Topics in Logic and Complexity
Handout 7

Anuj Dawar

MPhil Advanced Computer Science, Lent 2010

Connectivity

Consider the signature $(E, <)$.

Consider structures $G = (V, E, <)$ in which E is a graph relation and $<$ is a linear order.

There is no first order sentence γ in this signature such that

$$G \models \gamma \text{ if, and only if, } (V, E) \text{ is connected.}$$

Proof

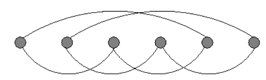
Suppose there was such a formula γ.

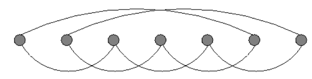
Let γ' be the formula obtained by replacing every occurrence of $E(x, y)$ in γ by the following formula

$$\begin{aligned} & y = x + 2 \vee \\ & (x = \max \wedge y = \min + 1) \vee \\ & (y = \min \wedge x = \max - 1). \end{aligned}$$

Then, $\neg\gamma'$ defines evenness on linear orders!

Proof





We obtain two *disjoint* cycles on linear orders of even length, and a *single* cycle on linear orders of odd length.

Reduction

The above is, in fact, a *first-order definable reduction* from the problem of evenness of linear orders to the problem of connectivity of ordered graphs.

It follows from the above that there is no first order formula that can express the *transitive closure* query on graphs.

Any such formula would also work on ordered *graphs.*

Gaifman Graphs and Neighbourhoods

On a structure $\mathbb{A}$, define the binary relation:

$E(a_1, a_2)$ if, and only if, there is some relation R and some tuple $\mathbf{a}$ containing both a_1 and a_2 with $R(\mathbf{a})$.

The graph $G\mathbb{A} = (A, E)$ is called the *Gaifman graph* of $\mathbb{A}$.

$dist(a, b)$ — the distance between a and b in the graph (A, E).

$\mathrm{Nbd}_r^{\mathbb{A}}(a)$ — the substructure of $\mathbb{A}$ given by the set:

$$\{b \mid dist(a, b) \leq r\}$$

Hanf Locality Theorem

We say $\mathbb{A}$ and $\mathbb{B}$ are *Hanf equivalent* with radius r ($\mathbb{A} \simeq_r \mathbb{B}$) if, for every $a \in A$ the two sets

$$\{a' \in a \mid \mathrm{Nbd}_r^{\mathbb{A}}(a) \cong \mathrm{Nbd}_r^{\mathbb{A}}(a')\} \quad \text{and} \quad \{b \in B \mid \mathrm{Nbd}_r^{\mathbb{A}}(a) \cong \mathrm{Nbd}_r^{\mathbb{B}}(b)\}$$

have the same cardinality

and, similarly for every $b \in B$.

Theorem (Hanf)
For every vocabulary σ and every p there is $r \leq 3^p$ such that for any σ-structures $\mathbb{A}$ and $\mathbb{B}$: if $\mathbb{A} \simeq_r \mathbb{B}$ then $\mathbb{A} \equiv_p \mathbb{B}$.

In other words, if $r \geq 3^p$, the equivalence relation $\simeq_r$ is a refinement of $\equiv_p$.

Hanf Locality

Duplicator's strategy is to maintain the following condition:

After k moves, if $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ have been selected, then

$$\bigcup_i \mathrm{Nbd}_{3^{p-k}}^{\mathbb{A}}(a_i) \cong \bigcup_i \mathrm{Nbd}_{3^{p-k}}^{\mathbb{B}}(b_i)$$

If *Spoiler* plays on a within distance $2 \cdot 3^{p-k-1}$ of a previously chosen point, play according to the isomorphism, otherwise, find b such that

$$\mathrm{Nbd}_{3^{p-k-1}}(a) \cong \mathrm{Nbd}_{3^{p-k-1}}(b)$$

and b is not within distance $2 \cdot 3^{p-k-1}$ of a previously chosen point.

Such a b is guaranteed by $\simeq_r$.

Uses of Hanf locality

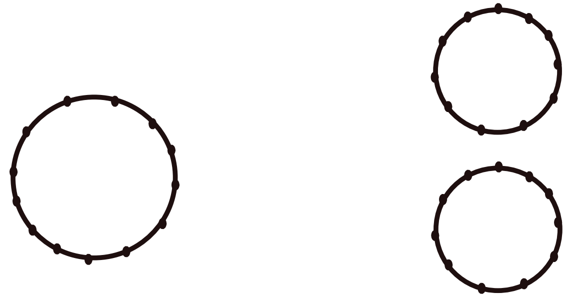
The Hanf locality theorem immediately yields, as special cases, the proofs of undefinability of:

- *connectivity*;
- *2-colourability*
- *acyclicity*
- *planarity*

A simple illustration can suffice.

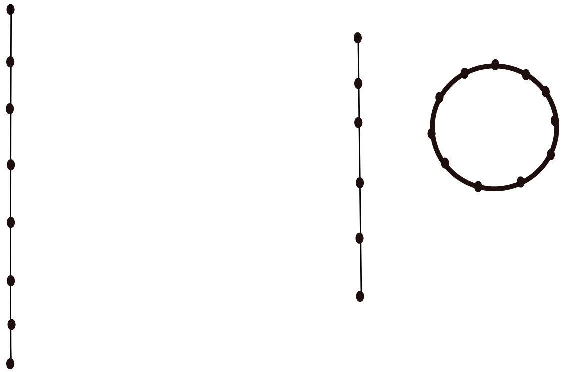
Connectivity

To illustrate the undefinability of *connectivity* and *2-colourability*, consider on the one hand the graph consisting of a single cycle of length $4r + 6$ and, on the other hand, a graph consisting of two disjoint cycles of length $2r + 3$.



