Glossary of mathematical notation and terminology

Set membership $\boxed{x \in X}$ means x is an element of the set X. (Non-membership is written $x \notin X$.)

Set inclusion $\boxed{X \subseteq Y}$ means every element of X is an element of Y; X is a **subset** of Y.

Set equality $\boxed{X = Y}$ means every element of X is an element of Y and every element of Y is an element of X.

Set comprehension $\boxed{\{x \in X \mid \text{'statement about } x\text{'}\}}$ denotes the subset of X whose elements satisfy 'statement about x'.

Listed sets $\boxed{\{x_1, x_2, \ldots, x_n\}}$ denotes the set whose elements are $x_1, x_2, \ldots, x_n$ ($n \geq 1$); in the case $n = 1$ we get the **singleton set** $\{x\}$, whose unique element is x.

Empty set $\boxed{\emptyset}$ denotes the set containing no elements; it is sometimes written as $\{\}$.

The set of natural numbers $\boxed{\mathbb{N}}$ has elements $0, 1, 2, 3, \ldots$.

Intersection $\boxed{X \cap Y}$ is defined by: $x \in X \cap Y$ if and only if $x \in X$ and $x \in Y$.

Union $\boxed{X \cup Y}$ is defined by: $x \in X \cup Y$ if and only if $x \in X$ or $x \in Y$.

(Relative) Complement $\boxed{X \setminus Y}$ is defined by: $x \in X \setminus Y$ if and only if $x \in X$ and $x \notin Y$.

Cartesian product $\boxed{X \times Y}$ denotes the set of all **ordered pairs** (x, y), with $x \in X$ and $y \in Y$. (By definition, two such ordered pairs, (x, y) and (x', y') are equal if and only if $x = x'$ and $y = y'$.) More generally the cartesian product $X_1 \times \cdots \times X_n$ of sets X_1, …, X_n, consists of all **ordered n-tuples** $(x_1, \ldots, x_n)$, where $x_i \in X_i$ for each $i = 1, \ldots, n$. When $X_1 = \cdots = X_n = X$, we write X^n for the **n-fold cartesian product** of a set X.

Finite lists $\boxed{X^*}$ denotes the set of all **lists** of elements of X of any finite length $n = 0, 1, 2, \ldots$. A list $(x_1, \ldots, x_n)$ of length $n \geq 1$ is just an element of the n-fold cartesian product X^n. The unique **list of length** 0 is written nil.

Partial functions $\boxed{\mathrm{Pfn}(X, Y)}$ denotes the set of all **partial functions from** X **to** Y and consists of all subsets f of the cartesian product $X \times Y$ that satisfy

f **is single-valued:** for all $x \in X$ and $y \in Y$, if $(x, y) \in f$ and $(x, y') \in f$, then $y = y'$.

We will use the following notation for partial functions:

‘$f(x) = y$’ means ‘$(x, y) \in f$’

‘$f(x)\downarrow$’ means ‘*for some* $y \in Y$, $(x, y) \in f$’ (and is read ‘$f(x)$ **is defined**’)

‘$f(x)\uparrow$’ means ‘*there is no* $y \in Y$ *with* $(x, y) \in f$’ (and is read ‘$f(x)$ **is undefined**’)

An n**-ary partial function** from X to Y is just a partial function from the n-fold cartesian product X^n to Y. Stretching the English language to breaking point, one sometimes says of such an $f \in \mathrm{Pfn}(X^n, Y)$ that it is a partial function of **arity** n. In this context, **unary** means 1-ary, **binary** means 2-ary, **ternary** means 3-ary, etc (?).

(Total) Functions $\boxed{\mathrm{Fun}(X, Y)}$ denotes the set of all **functions from** X **to** Y and consists of all partial functions f from X to Y that satisfy

f **is total:** for all $x \in X$, $f(x)\downarrow$.

In this case, for each $x \in X$ we write $f(x)$ for the unique $y \in Y$ such that $(x, y) \in f$. A function $f \in \mathrm{Fun}(X, Y)$ is

injective (or **one-to-one**) if and only if $f(x) = f(x')$ implies $x = x'$, for all $x, x' \in X$;

surjective (or **onto**) if and only if for all $y \in Y$, there is some $x \in X$ with $f(x) = y$;

a bijection (or **a one-to-one correspondence**) if and only if it is both injective and surjective.

Mathematical Induction To prove that a property $P(x)$ holds for all natural numbers $x = 0, 1, 2, 3, \ldots$, it suffi ces to prove the following two statements:

Base case: $P(0)$ is true;

Induction step: for an arbitrary number x, if $P(x)$ is true then so is $P(x + 1)$.