Uncertainty V: probabilistic reasoning through time

We now examine:

- How an agent might operate by keeping track of the state of its environment in an uncertain world, and how alterations in world state and uncertainty in observing the world can be modelled using probability distributions.
- How inferences can be performed regarding the current state, past state and future states.
- The *Viterbi algorithm* for computing the most likely sequence.
- A slightly simplified system within this framework called a *hidden Markov model* (HMM), and the way in which some inference tasks can be simplified in the HMM case.

Reading: Russell and Norvig, chapter 15.

Probabilistic reasoning through time

A fundamental idea throughout the AI courses has been that an agent should keep track of the state of the environment:

- The environment's state changes over time.
- The knowledge of how the state changes may be uncertain.
- The agent's perception of the state of the environment may be uncertain.

For all the usual reasons related to *uncertainty*, we need to move beyond logic, situation calculus *etc*.

States and evidence

We model the (unobservable) state of the environment as follows:

- We use a *sequence*
$$(S_0, S_1, S_2, \ldots)$$
of *sets* of *random variables* (RVs).
- Each S_t is a *set* of RVs
$$S_t = \{S_t^{(1)}, \ldots, S_t^{(n)}\}$$
denoting the state of the environment at time t, where $t = 0, 1, 2, \ldots$.

Think of the state as changing over time.

$$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots$$

States and evidence

At each time t there is also an *observable* set
$$E_t = \{E_t^{(1)}, \ldots, E_t^{(m)}\}$$
of random variables denoting the *evidence that an agent obtains about the state* at time t.

As usual capitals denote RVs and lower case denotes actual values. So actual values for the assorted RVs are denoted
$$S_t = \{s_t^{(1)}, \ldots, s_t^{(n)}\} = s_t$$
$$E_t = \{e_t^{(1)}, \ldots, e_t^{(m)}\} = e_t$$

Stationary and Markov processes

As t can in principle increase without bound we now need some simplifying assumptions.

Assumption 1: We deal with *stationary processes*: probability distributions do not change over time.

Assumption 2: We deal with *Markov processes*

$$\Pr(S_t|S_{0:t-1}) = \Pr(S_t|S_{t-1}) \tag{1}$$

where $S_{0:t-1} = (S_0, S_1, \ldots, S_{t-1})$.

(Strictly speaking this is a *first order Markov Process*, and we'll only consider these.)

$\Pr(S_t|S_{t-1})$ is called the *transition model*.

Stationary and Markov processes

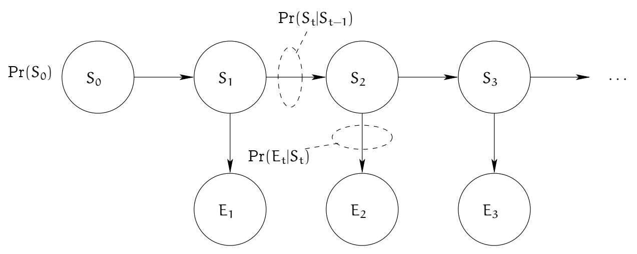
Assumption 3: We assume that evidence only depends on the current state

$$\Pr(E_t|S_{0:t}, E_{1:t-1}) = \Pr(E_t|S_t) \tag{2}$$

Then

$\Pr(E_t|S_t)$ is called the *sensor model*.

Stationary and Markov processes



$\Pr(S_0)$ is the *prior probability* of the starting state. We need this as there has to be some way of getting the process started.

The full joint distribution

Given:

1. The prior $\Pr(S_0)$.
2. The transition model $\Pr(S_t|S_{t-1})$.
3. The sensor model $\Pr(E_t|S_t)$.

along with the assumptions of stationarity and the assumptions of independence in equations 1 and 2 we have

$$\boxed{\Pr(S_0, S_1, \ldots, S_t, E_1, E_2, \ldots, E_t) = \Pr(S_0)\prod_{i=1}^{t}\Pr(S_i|S_{i-1})\Pr(E_i|S_i)}.$$

This follows from basic probability theory as for example

$$\begin{aligned}\Pr(S_0, S_1, S_2, E_1, E_2) &= \Pr(E_2|S_{0:2}, E_1)\Pr(S_2|S_{0:1}, E_1)\Pr(E_1|S_{0:1})\Pr(S_1|S_0)\mathrm{P}\\ &= \Pr(E_2|S_2)\Pr(S_2|S_1)\Pr(E_1|S_1)\Pr(S_1|S_0)\Pr(S_0)\end{aligned}$$

Example: two biased coins

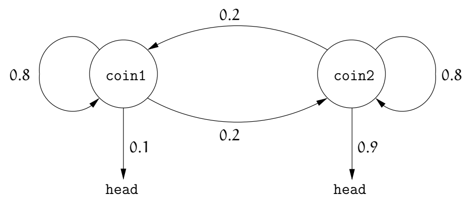
Here's a simple example with only two states and two observations.

I have two biased coins.

I flip one and tell you the outcome.

