Topics in Logic and Complexity
Handout 10

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Complexity of LFP

Any *query* definable in LFP is decidable by a *deterministic* machine in *polynomial time*.

To be precise, we can show, by induction on the structure of the formula $\phi(\mathbf{x})$ that for each formula ϕ there is a t such that

$$\mathbb{A} \models \phi[\mathbf{a}]$$

is decidable in time $O(n^t)$ where n is the number of elements of $\mathbb{A}$.

We prove this by induction on the structure of the formula.

Complexity of LFP

- Atomic formulas by direct lookup ($O(n^a)$ time, where a is the maximum arity of any predicate symbol in σ).
- Boolean connectives are easy.

  If $\mathbb{A} \models \phi_1$ can be decided in time $O(n^{t_1})$ and $\mathbb{A} \models \phi_2$ in time $O(n^{t_2})$, then $\mathbb{A} \models \phi_1 \wedge \phi_2$ can be decided in time $O(n^{\max(t_1,t_2)})$
- If $\phi \equiv \exists x\, \psi$ then for each $a \in \mathbb{A}$ check whether

$$(\mathbb{A}, c \mapsto a) \models \psi[c/x],$$

  where c is a new constant symbol. If $\mathbb{A} \models \psi$ can be decided in time $O(n^t)$, then $\mathbb{A} \models \phi$ can be decided in time $O(n^{t+1})$.

Complexity of LFP

Suppose $\phi \equiv \mathbf{lfp}_{R,\mathbf{x}}\psi(\mathbf{t})$ (R is l-ary)

To decide $\mathbb{A} \models \phi[\mathbf{a}]$:

$R := \emptyset$
for $i := 1$ **to** n^l **do**
  $R := R \cup F_\psi(R)$
end
if $\mathbf{a} \in R$ **then** accept **else** reject

Complexity of LFP

To compute $F_\psi(R)$

For every tuple $\mathbf{a} \in A^l$, determine whether $(\mathbb{A}, R) \models \psi[\mathbf{a}]$.

If deciding $(\mathbb{A}, R) \models \psi$ takes time $O(n^t)$, then each assignment to R inside the loop requires time $O(n^{l+t})$. The total time taken to execute the loop is then $O(n^{2l+t})$. Finally, the last line can be done by a search through R in time $O(n^l)$. The total running time is, therefore, $O(n^{2l+t})$.

The *space* required is $O(n^l)$.

Capturing P

For any ϕ of LFP, the language $\{[\mathbb{A}]_< \mid \mathbb{A} \models \phi\}$ is in P.

Suppose ρ is a signature that contains a *binary relation symbol* $<$, possibly along with other symbols.

Let $\mathcal{O}_\rho$ denote those structures $\mathbb{A}$ in which $<$ is a *linear order* of the universe.

For any language $L \in$ P, there is a sentence ϕ of LFP that defines the class of structures

$$\{\mathbb{A} \in \mathcal{O}_\rho \mid [\mathbb{A}]_{<^{\mathbb{A}}} \in L\}$$

(Immerman; Vardi 1982)

Capturing P

Recall the proof of *Fagin's Theorem*, that ESO captures NP.

Given a machine M and an integer k, there is a *first-order* formula $\phi_{M,k}$ such that

$$\mathbb{A} \models \exists < \exists T_{\sigma_1} \cdots T_{\sigma_s} \exists S_{q_1} \cdots S_{q_m} \exists H\ \phi_{M,k}$$

if, and only if, M accepts $[\mathbb{A}]_<$ in time n^k, for some order $<$.

If we *fix* the order $<$ as part of the structure $\mathbb{A}$, we do not need the outermost quantifier.

Moreover, for a *deterministic* machine M, the relations $T_{\sigma_1} \ldots T_{\sigma_s}, S_{q_1} \ldots S_{q_m}, H$ can be defined *inductively*.

