Recall the denotational semantics of λ-terms in a domain satisfying $i : (D \to D)_\bot \cong D$

Theorem If (D, i) is a minimal invariant, then for all closed λ-terms e

$$[\![e]\!] \neq \bot \supset \exists c.\ e \Rightarrow c$$

and hence $[\![-]\!]$ is computationally adequate:

$$[\![e]\!] \sqsubseteq [\![e']\!] \supset e \leq_{ctx} e'$$

Proof

It suffices to construct a binary relation

$$\lhd \subseteq D \times \Lambda_o$$

closed λ-terms

satisfying

$$d \lhd e \equiv d = \bot \vee \exists f, x, e_1.$$
$$d = fun(f) \;\&\; e \Rightarrow \lambda x.e_1 \;\&$$
$$\forall d', e'.\; d' \lhd e' \supset f(d) \lhd e_1[e'/x]$$

infix notation for $(d,e) \in \lhd$

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$$d \lhd e \equiv d = \bot \vee \exists f, x, e_1.$$
$$d = \text{fun}(f) \;\&\; e \Rightarrow \lambda x. e_1 \;\&$$
$$\forall d', e'.\; d' \lhd e' \supset f(d') \lhd e_1[e'/x]$$

$\boxed{\begin{array}{l}\text{fun} : (D \to D) \to D \\ \text{is restriction of} \quad i : (D \to D)_\bot \cong D \\ \text{to non-}\bot\text{ elements of } (D \to D)_\bot\end{array}}$

Proof

It suffices to construct a binary relation

$$\lhd \subseteq D \times \Lambda_0$$

satisfying

$$\boxed{\begin{aligned} d \lhd e \equiv\ & d = \bot \vee \exists f, x, e_1.\\ & d = fun(f)\ \&\ e \Rightarrow \lambda x. e_1\ \& \\ & \forall d', e'.\ d' \lhd e' \supset f(d) \lhd e_1[e'/x] \end{aligned}} \quad (*)$$

for then $\lhd$ satisfies ...

① If $d \triangleleft e$ and $\forall c.\ e \Rightarrow c \supset e' \Rightarrow c$, then $d \triangleleft e'$.

(Proof: immediate from property (*) of $\triangleleft$.)

① If $d \lhd e$ and $\forall c.\ e \Rightarrow c \supset e' \Rightarrow c$, then $d \lhd e'$.

② $d \lhd e \ \&\ d' \lhd e' \supset \mathrm{app}(d, d') \lhd e e'$

$$\boxed{\begin{array}{l} \mathrm{app} : D \times D \longrightarrow D \\ \mathrm{app}(d, d') \triangleq \begin{cases} f(d') & \text{if } d = \mathrm{fun}(f) \\ \perp & \text{if } d = \perp \end{cases} \end{array}}$$

Property ② follows from (*) using the definition of app and property ①.

① If $d \lhd e$ and $\forall c.\ e \Rightarrow c \supset e' \Rightarrow c$, then $d \lhd e'$.

② $d \lhd e \ \&\ d' \lhd e' \supset app(d, d') \lhd e e'$

③ (Fundamental Property for the logical relation $\lhd$)
For all (possibly open) λ-terms e, with free vars in $\{x_1, \dots, x_n\}$ say, all environments $\rho \in D^V$ and all $e_1, \dots, e_n \in \Lambda_0$,

if $\rho(x_1) \lhd e_1 \ \& \cdots \&\ \rho(x_n) \lhd e_n$, then

$$[\![e]\!]\rho \lhd e[e_1/x_1, \dots, e_n/x_n].$$

In particular, $\boxed{\text{for all } e \in \Lambda_0,\ [\![e]\!] \lhd e}$.

(Proof by induction on structure of e, using ② for application terms & $(*)$ for λ-abstractions.)

① If $d \lhd e$ and $\forall c.\ e \Rightarrow c \supset e' \Rightarrow c$, then $d \lhd e'$.

② $d \lhd e$ & $d' \lhd e'$ $\supset$ $app(d, d') \lhd ee'$

③ for all $e \in \Lambda_o$, $[\![e]\!] \lhd e$.

From ③ we get

$$[\![e]\!] \neq \perp \supset [\![e]\!] = fun(f), \text{ some } f \in D \to D$$

$$\supset e \Rightarrow c, \text{ some } c$$

by (*) for $[\![e]\!] \lhd e$

as required. $\square$

Proof

It suffices to construct a binary relation

$$\lhd \subseteq D \times \Lambda_0$$

satisfying

$$\boxed{\begin{aligned} d \lhd e \equiv\ & d = \bot \vee \exists f, x, e_1. \\ & d = \mathrm{fun}(f)\ \&\ e \Rightarrow \lambda x.e_1\ \& \\ & \forall d', e'.\ d' \lhd e' \supset f(d) \lhd e_1[e'/x] \end{aligned}} \quad (*)$$

But why does such a relation exist?

not a simple inductive definition, because

−ve

+ve

Detour: Complete lattices

If a poset $(P, \sqsubseteq)$ has least upper bounds $\bigsqcup S$ for all subsets $S \subseteq P$

a.k.a. "joins"

- $\forall x \in S.\ x \sqsubseteq \bigsqcup S$
- $(\forall x \in S.\ x \sqsubseteq y) \supset \bigsqcup S \sqsubseteq y$

$\bigsqcup S$ is an upper bound for S

$\bigsqcup S$ is smaller than any upper bound for S

Detour : Complete lattices

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- $\forall x \in S.\ \ x \sqsubseteq \bigsqcup S$
- $(\forall x \in S.\ x \sqsubseteq y) \supset \bigsqcup S \sqsubseteq y$

a.k.a. "meets"

then it also has greatest lower bounds $\sqcap S$ for all subsets $S \subseteq P$, since we can take $\sqcap S$ to be

$$\bigsqcup \{ y \in P \mid \forall x \in S.\ y \sqsubseteq x \}.$$

We call $(P, \sqsubseteq)$ a complete lattice in this case.

