Lecture 4

Scott Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \; (d \in S \;\Rightarrow\; f(d) \in S) \ .$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n\geq 0} d_n\right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

  $$\downarrow\!(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of D is chain-closed.

- The subsets

  $$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

  and

  $$\{(x, y) \in D \times D \mid x = y\}$$

  of $D \times D$ are chain-closed.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in \downarrow(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in \downarrow(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in \downarrow(d) \ .$$

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.
- If $\{\, S_i \,\}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.
- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

Example (III): Partial correctness

Let $\mathcal{F} : \mathit{State} \rightharpoonup \mathit{State}$ be the denotation of

$$\textbf{while}\ X > 0\ \textbf{do}\ (Y := X * Y; X := X - 1)\ .$$

Then

$$\forall x, y \geq 0.\ \mathcal{F}(x, y)\downarrow \implies \mathcal{F}(x, y) = (0, !x \cdot y)$$

where (a, b) denotes the state s such that $s(X) = a$ and $s(Y) = b$.

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$ is given by

$$f(w) = \lambda(x, y) \in \mathit{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(\mathit{State} \rightharpoonup \mathit{State})$ given by

$$S = \{ w \mid \forall x, y \geq 0.\ w(x, y)\downarrow \Rightarrow w(x, y) = (0, !x \cdot y) \}$$

and show that

$$w \in S \implies f(w) \in S\ .$$