# Complexity Theory
# Lecture 12

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## Can you ...

- show sorting is $\Omega(n \log n)$?
- define the class P?
- define the class NP?
- show 3SAT is NP-complete? (at least at a high level)
- show that TAUTOLOGY is in Co-NP?
- define a one-way function?
- understand every relationship in

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXP?$$

# Can you do these reductions?

- 3SAT $\leq_P$ IND
- IND $\leq_P$ CLIQUE
- 3SAT $\leq_P$ 3-Colourability
- 3SAT $\leq_P$ HAM
- HAM $\leq_P$ TSP
- 3SAT $\leq_P$ 3DM
- 3DM $\leq_P$ XSC (Exact Set Cover)
- XSC $\leq_P$ SC (Set Cover)
- XSC $\leq_P$ KNAPSACK



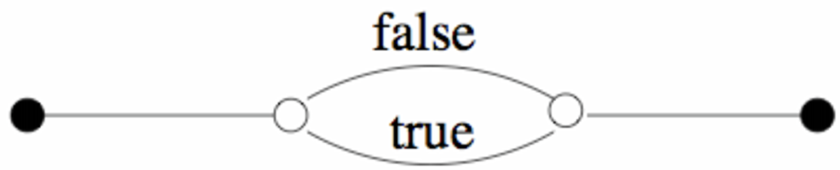
# (Undirected) Hamiltonian Path problem (HAM-PATH)

**HAM-PATH**

Given a graph $G = (V, E)$, does it contain a path that visits every node exactly once?

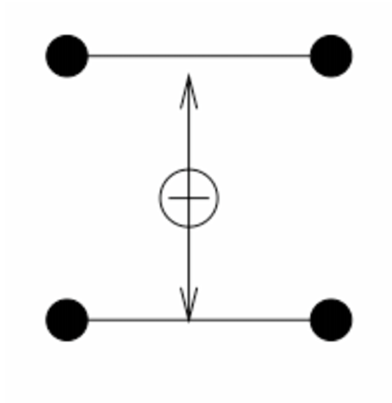
HAM-PATH is NP-complete.

Proof (Papadimitriou, pages 193 to 198): The problem is in NP since we can guess a path and check it in polynomial time.
To show that it is NP-complete we do a reduction from 3SAT.
First, we need a gadget to represent each variable $x$:



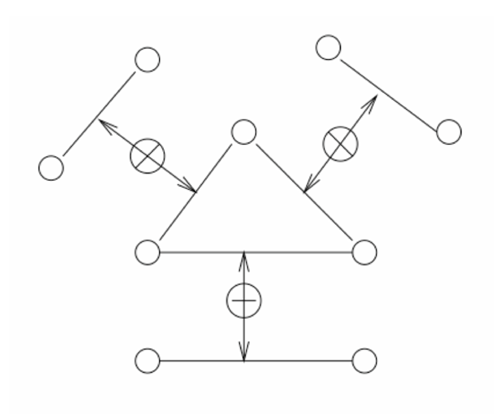

We construct a chain of these gadgets, one for each variable.

# The exclusive or (XOR) gadget



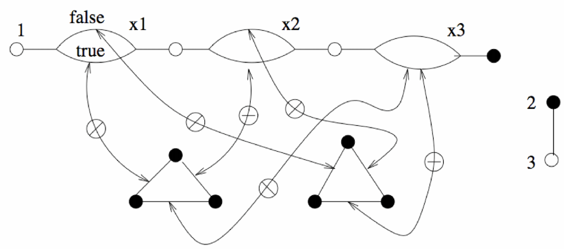



# Representing clauses



For each clause of three literals $(l_1 \vee l_2 \vee l_3)$, we construct a (virtual) triangle where each "edge" is associated with a literal, and connected with an XOR gadget to the virtual link that of the literal's variable that would make the literal true.

# Example




$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

The initial end-points are connected to a new node with label 1, the terminal end-points are connected to a dotted node (All dotted nodes will be connected in one large clique.) Finally, we add one arc with a dotted node and a node labeled 3. Thus, any Hamiltonian path must link nodes 1 and 3.