Detour: Complete lattices

Knaster-Tarski Fixed Point Theorem

If $f : P \to P$ is a monotone function on a complete lattice, then it has a least (pre-) fixed point.

Proof Consider $fix(f) \triangleq \sqcap S$ where $S \triangleq \{x \in P \mid f(x) \sqsubseteq x\}$. Then $f(x) \sqsubseteq x$ implies $x \in S$ so $fix(f) = \sqcap S \sqsubseteq x$ & hence $f(fix(f)) \sqsubseteq f(x) \sqsubseteq x$. Thus $f(fix(f))$ is a lower bound for S & hence $f(fix(f)) \sqsubseteq \sqcap S = fix(f)$. So $fix(f) \in S$ — i.e. $fix(f)$ is a pre-fixed point of f; by construction, it's the least such. $\square$

Construction of $\lhd$ satisfying (*)

Call a relation $R \subseteq D \times \Lambda_0$ **admissible** if it satisfies for all $e \in \Lambda_0$

- $\perp \, R \, e$
- for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in D, if $d_i \, R \, e$ for all $i = 0, 1, 2, \ldots$, then $(\bigsqcup_i d_i) \, R \, e$.

Note ~~that~~

$$\mathcal{R} \triangleq \{ R \subseteq D \times \Lambda_0 \mid R \text{ admissible} \}$$

is closed under arbitrary intersections, hence $(\mathcal{R}, \subseteq)$ is a complete lattice.

So $\quad \mathcal{R}^{op} \times \mathcal{R} \triangleq \{(R', R) \mid R' \,\&\, R \text{ admissible}\}$

(partial order $(R', R) \leqslant (S', S) \equiv S' \subseteq R' \,\&\, R \subseteq S$)

is also a complete lattice. Consider:

$$\Phi : \mathcal{R}^{op} \times \mathcal{R} \longrightarrow \mathcal{R}$$

defined by

$$\Phi(R', R) \equiv \{(d, e) \mid$$
$$d = \perp \;\vee\; \exists f, x, e_1 .$$
$$d = fun(f) \;\&\; e \Rightarrow \lambda x . e_1 \;\&$$
$$\forall (d', e') \in R' .\; (f(d'), e_1[e'/x]) \in R \;\}$$

Note that Φ is monotone & hence so is

$$\Phi^{\S} : \mathcal{R}^{op} \times \mathcal{R} \to \mathcal{R}^{op} \times \mathcal{R}$$
$$(R', R) \mapsto (\Phi(R, R'), \Phi(R', R))$$

By the Knaster-Tarski Fixed Point Theorem, $\Phi^{\S}$ has a least fixed point, $(\lhd', \lhd)$ say.

$$\boxed{\underline{\text{Claim}}: \lhd' = \lhd}$$

If so, then $\lhd = \Phi(\lhd, \lhd)$ which is exactly property $(*)$ for $\lhd$, as required. $\square$

Proof of claim

Since $(\lhd', \lhd)$ is a fixed point for $\Phi^{\S}$ it satisfies

$$\Phi(\lhd', \lhd) = \lhd \;\&\; \Phi(\lhd, \lhd') = \lhd'$$

Hence $(\lhd, \lhd')$ is also a fixed point for $\Phi^{\S}$.

Then since $(\lhd', \lhd)$ is the <u>least</u> fixed point

$$(\lhd', \lhd) \leqslant (\lhd, \lhd') \text{ in } \mathcal{R}^{op} \times \mathcal{R}$$

i.e. $\lhd \subseteq \lhd'$.

So we just have to show $\lhd' \subseteq \lhd$,

i.e. $\forall d, e.\; d \lhd' e \supset d \lhd e$

It's now we use the min. inv. property of D.

Proof of claim

Since $i : (D \to D)_\perp \cong D$ is a minimal invariant, we have $id_D = \bigsqcup_n \pi_n$ where $\pi_0 = \perp$ and

$$\pi_{n+1}(d) = \begin{cases} fun(\pi_n \circ f \circ \pi_n) & \text{if } d = fun(f) \\ \perp & \text{if } d = \perp \end{cases}$$

Proof of claim

Since $i : (D \to D)_\perp \cong D$ is a minimal invariant, we have $id_D = \bigsqcup_n \pi_n$ where $\pi_0 = \perp$ and

$$\pi_{n+1}(d) = \begin{cases} fun(\pi_n \circ f \circ \pi_n) & \text{if } d = fun(f) \\ \perp & \text{if } d = \perp \end{cases}$$

Then $\boxed{\forall n.\ \forall d, e.\ d \lhd' e \supset \pi_n(d) \lhd e}$

follows by induction on n, using $\begin{cases} \lhd = \Phi(\lhd', \lhd) \\ \lhd' = \Phi(\lhd, \lhd') \end{cases}$.

Hence $d \lhd' e \supset \forall n.\ \pi_n(d) \lhd e$

$\supset \left(\bigsqcup_n \pi_n(d)\right) \lhd e$ since $\lhd$ admissible

$\supset d \lhd e$ $\square$