Lecture 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \,.$$

NB. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x \not\Downarrow_{\tau\to\tau} \mathbf{fn}\, x : \tau.\, x$$

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   - Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$, $\mathbf{fn}\, x : \tau\, .\, M'$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall\, V\ ([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

Definition of $d \lhd_\gamma M \ \ (d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$

for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \big(n \in \mathbb{N} \ \Rightarrow \ M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0})\big)$$

$$b \lhd_{bool} M \quad \stackrel{\text{def}}{\Leftrightarrow} \quad (b = true \ \Rightarrow \ M \Downarrow_{bool} \mathbf{true})$$
$$\& \ (b = false \ \Rightarrow \ M \Downarrow_{bool} \mathbf{false})$$

Proof of: $[\![M]\!] \lhd_\gamma M$ implies adequacy

Case $\gamma = nat$.

$$[\![M]\!] = [\![V]\!]$$

$$\Longrightarrow \; [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] \qquad \text{for some } n \in \mathbb{N}$$

$$\Longrightarrow \; n = [\![M]\!] \lhd_\gamma M$$

$$\Longrightarrow \; M \Downarrow \mathbf{succ}^n(\mathbf{0}) \qquad \text{by definition of } \lhd_{nat}$$

Case $\gamma = bool$ is similar.

Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$\leadsto$ *logical* definition

Definition of

$$f \lhd_{\tau\to\tau'} M \quad \big(f \in (\llbracket\tau\rrbracket \to \llbracket\tau'\rrbracket), M \in \mathrm{PCF}_{\tau\to\tau'}\big)$$

$$f \lhd_{\tau\to\tau'} M$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \forall\, x \in \llbracket\tau\rrbracket, N \in \mathrm{PCF}_{\tau}$$

$$(x \lhd_{\tau} N \;\Rightarrow\; f(x) \lhd_{\tau'} M\, N)$$

Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fix}(M')$.

$\rightsquigarrow$ *admissibility* property

Admissibility property

Lemma. *For all types* τ *and* $M \in \mathrm{PCF}_\tau$*, the set*

$$\{\, d \in [\![\tau]\!] \mid d \lhd_\tau M \,\}$$

is an admissible subset of $[\![\tau]\!]$*.*

Further properties

Lemma. *For all types* τ*, elements* $d, d' \in [\![\tau]\!]$*, and terms* $M, N, V \in \mathrm{PCF}_\tau$*,*

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$*.*
2. *If* $d \lhd_\tau M$ *and* $\forall V\,(M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$ *.*

Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fn}\, x : \tau \,.\, M'$.

  $\rightsquigarrow$ *substitutivity* property for open terms

Fundamental property

Theorem. *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

NB. The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

Fundamental property of the relations $\lhd_\tau$

Proposition. *If* $\Gamma \vdash M : \tau$ *is a valid PCF typing, then for all* Γ*-environments* ρ *and all* Γ*-substitutions* σ

$$\rho \lhd_\Gamma \sigma \;\Rightarrow\; [\![\Gamma \vdash M]\!](\rho) \lhd_\tau M[\sigma]$$

- $\rho \lhd_\Gamma \sigma$ means that $\rho(x) \lhd_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for x in M, each $x \in dom(\Gamma)$.

Contextual preorder between PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \leq_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \implies \mathcal{C}[M_2] \Downarrow_\gamma V \ .$$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall\, V \in \mathrm{PCF}_\gamma\, (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V)\ .$$

At a function type $\tau \to \tau'$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall\, M \in \mathrm{PCF}_\tau\, (M_1\, M \leq_{\mathrm{ctx}} M_2\, M : \tau')\ .$$