Fundamental Properties of Lambda-calculus

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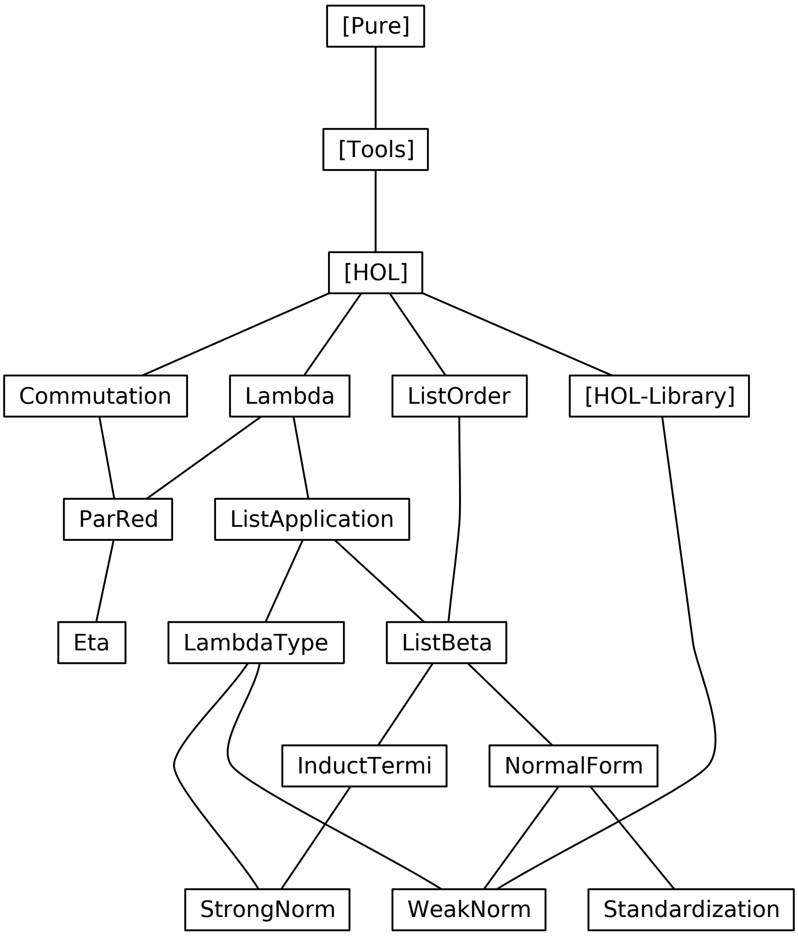
[Pure]
[Tools]
[HOL]
Commutation
Lambda
ListOrder
[HOL-Library]
ParRed
ListApplication
Eta
LambdaType
ListBeta
InductTermi
NormalForm
StrongNorm
WeakNorm
Standardization

1 Basic definitions of Lambda-calculus

```
theory Lambda
imports Main
begin

declare [[syntax-ambiguity-warning = false]]
```

1.1 Lambda-terms in de Bruijn notation and substitution

```
datatype dB =
    Var nat
  | App dB dB (infixl ° 200)
  | Abs dB

primrec
  lift :: [dB, nat] => dB
where
    lift (Var i) k = (if i < k then Var i else Var (i + 1))
  | lift (s ° t) k = lift s k ° lift t k
  | lift (Abs s) k = Abs (lift s (k + 1))

primrec
  subst :: [dB, dB, nat] => dB  (-[-'/-] [300, 0, 0] 300)
where
    subst-Var: (Var i)[s/k] =
      (if k < i then Var (i − 1) else if i = k then s else Var i)
  | subst-App: (t ° u)[s/k] = t[s/k] ° u[s/k]
  | subst-Abs: (Abs t)[s/k] = Abs (t[lift s 0 / k+1])

declare subst-Var [simp del]
```

Optimized versions of *subst* and *lift*.

```
primrec
  liftn :: [nat, dB, nat] => dB
where
    liftn n (Var i) k = (if i < k then Var i else Var (i + n))
  | liftn n (s ° t) k = liftn n s k ° liftn n t k
  | liftn n (Abs s) k = Abs (liftn n s (k + 1))

primrec
  substn :: [dB, dB, nat] => dB
where
    substn (Var i) s k =
      (if k < i then Var (i − 1) else if i = k then liftn k s 0 else Var i)
  | substn (t ° u) s k = substn t s k ° substn u s k
  | substn (Abs t) s k = Abs (substn t s (k + 1))
```

1.2 Beta-reduction

inductive *beta* :: [*dB*, *dB*] => *bool* (**infixl** $\rightarrow_\beta$ *50*)
where
beta [*simp*, *intro*!]: *Abs s* ° *t* $\rightarrow_\beta$ *s*[*t*/*0*]
| *appL* [*simp*, *intro*!]: *s* $\rightarrow_\beta$ *t* ==> *s* ° *u* $\rightarrow_\beta$ *t* ° *u*
| *appR* [*simp*, *intro*!]: *s* $\rightarrow_\beta$ *t* ==> *u* ° *s* $\rightarrow_\beta$ *u* ° *t*
| *abs* [*simp*, *intro*!]: *s* $\rightarrow_\beta$ *t* ==> *Abs s* $\rightarrow_\beta$ *Abs t*

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** $\rightarrow_\beta{}^*$ *50*) **where**
s $\rightarrow_\beta{}^*$ *t* == *beta*** *s t*

inductive-cases *beta-cases* [*elim*!]:
Var i $\rightarrow_\beta$ *t*
Abs r $\rightarrow_\beta$ *s*
s ° *t* $\rightarrow_\beta$ *u*

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
s $\rightarrow_\beta{}^*$ *s'* ==> *Abs s* $\rightarrow_\beta{}^*$ *Abs s'*
⟨*proof*⟩

lemma *rtrancl-beta-AppL*:
s $\rightarrow_\beta{}^*$ *s'* ==> *s* ° *t* $\rightarrow_\beta{}^*$ *s'* ° *t*
⟨*proof*⟩

lemma *rtrancl-beta-AppR*:
t $\rightarrow_\beta{}^*$ *t'* ==> *s* ° *t* $\rightarrow_\beta{}^*$ *s* ° *t'*
⟨*proof*⟩

lemma *rtrancl-beta-App* [*intro*]:
[| *s* $\rightarrow_\beta{}^*$ *s'*; *t* $\rightarrow_\beta{}^*$ *t'* |] ==> *s* ° *t* $\rightarrow_\beta{}^*$ *s'* ° *t'*
⟨*proof*⟩

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: (*Var k*)[*u*/*k*] = *u*
⟨*proof*⟩

lemma *subst-gt* [*simp*]: *i* < *j* ==> (*Var j*)[*u*/*i*] = *Var* (*j* − *1*)
⟨*proof*⟩

lemma *subst-lt* [*simp*]: *j* < *i* ==> (*Var j*)[*u*/*i*] = *Var j*
⟨*proof*⟩

lemma *lift-lift*:
$i < k + 1 \Longrightarrow lift\ (lift\ t\ i)\ (Suc\ k) = lift\ (lift\ t\ k)\ i$
⟨*proof*⟩

lemma *lift-subst* [*simp*]:
$j < i + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ (i + 1))\ [lift\ s\ i\ /\ j]$
⟨*proof*⟩

lemma *lift-subst-lt*:
$i < j + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ i)\ [lift\ s\ i\ /\ j + 1]$
⟨*proof*⟩

lemma *subst-lift* [*simp*]:
$(lift\ t\ k)[s/k] = t$
⟨*proof*⟩

lemma *subst-subst*:
$i < j + 1 \Longrightarrow t[lift\ v\ i\ /\ Suc\ j][u[v/j]/i] = t[u/i][v/j]$
⟨*proof*⟩