Capturing P

$$\begin{aligned}
&T_a(\mathbf{x}, \mathbf{y}) \Leftrightarrow \\
&(\mathbf{x} = \mathbf{1} \land \mathrm{Init}_a(\mathbf{y})) \lor \\
&\ \exists \mathbf{t} \exists \mathbf{h} \bigvee\nolimits_q \quad (\mathbf{x} = \mathbf{t} + 1 \land S_q(\mathbf{t}, \mathbf{h}) \land \\
&\qquad\qquad [(\mathbf{h} = \mathbf{y} \land \bigvee\nolimits_{\{b,d,q' \mid \Delta(q,b,q',a,d)\}} T_b(\mathbf{t}, \mathbf{y}) \lor \\
&\qquad\qquad \mathbf{h} \neq \mathbf{y} \land T_a(\mathbf{t}, \mathbf{y})]);
\end{aligned}$$

where $\mathrm{Init}_a(\mathbf{y})$ is the formula that defines the positions in which the symbol a appears in the input.

Capturing P

$S_q(\mathbf{x},\mathbf{y}) \Leftrightarrow$
$(\mathbf{x} = \mathbf{1} \wedge \mathbf{y} = \mathbf{1} \wedge q = q_0) \vee$
$\exists \mathbf{t} \exists \mathbf{h} \quad \bigvee_{\{a,b,q' \mid \Delta(q',a,q,b,R)\}} \big(\mathbf{x} = \mathbf{t}+1 \wedge S_{q'}(\mathbf{t},\mathbf{h}) \wedge T_a(\mathbf{t},\mathbf{h}) \wedge \mathbf{y} = \mathbf{h}+1)\big)$
$\bigvee_{\{a,b,q' \mid \Delta(q',a,q,b,L)\}} \big(\mathbf{x} = \mathbf{t}+1 \wedge S'_q(\mathbf{t},\mathbf{h}) \wedge T_a(\mathbf{t},\mathbf{h}) \wedge \mathbf{h} = \mathbf{y}+1)\big).$

Unordered Structures

In the absence of an *order relation*, there are properties in P that are not definable in LFP.

There is no sentence of LFP which defines the structures with an *even* number of elements.

Evenness

Let $\mathcal{E}$ be the collection of all structures in the empty signature.

In order to prove that *evenness* is not defined by any LFP sentence, we show the following.

Lemma
For every LFP formula ϕ there is a first order formula ψ, such that for all structures $\mathbb{A}$ in $\mathcal{E}$, $\mathbb{A} \models (\phi \leftrightarrow \psi)$.

Unordered Structures

Let $\psi(\mathbf{x},\mathbf{y})$ be a first order formula.

$\mathbf{lfp}_{R,\mathbf{x}}\psi$ defines the relation

$$F^{\infty}_{\psi,\mathbf{b}} = \bigcup_{i \in \mathbb{N}} F^{i}_{\psi,\mathbf{b}}$$

for a fixed interpretation of the variables $\mathbf{y}$ by the tuple of parameters $\mathbf{b}$.

For each i, there is a first order formula ψ^i such that on any structure $\mathbb{A}$,

$$F^{i}_{\psi,\mathbf{b}} = \{\mathbf{a} \mid \mathbb{A} \models \psi^i[\mathbf{a},\mathbf{b}]\}.$$

Defining the Stages

These formulas are obtained by *induction*.

ψ^1 is obtained from ψ by replacing all occurrences of subformulas of the form $R(\mathbf{t})$ by $t \neq t$.

ψ^{i+1} is obtained by replacing in ψ, all subformulas of the form $R(\mathbf{t})$ by $\psi^i(\mathbf{t}, \mathbf{y})$

Let $\mathbf{b}$ be an l-tuple, and $\mathbf{a}$ and $\mathbf{c}$ two k-tuples in a structure $\mathbb{A}$ such that

there is an automorphism $\imath$ of $\mathbb{A}$ (i.e. an *isomorphism* from $\mathbb{A}$ to itself) such that

- $\imath(\mathbf{b}) = \mathbf{b}$
- $\imath(\mathbf{a}) = \mathbf{c}$

Then,

$$\mathbf{a} \in F^i_{\psi,\mathbf{b}} \quad \text{if, and only if,} \quad \mathbf{c} \in F^i_{\psi,\mathbf{b}}.$$

Bounding the Induction

This defines an *equivalence relation* $\mathbf{a} \sim_{\mathbf{b}} \mathbf{c}$.

If there are p distinct equivalence classes, then

$$F^\infty_{\psi,\mathbf{b}} = F^p_{\psi,\mathbf{b}}$$

In $\mathcal{E}$ there is a uniform bound p, that does not depend on the size of the structure.

Reading List for this Handout

1. Libkin. Chapter 10.
2. Grädel et al. Section 3.3.