# Eulerian Path Problem is in P

## Eulerian Path Problem

Given a graph $G = (V, E)$, does it contain a path that visits every edge exactly once?

1. Pick any vertex to start.
2. From the current vertex, pick any edge, but never cross a *bridge* in the *reduced graph* (the graph with marked edges deleted), unless there is no other choice. A bridge is an edge whose deletion would increase the number of connected components.
3. Mark the edge (so will not use it again).
4. *Traverse* the edge, picking the node at the other end.
5. Repeat steps 2 through 4, until back at the starting point (or failure).

## Exercise

Prove that this algorithm is correct.

# Directed Hamiltonian Path problem (DHAM-PATH)

**HAM-PATH**

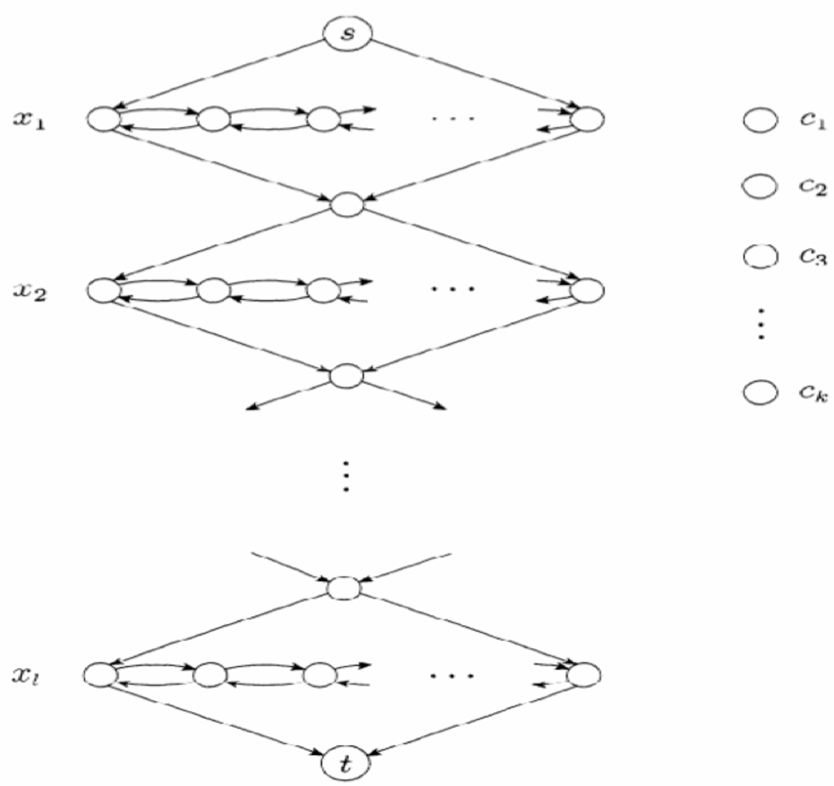
Given a directed graph $G = (V, E)$, does it contain a path that visits every node exactly once?

DHAM-PATH is NP-complete.

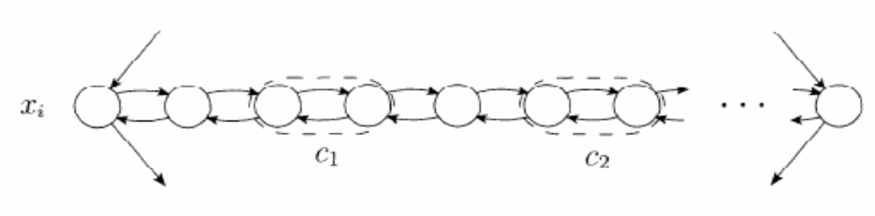
The problem is in NP since we can guess a path and check it in polynomial time.
To show that it is NP-complete we do a reduction from 3SAT.



# We construct a graph that looks like this ...



## Each variable diamond has a chain ...






## ... chains are connected to clause nodes