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $liftn\ 0\ t\ k = t$
⟨*proof*⟩

lemma *liftn-lift* [*simp*]: $liftn\ (Suc\ n)\ t\ k = lift\ (liftn\ n\ t\ k)\ k$
⟨*proof*⟩

lemma *substn-subst-n* [*simp*]: $substn\ t\ s\ n = t[liftn\ n\ s\ 0\ /\ n]$
⟨*proof*⟩

theorem *substn-subst-0*: $substn\ t\ s\ 0 = t[s/0]$
⟨*proof*⟩

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
$r \to_\beta s ==> r[t/i] \to_\beta s[t/i]$
⟨*proof*⟩

theorem *subst-preserves-beta'*: $r \to_\beta{}^* s ==> r[t/i] \to_\beta{}^* s[t/i]$
⟨*proof*⟩

theorem *lift-preserves-beta* [*simp*]:
$r \to_\beta s ==> lift\ r\ i \to_\beta lift\ s\ i$
⟨*proof*⟩

theorem *lift-preserves-beta'*: $r \to_\beta{}^* s$ ==> *lift* $r\ i \to_\beta{}^*$ *lift* $s\ i$
⟨*proof*⟩

theorem *subst-preserves-beta2* [*simp*]: $r \to_\beta s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
⟨*proof*⟩

theorem *subst-preserves-beta2'*: $r \to_\beta{}^* s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
⟨*proof*⟩

end

2 Abstract commutation and confluence notions

theory *Commutation*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

2.1 Basic definitions

definition
square :: ['a => 'a => *bool*, 'a => 'a => *bool*, 'a => 'a => *bool*, 'a => 'a => *bool*] => *bool* **where**
square $R\ S\ T\ U$ =
$(\forall x\ y.\ R\ x\ y$ --> $(\forall z.\ S\ x\ z$ --> $(\exists u.\ T\ y\ u \wedge U\ z\ u)))$

definition
commute :: ['a => 'a => *bool*, 'a => 'a => *bool*] => *bool* **where**
commute $R\ S$ = *square* $R\ S\ S\ R$

definition
diamond :: ('a => 'a => *bool*) => *bool* **where**
diamond R = *commute* $R\ R$

definition
Church-Rosser :: ('a => 'a => *bool*) => *bool* **where**
Church-Rosser R =
$(\forall x\ y.\ (sup\ R\ (R^{-1-1}))^{**}\ x\ y \longrightarrow (\exists z.\ R^{**}\ x\ z \wedge R^{**}\ y\ z))$

abbreviation
confluent :: ('a => 'a => *bool*) => *bool* **where**
confluent R == *diamond* (R^{**})

2.2 Basic lemmas

square

lemma *square-sym*: *square* $R\ S\ T\ U$ ==> *square* $S\ R\ U\ T$

⟨*proof*⟩

lemma *square-subset*:
  [| *square R S T U*; $T \leq T'$ |] ==> *square R S T' U*
⟨*proof*⟩

lemma *square-reflcl*:
  [| *square R S T* ($R^{==}$); $S \leq T$ |] ==> *square* ($R^{==}$) *S T* ($R^{==}$)
⟨*proof*⟩

lemma *square-rtrancl*:
  square R S S T ==> *square* (R^{**}) *S S* (T^{**})
⟨*proof*⟩

lemma *square-rtrancl-reflcl-commute*:
  square R S (S^{**}) ($R^{==}$) ==> *commute* (R^{**}) (S^{**})
⟨*proof*⟩

commute

lemma *commute-sym*: *commute R S* ==> *commute S R*
⟨*proof*⟩

lemma *commute-rtrancl*: *commute R S* ==> *commute* (R^{**}) (S^{**})
⟨*proof*⟩

lemma *commute-Un*:
  [| *commute R T*; *commute S T* |] ==> *commute* (*sup R S*) *T*
⟨*proof*⟩

diamond, *confluence*, and *union*

lemma *diamond-Un*:
  [| *diamond R*; *diamond S*; *commute R S* |] ==> *diamond* (*sup R S*)
⟨*proof*⟩

lemma *diamond-confluent*: *diamond R* ==> *confluent R*
⟨*proof*⟩

lemma *square-reflcl-confluent*:
  square R R ($R^{==}$) ($R^{==}$) ==> *confluent R*
⟨*proof*⟩

lemma *confluent-Un*:
  [| *confluent R*; *confluent S*; *commute* (R^{**}) (S^{**}) |] ==> *confluent* (*sup R S*)
⟨*proof*⟩

lemma *diamond-to-confluence*:
  [| *diamond R*; $T \leq R$; $R \leq T^{**}$ |] ==> *confluent T*
⟨*proof*⟩

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: $\textit{Church-Rosser}\ R = \textit{confluent}\ R$
⟨*proof*⟩

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
assumes *wf*: $\textit{wfP}\ (R^{-1-1})$
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible using *blast*.

theorem *newman'*:
assumes *wf*: $\textit{wfP}\ (R^{-1-1})$
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

Using the coherent logic prover, the proof of the induction step is completely automatic.

lemma *eq-imp-rtranclp*: $x = y \Longrightarrow r^{**}\ x\ y$
⟨*proof*⟩

theorem *newman''*:
assumes *wf*: $\textit{wfP}\ (R^{-1-1})$
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

inductive *par-beta* :: [*dB*, *dB*] => *bool* (**infixl** => *50*)
where
var [*simp*, *intro*!]: *Var n* => *Var n*
| *abs* [*simp*, *intro*!]: *s* => *t* ==> *Abs s* => *Abs t*
| *app* [*simp*, *intro*!]: [| s => s'; t => t' |] ==> $s \circ t$ => $s' \circ t'$
| *beta* [*simp*, *intro*!]: [| s => s'; t => t' |] ==> (*Abs s*) $\circ$ t => $s'[t'/0]$

inductive-cases *par-beta-cases* [*elim*!]:
Var n => *t*
Abs s => *Abs t*
(*Abs s*) $\circ$ *t* => *u*
s $\circ$ *t* => *u*
Abs s => *t*

3.2 Inclusions

beta $\subseteq$ *par-beta* $\subseteq$ $beta^*$

lemma *par-beta-varL* [*simp*]:
(*Var n* => *t*) = (*t* = *Var n*)
⟨*proof*⟩

lemma *par-beta-refl* [*simp*]: *t* => *t*
⟨*proof*⟩

lemma *beta-subset-par-beta*: *beta* <= *par-beta*
⟨*proof*⟩

lemma *par-beta-subset-beta*: *par-beta* $\leq$ $beta^{**}$
⟨*proof*⟩

3.3 Misc properties of *par-beta*

lemma *par-beta-lift* [*simp*]:
t => t' $\Longrightarrow$ *lift t n* => *lift* t' *n*
⟨*proof*⟩

lemma *par-beta-subst*:
s => s' $\Longrightarrow$ t => t' $\Longrightarrow$ $t[s/n]$ => $t'[s'/n]$
⟨*proof*⟩

