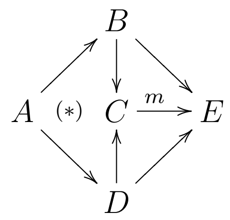
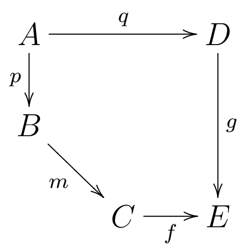
Category Theory for Computer Science
Mid-Term Problems

November 2009

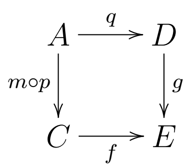
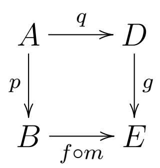
Problem 1.
(a) Consider the commutative diagram



where m is a mono. Prove that if $(*)$ is a pullback then so is the outer square.
(b) Consider the commutative diagram

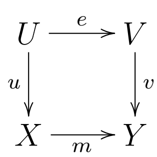


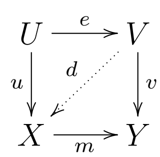
where m is a mono.

Prove that if  is a pullback, then so is 

Problem 2.

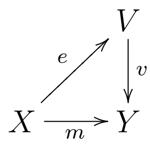
- A mono $m : X \to Y$ is *regular* if there exists a diagram $Y \overset{f}{\underset{g}{\rightrightarrows}} Z$ for which m is an equalizer.
- A mono $m : X \to Y$ is *strong*, if for every commutative diagram as on the left below where $e : U \to V$ is epi





there exists exactly one arrow $d : V \to X$ as on the right above such that both triangles commute.

- A mono $m : X \to Y$ is *extremal* if for every commutative diagram



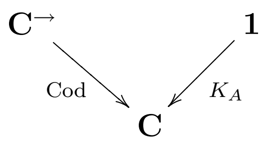
where $e : X \to V$ is epi, e is an isomorphism.

Prove that the following implications between properties of monomorphisms hold in any category:

$$\text{section} \Longrightarrow \text{regular} \Longrightarrow \text{strong} \Longrightarrow \text{extremal} \ .$$

(None of these is in general an equivalence, but that is another story.)

Problem 3. Given an object A in a small category $\mathbf{C}$, consider the following diagram in the category $\mathbf{Cat}$ of small categories and functors:



where:

- $\mathbf{C}^{\to}$ is the arrow category of $\mathbf{C}$ and Cod is the *codomain functor*, defined by:
  - $\mathrm{Cod}(f) = \mathrm{cod}(f)$ on objects in $\mathbf{C}^{\to}$,
  - $\mathrm{Cod}(h, k) = k$ on arrows in $\mathbf{C}^{\to}$.
- $\mathbf{1}$ is the category with one object $*$ and one arrow 1_*, and K_A is the constant functor defined by:
  - $K_A(*) = A$,
  - $K_A(1_*) = 1_A$.

Prove that the slice category $\mathbf{C}/A$, together with suitable functors to $\mathbf{C}^{\to}$ and $\mathbf{1}$, is a pullback of the above diagram.

Problem 4. Fix a set X with $* \notin X$.

For $n \in \mathbb{N}$, define

$$X_n \overset{\text{def}}{=} \left\{ (x_1, \ldots, x_n) \in \big(X \cup \{*\}\big)^n \;\middle|\; x_i = * \implies x_{i+1} = * \text{ for all } i = 1, \ldots, n-1 \right\} .$$

(a) Consider the following infinite diagram in the category **Sets** of sets and functions:

$$X_1 \xrightarrow{e_1} X_2 \xrightarrow{e_2} \cdots \longrightarrow X_n \xrightarrow{e_n} X_{n+1} \longrightarrow \cdots \qquad (n \in \mathbb{N}) \tag{1}$$

where $e_n : X_n \to X_{n+1}$ is the function given by

$$e_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, *) .$$

Give a simple description of a colimit of (1), and prove its universal property.

(b) Consider the following infinite diagram in the category **Sets** of sets and functions:

$$X_1 \xleftarrow[p_1]{} X_2 \xleftarrow[p_2]{} \cdots \longleftarrow X_n \xleftarrow[p_n]{} X_{n+1} \longleftarrow \cdots \qquad (n \in \mathbb{N}) \tag{2}$$

where $p_n : X_{n+1} \to X_n$ is the function given by

$$p_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n) .$$

Give a simple description of a limit of (2), and prove its universal property.