

Isabelle/HOLCF Tutorial

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1 Domain package examples

```
theory Domain_ex
imports HOLCF
begin
```

Domain constructors are strict by default.

```
domain d1 = d1a | d1b "d1" "d1"
```

```
lemma "d1b. $\perp$ .y =  $\perp$ " by simp
```

Constructors can be made lazy using the *lazy* keyword.

```
domain d2 = d2a | d2b (lazy "d2")
```

```
lemma "d2b.x  $\neq$   $\perp$ " by simp
```

Strict and lazy arguments may be mixed arbitrarily.

domain $d3 = d3a \mid d3b$ (**lazy** "d2") "d2"

lemma " $P (d3b \cdot x \cdot y = \perp) \longleftrightarrow P (y = \perp)$ " **by** *simp*

Selectors can be used with strict or lazy constructor arguments.

domain $d4 = d4a \mid d4b$ (**lazy** $d4b_left :: "d2"$) ($d4b_right :: "d2"$)

lemma " $y \neq \perp \implies d4b_left \cdot (d4b \cdot x \cdot y) = x$ " **by** *simp*

Mixfix declarations can be given for data constructors.

domain $d5 = d5a \mid d5b$ (**lazy** "d5") "d5" (**infixl** ":#:" 70)

lemma " $d5a \neq x \text{ :#} : y \text{ :#} : z$ " **by** *simp*

Mixfix declarations can also be given for type constructors.

domain ('a, 'b) *lazypair* (**infixl** "::*:" 25) =
lpair (**lazy** *lfst* :: 'a) (**lazy** *lsnd* :: 'b) (**infixl** "::*:" 75)

lemma " $\forall p :: ('a :: 'b). p \sqsubseteq lfst \cdot p :: lsnd \cdot p$ "
by (*rule allI, case_tac p, simp_all*)

Non-recursive constructor arguments can have arbitrary types.

domain ('a, 'b) $d6 = d6$ "*int lift*" "'a \oplus 'b u" (**lazy** "('a :: 'b) \times ('b \rightarrow 'a)")

Indirect recursion is allowed for sums, products, lifting, and the continuous function space. However, the domain package does not generate an induction rule in terms of the constructors.

domain 'a $d7 = d7a$ "'a $d7 \oplus$ *int lift*" | $d7b$ "'a \otimes 'a $d7$ " | $d7c$ (**lazy** "'a $d7 \rightarrow$ 'a")

— Indirect recursion detected, skipping proofs of (co)induction rules

Note that $d7.induct$ is absent.

Indirect recursion is also allowed using previously-defined datatypes.

domain 'a *slist* = *SNil* | *SCons* 'a "'a *slist*"

domain 'a *stree* = *STip* | *SBranch* "'a *stree slist*"

Mutually-recursive datatypes can be defined using the *and* keyword.

domain $d8 = d8a \mid d8b$ "d9" **and** $d9 = d9a \mid d9b$ (**lazy** "d8")

Non-regular recursion is not allowed.

Mutually-recursive datatypes must have all the same type arguments, not necessarily in the same order.

domain ('a, 'b) *list1* = *Nil1* | *Cons1* 'a "('b, 'a) *list2*"

```
and ('b, 'a) list2 = Nil2 | Cons2 'b "( 'a, 'b) list1"
```

Induction rules for flat datatypes have no admissibility side-condition.

```
domain 'a flattree = Tip | Branch "'a flattree" "'a flattree"
```

```
lemma "[[P ⊥; P Tip; ∧x y. [x ≠ ⊥; y ≠ ⊥; P x; P y]] ⇒ P (Branch.x.y)]
⇒ P x"
```

```
by (rule flattree.induct) — no admissibility requirement
```

Trivial datatypes will produce a warning message.

```
domain triv = Triv triv triv
— domain Domain_ex.triv is empty!
```

```
lemma "(x::triv) = ⊥" by (induct x, simp_all)
```

