Recall:

$\mathrm{Dom}_\perp$, $\mathrm{Dom}_\perp^{op}$ & $\mathrm{Dom}_\perp^{op} \times \mathrm{Dom}_\perp$ are examples of

cpo-enriched category

- an ordinary category $\mathcal{C}$, plus
- cpo structure on each hom $\mathcal{C}(A,B)$ such that composition
$$\begin{aligned} \mathcal{C}(A,B) \times \mathcal{C}(B,C) &\longrightarrow \mathcal{C}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$
is a <u>continuous</u> function

Functors $F : \mathcal{C} \to \mathcal{C}'$ are cpo-enriched, or locally continuous, if each function
$$\begin{aligned} \mathcal{C}(A,B) &\longrightarrow \mathcal{C}'(FA,FB) \\ f &\longmapsto F(f) \end{aligned}$$
is continuous.

$\mathrm{Dom}_\perp$, $\mathrm{Dom}_\perp^{op}$ & $\mathrm{Dom}_\perp^{op} \times \mathrm{Dom}_\perp$ are also examples of

$\mathrm{Dom}_\perp$-enriched category

- an ordinary category $\mathcal{C}$, plus
- domain structure on each hom $\mathcal{C}(A,B)$ such that composition induces
$$\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$$
in $\mathrm{Dom}_\perp$.

equivalent to requiring
$$\begin{aligned} \mathcal{C}(A,B) &\to \mathcal{C}(A',B) \\ g &\longmapsto g \circ f \end{aligned} \qquad \begin{aligned} \mathcal{C}(A,B) &\to \mathcal{C}(A,B') \\ g &\longmapsto h \circ g \end{aligned}$$
to be <u>strict cts</u> for each $f \in \mathcal{C}(A',A)$, $h \in \mathcal{C}(B,B')$

Theorem (Freyd 1992)

If $\mathcal{D}$ is $\mathbf{Dom}_\perp$-enriched, $F : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D}$ is locally continuous and $i : F(D, D) \cong D$ is a minimal invariant, then

$$F^{\S} : \mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{D}^{op} \times \mathcal{D}$$
$$(D', D) \longmapsto (F(D, D'), F(D', D))$$

has a "regular, free di-algebra" given by (D, i) — in other words ...

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$$(i^{-1}, i) : F^{\S}(D, D) \to (D, D)$$

Initial algebra property

for all $(f,g): F^{\S}(A,B) \to (A,B)$

there exists a unique $(h,k) : (D,D) \to (A,B)$

in $\mathcal{D}^{op} \times \mathcal{D}$ such that

$$\begin{tikzcd}
F^{\S}(D,D) \arrow[r, "{(i^{-1},i)}"] \arrow[d, "{F^{\S}(h,k)}"'] & (D,D) \arrow[d, "{(h,k)}"] \\
F^{\S}(A,B) \arrow[r, "{(f,g)}"'] & (A,B)
\end{tikzcd}$$

commutes.

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has an initial algebra given by

$$(i^{-1}, i) : F^{\S}(D,D) \to (D,D)$$

and a final coalgebra given by

$$(i, i^{-1}) : (D,D) \to F^{\S}(D,D)$$

Final coalgebra property

for all $(g,f):(B,A)\to F^{\S}(B,A)$

there exists a unique $(k,h):(B,A)\to(D,D)$

in $\mathcal{D}^{op}\times\mathcal{D}$ such that

$$\begin{tikzcd}
(B,A) \arrow[r, "{(g,f)}"] \arrow[d, "{(k,h)}"'] & F^{\S}(B,A) \arrow[d, "{F^{\S}(k,h)}"] \\
(D,D) \arrow[r, "{(i,i^{-1})}"'] & F^{\S}(D,D)
\end{tikzcd}$$

commutes.

Theorem (Freyd 1992)

If $\mathcal{D}$ is $\mathrm{Dom}_\perp$-enriched, $F:\mathcal{D}^{op}\times\mathcal{D}\to\mathcal{D}$ is locally continuous and $i:F(D,D)\cong D$ is a minimal invariant, then

$$F^{\S}:\mathcal{D}^{op}\times\mathcal{D}\longrightarrow\mathcal{D}^{op}\times\mathcal{D}$$
$$(D',D)\longmapsto(F(D,D'),F(D',D))$$

has a "regular, free di-algebra" given by (D,i) — in other words ...

Regular free di-algebra property of $i : F(D,D) \cong D$:

for all $\begin{cases} f : A \to F(B,A) \\ g : F(A,B) \to B \end{cases}$ in $\mathcal{D}$, there exist

unique $\begin{cases} h : A \to D \\ k : D \to B \end{cases}$ making

$$\begin{tikzcd} D \arrow[r, "i^{-1}"] & F(D,D) \\ A \arrow[u, "h"] \arrow[r, "f"'] & F(B,A) \arrow[u, "F(k,h)"'] \end{tikzcd} \quad \& \quad \begin{tikzcd} F(D,D) \arrow[r, "i"] \arrow[d, "F(h,k)"'] & D \arrow[d, "k"] \\ F(A,B) \arrow[r, "g"'] & B \end{tikzcd}$$

commute.

