From left to right adjoints

Let $F : \mathbf{C} \to \mathbf{D}$ be left adjoint to $G : \mathbf{D} \to \mathbf{C}$ with unit $\eta : \mathrm{Id}_{\mathbf{C}} \to GF$.

For any $D \in |\mathbf{D}|$, define $\epsilon_D : FGD \to D$ by $\epsilon_D = (1_{GD})^\sharp$

$$\begin{tikzcd}
 & \mathbf{C} & & \mathbf{D} \arrow[ll, "G"'] \\
GD \arrow[r, "\eta_{GD}"] \arrow[dr, "1_{GD}"'] & GFGD \arrow[d, "G\epsilon_D"] & & FGD \arrow[d, dotted, "\exists!"', "\epsilon_D = 1^\sharp_{GD}"] \\
 & GD & & D
\end{tikzcd}$$

Fact: $\epsilon : FG \to \mathrm{Id}_D$ is a natural transformation.

Fact: G is right adjoint to F with counit ϵ.

Moreover, $G\epsilon \circ \eta G = \mathrm{id}_G$ and $\epsilon F \circ F\eta = \mathrm{id}_F$.

Adjunctions

Defn: An adjunction between categories $\mathbf{C}$ and $\mathbf{D}$ consists of:

- functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$,
- natural transformations $\eta : \mathrm{Id}_{\mathbf{C}} \to GF$, $\epsilon : FG \to \mathrm{Id}_D$,

  (unit) (counit)

such that $G\epsilon \circ \eta G = \mathrm{id}_G$ and $\epsilon F \circ F\eta = \mathrm{id}_F$.

Such an adjunction is denoted $F \dashv G$.

Fact: In an adjunction $F \dashv G$, F is left adjoint to G and G is right adjoint to F.

Examples:

- An adjunction where η, ϵ are natural isomorphisms is an equivalence of categories.
- Adjunctions between posets are Galois connections.

Transposing along adjunctions

Fact: Any adjunction $F \dashv G$ yields a bijection:

$\hom_{\mathbf{C}}(C, GD) \cong \hom_{\mathbf{D}}(FC, D)$ for any $C \in |\mathbf{C}|, D \in |\mathbf{D}|$

Moreover, this bijection is natural in C and D:

$$\mathrm{Hom}_{\mathbf{C}}(-, G-) \cong \mathrm{Hom}_{\mathbf{D}}(F-, -) \qquad (\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Sets})$$

Fact: For any functors $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$, a natural bijection as above

induces an adjunction $F \dashv G$.

Equivalent defn: An adjunction is a pair of functors $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$

with a bijection $\hom_{\mathbf{C}}(C, GD) \cong \hom_{\mathbf{D}}(FC, D)$

natural in C and D.

right adjoint

left adjoint