I then either stay with the same coin, or swap them.

This continues, producing a succession of outcomes.



Example: two biased coins

We'll use the following numbers:

- The prior $\Pr(S_0 = \texttt{coin1}) = 0.5$.
- The transition model

$$\Pr(S_t = \texttt{coin1}|S_{t-1} = \texttt{coin1}) = \Pr(S_t = \texttt{coin2}|S_{t-1} = \texttt{coin2}) = 0.8$$
$$\Pr(S_t = \texttt{coin1}|S_{t-1} = \texttt{coin2}) = \Pr(S_t = \texttt{coin2}|S_{t-1} = \texttt{coin1}) = 0.2$$

- The sensor model

$$\Pr(E_t = \texttt{head}|S_t = \texttt{coin1}) = 0.1$$
$$\Pr(E_t = \texttt{head}|S_t = \texttt{coin2}) = 0.9$$

Example: two biased coins

This is straighforward to simulate.

Here's an example of what happens:

```
[C2,C2,C1,C1,C1,C1,C1,C1,C1,C1,C1,C1,C2,C1,C1,C1,C1,C1,C1,C1,C1,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2,C2]
```

$$\Downarrow$$

```
[Hd,Tl,Tl,Tl,Hd,Tl,Hd,Tl,Tl,Tl,Hd,Tl,Hd,Tl,Tl,Tl,Tl,Tl,Hd,Tl,Tl,Hd,Hd,Hd,Hd,Hd,Hd,Hd,Hd,Tl,Hd,Hd,Hd,Hd,Hd,Hd,Hd,Hd,Tl,Hd]
```

As expected, we tend to see runs of a single coin, and might expect to be able to guess which is being used as one favours heads and the other tails.

Example: 2008, paper 9, question 5

A friend of mine likes to climb on the roofs of Cambridge. To make a good start to the coming week, he climbs on a Sunday with probability 0.98. Being concerned for his own safety, he is less likely to climb today if he climbed yesterday, so

$$\Pr(\texttt{climb today}|\texttt{climb yesterday}) = 0.4$$

If he did not climb yesterday then he is very unlikely to climb today, so

$$\Pr(\texttt{climb today}|\neg\texttt{climb yesterday}) = 0.1$$

Unfortunately, he is not a very good climber, and is quite likely to injure himself if he goes climbing, so

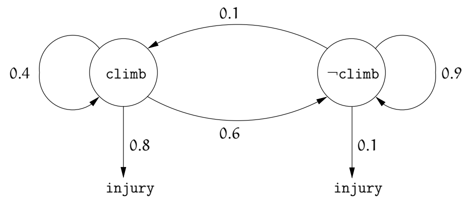
$$\Pr(\texttt{injury}|\texttt{climb today}) = 0.8$$

whereas

$$\Pr(\texttt{injury}|\neg\texttt{climb today}) = 0.1$$

Example: 2008, paper 9, question 5

This has a similar corresponding diagram:



We'll look at the rest of this exam question later.

Performing inference

There are four basic inference tasks that we might want to perform.

In each of the following cases, assume that we have observed the evidence

$$E_{1:t} = e_{1:t}$$

Task 1: *filtering*

Deduce what state we might now be in by computing

$$\Pr(S_t|e_{1:t}).$$

In the coin tossing question: *"If you've seen all the outcomes so far, infer which coin was used last"*.

In the exam question: *"If you observed all the injuries so far, infer whether my friend climbed today"*.

Performing inference

Task 2: *prediction*

Deduce what state we might be in some time in the future by computing

$$\Pr(S_{t+T}|e_{1:t}) \text{ for some } T > 0.$$

In the coin tossing question: *"If you've seen all the outcomes so far, infer which coin will be tossed T steps in the future"*.

In the exam question: *"If you've observed all the injuries so far, infer whether my friend will go climbing T nights from now"*.

Performing inference

Task 3: *Smoothing*

Deduce what state we might have been in at some point in the past by computing

$$\Pr(S_t|e_{1:T}) \text{ for } 0 \leq t < T.$$

In the coin tossing question: *"If you've seen all the outcomes so far, infer which coin was tossed at time t in the past"*.

In the exam question: *"If you've observed all the injuries so far, infer whether my friend climbed on night t in the past"*.

Performing inference

Task 4: *Find the most likely explanation*

Deduce the most likely sequence of states so far by computing

$$\underset{s_{1:t}}{\text{argmax}} \Pr(s_{1:t}|e_{1:t})$$

In the coin tossing question: *"If you've seen all the outcomes so far, infer the most probable sequence of coins used"*.

In the exam question: *"If you've observed all the injuries so far, infer the most probable collection of nights on which my friend climbed"*.

Filtering

We want to compute $\Pr(S_t|e_{1:t})$. This is often called the *forward message* and denoted

$$f_{1:t} = \Pr(S_t|e_{1:t})$$

for reasons that are about to become clear.

Remember that S_t is an RV and so $f_{1:t}$ is a *probability distribution* containing a probability for each possible value of S_t.