3.4 Confluence (directly)

lemma *diamond-par-beta*: *diamond par-beta*
⟨*proof*⟩

3.5 Complete developments

fun

cd :: *dB* => *dB*
where
cd (*Var n*) = *Var n*
| *cd* (*Var n* ° *t*) = *Var n* ° *cd t*
| *cd* ((*s1* ° *s2*) ° *t*) = *cd* (*s1* ° *s2*) ° *cd t*
| *cd* (*Abs u* ° *t*) = (*cd u*)[*cd t*/*0*]
| *cd* (*Abs s*) = *Abs* (*cd s*)

lemma *par-beta-cd*: *s* => *t* ⟹ *t* => *cd s*
⟨*proof*⟩

3.6 Confluence (via complete developments)

lemma *diamond-par-beta2*: *diamond par-beta*
⟨*proof*⟩

theorem *beta-confluent*: *confluent beta*
⟨*proof*⟩

end

4 Eta-reduction

theory *Eta* **imports** *ParRed* **begin**

4.1 Definition of eta-reduction and relatives

primrec
free :: *dB* => *nat* => *bool*
where
free (*Var j*) *i* = (*j* = *i*)
| *free* (*s* ° *t*) *i* = (*free s i* ∨ *free t i*)
| *free* (*Abs s*) *i* = *free s* (*i* + *1*)

inductive
eta :: [*dB*, *dB*] => *bool* (**infixl** $\to_\eta$ *50*)
where
eta [*simp*, *intro*]: ¬ *free s 0* ==> *Abs* (*s* ° *Var 0*) $\to_\eta$ *s*[*dummy*/*0*]
| *appL* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *s* ° *u* $\to_\eta$ *t* ° *u*
| *appR* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *u* ° *s* $\to_\eta$ *u* ° *t*
| *abs* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *Abs s* $\to_\eta$ *Abs t*

abbreviation
eta-reds :: [*dB*, *dB*] => *bool* (**infixl** ${\to_\eta}^*$ *50*) **where**
s ${\to_\eta}^*$ *t* == *eta*** *s t*

abbreviation
eta-red0 :: [*dB*, *dB*] => *bool* (**infixl** ${\to_\eta}^=$ *50*) **where**
s ${\to_\eta}^=$ *t* == *eta*$^{==}$ *s t*

inductive-cases *eta-cases* [*elim*!]:
$Abs\ s \to_\eta z$
$s \circ t \to_\eta u$
$Var\ i \to_\eta t$

4.2 Properties of *eta*, *subst* and *free*

lemma *subst-not-free* [*simp*]: $\neg$ *free s i* $\Longrightarrow s[t/i] = s[u/i]$
⟨*proof*⟩

lemma *free-lift* [*simp*]:
free (*lift t k*) $i = (i < k \wedge$ *free t i* $\vee k < i \wedge$ *free t* $(i - 1))$
⟨*proof*⟩

lemma *free-subst* [*simp*]:
free $(s[t/k])\ i =$
(*free s k* $\wedge$ *free t i* $\vee$ *free s* (*if* $i < k$ *then* i *else* $i + 1$))
⟨*proof*⟩

lemma *free-eta*: $s \to_\eta t$ ==> *free t i* = *free s i*
⟨*proof*⟩

lemma *not-free-eta*:
[| $s \to_\eta t$; $\neg$ *free s i* |] ==> $\neg$ *free t i*
⟨*proof*⟩

lemma *eta-subst* [*simp*]:
$s \to_\eta t$ ==> $s[u/i] \to_\eta t[u/i]$
⟨*proof*⟩

theorem *lift-subst-dummy*: $\neg$ *free s i* $\Longrightarrow$ *lift* $(s[dummy/i])\ i = s$
⟨*proof*⟩

4.3 Confluence of *eta*

lemma *square-eta*: *square eta eta* $(eta^{==})$ $(eta^{==})$
⟨*proof*⟩

theorem *eta-confluent*: *confluent eta*
⟨*proof*⟩

4.4 Congruence rules for eta^*

lemma *rtrancl-eta-Abs*: $s \to_\eta{}^* s'$ ==> $Abs\ s \to_\eta{}^* Abs\ s'$
⟨*proof*⟩

lemma *rtrancl-eta-AppL*: $s \to_\eta{}^* s'$ ==> $s \circ t \to_\eta{}^* s' \circ t$
⟨*proof*⟩

lemma *rtrancl-eta-AppR*: $t \to_\eta{}^* t' ==> s \circ t \to_\eta{}^* s \circ t'$
⟨*proof*⟩

lemma *rtrancl-eta-App*:
$[| s \to_\eta{}^* s'; t \to_\eta{}^* t' |] ==> s \circ t \to_\eta{}^* s' \circ t'$
⟨*proof*⟩

4.5 Commutation of *beta* and *eta*

lemma *free-beta*:
$s \to_\beta t ==> \mathit{free}\ t\ i \Longrightarrow \mathit{free}\ s\ i$
⟨*proof*⟩

lemma *beta-subst* [*intro*]: $s \to_\beta t ==> s[u/i] \to_\beta t[u/i]$
⟨*proof*⟩

lemma *subst-Var-Suc* [*simp*]: $t[\mathit{Var}\ i/i] = t[\mathit{Var}(i)/i + 1]$
⟨*proof*⟩

lemma *eta-lift* [*simp*]: $s \to_\eta t ==> \mathit{lift}\ s\ i \to_\eta \mathit{lift}\ t\ i$
⟨*proof*⟩

lemma *rtrancl-eta-subst*: $s \to_\eta t \Longrightarrow u[s/i] \to_\eta{}^* u[t/i]$
⟨*proof*⟩

lemma *rtrancl-eta-subst'*:
fixes $s\ t :: \mathit{dB}$
assumes *eta*: $s \to_\eta{}^* t$
shows $s[u/i] \to_\eta{}^* t[u/i]$ ⟨*proof*⟩

lemma *rtrancl-eta-subst''*:
fixes $s\ t :: \mathit{dB}$
assumes *eta*: $s \to_\eta{}^* t$
shows $u[s/i] \to_\eta{}^* u[t/i]$ ⟨*proof*⟩

lemma *square-beta-eta*: $\mathit{square\ beta\ eta}\ (\mathit{eta}^{**})\ (\mathit{beta}^{==})$
⟨*proof*⟩

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
⟨*proof*⟩

4.6 Implicit definition of *eta*

$\mathit{Abs}\ (\mathit{lift}\ s\ 0 \circ \mathit{Var}\ 0) \to_\eta s$

lemma *not-free-iff-lifted*:
$(\neg\ \mathit{free}\ s\ i) = (\exists\, t.\ s = \mathit{lift}\ t\ i)$
⟨*proof*⟩

theorem *explicit-is-implicit*:

$(\forall\, s\ u.\ (\neg\ \mathit{free}\ s\ 0)\ \text{-->}\ R\ (\mathit{Abs}\ (s\ \circ\ \mathit{Var}\ 0))\ (s[u/0])) =$
$(\forall\, s.\ R\ (\mathit{Abs}\ (\mathit{lift}\ s\ 0\ \circ\ \mathit{Var}\ 0))\ s)$
⟨*proof*⟩