Lazy constructor arguments may have unpointed types.

```
domain natlist = nnil | ncons (lazy "nat discr") natlist
```

Class constraints may be given for type parameters on the LHS.

```
domain ('a::predomain) box = Box (lazy 'a)
```

```
domain ('a::countable) stream = snil | scon (lazy "'a discr") "'a stream"
```

1.1 Generated constants and theorems

```
domain 'a tree = Leaf (lazy 'a) | Node (left :: "'a tree") (right ::
"'a tree")
```

```
lemmas tree_abs_bottom_iff =
iso.abs_bottom_iff [OF iso.intro [OF tree.abs_iso tree.rep_iso]]
```

Rules about isomorphism

```
term tree_rep
term tree_abs
thm tree.rep_iso
thm tree.abs_iso
thm tree.iso_rews
```

Rules about constructors

```
term Leaf
term Node
thm Leaf_def Node_def
thm tree.nchotomy
thm tree.exhaust
thm tree.compacts
thm tree.con_rews
thm tree.dist_les
thm tree.dist_eqs
```

```
thm tree.inverts  
thm tree.injects
```

Rules about case combinator

```
term tree_case  
thm tree.tree_case_def  
thm tree.case_rews
```

Rules about selectors

```
term left  
term right  
thm tree.sel_rews
```

Rules about discriminators

```
term is_Leaf  
term is_Node  
thm tree.dis_rews
```

Rules about monadic pattern match combinators

```
term match_Leaf  
term match_Node  
thm tree.match_rews
```

Rules about take function

```
term tree_take  
thm tree.take_def  
thm tree.take_0  
thm tree.take_Suc  
thm tree.take_rews  
thm tree.chain_take  
thm tree.take_take  
thm tree.deflation_take  
thm tree.take_below  
thm tree.take_lemma  
thm tree.lub_take  
thm tree.reach  
thm tree.finite_induct
```

Rules about finiteness predicate

```
term tree_finite  
thm tree.finite_def  
thm tree.finite
```

Rules about bisimulation predicate

```
term tree_bisim  
thm tree.bisim_def  
thm tree.coinduct
```

Induction rule

```
thm tree.induct
```

1.2 Known bugs

Declaring a mixfix with spaces causes some strange parse errors.

```
end
```

2 Fixrec package examples

```
theory Fixrec_ex
imports HOLCF
begin
```

2.1 Basic *fixrec* examples

Fixrec patterns can mention any constructor defined by the domain package, as well as any of the following built-in constructors: Pair, spair, sinl, sinr, up, ONE, TT, FF.

Typical usage is with lazy constructors.

```
fixrec down :: "'a u → 'a"
where "down·(up·x) = x"
```

With strict constructors, rewrite rules may require side conditions.

```
fixrec from_sinl :: "'a ⊕ 'b → 'a"
where "x ≠ ⊥ ⇒ from_sinl·(sinl·x) = x"
```

Lifting can turn a strict constructor into a lazy one.

```
fixrec from_sinl_up :: "'a u ⊕ 'b → 'a"
where "from_sinl_up·(sinl·(up·x)) = x"
```

Fixrec also works with the HOL pair constructor.

```
fixrec down2 :: "'a u × 'b u → 'a × 'b"
where "down2·(up·x, up·y) = (x, y)"
```

2.2 Examples using *fixrec_simp*

A type of lazy lists.

```
domain 'a llist = lNil | lCons (lazy 'a) (lazy "'a llist")
```

A zip function for lazy lists.

Notice that the patterns are not exhaustive.

```
fixrec
```

```

  lzip :: "'a llist → 'b llist → ('a × 'b) llist"
where
  "lzip.(lCons.x.xs).(lCons.y.ys) = lCons.(x, y).(lzip.xs.ys)"
| "lzip.lNil.lNil = lNil"

```

fixrec_simp is useful for producing strictness theorems.