It turns out that this can be done in a simple manner with a *recursive estimation*. Obtain the result at time $t+1$:

1. using the result from time t and...
2. ...incorporating new evidence e_{t+1}.
$$f_{1:t+1} = g(e_{t+1}, f_{1:t})$$
for a suitable function g that we'll now derive.

Filtering

Step 1:

Project the current state distribution forward

$$\begin{aligned}\Pr(S_{t+1}|e_{1:t+1}) &= \Pr(S_{t+1}|e_{1:t}, e_{t+1}) \\ &= c\Pr(e_{t+1}|S_{t+1}, e_{1:t})\Pr(S_{t+1}|e_{1:t}) \\ &= c\underbrace{\Pr(e_{t+1}|S_{t+1})}_{\text{Sensor model}}\underbrace{\Pr(S_{t+1}|e_{1:t})}_{\text{Needs more work}}\end{aligned}$$

where as usual c is a constant that normalises the distribution. Here,

- The first line does nothing but split $e_{1:t+1}$ into e_{t+1} and $e_{1:t}$.
- The second line is an application of Bayes' theorem.
- The third line uses *assumption 3* regarding sensor models.

Filtering

Step 2:

To obtain $\Pr(S_{t+1}|e_{1:t})$

$$\begin{aligned}\Pr(S_{t+1}|e_{1:t}) &= \sum_{s_t} \Pr(S_{t+1}, s_t|e_{1:t}) \\ &= \sum_{s_t} \Pr(S_{t+1}|s_t, e_{1:t})\Pr(s_t|e_{1:t}) \\ &= \sum_{s_t} \underbrace{\Pr(S_{t+1}|s_t)}_{\text{Transition model}} \underbrace{\Pr(s_t|e_{1:t})}_{\text{Available from previous step}}\end{aligned}$$

Here,

- The first line uses marginalisation.
- The second line uses the basic equation $\Pr(A, B) = \Pr(A|B)\Pr(B)$.
- The third line uses *assumption 2* regarding transition models.

Filtering

Pulling it all together

$$\boxed{\Pr(S_{t+1}|e_{1:t+1}) = c\underbrace{\Pr(e_{t+1}|S_{t+1})}_{\text{Sensor model}} \sum_{s_t} \underbrace{\Pr(S_{t+1}|s_t)}_{\text{Transition model}} \underbrace{\Pr(s_t|e_{1:t})}_{\text{From previous step}}} \quad (3)$$

This will be shortened to

$$f_{1:t+1} = c\text{FORWARD}(e_{t+1}, f_{1:t})$$

Here

- $f_{1:t}$ is a shorthand for $\Pr(S_t|e_{1:t})$.
- $f_{1:t}$ is often interpreted as a *message* being passed forward.
- The process is started using the *prior*.

Prediction

Prediction is somewhat simpler as

$$\begin{aligned}
\boxed{\underbrace{\Pr(S_{t+T+1}|e_{1:t})}_{\text{Prediction at t+T+1}}} &= \sum_{s_{t+T}} \Pr(S_{t+T+1}, s_{t+T}|e_{1:t}) \\
&= \sum_{s_{t+T}} \Pr(S_{t+T+1}|s_{t+T}, e_{1:t})\Pr(s_{t+T}|e_{1:t}) \\
&= \boxed{\sum_{s_{t+T}} \underbrace{\Pr(S_{t+T+1}|s_{t+T})}_{\text{Transition model}} \underbrace{\Pr(s_{t+T}|e_{1:t})}_{\text{Prediction at t+T}}}
\end{aligned}$$

However we do not get to make accurate predictions arbitrarily far into the future!

Smoothing

For smoothing, we want to calculate $\Pr(S_t|e_{1:T})$ for $0 \leq t < T$.

Again, we can do this in two steps.

Step 1:

$$\begin{aligned}
\Pr(S_t|e_{1:T}) &= \Pr(S_t|e_{1:T}, e_{t+1:T}) \\
&= c\Pr(S_t|e_{1:t})\Pr(e_{t+1:T}|S_t, e_{1:t}) \\
&= c\Pr(S_t|e_{1:t})\Pr(e_{t+1:T}|S_t) \\
&= cf_{1:t}b_{t+1:T}
\end{aligned}$$

Here

- $f_{1:t}$ is the forward message defined earlier.
- $b_{t+1:T}$ is a shorthand for $\Pr(e_{t+1:T}|S_t)$ to be regarded as *a message being passed backward*.

