Lecture 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \,.$$

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but

$$\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau .\, x$$

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1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
   - ▶ Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$, $\mathbf{fn}\, x : \tau \,.\, M'$.

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Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
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2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V\,([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

Definition of $d \lhd_\gamma M$ $(d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$ for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \overset{\text{def}}{\Leftrightarrow} (n \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0}))$$

$$b \lhd_{bool} M \overset{\text{def}}{\Leftrightarrow} (b = \mathit{true} \Rightarrow M \Downarrow_{bool} \mathbf{true}) \;\&\; (b = \mathit{false} \Rightarrow M \Downarrow_{bool} \mathbf{false})$$

Proof of: $[\![M]\!] \lhd_\gamma M$ implies adequacy

Case $\gamma = nat$.

$$\begin{aligned}
&[\![M]\!] = [\![V]\!] \\
&\implies [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
&\implies n = [\![M]\!] \lhd_\gamma M \\
&\implies M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}$$

Case $\gamma = bool$ is similar.

Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$\rightsquigarrow$ *logical* definition

Definition of $f \lhd_{\tau \to \tau'} M \;\big(f \in (\llbracket\tau\rrbracket \to \llbracket\tau'\rrbracket), M \in \mathrm{PCF}_{\tau\to\tau'}\big)$

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$$\begin{aligned} & f \lhd_{\tau\to\tau'} M \\ & \quad \overset{\text{def}}{\Leftrightarrow} \; \forall\, x \in \llbracket\tau\rrbracket, N \in \mathrm{PCF}_\tau \\ & \qquad\qquad (x \lhd_\tau N \;\Rightarrow\; f(x) \lhd_{\tau'} M\, N) \end{aligned}$$

Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fix}(M')$.

$\rightsquigarrow$ *admissibility* property

Admissibility property

Lemma. *For all types* τ *and* $M \in \mathrm{PCF}_\tau$*, the set*

$$\{ d \in [\![\tau]\!] \mid d \lhd_\tau M \}$$

is an admissible subset of $[\![\tau]\!]$*.*

Further properties

Lemma. *For all types* τ*, elements* $d, d' \in [\![\tau]\!]$*, and terms* $M, N, V \in \mathrm{PCF}_\tau$*,*

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$*.*
2. *If* $d \lhd_\tau M$ *and* $\forall V\ (M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$ *.*

Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fn}\, x : \tau \,.\, M'$.

$\rightsquigarrow$ *substitutivity* property for open terms

Fundamental property

Theorem. *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$*, if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$ *.*

Fundamental property

Theorem. *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

NB. The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

Contextual preorder from formal approximation

Proposition. *For all PCF types* τ *and all closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$,

$$[\![M_1]\!] \lhd_\tau M_2 \iff M_1 \leq_{\mathrm{ctx}} M_2 : \tau \ .$$

Contextual preorder between PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \leq_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \implies \mathcal{C}[M_2] \Downarrow_\gamma V \ .$$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \ (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \ .$$

At a function type $\tau \to \tau'$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \ (M_1\, M \leq_{\mathrm{ctx}} M_2\, M : \tau') \ .$$