A **locally continuous functor** $F : \mathbf{Dom}_\bot^{op} \times \mathbf{Dom}_\bot \to \mathbf{Dom}_\bot$

is given by

$$\boxed{\begin{array}{lll} \bullet \text{ domains } D, E & \longmapsto & \text{domain } F(D,E) \\ \bullet \text{ strict cts functions} & & \text{strict cts function} \\ \quad f \in D' \multimap D & \longmapsto & F(f,g) \in F(D,E) \multimap F(D',E') \\ \quad g \in E \multimap E' & & \end{array}}$$

satisfying

$$\left.\begin{array}{l} \bullet\ F(id, id) = id \\[1em] \bullet\ \begin{tikzcd} F(D,E) \arrow[r, "{F(f,g)}"] \arrow[rr, bend right, "{F(f \circ f', g' \circ g)}"'] & F(D',E') \arrow[r, "{F(f',g')}"] & F(D'',E'') \end{tikzcd} \end{array}\right\} \text{functoriality}$$

A **locally continuous functor** $F : \mathrm{Dom}_\perp^{op} \times \mathrm{Dom}_\perp \to \mathrm{Dom}_\perp$

is given by

- domains $D, E \longmapsto$ domain $F(D,E)$
- strict cts functions $f \in D' \multimap D$, $g \in E \multimap E'$ $\longmapsto$ strict cts function $F(f,g) \in F(D,E) \multimap F(D',E')$

satisfying

monotonicity

- $f \sqsubseteq f' \ \&\ g \sqsubseteq g' \supset F(f,g) \sqsubseteq F(f',g')$
- $F(\bigsqcup_n f_n, \bigsqcup_m g_m) = \bigsqcup_k F(f_k, g_k)$

continuity

Minimal invariants

An invariant for locally cts functor $F : \mathrm{Dom}_\perp^{op} \times \mathrm{Dom}_\perp \to \mathrm{Dom}_\perp$ is given by

$$\boxed{\text{domain } D + \text{isomorphism } i : F(D,D) \cong D}$$

(D, i) is a minimal invariant if

least fixed point of $(D \multimap D) \longrightarrow (D \multimap D)$
$e \mapsto i \circ F(e,e) \circ i^{-1}$

is the identity id_D.

Minimal invariants

An invariant for locally cts functor $F : \mathbf{Dom}_\perp^{op} \times \mathbf{Dom}_\perp \to \mathbf{Dom}_\perp$ is given by

$$\boxed{\text{domain } D + \text{isomorphism } i : F(D,D) \cong D}$$

(D, i) is a minimal invariant if $id_D = \bigsqcup_{n \geq 0} \pi_n$ in $D \multimap D$, where

$$\begin{cases} \pi_0 \triangleq \perp_{D \multimap D} = \lambda d \in D.\, \perp_D \\ \pi_{n+1} \triangleq i \circ F(\pi_n, \pi_n) \circ i^{-1} \end{cases}$$

Main Theorem

Every locally continuous $F : \mathrm{Dom}_\perp^{op} \times \mathrm{Dom}_\perp \to \mathrm{Dom}_\perp$ possesses a minimal invariant $i : F(D,D) \cong D$

Existence

and it is unique up to isomorphism : if $i' : F(D',D') \cong D'$ is another, then there is an isomorphism $\delta : D \cong D'$ such that

$$\begin{tikzcd} F(D,D) \arrow[r, "i"] \arrow[d, "F(\delta^{-1},\delta)"', "\cong"] & D \arrow[d, "\delta", "\cong"'] \\ F(D',D') \arrow[r, "i'"'] & D' \end{tikzcd}$$

commutes.