Note that pattern matching is done in left-to-right order.

```

lemma lzip_stricts [simp]:
  "lzip.⊥.ys = ⊥"
  "lzip.lNil.⊥ = ⊥"
  "lzip.(lCons.x.xs).⊥ = ⊥"
by fixrec_simp+

```

fixrec_simp can also produce rules for missing cases.

```

lemma lzip_undefs [simp]:
  "lzip.lNil.(lCons.y.ys) = ⊥"
  "lzip.(lCons.x.xs).lNil = ⊥"
by fixrec_simp+

```

2.3 Pattern matching with bottoms

As an alternative to using *fixrec_simp*, it is also possible to use bottom as a constructor pattern. When using a bottom pattern, the right-hand-side must also be bottom; otherwise, *fixrec* will not be able to prove the equation.

```

fixrec
  from_sinr_up :: "'a ⊕ 'b⊥ → 'b"
where
  "from_sinr_up.⊥ = ⊥"
| "from_sinr_up.(sinr.(up.x)) = x"

```

If the function is already strict in that argument, then the bottom pattern does not change the meaning of the function. For example, in the definition of *from_sinr_up*, the first equation is actually redundant, and could have been proven separately by *fixrec_simp*.

A bottom pattern can also be used to make a function strict in a certain argument, similar to a bang-pattern in Haskell.

```

fixrec
  seq :: "'a → 'b → 'b"
where
  "seq.⊥.y = ⊥"
| "x ≠ ⊥ ⇒ seq.x.y = y"

```

2.4 Skipping proofs of rewrite rules

Another zip function for lazy lists.

Notice that this version has overlapping patterns. The second equation cannot be proved as a theorem because it only applies when the first pattern fails.

fixrec

```
  lzip2 :: "'a llist → 'b llist → ('a × 'b) llist"
where
  "lzip2·(lCons·x·xs)·(lCons·y·ys) = lCons·(x, y)·(lzip2·xs·ys)"
| (unchecked) "lzip2·xs·ys = lNil"
```

Usually `fixrec` tries to prove all equations as theorems. The "unchecked" option overrides this behavior, so `fixrec` does not attempt to prove that particular equation.

Simp rules can be generated later using `fixrec_simp`.

```
lemma lzip2_simps [simp]:
  "lzip2·(lCons·x·xs)·lNil = lNil"
  "lzip2·lNil·(lCons·y·ys) = lNil"
  "lzip2·lNil·lNil = lNil"
by fixrec_simp+
```

```
lemma lzip2_stricts [simp]:
  "lzip2·⊥·ys = ⊥"
  "lzip2·(lCons·x·xs)·⊥ = ⊥"
by fixrec_simp+
```

2.5 Mutual recursion with `fixrec`

Tree and forest types.

```
domain 'a tree = Leaf (lazy 'a) | Branch (lazy "'a forest")
and 'a forest = Empty | Trees (lazy "'a tree") "'a forest"
```

To define mutually recursive functions, give multiple type signatures separated by the keyword `and`.

fixrec

```
  map_tree :: "('a → 'b) → ('a tree → 'b tree)"
and
  map_forest :: "('a → 'b) → ('a forest → 'b forest)"
where
  "map_tree·f·(Leaf·x) = Leaf·(f·x)"
| "map_tree·f·(Branch·ts) = Branch·(map_forest·f·ts)"
| "map_forest·f·Empty = Empty"
| "ts ≠ ⊥ ⇒
  map_forest·f·(Trees·t·ts) = Trees·(map_tree·f·t)·(map_forest·f·ts)"
```

```
lemma map_tree_strict [simp]: "map_tree·f·⊥ = ⊥"
by fixrec_simp
```

```
lemma map_forest_strict [simp]: "map_forest.f.⊥ = ⊥"
by fixrec_simp
```

2.6 Looping simp rules

The defining equations of a fixrec definition are declared as simp rules by default. In some cases, especially for constants with no arguments or functions with variable patterns, the defining equations may cause the simplifier to loop. In these cases it will be necessary to use a `[simp del]` declaration.

```
fixrec
  repeat :: "'a → 'a llist"
where
  [simp del]: "repeat.x = lCons.x.(repeat.x)"
```