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
fixes s :: *dB*
assumes *free*: $\neg\ \mathit{free}\ s\ 0$
and *s*: $s[\mathit{dummy}/0] \Rightarrow u$
shows $\exists\, t'.\ \mathit{Abs}\ (s\ \circ\ \mathit{Var}\ 0) \Rightarrow t' \wedge t' \to_{\eta}{}^{*}\ u$
⟨*proof*⟩

theorem *eta-par-beta*:
assumes *st*: $s \to_{\eta}\ t$
and *tu*: $t \Rightarrow u$
shows $\exists\, t'.\ s \Rightarrow t' \wedge t' \to_{\eta}{}^{*}\ u$ ⟨*proof*⟩

theorem *eta-postponement'*:
assumes *eta*: $s \to_{\eta}{}^{*}\ t$ **and** *beta*: $t \Rightarrow u$
shows $\exists\, t'.\ s \Rightarrow t' \wedge t' \to_{\eta}{}^{*}\ u$ ⟨*proof*⟩

theorem *eta-postponement*:
assumes $(\mathit{sup}\ \mathit{beta}\ \mathit{eta})^{**}\ s\ t$
shows $(\mathit{beta}^{**}\ \mathit{OO}\ \mathit{eta}^{**})\ s\ t$ ⟨*proof*⟩

end

5 Application of a term to a list of terms

theory *ListApplication* **imports** *Lambda* **begin**

abbreviation
list-application :: *dB* => *dB list* => *dB* (**infixl** °° *150*) **where**
t °° ts == *foldl* (°) t ts

lemma *apps-eq-tail-conv* [*iff*]: (r °° ts = s °° ts) = (r = s)
⟨*proof*⟩

lemma *Var-eq-apps-conv* [*iff*]: (*Var* m = s °° ss) = (*Var* m = s ∧ ss = [])
⟨*proof*⟩

lemma *Var-apps-eq-Var-apps-conv* [*iff*]:
(*Var* m °° rs = *Var* n °° ss) = (m = n ∧ rs = ss)
⟨*proof*⟩

lemma *App-eq-foldl-conv*:
$(r \circ s = t \circ\circ ts) =$
$(\mathit{if}\ ts = []\ \mathit{then}\ r \circ s = t$
$\mathit{else}\ (\exists\, ss.\ ts = ss\ @\ [s] \wedge r = t \circ\circ ss))$
$\langle proof \rangle$

lemma *Abs-eq-apps-conv* [*iff*]:
$(Abs\ r = s \circ\circ ss) = (Abs\ r = s \wedge ss = [])$
$\langle proof \rangle$

lemma *apps-eq-Abs-conv* [*iff*]: $(s \circ\circ ss = Abs\ r) = (s = Abs\ r \wedge ss = [])$
$\langle proof \rangle$

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
$(Abs\ r \circ\circ rs = Abs\ s \circ\circ ss) = (r = s \wedge rs = ss)$
$\langle proof \rangle$

lemma *Abs-App-neq-Var-apps* [*iff*]:
$Abs\ s \circ t \neq Var\ n \circ\circ ss$
$\langle proof \rangle$

lemma *Var-apps-neq-Abs-apps* [*iff*]:
$Var\ n \circ\circ ts \neq Abs\ r \circ\circ ss$
$\langle proof \rangle$

lemma *ex-head-tail*:
$\exists\, ts\ h.\ t = h \circ\circ ts \wedge ((\exists\, n.\ h = Var\ n) \vee (\exists\, u.\ h = Abs\ u))$
$\langle proof \rangle$

lemma *size-apps* [*simp*]:
$size\ (r \circ\circ rs) = size\ r + foldl\ (+)\ 0\ (map\ size\ rs) + length\ rs$
$\langle proof \rangle$

lemma *lem0*: $[|\ (0::nat) < k;\ m <= n\ |] ==> m < n + k$
$\langle proof \rangle$

lemma *lift-map* [*simp*]:
$lift\ (t \circ\circ ts)\ i = lift\ t\ i \circ\circ map\ (\lambda t.\ lift\ t\ i)\ ts$
$\langle proof \rangle$

lemma *subst-map* [*simp*]:
$subst\ (t \circ\circ ts)\ u\ i = subst\ t\ u\ i \circ\circ map\ (\lambda t.\ subst\ t\ u\ i)\ ts$
$\langle proof \rangle$

lemma *app-last*: $(t \circ\circ ts) \circ u = t \circ\circ (ts\ @\ [u])$
$\langle proof \rangle$

A customized induction schema for $\circ\circ$.

lemma *lem*:
  assumes !!*n ts*. $\forall t \in set\ ts.\ P\ t$ ==> $P\ (Var\ n\ {}^{\circ\circ}\ ts)$
    and !!*u ts*. [| $P\ u$; $\forall t \in set\ ts.\ P\ t$ |] ==> $P\ (Abs\ u\ {}^{\circ\circ}\ ts)$
  shows $size\ t = n \Longrightarrow P\ t$
  ⟨*proof*⟩

theorem *Apps-dB-induct*:
  assumes !!*n ts*. $\forall t \in set\ ts.\ P\ t$ ==> $P\ (Var\ n\ {}^{\circ\circ}\ ts)$
    and !!*u ts*. [| $P\ u$; $\forall t \in set\ ts.\ P\ t$ |] ==> $P\ (Abs\ u\ {}^{\circ\circ}\ ts)$
  shows $P\ t$
  ⟨*proof*⟩

end

6 Simply-typed lambda terms

theory *LambdaType* **imports** *ListApplication* **begin**

6.1 Environments

definition
  $shift :: (nat \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a \Rightarrow nat \Rightarrow 'a$ (-⟨-:-⟩ [90, 0, 0] 91) **where**
  $e\langle i{:}a\rangle = (\lambda j.\ if\ j < i\ then\ e\ j\ else\ if\ j = i\ then\ a\ else\ e\ (j - 1))$

lemma *shift-eq* [*simp*]: $i = j \Longrightarrow (e\langle i{:}T\rangle)\ j = T$
  ⟨*proof*⟩

lemma *shift-gt* [*simp*]: $j < i \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ j$
  ⟨*proof*⟩

lemma *shift-lt* [*simp*]: $i < j \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ (j - 1)$
  ⟨*proof*⟩

lemma *shift-commute* [*simp*]: $e\langle i{:}U\rangle\langle 0{:}T\rangle = e\langle 0{:}T\rangle\langle Suc\ i{:}U\rangle$
  ⟨*proof*⟩

6.2 Types and typing rules

datatype *type* =
    Atom nat
  | *Fun type type*   (**infixr** ⇒ *200*)

inductive $typing :: (nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ (- ⊢ - : - [50, 50, 50] 50)
  where
    Var [*intro*!]: $env\ x = T \Longrightarrow env \vdash Var\ x : T$
  | *Abs* [*intro*!]: $env\langle 0{:}T\rangle \vdash t : U \Longrightarrow env \vdash Abs\ t : (T \Rightarrow U)$
  | *App* [*intro*!]: $env \vdash s : T \Rightarrow U \Longrightarrow env \vdash t : T \Longrightarrow env \vdash (s\ {}^{\circ}\ t) : U$

inductive-cases *typing-elims* [*elim*!]:

$e \vdash Var\ i : T$
$e \vdash t \circ u : T$
$e \vdash Abs\ t : T$

primrec
typings :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
where
typings e [] *Ts* = (*Ts* = [])
| *typings e* (*t* # *ts*) *Ts* =
(*case Ts of*
[] ⇒ *False*
| *T* # *Ts* ⇒ $e \vdash t : T$ ∧ *typings e ts Ts*)

abbreviation
typings-rel :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
(- ⊩ - : - [*50*, *50*, *50*] *50*) **where**
env ⊩ *ts* : *Ts* == *typings env ts Ts*

abbreviation
funs :: *type list* ⇒ *type* ⇒ *type* (**infixr** ⇛ *200*) **where**
Ts ⇛ *T* == *foldr Fun Ts T*

