

# ***Lecture 4***

**Scott Induction**

## Scott's Fixed Point Induction Principle

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Let  $f : D \rightarrow D$  be a continuous function on a domain  $D$ .

For any admissible subset  $S \subseteq D$ , to prove that the least fixed point of  $f$  is in  $S$ , i.e. that

$$\text{fix}(f) \in S ,$$

it suffices to prove

$$\forall d \in D (d \in S \Rightarrow f(d) \in S) .$$

## Chain-closed and admissible subsets

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Let  $D$  be a cpo. A subset  $S \subseteq D$  is called **chain-closed** iff for all chains  $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$  in  $D$

$$(\forall n \geq 0. d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If  $D$  is a domain,  $S \subseteq D$  is called **admissible** iff it is a chain-closed subset of  $D$  and  $\perp \in S$ .

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A property  $\Phi(d)$  of elements  $d \in D$  is called **chain-closed** (resp. **admissible**) iff  $\{d \in D \mid \Phi(d)\}$  is a **chain-closed** (resp. **admissible**) subset of  $D$ .

## Building chain-closed subsets (I)

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Let  $D, E$  be cpos.

**Basic relations:**

- For every  $d \in D$ , the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\}$$

of  $D$  is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of  $D \times D$  are chain-closed.

## Example (I): Least pre-fixed point property

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Let  $D$  be a domain and let  $f : D \rightarrow D$  be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let  $d \in D$  be a pre-fixed point of  $f$ . Then,

$$\begin{aligned} x \in \downarrow(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in \downarrow(d) \end{aligned}$$

Hence,

$$\text{fix}(f) \in \downarrow(d) .$$

## Building chain-closed subsets (II)

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### Inverse image:

Let  $f : D \rightarrow E$  be a continuous function.

If  $S$  is a chain-closed subset of  $E$  then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of  $D$ .

## Example (II)

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Let  $D$  be a domain and let  $f, g : D \rightarrow D$  be continuous functions such that  $f \circ g \sqsubseteq g \circ f$ . Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \text{fix}(f) \sqsubseteq \text{fix}(g) .$$

Proof by Scott induction.

Consider the admissible property  $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$  of  $D$ .

Since

$$f(x) \sqsubseteq g(x) \implies g(f(x)) \sqsubseteq g(g(x)) \implies f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)) .$$

## Building chain-closed subsets (III)

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### Logical operations:

- If  $S, T \subseteq D$  are chain-closed subsets of  $D$  then
$$S \cup T \quad \text{and} \quad S \cap T$$
are chain-closed subsets of  $D$ .
- If  $\{S_i\}_{i \in I}$  is a family of chain-closed subsets of  $D$  indexed by a set  $I$ , then  $\bigcap_{i \in I} S_i$  is a chain-closed subset of  $D$ .
- If a property  $P(x, y)$  determines a chain-closed subset of  $D \times E$ , then the property  $\forall x \in D. P(x, y)$  determines a chain-closed subset of  $E$ .



## Example (III): Partial correctness

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Let  $\mathcal{F} : \text{State} \rightarrow \text{State}$  be the denotation of

**while**  $X > 0$  **do** ( $Y := X * Y; X := X - 1$ ).

For all  $x, y \geq 0$ ,

$$\begin{aligned} \mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow \\ \implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y]. \end{aligned}$$

Recall that

$$\mathcal{F} = \text{fix}(f)$$

where  $f : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$  is given by

$$f(w) = \lambda(x, y) \in \text{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of  $(State \rightarrow State)$  given by

$$S = \left\{ w \mid \begin{array}{l} \forall x, y \geq 0. \\ w[X \mapsto x, Y \mapsto y] \downarrow \\ \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S .$$