Uniqueness

Uniqueness

Given two min. invariants $\begin{cases} i : F(D, D) \cong D \\ i' : F(D', D') \cong D' \end{cases}$

consider

$$\begin{tikzcd}
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"] \arrow[d, "\Phi"'] & (D' \multimap D') \times (D \multimap D) \arrow[d, "\Psi"] \\
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"'] & (D' \multimap D') \times (D \multimap D)
\end{tikzcd}$$

where $\begin{cases} s(\delta', \delta) \triangleq (\delta \circ \delta', \delta' \circ \delta) \\ \Phi(\delta', \delta) \triangleq (i \circ F(\delta, \delta') \circ i'^{-1}, i' \circ F(\delta', \delta) \circ i^{-1}) \\ \Psi(e', e) \triangleq (i' \circ F(e', e') \circ i'^{-1}, i \circ F(e, e) \circ i^{-1}) \end{cases}$

$$\boxed{\begin{tikzcd}
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"] \arrow[d, "\Phi"'] & (D' \multimap D') \times (D \multimap D) \arrow[d, "\Psi"] \\
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"'] & (D' \multimap D') \times (D \multimap D)
\end{tikzcd}}$$

Since $\Psi \circ s = s \circ \Phi$ & s is strict, by Plotkin's Uniformity Principle $(\delta', \delta) \triangleq \mathrm{fix}(\Phi)$ satisfies

$$(\delta \circ \delta', \delta' \circ \delta) = s(\delta', \delta) = s(\mathrm{fix}\,\Phi) = \mathrm{fix}(\Psi).$$

$$
\begin{tikzcd}
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"] \arrow[d, "\Phi"'] & (D' \multimap D') \times (D \multimap D) \arrow[d, "\Psi"] \\
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"'] & (D' \multimap D') \times (D \multimap D)
\end{tikzcd}
$$

Since $\Psi \circ s = s \circ \Phi$ & s is strict, by Plotkin's Uniformity Principle $(\delta', \delta) \triangleq \mathit{fix}(\Phi)$ satisfies

$$(\delta \circ \delta', \delta' \circ \delta) = s(\delta', \delta) = s(\mathit{fix}\,\Phi) = \mathit{fix}(\Psi).$$

But $\mathit{fix}(\Psi) = (\mathit{fix}(\lambda e'.\, i' \circ F(e', e') \circ i'^{-1}), \mathit{fix}(\lambda e.\, i \circ F(e,e) \circ i^{-1}))$

Exercise: prove $\mathit{fix}(f' \times f) = (\mathit{fix}(f'), \mathit{fix}(f))$ for any $f' \in D \to D$ & $f \in D \to D$

$$
\begin{tikzcd}
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"] \arrow[d, "\Phi"'] & (D' \multimap D') \times (D \multimap D) \arrow[d, "\Psi"] \\
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"'] & (D' \multimap D') \times (D \multimap D)
\end{tikzcd}
$$

Since $\Psi \circ s = s \circ \Phi$ & s is strict, by Plotkin's Uniformity Principle $(\delta', \delta) \triangleq fix(\Phi)$ satisfies

$$(\delta \circ \delta', \delta' \circ \delta) = s(\delta', \delta) = s(fix\, \Phi) = fix(\Psi).$$

But $fix(\Psi) = (fix(\lambda e'.\, i' \circ F(e', e') \circ i'^{-1}), fix(\lambda e.\, i \circ F(e, e) \circ i^{-1}))$

$= (id_{D'}, id_D)$ by min. inv. property of D' & D

$$\boxed{\begin{tikzcd}
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"] \arrow[d, "\Phi"'] & (D' \multimap D') \times (D \multimap D) \arrow[d, "\Psi"] \\
(D' \multimap D) \times (D \multimap D') \arrow[r, "s"'] & (D' \multimap D') \times (D \multimap D)
\end{tikzcd}}$$

Since $\Psi \circ s = s \circ \Phi$ & s is strict, by Plotkin's Uniformity Principle $(\delta', \delta) \triangleq fix(\Phi)$ satisfies

$$(\delta \circ \delta', \delta' \circ \delta) = s(\delta', \delta) = s(fix\, \Phi) = fix(\Psi).$$

But $fix(\Psi) = (fix(\lambda e'. i' \circ F(e', e') \circ i'^{-1}), fix(\lambda e. i \circ F(e,e) \circ i^{-1}))$