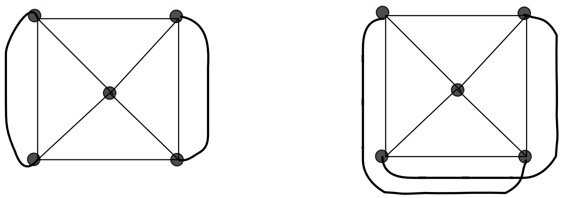
Acyclicity

A figure illustrating that *acyclicity* is not first-order definable.



Planarity

A figure illustrating that *planarity* is not first-order definable.



Monadic Second Order Logic

MSO consists of those second order formulas in which all relational variables are *unary*.

> *That is, we allow quantification over* sets *of elements, but not other relations.*

Any MSO formula can be put in prenex normal form with second-order quantifiers preceding first order ones.

Mon.Σ^1_1 — MSO formulas with only *existential* second-order quantifiers in prenex normal form.

Mon.Π^1_1 — MSO formulas with only *universal* second-order quantifiers in prenex normal form.

Undefinability in MSO

The method of games and *locality* can also be used to show *inexpressibility* results in MSO.

In particular,

There is a Mon.Σ^1_1 query that is not definable in Mon.Π^1_1

(Fagin 1974)

Note: A similar result without the *monadic* restriction would imply that NP $\neq$ co-NP and therefore that P $\neq$ NP.

Connectivity

Recall that *connectivity* of graphs can be defined by a Mon.Π^1_1 sentence.

$$\forall S(\exists x\, Sx \wedge (\forall x \forall y\, (Sx \wedge Exy) \rightarrow Sy)) \rightarrow \forall x\, Sx$$

and by a Σ^1_1 sentence (simply because it is in NP.

We now aim to show that *connectivity* is not definable by a Mon.Σ^1_1 sentence.

MSO Game

The m-round monadic Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- At the ith round, *Spoiler* chooses one of the structures (say $\mathbb{B}$) and plays either a point move or a set move.

  In a point move, he chooses one of the elements of the chosen structure (say b_i) – *Duplicator* must respond with an element of the other structure (say a_i).

  In a set move, he chooses a subset of the universe of the chosen structure (say S_i) – *Duplicator* must respond with a subset of the other structure (say R_i).

MSO Game

- If, after m rounds, the map

$$a_i \mapsto b_i$$

is a partial isomorphism between

$$(\mathbb{A}, R_1, \ldots, R_q) \text{ and } (\mathbb{B}, S_1, \ldots, S_q)$$

then *Duplicator* has won the game, otherwise *Spoiler* has won.

MSO Game

If we define the *quantifier rank* of an MSO formula by adding the following inductive rule to those for a formula of FO:

if $\phi = \exists S \psi$ or $\phi = \forall S \psi$ then $\mathrm{qr}(\phi) = \mathrm{qr}(\psi) + 1$

then, we have

Duplicator has a winning strategy in the m-round monadic Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ if, and only if, for every sentence ϕ of MSO with $\mathrm{qr}(\phi) \leq m$

$$\mathbb{A} \models \phi \quad \text{if, and only if,} \quad \mathbb{B} \models \phi$$

Existential Game

The m, p-move existential game on ($\mathbb{A}$,$\mathbb{B}$):

- First *Spoiler* makes m set moves on $\mathbb{A}$, and *Duplicator* replies on $\mathbb{B}$.
- This is followed by an Ehrenfeucht game with p point moves.

If *Duplicator* has a winning strategy, then for every Mon.Σ^1_1 sentence:

$$\phi \equiv \exists R_1 \ldots \exists R_m \, \psi$$

with $\mathrm{qr}(\psi) = p$,

$$\text{if } \mathbb{A} \models \phi \text{ then } \mathbb{B} \models \phi$$

Variation

To show that a Boolean query Q is not Mon.Σ^1_1 definable, find for each m and p

- $\mathbb{A} \in Q$; and
- $\mathbb{B} \not\in P$; such that
- *Duplicator* wins the m, p move game on $(\mathbb{A}, \mathbb{B})$.

Or,

- *Duplicator* chooses $\mathbb{A}$.
- *Spoiler* colours $\mathbb{A}$ (with 2^m colours).
- *Duplicator* chooses $\mathbb{B}$ and colours it.
- They play a p-round Ehrenfeucht game.

Application

Write C_n for the graph that is a simple *cycle* of length n.

For n sufficiently large, and any *colouring* of C_n, we can find an $n' < n$ and a colouring of

> $C_{n'} \oplus C_{n-n'}$ the disjoint union of two cycles—one of length n', the other of length $n - n'$

So that the graphs C_n and $C_{n'} \oplus C_{n-n'}$ are $\simeq_r$ equivalent.

Taking $n > (2r+1)^{2^m+2}$ suffices.

Reading List for this Handout

1. Ebbinghaus and Flum. Section 2.4.
2. Libkin. Chapter 4.
3. Grädel et al. Section 2.3 and 2.5