6.3 Some examples

schematic-goal $e \vdash Abs\ (Abs\ (Abs\ (Var\ 1 \circ (Var\ 2 \circ Var\ 1 \circ Var\ 0)))) : ?T$
⟨*proof*⟩

schematic-goal $e \vdash Abs\ (Abs\ (Abs\ (Var\ 2 \circ Var\ 0 \circ (Var\ 1 \circ Var\ 0)))) : ?T$
⟨*proof*⟩

6.4 Lists of types

lemma *lists-typings*:
e ⊩ *ts* : *Ts* ⟹ *listsp* (λ*t*. ∃ *T*. $e \vdash t : T$) *ts*
⟨*proof*⟩

lemma *types-snoc*: *e* ⊩ *ts* : *Ts* ⟹ $e \vdash t : T$ ⟹ *e* ⊩ *ts* @ [*t*] : *Ts* @ [*T*]
⟨*proof*⟩

lemma *types-snoc-eq*: *e* ⊩ *ts* @ [*t*] : *Ts* @ [*T*] =
(*e* ⊩ *ts* : *Ts* ∧ $e \vdash t : T$)
⟨*proof*⟩

lemma *rev-exhaust2* [*extraction-expand*]:
obtains (*Nil*) *xs* = [] | (*snoc*) *ys y* **where** *xs* = *ys* @ [*y*]
— Cannot use *rev-exhaust* from the *List* theory, since it is not constructive
⟨*proof*⟩

lemma *types-snocE*:
assumes ‹*e* ⊩ *ts* @ [*t*] : *Ts*›

obtains Us **and** U **where** ‹$Ts = Us$ @ $[U]$› ‹$e \Vdash ts : Us$› ‹$e \vdash t : U$›
⟨*proof*⟩

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \,^{\circ\circ}\, ts : T \Longrightarrow \exists\, Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
⟨*proof*⟩

lemma *list-app-typeE*:
$e \vdash t \,^{\circ\circ}\, ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \,^{\circ\circ}\, ts : T$
⟨*proof*⟩

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \,^{\circ\circ}\, ts : T \Longrightarrow e \vdash Var\ i \,^{\circ\circ}\, ts : U \Longrightarrow T = U$
⟨*proof*⟩

lemma *var-app-types*: $e \vdash Var\ i \,^{\circ\circ}\, ts \,^{\circ\circ}\, us : T \Longrightarrow e \Vdash ts : Ts \Longrightarrow$
$e \vdash Var\ i \,^{\circ\circ}\, ts : U \Longrightarrow \exists\, Us.\ U = Us \Rrightarrow T \wedge e \Vdash us : Us$
⟨*proof*⟩

lemma *var-app-typesE*: $e \vdash Var\ i \,^{\circ\circ}\, ts : T \Longrightarrow$
$(\bigwedge Ts.\ e \vdash Var\ i : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *abs-typeE*: $e \vdash Abs\ t : T \Longrightarrow (\bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro*!]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash lift\ t\ i : T$
⟨*proof*⟩

lemma *lift-types*:
$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (map\ (\lambda t.\ lift\ t\ i)\ ts) : Ts$
⟨*proof*⟩

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
⟨*proof*⟩

lemma *substs-lemma*:
$e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
$e \Vdash (map\ (\lambda t.\ t[u/i])\ ts) : Ts$
$\langle proof\rangle$

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \rightarrow_\beta t' \Longrightarrow e \vdash t' : T$
$\langle proof\rangle$

theorem *subject-reduction'*: $t \rightarrow_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
$\langle proof\rangle$

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
assumes
$(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
$(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
shows $P\ T$
$\langle proof\rangle$

end

7 Lifting an order to lists of elements

theory *ListOrder*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

Lifting an order to lists of elements, relating exactly one element.

definition
step1 :: ('*a* => '*a* => *bool*) => '*a list* => '*a list* => *bool* **where**
step1 r =
$(\lambda ys\ xs.\ \exists us\ z\ z'\ vs.\ xs = us\ @\ z\ \#\ vs \wedge r\ z'\ z \wedge ys =$
$us\ @\ z'\ \#\ vs)$

lemma *step1-converse* [*simp*]: $step1\ (r^{-1-1}) = (step1\ r)^{-1-1}$
$\langle proof\rangle$

lemma *in-step1-converse* [*iff*]: $(step1\ (r^{-1-1})\ x\ y) = ((step1\ r)^{-1-1}\ x\ y)$
$\langle proof\rangle$

lemma *not-Nil-step1* [*iff*]: $\neg\ step1\ r\ []\ xs$
$\langle proof\rangle$

lemma *not-step1-Nil* [*iff*]: ¬ *step1 r xs* []
⟨*proof*⟩

lemma *Cons-step1-Cons* [*iff*]:
(*step1 r* (*y* # *ys*) (*x* # *xs*)) =
(*r y x* ∧ *xs* = *ys* ∨ *x* = *y* ∧ *step1 r ys xs*)
⟨*proof*⟩

lemma *append-step1I*:
step1 r ys xs ∧ *vs* = *us* ∨ *ys* = *xs* ∧ *step1 r vs us*
==> *step1 r* (*ys* @ *vs*) (*xs* @ *us*)
⟨*proof*⟩

lemma *Cons-step1E* [*elim*!]:
assumes *step1 r ys* (*x* # *xs*)
and !!*y*. *ys* = *y* # *xs* ⟹ *r y x* ⟹ *R*
and !!*zs*. *ys* = *x* # *zs* ⟹ *step1 r zs xs* ⟹ *R*
shows *R*
⟨*proof*⟩

lemma *Snoc-step1-SnocD*:
step1 r (*ys* @ [*y*]) (*xs* @ [*x*])
==> (*step1 r ys xs* ∧ *y* = *x* ∨ *ys* = *xs* ∧ *r y x*)
⟨*proof*⟩

lemma *Cons-acc-step1I* [*intro*!]:
Wellfounded.accp r x ==> *Wellfounded.accp* (*step1 r*) *xs* ⟹ *Wellfounded.accp* (*step1 r*) (*x* # *xs*)
⟨*proof*⟩

lemma *lists-accD*: *listsp* (*Wellfounded.accp r*) *xs* ==> *Wellfounded.accp* (*step1 r*) *xs*
⟨*proof*⟩

lemma *ex-step1I*:
[| *x* ∈ *set xs*; *r y x* |]
==> ∃ *ys*. *step1 r ys xs* ∧ *y* ∈ *set ys*
⟨*proof*⟩

lemma *lists-accI*: *Wellfounded.accp* (*step1 r*) *xs* ==> *listsp* (*Wellfounded.accp r*) *xs*
⟨*proof*⟩