We can derive other non-looping simp rules for `repeat` by using the `subst` method with the `repeat.simps` rule.

```
lemma repeat_simps [simp]:
  "repeat.x ≠ ⊥"
  "repeat.x ≠ lNil"
  "repeat.x = lCons.y.ys ↔ x = y ∧ repeat.x = ys"
by (subst repeat.simps, simp)+
```

```
lemma llist_case_repeat [simp]:
  "lList.case.z.f.(repeat.x) = f.x.(repeat.x)"
by (subst repeat.simps, simp)
```

For mutually-recursive constants, looping might only occur if all equations are in the simpset at the same time. In such cases it may only be necessary to declare `[simp del]` on one equation.

```
fixrec
  inf_tree :: "'a tree" and inf_forest :: "'a forest"
where
  [simp del]: "inf_tree = Branch.inf_forest"
| "inf_forest = Trees.inf_tree.(Trees.inf_tree.Empty)"
```

2.7 Using fixrec inside locales

```
locale test =
  fixes foo :: "'a → 'a"
  assumes foo_strict: "foo.⊥ = ⊥"
begin
```

```
fixrec
  bar :: "'a u → 'a"
where
  "bar.(up.x) = foo.x"
```

```
lemma bar_strict: "bar.⊥ = ⊥"
```



```
by fixrec_simp
```

```
end
```

```
end
```

3 Definitional domain package

```
theory New_Domain
imports HOLCF
begin
```

UPDATE: The definitional back-end is now the default mode of the domain package. This file should be merged with *Domain_ex.thy*.

Provided that *domain* is the default sort, the *new_domain* package should work with any type definition supported by the old domain package.

```
domain 'a llist = LNil | LCons (lazy 'a) (lazy "'a llist")
```

The difference is that the new domain package is completely definitional, and does not generate any axioms. The following type and constant definitions are not produced by the old domain package.

```
thm type_definition_llist
thm llist_abs_def llist_rep_def
```

The new domain package also adds support for indirect recursion with user-defined datatypes. This definition of a tree datatype uses indirect recursion through the lazy list type constructor.

```
domain 'a ltree = Leaf (lazy 'a) | Branch (lazy "'a ltree llist")
```

For indirect-recursive definitions, the domain package is not able to generate a high-level induction rule. (It produces a warning message instead.) The low-level reach lemma (now proved as a theorem, no longer generated as an axiom) can be used to derive other induction rules.

```
thm ltree.reach
```

The definition of the take function uses map functions associated with each type constructor involved in the definition. A map function for the lazy list type has been generated by the new domain package.

```
thm ltree.take_rews
thm llist_map_def
```

```
lemma ltree_induct:
  fixes P :: "'a ltree  $\Rightarrow$  bool"
  assumes adm: "adm P"
  assumes bot: "P  $\perp$ "
```

```

assumes Leaf: " $\bigwedge x. P (Leaf \cdot x)$ "
assumes Branch: " $\bigwedge f l. \forall x. P (f \cdot x) \implies P (Branch \cdot (l \text{list\_map } f \cdot l))$ "
shows "P x"
proof -
  have "P ( $\sqcup i. ltree\_take i \cdot x$ )"
  using adm
  proof (rule admD)
    fix i
    show "P (ltree_take i \cdot x)"
    proof (induct i arbitrary: x)
      case (0 x)
      show "P (ltree_take 0 \cdot x)" by (simp add: bot)
    next
      case (Suc n x)
      show "P (ltree_take (Suc n) \cdot x)"
      apply (cases x)
      apply (simp add: bot)
      apply (simp add: Leaf)
      apply (simp add: Branch Suc)
      done
    qed
  qed (simp add: ltree.chain_take)
  thus ?thesis
  by (simp add: ltree.reach)
qed
end

```