$= (id_{D'}, id_D)$ by min. inv. property of D' & D

So $\delta \circ \delta' = id_{D'}$ & $\delta' \circ \delta = id_D$

i.e. $\delta : D \to D'$ is an iso (with inverse δ').

$$
\begin{aligned}
(\delta^{-1}, \delta) &= (\delta', \delta) && \text{from above} \\
&= \mathrm{fix}(\Phi) && \text{by definition of } \delta' \,\&\, \delta \\
&= \Phi(\mathrm{fix}(\Phi)) && \text{fixed point!}
\end{aligned}
$$

$$(\delta^{-1}; \delta) = (\delta'; \delta) \quad \text{(from above)}$$

$$\overset{(1)}{=} \mathrm{fix}(\Phi) \quad \text{(by definition of } \delta' \text{ \& } \delta\text{)}$$

$$= \Phi(\mathrm{fix}(\Phi)) \quad \text{fixed point!}$$

$$= \Phi(\delta^{-1}; \delta) \quad \text{by (1)}$$

$$(\delta^{-1}, \delta) = (\delta', \delta) \quad \text{(from above)}$$

$$\stackrel{(1)}{=} \mathrm{fix}(\Phi) \quad \text{(by definition of } \delta' \text{ \& } \delta\text{)}$$

$$= \Phi(\mathrm{fix}(\Phi)) \quad \text{fixed point!}$$

$$= \Phi(\delta^{-1}, \delta) \quad \text{by (1)}$$

$$= (\dots, i' \circ F(\delta^{-1}, \delta) \circ i^{-1}) \quad \text{by def}^n \text{ of } \Phi$$

so $\delta = i' \circ F(\delta^{-1}, \delta) \circ i^{-1}$, hence

$$\begin{tikzcd}
F(D, D) \arrow[r, "i"] \arrow[d, "F(\delta^{-1}, \delta)"', "\cong"] & D \arrow[d, "\cong"', "\delta"] \\
F(D', D') \arrow[r, "i'"'] & D'
\end{tikzcd}$$

as required for uniqueness. $\square$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

countable product of domains F_n defined by

$$\begin{cases} F_0 = \{\perp\} \\ F_{n+1} = F(F_n, F_n) \end{cases}$$

Elements of $\prod_{n<\omega} F_n$ are tuples $d = (d_n \mid n<\omega)$ of elements $d_n \in F_n$.

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \{ d \in \prod_{n<\omega} F_n \mid \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \}}$$

strict continuous functions $\varphi_{m,n} \in F_m \multimap F_n$

defined by:
$$\begin{cases} \varphi_{0,n} \triangleq \bot \\ \varphi_{m,0} \triangleq \bot \\ \varphi_{m+1,n+1} \triangleq F(\varphi_{n,m}, \varphi_{m,n}) \end{cases}$$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

D is a domain because it is a subset of $\prod_{n<\omega} F_n$

which $\begin{cases} \text{is closed under lubs of chains} \\ \text{contains the least element.} \end{cases}$ exercise

- $\bot_D = (\bot_{F_n} \mid n<\omega)$
- $d \sqsubseteq d'$ in D iff $d_n \sqsubseteq d'_n$ in F_n for all $n<\omega$
- $\bigsqcup_{k<\omega} d_k$ in D is $\left(\bigsqcup_{k<\omega} (d_k)_n \mid n<\omega \right)$

Lemmas about $\varphi_{m,n} \in F_m \to F_n$

- $$\varphi_{m,m} = \mathrm{id}_{F_m}$$
- $$\varphi_{k,n} \circ \varphi_{m,k} \subseteq \varphi_{m,n}$$
- $$\varphi_{k,n} \circ \varphi_{m,k} = \varphi_{m,n} \quad \text{if } k > \min\{m,n\}$$

(Exercise: prove these by induction over $\mathbb{N}$.)