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
list-beta :: *dB list* => *dB list* => *bool* (**infixl** => *50*) **where**
rs => *ss* == *step1 beta rs ss*

lemma *head-Var-reduction*:
Var n °° *rs* $\rightarrow_\beta$ *v* ⟹ ∃ *ss*. *rs* => *ss* ∧ *v* = *Var n* °° *ss*
⟨*proof*⟩

lemma *apps-betasE* [*elim*!]:
assumes *major*: *r* °° *rs* $\rightarrow_\beta$ *s*
and *cases*: !!*r'*. [| *r* $\rightarrow_\beta$ *r'*; *s* = *r'* °° *rs* |] ==> *R*
!!*rs'*. [| *rs* => *rs'*; *s* = *r* °° *rs'* |] ==> *R*
!!*t u us*. [| *r* = *Abs t*; *rs* = *u* # *us*; *s* = *t*[*u*/*0*] °° *us* |] ==> *R*
shows *R*
⟨*proof*⟩

lemma *apps-preserves-beta* [*simp*]:
r $\rightarrow_\beta$ *s* ==> *r* °° *ss* $\rightarrow_\beta$ *s* °° *ss*
⟨*proof*⟩

lemma *apps-preserves-beta2* [*simp*]:
r $\rightarrow_\beta{}^*$ *s* ==> *r* °° *ss* $\rightarrow_\beta{}^*$ *s* °° *ss*
⟨*proof*⟩

lemma *apps-preserves-betas* [*simp*]:
rs => *ss* ⟹ *r* °° *rs* $\rightarrow_\beta$ *r* °° *ss*
⟨*proof*⟩

end

9 Inductive characterization of terminating lambda terms

theory *InductTermi* **imports** *ListBeta* **begin**

9.1 Terminating lambda terms

inductive *IT* :: *dB* => *bool*
where
Var [*intro*]: *listsp IT rs* ==> *IT* (*Var n* °° *rs*)
| *Lambda* [*intro*]: *IT r* ==> *IT* (*Abs r*)
| *Beta* [*intro*]: *IT* ((*r*[*s*/*0*]) °° *ss*) ==> *IT s* ==> *IT* ((*Abs r* ° *s*) °° *ss*)

9.2 Every term in *IT* terminates

lemma *double-induction-lemma* [*rule-format*]:
termip beta s ==> ∀ *t*. *termip beta t* −−>

$(\forall\, r\ ss.\ t = r[s/0]\ {}^{\circ\circ}\ ss\ {-}{-}{>}\ termip\ beta\ (Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss))$
⟨*proof*⟩

lemma *IT-implies-termi*: *IT t ==> termip beta t*
⟨*proof*⟩

9.3 Every terminating term is in *IT*

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: $Var\ n\ {}^{\circ\circ}\ ss \neq Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ts$
⟨*proof*⟩

lemma [*simp*]:
$(Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss = Abs\ r'\ {}^{\circ}\ s'\ {}^{\circ\circ}\ ss') = (r = r' \wedge s = s' \wedge ss = ss')$
⟨*proof*⟩

inductive-cases [*elim*!]:
$IT\ (Var\ n\ {}^{\circ\circ}\ ss)$
$IT\ (Abs\ t)$
$IT\ (Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ts)$

theorem *termi-implies-IT*: *termip beta r ==> IT r*
⟨*proof*⟩

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *LambdaType InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

lemma *lift-IT* [*intro*!]: $IT\ t \Longrightarrow IT\ (lift\ t\ i)$
⟨*proof*⟩

lemma *lifts-IT*: $listsp\ IT\ ts \Longrightarrow listsp\ IT\ (map\ (\lambda t.\ lift\ t\ 0)\ ts)$
⟨*proof*⟩

lemma *subst-Var-IT*: $IT\ r \Longrightarrow IT\ (r[Var\ i/j])$
⟨*proof*⟩

lemma *Var-IT*: $IT\ (Var\ n)$
⟨*proof*⟩

lemma *app-Var-IT*: $IT\ t \Longrightarrow IT\ (t \circ Var\ i)$
⟨*proof*⟩

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:
$\bigwedge t\ e\ T\ u\ i.\ IT\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
$IT\ u \Longrightarrow e \vdash u : U \Longrightarrow IT\ (t[u/i])$
(**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
⟨*proof*⟩

10.3 Well-typed terms are strongly normalizing

lemma *type-implies-IT*:
assumes $e \vdash t : T$
shows $IT\ t$
⟨*proof*⟩

theorem *type-implies-termi*: $e \vdash t : T \Longrightarrow termip\ beta\ t$
⟨*proof*⟩

end

11 Inductive characterization of lambda terms in normal form

theory *NormalForm*
imports *ListBeta*
begin

11.1 Terms in normal form

definition
$listall :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
$listall\ P\ xs \equiv (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand-def*]

theorem *listall-nil*: $listall\ P\ []$
⟨*proof*⟩

theorem *listall-nil-eq* [*simp*]: $listall\ P\ [] = True$
⟨*proof*⟩

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
⟨*proof*⟩

theorem *listall-cons-eq* [*simp*]: $listall\ P\ (x\ \#\ xs) = (P\ x \wedge listall\ P\ xs)$

⟨*proof*⟩

lemma *listall-conj1*: *listall* $(\lambda x.\ P\ x \wedge Q\ x)$ *xs* $\Longrightarrow$ *listall P xs*
⟨*proof*⟩

lemma *listall-conj2*: *listall* $(\lambda x.\ P\ x \wedge Q\ x)$ *xs* $\Longrightarrow$ *listall Q xs*
⟨*proof*⟩

lemma *listall-app*: *listall P* (*xs* @ *ys*) = (*listall P xs* $\wedge$ *listall P ys*)
⟨*proof*⟩

lemma *listall-snoc* [*simp*]: *listall P* (*xs* @ [*x*]) = (*listall P xs* $\wedge$ *P x*)
⟨*proof*⟩

lemma *listall-cong* [*cong*, *extraction-expand*]:
xs = *ys* $\Longrightarrow$ *listall P xs* = *listall P ys*
— Currently needed for strange technical reasons
⟨*proof*⟩

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: *listall P xs* = *listsp P xs*
⟨*proof*⟩

inductive *NF* :: *dB* ⇒ *bool*
where
App: *listall NF ts* $\Longrightarrow$ *NF* (*Var x* °° *ts*)
| *Abs*: *NF t* $\Longrightarrow$ *NF* (*Abs t*)
monos *listall-def*

lemma *nat-eq-dec*: $\bigwedge n$::*nat*. $m = n \vee m \neq n$
⟨*proof*⟩

lemma *nat-le-dec*: $\bigwedge n$::*nat*. $m < n \vee \neg\ (m < n)$
⟨*proof*⟩

lemma *App-NF-D*: **assumes** *NF*: *NF* (*Var n* °° *ts*)
shows *listall NF ts* ⟨*proof*⟩

11.2 Properties of *NF*

lemma *Var-NF*: *NF* (*Var n*)
⟨*proof*⟩

lemma *Abs-NF*:
assumes *NF*: *NF* (*Abs t* °° *ts*)
shows *ts* = [] ⟨*proof*⟩

lemma *subst-terms-NF*: *listall NF ts* $\Longrightarrow$
listall $(\lambda t.\ \forall i\ j.\ NF\ (t[Var\ i/j]))$ *ts* $\Longrightarrow$

$listall\ NF\ (map\ (\lambda t.\ t[Var\ i/j])\ ts)$
⟨*proof*⟩

lemma *subst-Var-NF*: $NF\ t \Longrightarrow NF\ (t[Var\ i/j])$
⟨*proof*⟩

lemma *app-Var-NF*: $NF\ t \Longrightarrow \exists t'.\ t \circ Var\ i \rightarrow_\beta{}^* t' \wedge NF\ t'$
⟨*proof*⟩

lemma *lift-terms-NF*: $listall\ NF\ ts \Longrightarrow$
$listall\ (\lambda t.\ \forall i.\ NF\ (lift\ t\ i))\ ts \Longrightarrow$
$listall\ NF\ (map\ (\lambda t.\ lift\ t\ i)\ ts)$
⟨*proof*⟩

lemma *lift-NF*: $NF\ t \Longrightarrow NF\ (lift\ t\ i)$
⟨*proof*⟩

NF characterizes exactly the terms that are in normal form.