Smoothing

Step 2:

$$\begin{aligned}
\boxed{b_{t+1:T}} = \Pr(e_{t+1:T}|S_t) &= \sum_{s_{t+1}} \Pr(e_{t+1:T}, s_{t+1}|S_t) \\
&= \sum_{s_{t+1}} \Pr(e_{t+1:T}|s_{t+1})\Pr(s_{t+1}|S_t) \\
&= \sum_{s_{t+1}} \Pr(e_{t+1}, e_{t+2:T}|s_{t+1})\Pr(s_{t+1}|S_t) \\
&= \sum_{s_{t+1}} \underbrace{\Pr(e_{t+1}|s_{t+1})}_{\text{Sensor model}}\underbrace{\Pr(e_{t+2:T}|s_{t+1})}_{b_{t+2:T}} \underbrace{\Pr(s_{t+1}|S_t)}_{\text{Transition mode}} \\
&= \boxed{\text{BACKWARD}(e_{t+1:T}, b_{t+2:T})}
\end{aligned} \quad (4)$$

This process is initialised with

$$b_{t+1:t} = \Pr(e_{T+1:T}|S_T) = (1, \ldots, 1)$$

The forward-backward algorithm

So: our original aim of computing $\Pr(S_t|e_{1:T})$ can be achieved using:

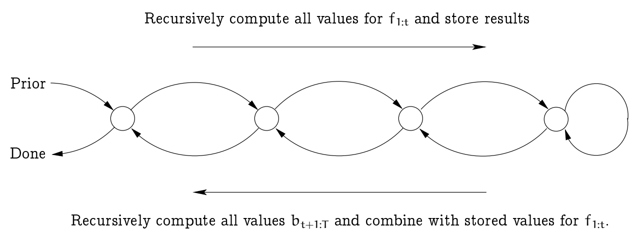
- a recursive process working from time 1 to time t (equation 3);
- a recursive process working from time T to time $t+1$ (equation 4).

This results in a process that is $O(T)$ given the evidence $e_{1:T}$ and smooths for a *single* point at time t.

To smooth at *all* points $1:T$ we can easily repeat the process obtaining $O(T^2)$.

Alternatively a very simple example of *dynamic programming* allows us to smooth at all points in $O(T)$ time.

The forward-backward algorithm


Recursively compute all values for $f_{1:t}$ and store results
Prior
Done
Recursively compute all values $b_{t+1:T}$ and combine with stored values for $f_{1:t}$.

Computing the most likely sequence: the Viterbi algorithm

In computing the most likely sequence the aim is to obtain

$$\underset{s_{1:t}}{\operatorname{argmax}} \Pr(s_{1:t}|e_{1:t})$$

Earlier we derived the joint distribution for all relevant variables

$$\boxed{\Pr(S_0, S_1, \ldots, S_t, E_1, E_2, \ldots, E_t) = \Pr(S_0) \prod_{i=1}^{t} \Pr(S_i|S_{i-1})\Pr(E_i|S_i)}$$

Computing the most likely sequence: the Viterbi algorithm

We therefore have

$$\boxed{\max_{s_{1:t}} \Pr(s_{1:t}, S_{t+1}|e_{1:t+1})}$$
$$= c \max_{s_{1:t}} \Pr(e_{t+1}|S_{t+1})\Pr(S_{t+1}|s_t)\Pr(s_{1:t}|e_{1:t})$$
$$= c\Pr(e_{t+1}|S_{t+1}) \max_{s_t} \left\{ \Pr(S_{t+1}|s_t) \boxed{\max_{s_{1:t-1}} \Pr(s_{1:t-1}, s_t|e_{1:t})} \right\}$$

This looks *a bit fierce*, despite the fact that:

- The second line is just Bayes' theorem applied to the joint distribution.
- The last line is just a re-arrangement of the second line.

Computing the most likely sequence: the Viterbi algorithm

There is however a way to visualise it that leads to a dynamic programming algorithm called the *Viterbi algorithm*.

Step 1: Simplify the notation.

- Assume there are n states $s_1, \ldots, s_n$ and m possible observations $e_1, \ldots, e_m$ at any given time.
- Denote $\Pr(S_t = s_j | S_{t-1} = s_i)$ by $p_{i,j}(t)$.
- Denote $\Pr(e_t | S_t = s_i)$ by $q_i(t)$.

It's important to remember in what follows that the *observations are known* but that we're *maximizing over all possible state sequences*.

Computing the most likely sequence: the Viterbi algorithm

The equation we're interested in is now of the form

$$\boxed{P = \prod_{t=1}^{T} p_{i,j}(t) q_i(t)}$$

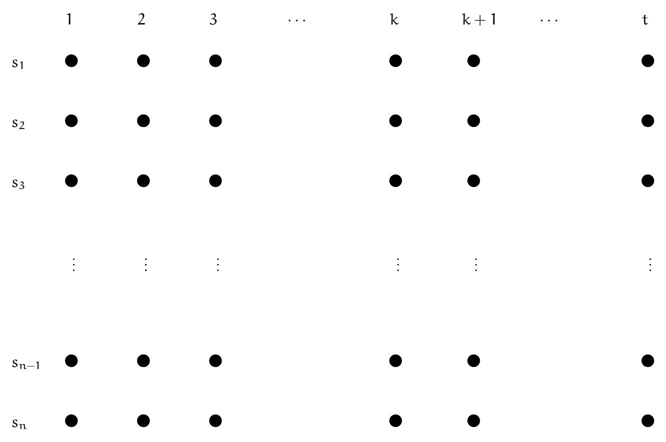
(The prior $\Pr(S_0)$ has been dropped out for the sake of clarity, but is easy to put back in in what follows.)