$\forall (f,g) \in \mathcal{D}(A, FBA) \times \mathcal{D}(FAB, B)$

$\exists! (h,k) \in \mathcal{D}(A,D) \times \mathcal{D}(D,B)$ s.t. $\begin{cases} i^{-1} \circ h = F(k,h) \circ f \\ k \circ i = g \circ F(h,k) \end{cases}$

Proof Existence:

Define $(h,k) \triangleq \mathrm{fix}(\varphi)$

where $\varphi : \mathcal{D}(A,D) \times \mathcal{D}(D,B) \longrightarrow \mathcal{D}(A,D) \times \mathcal{D}(D,B)$

is $(u,v) \mapsto (i \circ F(v,u) \circ f,\ g \circ F(u,v) \circ i^{-1})$

Since $(h,k) = \varphi(h,k)$, we get

$\begin{cases} h = i \circ F(k,h) \circ f \\ k = g \circ F(h,k) \circ i^{-1} \end{cases}$, so $\begin{cases} i^{-1} \circ h = F(k,h) \circ f \\ k \circ i = g \circ F(h,k) \end{cases}$

as required

$\forall (f,g) \in \mathcal{D}(A, FBA) \times \mathcal{D}(FAB, B)$

$\exists! \, (h,k) \in \mathcal{D}(A,D) \times \mathcal{D}(D,B)$ s.t. $\begin{cases} i^{-1} \circ h = F(k,h) \circ f \\ k \circ i = g \circ F(h,k) \end{cases}$

<u>Proof</u> Uniqueness: Suppose also have $\begin{cases} i^{-1} \cdot h' = F(k',h') \cdot f \\ k' \cdot i = g \cdot F(h',k') \end{cases}$

Recall that $id_D = \bigsqcup_n \pi_n$ where $\begin{cases} \pi_0 = \bot \\ \pi_{n+1} = i \cdot F(\pi_n, \pi_n) \cdot i^{-1} \end{cases}$.

<u>Claim</u> $\boxed{\forall n. \ \pi_n \circ h \sqsubseteq h' \ \& \ k \circ \pi_n \sqsubseteq k'}$

If so, then $\begin{cases} h = id_D \circ h = \bigsqcup_n \pi_n \circ h \sqsubseteq h' \\ k = k \circ id_D = \bigsqcup_n k \circ \pi_n \sqsubseteq k' \end{cases}$

and symmetrically $h' \sqsubseteq h$ & $k' \sqsubseteq k$ — so that $h = h'$ & $k = k'$, as required.

<u>Claim</u> $\boxed{\forall n. \ \pi_n \circ h \sqsubseteq h' \ \& \ k \circ \pi_n \sqsubseteq k'}$

Proof by induction on n:

<u>$n=0$</u>: $\begin{cases} \pi_0 \circ h = \bot \circ h = \bot \sqsubseteq h' \\ k \circ \pi_0 = k \circ \bot = \bot \sqsubseteq k' \end{cases}$

since k strict

<u>induction step</u>:

$$\begin{aligned} \pi_{n+1} \circ h &= i \cdot F(\pi_n, \pi_n) \circ i^{-1} \circ h \\ &= i \cdot F(\pi_n, \pi_n) \circ F(k,h) f \\ &= i \circ F(k \pi_n, \pi_n \cdot h) f \\ &\sqsubseteq i \circ F(k', h') \circ f \\ &= i \circ i^{-1} \circ h' \\ &= h' \end{aligned}$$

<by ind. hyp.>

$$\begin{aligned} k \pi_{n+1} &= k \cdot i \cdot F(\pi_n, \pi_n) \cdot i^{-1} \\ &= g \cdot F(h,k) \cdot F(\pi_n, \pi_n) \cdot i^{-1} \\ &= g \cdot F(\pi_n \cdot h, k \cdot \pi_n) \cdot i^{-1} \\ &\sqsubseteq g \cdot F(h', k') \cdot i^{-1} \\ &= k' \cdot i \cdot i^{-1} \\ &= k' \end{aligned}$$

Conclusion

Minimal invariant property of recursive domains can be stated independently of any particular construction of the recursively defined domain & characterizes it uniquely up to iso among all solutions of the associated domain equation

<u>Claim</u>: many applications of recursive domains follow directly from the min. inv. property.

Conclusion

Minimal invariant property of recursive domains can be stated independently of any particular construction of the recursively defined domain & characterizes it uniquely up to iso among all solutions of the associated domain equation

<u>Claim</u>: many applications of recursive domains follow directly from the min. inv. property.

- computational adequacy
- existence of logical relations
- induction/coinduction principles