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

$e_m \in D$ because of $\boxed{\forall k,m,n.\ \varphi_{k,n} \circ \varphi_{m,k} \sqsubseteq \varphi_{m,n}}$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

These satisfy:

(EP1) $p_n \circ e_m = \varphi_{m,n}$

(EP2) $p_n \circ e_n = id_{F_n}$

(EP3) $\bigsqcup_{n<\omega} e_n \circ p_n = id_D$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

These satisfy:

(EP1) $p_n \circ e_m = \varphi_{m,n}$ ← follows directly from def^n of p_n & e_n

(EP2) $p_n \circ e_n = id_{F_n}$

(EP3) $\bigsqcup_{n<\omega} e_n \circ p_n = id_D$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

These satisfy:

(EP1) $p_n \circ e_m = \varphi_{m,n}$

(EP2) $p_n \circ e_n = id_{F_n}$ since $\boxed{\varphi_{n,n} = id_{F_n}}$

(EP3) $\bigsqcup_{n<\omega} e_n \circ p_n = id_D$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \{ d \in \prod_{n<\omega} F_n \mid \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

These satisfy:

(EP1) $p_n \circ e_m = \varphi_{m,n}$

(EP2) $p_n \circ e_n = id_{F_n}$

(EP3) $\bigsqcup_{n<\omega} e_n \circ p_n = id_D$

use def^n of D plus

$$\boxed{\begin{array}{l} \forall k > \min\{m,n\}. \\ \varphi_{k,n} \circ \varphi_{m,k} = \varphi_{m,n} \end{array}}$$

to see that $e_0 p_0 \sqsubseteq e_1 p_1 \sqsubseteq \cdots$ & that lub is id_D

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

Then $\begin{cases} F(D,D) \xrightarrow{F(e_n,p_n)} F(F_n,F_n) = F_{n+1} \xrightarrow{e_{n+1}} D \\ D \xrightarrow{p_{n+1}} F_{n+1} = F(F_n,F_n) \xrightarrow{F(p_n,e_n)} F(D,D) \end{cases}$

satisfy $\begin{cases} \forall n.\ e_{n+1} \circ F(e_n,p_n) \sqsubseteq e_{n+2} \circ F(e_{n+1},p_{n+1}) \\ \forall n.\ F(p_n,e_n) \circ p_{n+1} \sqsubseteq F(p_{n+1},e_{n+1}) \circ p_{n+2} \end{cases}$

Existence: construction of min. inv. for F

$$\boxed{D \triangleq \left\{ d \in \prod_{n<\omega} F_n \;\middle|\; \forall m,n<\omega.\ \varphi_{m,n}(d_m) \sqsubseteq d_n \right\}}$$

Define $\begin{cases} p_n \in D \multimap F_n \\ e_m \in F_m \multimap D \end{cases}$ by $\begin{cases} p_n(d) \triangleq d_n \\ e_m(x) \triangleq (\varphi_{m,n}(x) \mid n<\omega) \end{cases}$

and then

$$\begin{cases} i \triangleq \bigsqcup_{n<\omega} e_{n+1} \circ F(e_n, p_n) \in F(D,D) \multimap D \\ i' \triangleq \bigsqcup_{n<\omega} F(p_n, e_n) \circ p_{n+1} \in D \multimap F(D,D) \end{cases}$$

Some Lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

Proof: $F(e_n, p_n) F(p_m, e_m) = F(p_m e_n, p_n e_m)$

$= F(\varphi_{n,m}, \varphi_{m,n})$ by def.ⁿ of p & e

$= \varphi_{m+1, n+1}$ by def.ⁿ of $\varphi_{-,-}$

$= p_{n+1} e_{m+1}$ by def.ⁿ of p & e

$\square$

Some lemmas

$$(*)\quad \boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1}}$$

<u>Proof:</u>

$$
\begin{aligned}
i \circ F(p_n, e_n) &\triangleq \left(\bigsqcup_k e_{k+1} \circ F(e_k, p_k) \right) \circ F(p_n, e_n) \\
&= \bigsqcup_k e_{k+1} F(e_k, p_k) F(p_n, e_n) \\
&= \bigsqcup_k e_{k+1} \, p_{k+1} \, e_{n+1} && \text{by } (*) \\
&= \left(\bigsqcup_k e_{k+1} p_{k+1} \right) \circ e_{n+1} \\
&= e_{n+1} && \text{by (EP3)}
\end{aligned}
$$