The equation P will be referred to in what follows.

It is in fact a *function* of *any given sequence of states*.

Computing the most likely sequence: the Viterbi algorithm

Step 2: Make a grid: columns denote time and rows denote state.



Computing the most likely sequence: the Viterbi algorithm

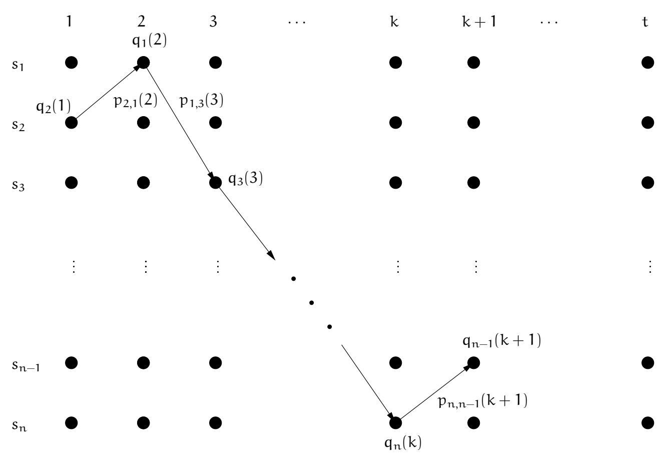
Step 3: Label the nodes:

- Say at time t the actual observation was e_t. Then label the node for s_i in column t with the value $q_i(t)$.
- Any sequence of states through time is now a path through the grid. So for any transition from s_i at time $t-1$ to s_j at time t label the transition with the value $p_{i,j}(t)$.

In the following diagrams we can often just write $p_{i,j}$ and q_i because the time is clear from the diagram.

So for instance...

Computing the most likely sequence: the Viterbi algorithm

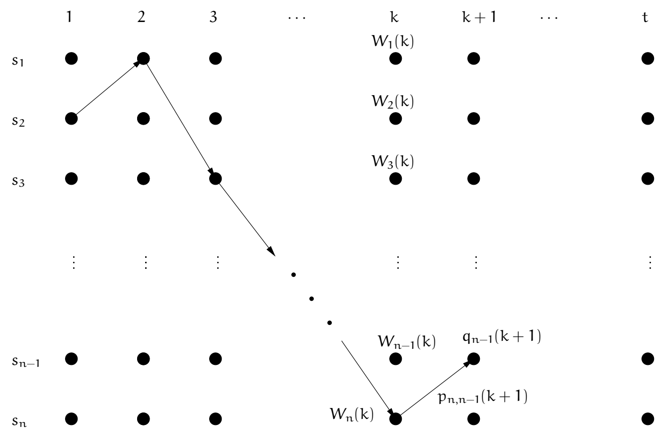


Computing the most likely sequence: the Viterbi algorithm

- The value of $P = \prod_{t=1}^{T} p_{i,j}(t) q_i(t)$ for any path through the grid is just the product of the corresponding labels that have been added.
- But we don't want to find the maximum by looking at all the possible paths because this would be time-consuming.
- The *Viterbi algorithm* computes the maximum by moving from one column to the next updating as it goes.
- Say you're at column k and *for each node* m in that column you know the *highest value* for the product to this point over *any possible path*. Call this:

$$\boxed{W_m(k) = \max_{s_{1:k}} \prod_{t=1}^{k} p_{i,j}(t) q_i(t)}$$

Computing the most likely sequence: the Viterbi algorithm



Computing the most likely sequence: the Viterbi algorithm

Here is the key point: you only need to know

- The values $W_i(k)$ for $i = 1, \ldots, n$ at time k.
- The numbers $p_{i,j}(k+1)$.
- The numbers $q_i(k+1)$.

to compute the values $W_i(k+1)$ for the next column $k+1$.

This is because

$$\boxed{W_i(k+1) = \max_j W_j(k) p_{j,i}(k+1) q_i(k+1)}$$

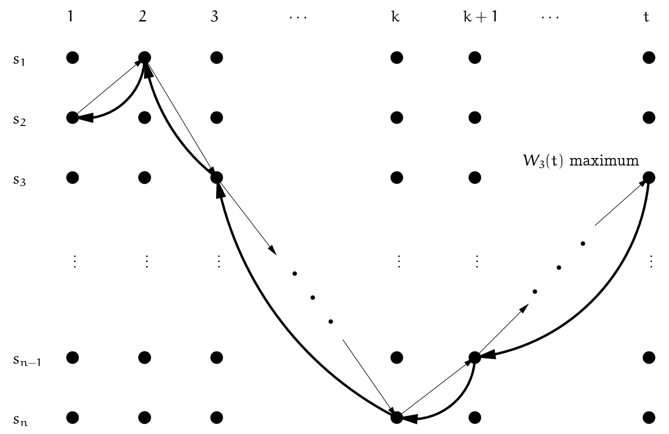
Computing the most likely sequence: the Viterbi algorithm

Once you get to the column for time t:

- The node with the largest value for $W_i(t)$ tells you the largest possible value of P.
- Provided you stored *the path taken to get there* you can *work backwards* to find *the corresponding sequence of states*.

