Complexity Theory
Lecture 7

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http://www.cl.cam.ac.uk/teaching/0910/Complexity/

Hamiltonian Graphs

Recall the definition of HAM—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a *Hamiltonian cycle* in G is a path in the graph, starting and ending at the same node, such that every node in V appears on the cycle *exactly once*.

A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM

Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.

Travelling Salesman

Recall the travelling salesman problem

Given

- V — a set of nodes.
- $c : V \times V \to \mathbb{N}$ — a cost matrix.

Find an ordering $v_1, \ldots, v_n$ of V for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.

Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem TSP consists of the set of triples

$$(V, c : V \times V \to \mathbb{N}, t)$$

such that there is a tour of the set of vertices V, which under the cost matrix c, has cost t or less.

Reduction

There is a simple reduction from HAM to TSP, mapping a graph (V, E) to the triple $(V, c : V \times V \to \mathbb{N}, n)$, where

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \textit{otherwise} \end{cases}$$

and n is the size of V.

Sets, Numbers and Scheduling

It is not just problems about formulas and graphs that turn out to be NP-complete.

Literally hundreds of naturally arising problems have been proved NP-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

3D Matching

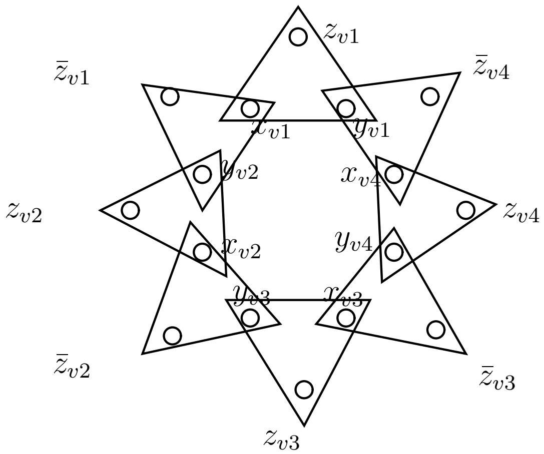
The decision problem of *3D Matching* is defined as:

> Given three disjoint sets X, Y and Z, and a set of triples $M \subseteq X \times Y \times Z$, does M contain a matching?
> I.e. is there a subset $M' \subseteq M$, such that each element of X, Y and Z appears in exactly one triple of M'?

We can show that 3DM is NP-complete by a reduction from 3SAT.

Reduction

If a Boolean expression ϕ in 3CNF has n variables, and m clauses, we construct for each variable v the following gadget.



In addition, for every clause c, we have two elements x_c and y_c.

If the literal v occurs in c, we include the triple

$$(x_c, y_c, z_{vc})$$

in M.

Similarly, if $\neg v$ occurs in c, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in M.

Finally, we include extra dummy elements in X and Y to make the numbers match up.

Exact Set Covering

Two other well known problems are proved NP-complete by immediate reduction from 3DM.

Exact Cover by 3-Sets is defined by:

> Given a set U with $3n$ elements, and a collection $S = \{S_1, \ldots, S_m\}$ of three-element subsets of U, is there a sub collection containing exactly n of these sets whose union is all of U?

The reduction from 3DM simply takes $U = X \cup Y \cup Z$, and S to be the collection of three-element subsets resulting from M.

Set Covering

More generally, we have the *Set Covering* problem:

> Given a set U, a collection of $S = \{S_1, \ldots, S_m\}$ subsets of U and an integer budget B, is there a collection of B sets in S whose union is U?

Knapsack

KNAPSACK is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems NP-complete.

In the problem, we are given n items, each with a positive integer value v_i and weight w_i.

We are also given a maximum total weight W, and a minimum total value V.

> Can we select a subset of the items whose total weight does not exceed W, and whose total value exceeds V?

Reduction

The proof that KNAPSACK is NP-complete is by a reduction from the problem of Exact Cover by 3-Sets.

Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of U, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of KNAPSACK with m elements each corresponding to one of the S_i, and having weight and value

$$\Sigma_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\Sigma_{j=0}^{3n-1} (m+1)^j$$