lemma *NF-eq*: $NF\ t = (\forall t'.\ \neg\ t \rightarrow_\beta t')$
⟨*proof*⟩

end

12 Standardization

theory *Standardization*
imports *NormalForm*
begin

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

declare *listrel-mono* [*mono-set*]

inductive
$sred :: dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\rightarrow_s$ *50*)
and $sredlist :: dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $[\rightarrow_s]$ *50*)
where
$s\ [\rightarrow_s]\ t \equiv listrelp\ (\rightarrow_s)\ s\ t$
| Var: $rs\ [\rightarrow_s]\ rs' \Longrightarrow Var\ x \circ\circ rs \rightarrow_s Var\ x \circ\circ rs'$
| Abs: $r \rightarrow_s r' \Longrightarrow ss\ [\rightarrow_s]\ ss' \Longrightarrow Abs\ r \circ\circ ss \rightarrow_s Abs\ r' \circ\circ ss'$
| $Beta$: $r[s/0] \circ\circ ss \rightarrow_s t \Longrightarrow Abs\ r \circ s \circ\circ ss \rightarrow_s t$

lemma *refl-listrelp*: $\forall x \in set\ xs.\ R\ x\ x \Longrightarrow listrelp\ R\ xs\ xs$
⟨*proof*⟩

lemma *refl-sred*: $t \rightarrow_s t$
⟨*proof*⟩

lemma *refl-sreds*: $ts\ [\rightarrow_s]\ ts$
⟨*proof*⟩

lemma *listrelp-conj1*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ R\ x\ y$
⟨*proof*⟩

lemma *listrelp-conj2*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ S\ x\ y$
⟨*proof*⟩

lemma *listrelp-app*:
assumes *xsys*: $listrelp\ R\ xs\ ys$
shows $listrelp\ R\ xs'\ ys' \Longrightarrow listrelp\ R\ (xs\ @\ xs')\ (ys\ @\ ys')$ ⟨*proof*⟩

lemma *lemma1*:
assumes *r*: $r \rightarrow_s r'$ **and** *s*: $s \rightarrow_s s'$
shows $r \circ s \rightarrow_s r' \circ s'$ ⟨*proof*⟩

lemma *lemma1'*:
assumes *ts*: $ts\ [\rightarrow_s]\ ts'$
shows $r \rightarrow_s r' \Longrightarrow r \circ\circ ts \rightarrow_s r' \circ\circ ts'$ ⟨*proof*⟩

lemma *lemma2-1*:
assumes *beta*: $t \rightarrow_\beta u$
shows $t \rightarrow_s u$ ⟨*proof*⟩

lemma *listrelp-betas*:
assumes *ts*: $listrelp\ (\rightarrow_\beta{}^*)\ ts\ ts'$
shows $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow t \circ\circ ts \rightarrow_\beta{}^* t' \circ\circ ts'$ ⟨*proof*⟩

lemma *lemma2-2*:
assumes *t*: $t \rightarrow_s u$
shows $t \rightarrow_\beta{}^* u$ ⟨*proof*⟩

lemma *sred-lift*:
assumes *s*: $s \rightarrow_s t$
shows $lift\ s\ i \rightarrow_s lift\ t\ i$ ⟨*proof*⟩

lemma *lemma3*:
assumes *r*: $r \rightarrow_s r'$
shows $s \rightarrow_s s' \Longrightarrow r[s/x] \rightarrow_s r'[s'/x]$ ⟨*proof*⟩

lemma *lemma4-aux*:
assumes *rs*: $listrelp\ (\lambda t\ u.\ t \rightarrow_s u \wedge (\forall r.\ u \rightarrow_\beta r \longrightarrow t \rightarrow_s r))\ rs\ rs'$
shows $rs' \Rightarrow ss \Longrightarrow rs\ [\rightarrow_s]\ ss$ ⟨*proof*⟩

lemma *lemma4*:

assumes *r*: $r \to_s r'$
shows $r' \to_\beta r'' \Longrightarrow r \to_s r''$ ⟨*proof*⟩

lemma *rtrancl-beta-sred*:
assumes *r*: $r \to_\beta{}^* r'$
shows $r \to_s r'$ ⟨*proof*⟩

12.2 Leftmost reduction and weakly normalizing terms

inductive
lred :: *dB* ⇒ *dB* ⇒ *bool* (**infixl** $\to_l$ *50*)
and *lredlist* :: *dB list* ⇒ *dB list* ⇒ *bool* (**infixl** $[\to_l]$ *50*)
where
$s\ [\to_l]\ t \equiv listrelp\ (\to_l)\ s\ t$
| *Var*: $rs\ [\to_l]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \to_l Var\ x\ {}^{\circ\circ}\ rs'$
| *Abs*: $r \to_l r' \Longrightarrow Abs\ r \to_l Abs\ r'$
| *Beta*: $r[s/0]\ {}^{\circ\circ}\ ss \to_l t \Longrightarrow Abs\ r \circ s\ {}^{\circ\circ}\ ss \to_l t$

lemma *lred-imp-sred*:
assumes *lred*: $s \to_l t$
shows $s \to_s t$ ⟨*proof*⟩

inductive *WN* :: *dB* => *bool*
where
Var: *listsp WN rs* ⟹ *WN* (*Var n* °° *rs*)
| *Lambda*: *WN r* ⟹ *WN* (*Abs r*)
| *Beta*: *WN* ((*r*[*s*/*0*]) °° *ss*) ⟹ *WN* ((*Abs r* ° *s*) °° *ss*)

lemma *listrelp-imp-listsp1*:
assumes *H*: *listrelp* (λ*x y*. *P x*) *xs ys*
shows *listsp P xs* ⟨*proof*⟩

lemma *listrelp-imp-listsp2*:
assumes *H*: *listrelp* (λ*x y*. *P y*) *xs ys*
shows *listsp P ys* ⟨*proof*⟩

lemma *lemma5*:
assumes *lred*: $r \to_l r'$
shows *WN r* **and** *NF r′* ⟨*proof*⟩

lemma *lemma6*:
assumes *wn*: *WN r*
shows $\exists r'.\ r \to_l r'$ ⟨*proof*⟩

lemma *lemma7*:
assumes *r*: $r \to_s r'$
shows $NF\ r' \Longrightarrow r \to_l r'$ ⟨*proof*⟩

lemma *WN-eq*: $WN\ t = (\exists t'.\ t \to_\beta{}^* t' \wedge NF\ t')$

$\langle proof \rangle$

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *LambdaType NormalForm HOL−Library.Realizers HOL−Library.Code-Target-Int*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
  assumes *f-compat*: $\bigwedge t\ t'.\ t \rightarrow_\beta{}^*\ t' \Longrightarrow f\ t \rightarrow_\beta{}^*\ f\ t'$
  and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
  and *uNF*: $NF\ u$ **and** *uT*: $e \vdash u : T$
  shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
    $listall\ (\lambda t.\ \forall e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
      $NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists t'.\ t[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'))\ as \Longrightarrow$
    $\exists as'.\ \forall j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \rightarrow_\beta{}^*$
      $Var\ j\ {}^{\circ\circ}\ map\ f\ as' \wedge NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
  (**is** $\bigwedge Us.\ \text{-} \Longrightarrow listall\ ?R\ as \Longrightarrow \exists as'.\ ?ex\ Us\ as\ as'$)
$\langle proof \rangle$