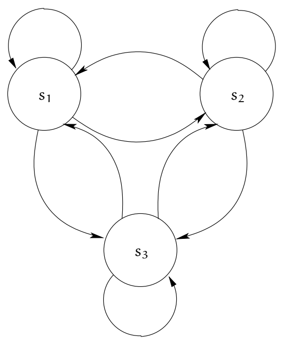
This is the *Viterbi algorithm*.

Computing the most likely sequence: the Viterbi algorithm



Hidden Markov models

Now for a specific case: hidden Markov models (HMMs). Here we have a *single*, *discrete* state variable S_i taking values $s_1, s_2, \ldots, s_n$. For example, with $n = 3$ we might have



	$\Pr(S_{t+1}\|S_t = s_1)$	$\Pr(S_{t+1}\|S_t = s_2)$	$\Pr(S_{t+1}\|S_t = s_3)$
s_1	0.3	0.2	0.2
s_2	0.1	0.6	0.3
s_3	0.6	0.2	0.5

Hidden Markov models

In this simplified case the conditional probabilities $\Pr(S_{t+1}|S_t)$ can be represented using the matrix

$$S_{ij} = \Pr(S_{t+1} = s_j | S_t = s_i)$$

or for the example on the previous slide

$$\mathbf{S} = \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} \begin{array}{l} \leftarrow \Pr(S|s_1) \\ \leftarrow \Pr(S|s_2) \\ \leftarrow \Pr(S|s_3) \end{array}$$

$$= \begin{pmatrix} \Pr(s_1|s_1) & \Pr(s_2|s_1) & \cdots & \Pr(s_n|s_1) \\ \Pr(s_1|s_2) & \Pr(s_2|s_2) & \cdots & \Pr(s_n|s_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Pr(s_1|s_n) & \Pr(s_2|s_n) & \cdots & \Pr(s_n|s_n) \end{pmatrix}$$

To save space, I am abbreviating $\Pr(S_{t+1} = s_i | S_t = s_j)$ to $\Pr(s_i|s_j)$.

Hidden Markov models

The computations we're making are always conditional on some actual observations $e_{1:T}$.

For each t we can therefore use the sensor model to define a further matrix $\mathbf{E}_t$:

- $\mathbf{E}_t$ is square and diagonal (all off-diagonal elements are 0);
- the ith element of the diagonal is $\Pr(e_t|S_t = s_i)$.

So in our present example with 3 states, there will be a matrix

$$\mathbf{E}_t = \begin{pmatrix} \Pr(e_t|s_1) & 0 & 0 \\ 0 & \Pr(e_t|s_2) & 0 \\ 0 & 0 & \Pr(e_t|s_3) \end{pmatrix}$$

for each $t = 1, \dots, T$.

Hidden Markov models

In the general case the equation for filtering was

$$\Pr(S_{t+1}|e_{1:t+1}) = c\Pr(e_{t+1}|S_{t+1}) \sum_{s_t} \Pr(S_{t+1}|s_t)\Pr(s_t|e_{1:t})$$

and the message $f_{1:t}$ was introduced as a representation of $\Pr(S_t|e_{1:t})$.

In the present case we can define $f_{1:t}$ to be the vector

$$f_{1:t} = \begin{pmatrix} \Pr(s_1|e_{1:t}) \\ \Pr(s_2|e_{1:t}) \\ \vdots \\ \Pr(s_n|e_{1:t}) \end{pmatrix}$$

Key point: *the filtering equation now reduces to nothing but matrix multiplication*.

What does matrix multiplication do?

What does matrix multiplication do? *It computes weighted summations*:

$$\mathbf{A}b = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m a_{1,i}b_i \\ \sum_{i=1}^m a_{2,i}b_i \\ \vdots \\ \sum_{i=1}^m a_{n,i}b_i \end{pmatrix}$$

So the point at the end of the last slide shouldn't come as a big surprise!

