisms $a = (R \rightarrowtail A \times A \xrightarrow{\pi_1} A)$ and $b = (R \rightarrowtail A \times A \xrightarrow{\pi_2} A)$.

- (Reflexivity.) There is a morphism $r : A \to R$ making the following diagram commute.

$$\begin{tikzcd} & A \arrow[d, "r"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\ & R \arrow[dl, "a"] \arrow[dr, "b"'] & \\ A & & A \end{tikzcd}$$

- (Symmetry.) There is a morphism $s : R \to R$ making the following diagram commute.

$$\begin{tikzcd} & R \arrow[d, "s"] \arrow[ddl, bend right, "b"'] \arrow[ddr, bend left, "a"] & \\ & R \arrow[dl, "a"] \arrow[dr, "b"'] & \\ A & & A \end{tikzcd}$$

- (Transitivity.) Let P be the pullback of b along a, as follows:

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ R \arrow[dr, "b"'] & & R \arrow[dl, "a"] \\ & A & \end{tikzcd}$$

  Transitivity requires that there is a morphism $t : P \to R$ making the following diagram commute.

$$\begin{tikzcd} & P \arrow[d, "t"] \arrow[ddl, bend right, "ap"'] \arrow[ddr, bend left, "bq"] & \\ & R \arrow[dl, "a"] \arrow[dr, "b"'] & \\ A & & A \end{tikzcd}$$