$\square$

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1} \;\&\; F(e_n, p_n) \circ i' = p_{n+1}}$$

proved similarly to this

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

(*) $$\boxed{i \circ F(p_n, e_n) = e_{n+1} \;\&\; F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D}$$

Proof: $ii' \triangleq i\left(\bigsqcup_m F(p_m, e_m)\, p_{m+1}\right)$

$= \bigsqcup_m e_{m+1}\, p_{m+1}$ by (*)

$= id$ by (EP3)

$\square$

Some lemmas

$$(*)\quad \boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1} \ \& \ F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D \quad \& \quad i' \circ i = id_{F(D,D)}}$$

Proof: $i'i \triangleq \left(\bigsqcup_m F(p_m, e_m) p_{m+1}\right)\left(\bigsqcup_n e_{n+1} F(e_n, p_n)\right)$

$$= \bigsqcup_k F(p_k, e_k) p_{k+1} e_{k+1} F(e_k, p_k)$$

$$= \bigsqcup_k F(p_k, e_k) F(e_k, p_k) F(p_k, e_k) F(e_k, p_k) \quad \text{by } (*)$$

$$= \bigsqcup_k F(e_k p_k, e_k p_k) F(e_k p_k, e_k p_k)$$

$$= F(id, id) F(id, id) \quad \text{by (EP3)}$$

$$= id$$

$\square$

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1} \ \& \ F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D \quad \& \quad i' \circ i = id_{F(D,D)}}$$

So $i : F(D,D) \to D$ is an iso with $i^{-1} = i'$ and we just need to prove the min. inv. property $id_D = \bigsqcup_n \pi_n$ where

$$\begin{cases} \pi_0 \triangleq \bot \\ \pi_{n+1} \triangleq i \circ F(\pi_n, \pi_n) \circ i^{-1} \end{cases}$$

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1} \ \& \ F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D \quad \& \quad i' \circ i = id_{F(D,D)}}$$

$$\boxed{e_n \circ p_n = \pi_n}$$

<u>Proof</u>: Since $F_0 = \{\bot\}$, $e_0 = \bot$ & $p_0 = \bot$, so $e_0 p_0 = \bot = \pi_0$.

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

(*) $$\boxed{i \circ F(p_n, e_n) = e_{n+1} \ \& \ F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D \quad \& \quad i' \circ i = id_{F(D,D)}}$$

$$\boxed{e_n \circ p_n = \pi_n}$$

Proof: Since $F_0 = \{\perp\}$, $e_0 = \perp$ & $p_0 = \perp$, so $e_0 p_0 = \perp = \pi_0$.

And if $e_n p_n = \pi_n$, then

$$e_{n+1} p_{n+1} = i F(p_n, e_n) F(e_n p_n) i' \quad \text{by } (*)$$

$$= i F(e_n p_n, e_n p_n) i'$$

$$= i F(\pi_n, \pi_n) i' \quad \text{by ind. hyp.}$$

$$= \pi_{n+1} \quad \text{since } i' = i^{-1}$$

$\square$

Some lemmas

$$\boxed{F(e_n, p_n) \circ F(p_m, e_m) = p_{n+1} \circ e_{m+1}}$$

$$\boxed{i \circ F(p_n, e_n) = e_{n+1} \;\&\; F(e_n, p_n) \circ i' = p_{n+1}}$$

$$\boxed{i \circ i' = id_D \quad \& \quad i' \circ i = id_{F(D,D)}}$$

$$\boxed{e_n \circ p_n = \pi_n}$$

and hence

$$\bigsqcup_n \pi_n = \bigsqcup_n e_n p_n = id_D \text{ by (EP3).}$$

So (D, i) is a min. inv. for F.

$\square$