Hidden Markov models

Now, note that if we have n states

$$\begin{aligned}
\mathbf{S}^T f_{1:t} &= \begin{pmatrix} \Pr(s_1|s_1) & \cdots & \Pr(s_1|s_n) \\ \Pr(s_2|s_1) & \cdots & \Pr(s_2|s_n) \\ \vdots & \ddots & \vdots \\ \Pr(s_n|s_1) & \cdots & \Pr(s_n|s_n) \end{pmatrix} \begin{pmatrix} \Pr(s_1|e_{1:t}) \\ \Pr(s_2|e_{1:t}) \\ \vdots \\ \Pr(s_n|e_{1:t}) \end{pmatrix} \\
&= \begin{pmatrix} \Pr(s_1|s_1)\Pr(s_1|e_{1:t}) + \cdots + \Pr(s_1|s_n)\Pr(s_n|e_{1:t}) \\ \Pr(s_2|s_1)\Pr(s_1|e_{1:t}) + \cdots + \Pr(s_2|s_n)\Pr(s_n|e_{1:t}) \\ \vdots \\ \Pr(s_n|s_1)\Pr(s_1|e_{1:t}) + \cdots + \Pr(s_n|s_n)\Pr(s_n|e_{1:t}) \end{pmatrix} \\
&= \begin{pmatrix} \sum_s \Pr(s_1|s)\Pr(s|e_{1:t}) \\ \sum_s \Pr(s_2|s)\Pr(s|e_{1:t}) \\ \vdots \\ \sum_s \Pr(s_n|s)\Pr(s|e_{1:t}) \end{pmatrix}
\end{aligned}$$

Hidden Markov models

And taking things one step further

$$\mathbf{E}_{t+1}\mathbf{S}^{\mathsf{T}}f_{1:t} = \begin{pmatrix} \Pr(e_{t+1}|s_1) & & 0 \\ & \ddots & \\ 0 & & \Pr(e_{t+1}|s_n) \end{pmatrix} \begin{pmatrix} \sum_s \Pr(s_1|s)\Pr(s|e_{1:t}) \\ \sum_s \Pr(s_2|s)\Pr(s|e_{1:t}) \\ \vdots \\ \sum_s \Pr(s_n|s)\Pr(s|e_{1:t}) \end{pmatrix}$$

$$= \begin{pmatrix} \Pr(e_{t+1}|s_1)\sum_s \Pr(s_1|s)\Pr(s|e_{1:t}) \\ \Pr(e_{t+1}|s_2)\sum_s \Pr(s_2|s)\Pr(s|e_{1:t}) \\ \vdots \\ \Pr(e_{t+1}|s_n)\sum_s \Pr(s_n|s)\Pr(s|e_{1:t}) \end{pmatrix}$$

Compare this with the equation for filtering

$$\Pr(S_{t+1}|e_{1:t+1}) = c\Pr(e_{t+1}|S_{t+1})\sum_{s_t}\Pr(S_{t+1}|s_t)\Pr(s_t|e_{1:t})$$

Hidden Markov models

Comparing the expression for $\mathbf{E}_{t+1}\mathbf{S}^{\mathsf{T}}f_{1:t}$ with the equation for filtering we see that

$$\boxed{f_{1:t+1} = c\mathbf{E}_{t+1}\mathbf{S}^{\mathsf{T}}f_{1:t}}$$

and a similar equation can be found for b

$$\boxed{b_{T+1:t} = \mathbf{S}\mathbf{E}_{T+1}b_{T+2:t}}$$

Exercise: derive this.

The fact that these can be expressed simply using only multiplication of vectors and matrices allows us to make an improvement to the forward-backward algorithm.

Hidden Markov models

The *forward-backward* algorithm works by:

- Moving up the sequence from 1 to T, computing and storing values for f.
- Moving down the sequence from T to 1 computing values for b and *combining* them with the stored values for f using the equation

$$\Pr(S_t|e_{1:T}) = cf_{1:t}b_{t+1:T}$$

Now in our simplified HMM case we have

$$f_{1:t+1} = c\mathbf{E}_{t+1}\mathbf{S}^{\mathsf{T}}f_{1:t}$$

or multiplying through by $(\mathbf{E}_{t+1}\mathbf{S}^{\mathsf{T}})^{-1}$ and re-arranging

$$\boxed{f_{1:t} = \frac{1}{c}(\mathbf{S}^{\mathsf{T}})^{-1}(\mathbf{E}_{t+1})^{-1}f_{1:t+1}}$$

Hidden Markov models

So as long as:

- We know the *final* value for f.
- $\mathbf{S}^{\mathsf{T}}$ has an inverse.
- Every observation has non-zero probability in every state.

We *don't* have to store T different values for f—we just work through, discarding intermediate values, to obtain the last value and then work backward.

Example: 2008, paper 9, question 5

A friend of mine likes to climb on the roofs of Cambridge. To make a good start to the coming week, he climbs on a Sunday with probability 0.98. Being concerned for his own safety, he is less likely to climb today if he climbed yesterday, so

$$\Pr(\texttt{climb today}|\texttt{climb yesterday}) = 0.4$$

If he did not climb yesterday then he is very unlikely to climb today, so

$$\Pr(\texttt{climb today}|\neg\texttt{climb yesterday}) = 0.1$$

Unfortunately, he is not a very good climber, and is quite likely to injure himself if he goes climbing, so

$$\Pr(\texttt{injury}|\texttt{climb today}) = 0.8$$

whereas

$$\Pr(\texttt{injury}|\neg\texttt{climb today}) = 0.1$$

Example: 2008, paper 9, question 5

You learn that on Monday and Tuesday evening he obtains an injury, but on Wednesday evening he does not. Use the filtering algorithm to compute the probability that he climbed on Wednesday.