lemma *subst-type-NF*:
  $\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists t'.\ t[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
  (**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
$\langle proof \rangle$
inductive *rtyping* :: $(nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ $(\text{-} \vdash_R \text{-} : \text{-}\ [50, 50, 50]\ 50)$
  where
    Var: $e\ x = T \Longrightarrow e \vdash_R Var\ x : T$
  | *Abs*: $e\langle 0{:}T\rangle \vdash_R t : U \Longrightarrow e \vdash_R Abs\ t : (T \Rightarrow U)$
  | *App*: $e \vdash_R s : T \Rightarrow U \Longrightarrow e \vdash_R t : T \Longrightarrow e \vdash_R (s \circ t) : U$

lemma *rtyping-imp-typing*: $e \vdash_R t : T \Longrightarrow e \vdash t : T$
  $\langle proof \rangle$

theorem *type-NF*:
  assumes $e \vdash_R t : T$
  shows $\exists t'.\ t \rightarrow_\beta{}^*\ t' \wedge NF\ t'$ $\langle proof \rangle$

13.2 Extracting the program

declare *NF.induct* [*ind-realizer*]
declare *rtranclp.induct* [*ind-realizer irrelevant*]
declare *rtyping.induct* [*ind-realizer*]
lemmas [*extraction-expand*] = *conj-assoc listall-cons-eq subst-all equal-allI*

extract *type-NF*

lemma *rtranclR-rtrancl-eq*: $rtranclpR\ r\ a\ b = r^{**}\ a\ b$
⟨*proof*⟩

lemma *NFR-imp-NF*: $NFR\ nf\ t \Longrightarrow NF\ t$
⟨*proof*⟩

The program corresponding to the proof of the central lemma, which performs substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$$\begin{array}{l}\bigwedge x.\ NFR\ x\ t \Longrightarrow \\ \quad e\langle i{:}U\rangle \vdash t : T \Longrightarrow \\ \quad (\bigwedge xa.\ NFR\ xa\ u \Longrightarrow \\ \qquad e \vdash u : U \Longrightarrow \\ \qquad t[u/i] \rightarrow_\beta{}^*\ fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge \\ \qquad NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U \\ T\ u\ x\ xa)))\end{array}$$

where *NFR* is the realizability predicate corresponding to the datatype *NFT*, which is inductively defined by the rules

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
  type-induct-P xc
   (λx H2 H2a xa xaa xb xc xd H.
       compat-NFT.rec-split-NFT default
        (λts xa xaa r xb xc xd xe H.
            var-app-typesE-P (xb⟨xe:x⟩) xa ts
             (λUs--. case nat-eq-dec xa xe of
                     Left ⇒ case ts of [] ⇒ (xd, H)
                             | a # list ⇒
                                case Us-- of [] ⇒ default
                                | T''-- # Ts-- ⇒
                                   let (x, y) =
                                         norm-list (λt. lift t 0) xd xb xe list Ts--
                                          (λt. lift-NF 0) H
                                          (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                       (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                       (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                       (xa, ya) =
                                         H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                       (x, y) =
                                         H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                          xb xc xa 0 (y 0) ya
                                   in (x, y)
                     | Right ⇒
                        let (x, y) =
                              let (x, y) =
                                    norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                     (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                              in (x, λx. y x)
                        in case nat-le-dec xe xa of
                            Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                            | Right ⇒ (dB.Var xa °° x, y xa)))
        (λt x r xa xaa xb xc H.
            abs-typeE-P xaa
             (λU V. let (x, y) =
                          let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xb 0) (Suc xc) (lift-NF 0 H)
                          in (dB.Abs x, NFT.Abs x y)
                    in (x, y)))
        H (λa. xaa a) xb xc xd)
   x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
   compat-NFT.rec-split-NFT default
    (λts x xa r xb xc.
        case nat-eq-dec x xc of
        Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
                  (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z))))
        | Right ⇒
            case nat-le-dec xc x of
            Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                      (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                        (listall-conj2-P-Q ts (λz. (xa z, r z))))
            | Right ⇒
                NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
                 (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z)))))
    (λt x r xa xaa. NFT.Abs (t[dB.Var (Suc xa)/Suc xaa]) (r (Suc xa) (Suc xaa))) H x xa

app-Var-NF ≡
λx. compat-NFT.rec-split-NFT default
      (λts xa xaa r.
          (dB.Var xa °° (ts @ [dB.Var x]),
           NFT.App (ts @ [dB.Var x]) xa
            (snd (listall-app-P ts)
              (listall-conj1-P-Q ts (λz. (xaa z, r z)),
               listall-cons-P (Var-NF x) listall-nil-eq-P))))
      (λt xa r. (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. compat-NFT.rec-split-NFT default
         (λts x xa r xb.
             case nat-le-dec x xb of
             Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                       (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                         (listall-conj2-P-Q ts (λz. (xa z, r z))))
             | Right ⇒
                 NFT.App (map (λt. lift t xb) ts) (Suc x)
                  (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z)))))
         (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rec-rtypingT (λe x T. (dB.Var x, Var-NF x))
      (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
      (λe s T U t x xa r ra.
          let (x, y) = r; (xa, ya) = ra;
              (x, y) =
                let (x, y) =
                      subst-type-NF (dB.Var 0 ° lift xa 0) e 0 (T ⇒ U) U x
                       (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                in (x, y)
          in (x, y))
      H
```

Figure 2: Program extracted from lemmas and main theorem

$\forall\, i{<}length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \to_{\beta}{}^{*}\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (s \circ t)\ U$

13.3 Generating executable code

instantiation *NFT* :: *default*
begin

definition *default* = *Dummy* ()

instance ⟨*proof*⟩

end

instantiation *dB* :: *default*
begin

definition *default* = *dB.Var 0*

instance ⟨*proof*⟩

end

instantiation *prod* :: (*default*, *default*) *default*
begin

definition *default* = (*default*, *default*)

instance ⟨*proof*⟩

end

instantiation *list* :: (*type*) *default*
begin

definition *default* = []

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *default*) *default*
begin

definition *default* = (λx. *default*)

instance ⟨*proof*⟩

end

definition *int-of-nat* :: *nat* ⇒ *int* **where**
int-of-nat = *of-nat*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

⟨*ML*⟩

end

References

[1] F. Joachimski and R. Matthes. Short proofs of normalization for the simply-typed λ-calculus, permutative conversions and Gödel's T. *Archive for Mathematical Logic*, 42(1):59–87, 2003.

[2] R. Loader. Notes on Simply Typed Lambda Calculus. Technical Report ECS-LFCS-98-381, Laboratory for Foundations of Computer Science, School of Informatics, University of Edinburgh, 1998.

[3] R. Matthes. Lambda Calculus: A Case for Inductive Definitions. In *Lecture notes of the 12th European Summer School in Logic, Language and Information (ESSLLI 2000)*. School of Computer Science, University of Birmingham, August 2000.

[4] M. Takahashi. Parallel reductions in λ-calculus. *Information and Computation*, 118(1):120–127, April 1995.