Initially

$$f_{1:0} = \begin{pmatrix} 0.98 \\ 0.02 \end{pmatrix}$$
$$S = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.9 \end{pmatrix}$$
$$E = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.1 \end{pmatrix}$$
$$E' = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$$

Example: 2008, paper 9, question 5

The update equation is

$$f_{1:t+1} = cE_{t+1}S^T f_{1:t}$$

so

$$f_{1:1} = \frac{c}{10,000}\begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 98 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.83874 \\ 0.16126 \end{pmatrix}$$

Repeating this twice more using E' rather than E the final time gives

$$f_{1:2} = \begin{pmatrix} 0.81268 \\ 0.18732 \end{pmatrix}$$
$$f_{1:3} = \begin{pmatrix} 0.10429 \\ 0.89571 \end{pmatrix}$$

so the answer is 0.1.

Example: 2008, paper 9, question 5

Over the course of the week, you also learn that he does not obtain an injury on Thursday or Friday. Use the smoothing algorithm to compute the probability that he climbed on Thursday.

The S, E and E' matrices are the same. The backward message starts as

$$b_{6:5} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the update equation is

$$b_{t:T} = SE_t b_{t+1:T}$$

Then working backwards

$$b_{5:5} = \frac{1}{100}\begin{pmatrix} 4 & 6 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.62 \\ 0.83 \end{pmatrix}$$

Example: 2008, paper 9, question 5

We also need one more forward step, which gives

$$f_{1:4} = \begin{pmatrix} 0.03249 \\ 0.96751 \end{pmatrix}$$

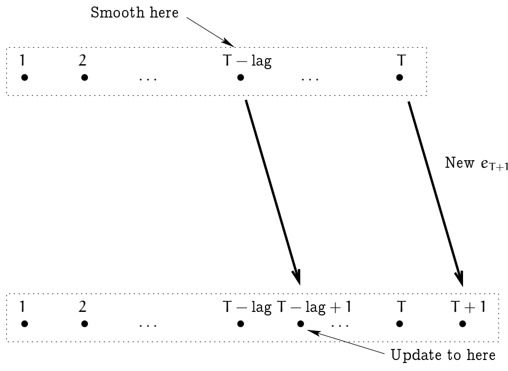
Finally

$$c f_{1:4} b_{5:5} = c \begin{pmatrix} 0.03249 \times 0.62 \\ 0.96751 \times 0.83 \end{pmatrix} = \begin{pmatrix} 0.02447 \\ 0.97553 \end{pmatrix}$$

giving the answer 0.02447.

Online smoothing

Say we want to smooth at a *fixed number of time steps*. We can also obtain a simple algorithm for updating the result each time a new e_{t+1} appears.



Online smoothing

As usual we need to calculate

$$c f_{1:T-\text{lag}} b_{T-\text{lag}+1:T}$$

to smooth at time $(T - \text{lag})$ if we've progressed to time T. So: assume $f_{1:T-\text{lag}}$ and $b_{T-\text{lag}+1:T}$ are known.

What can we now do when e_{T+1} arrives to obtain $f_{1:T-\text{lag}+1}$ and $b_{T-\text{lag}+2:T+1}$?

f is easy to update because as usual

$$f_{1:T-\text{lag}+1} = c\mathbf{E}_{T-\text{lag}+1}\mathbf{S}^T \underbrace{\boxed{f_{1:T-\text{lag}}}}_{\text{Known}}$$

Online smoothing

b is more tricky.

We know that

$$b_{T-\text{lag}+1:T} = \mathbf{SE}_{T-\text{lag}+1} b_{T-\text{lag}+2:T}$$

and continuing this recursion up to the end of the sequence at T gives

$$b_{T-\text{lag}+1:T} = \prod_{i=T-\text{lag}+1}^{T} \mathbf{SE}_i \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Online smoothing

Define

$$\boldsymbol{\beta}_{a:b} = \prod_{i=a}^{b} \mathbf{SE}_i$$

so

$$b_{T-lag+1:T} = \boldsymbol{\beta}_{T-lag+1:T} \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Online smoothing

Now when e_{T+1} arrives we have

$$\begin{aligned} b_{T-lag+2:T+1} &= \prod_{i=T-lag+2}^{T+1} \mathbf{SE}_i \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \boldsymbol{\beta}_{T-lag+2:T+1} \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \mathbf{E}^{-1}_{T-lag+1}\mathbf{S}^{-1}\boldsymbol{\beta}_{T-lag+1:T}\mathbf{SE}_{T+1} \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

Online smoothing

This leads to an easy way to update $\boldsymbol{\beta}$

$$\boxed{\boldsymbol{\beta}_{a+1:b+1} = \mathbf{E}^{-1}_a\mathbf{S}^{-1}\boldsymbol{\beta}_{a:b}\mathbf{SE}_{b+1}}$$

Using this gives the required update for b.