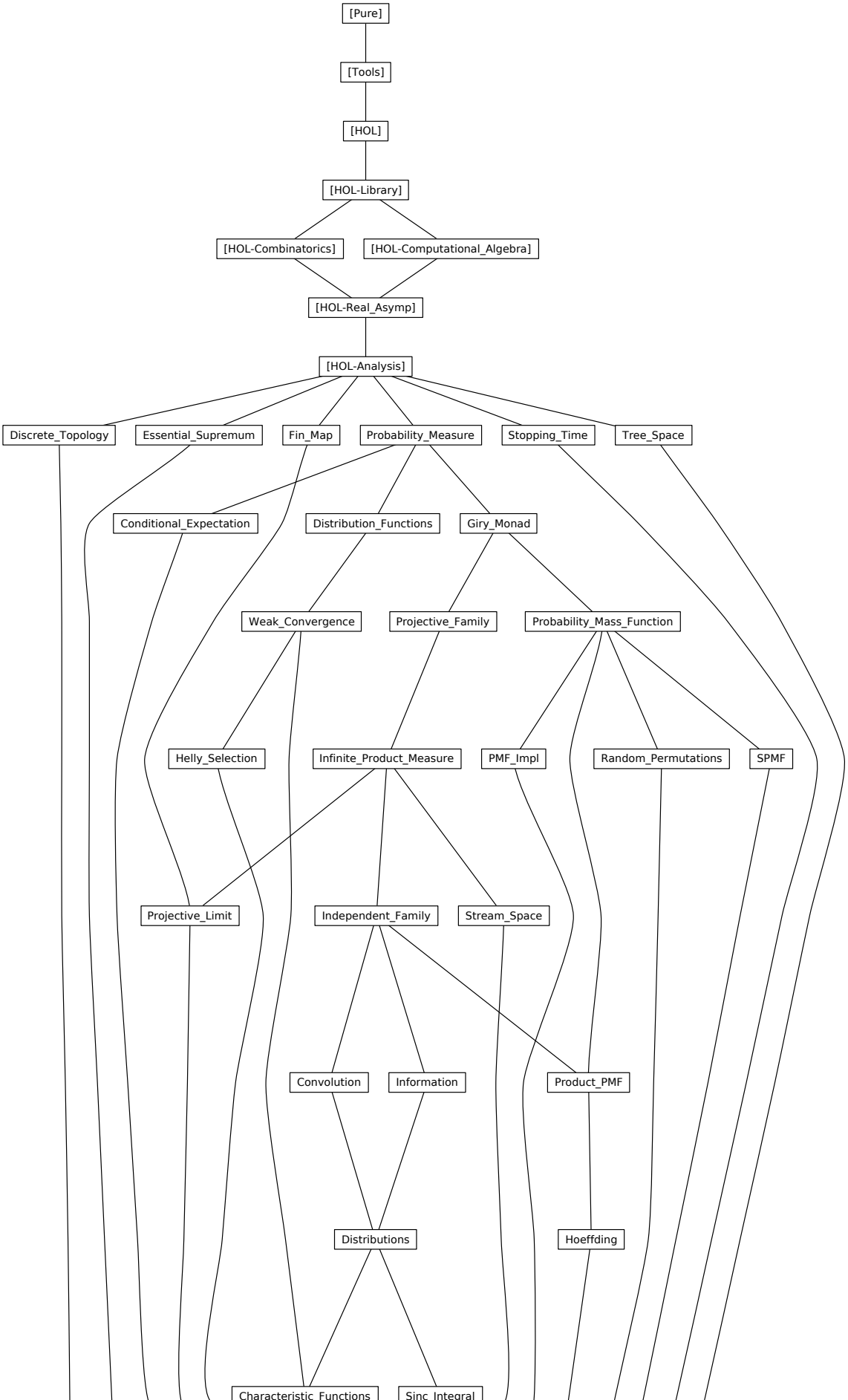
Measure and Probability Theory

January 18, 2026

Contents


[Pure]
[Tools]
[HOL]
[HOL-Library]
[HOL-Combinatorics]
[HOL-Computational_Algebra]
[HOL-Real_Asymp]
[HOL-Analysis]
Discrete_Topology
Essential_Supremum
Fin_Map
Probability_Measure
Stopping_Time
Tree_Space
Conditional_Expectation
Distribution_Functions
Giry_Monad
Weak_Convergence
Projective_Family
Probability_Mass_Function
Helly_Selection
Infinite_Product_Measure
PMF_Impl
Random_Permutations
SPMF
Projective_Limit
Independent_Family
Stream_Space
Convolution
Information
Product_PMF
Distributions
Hoeffding
Characteristic_Functions
Sinc_Integral

1 Probability measure

theory *Probability-Measure*
imports *HOL−Analysis.Analysis*
begin

locale *prob-space* = *finite-measure* +
assumes *emeasure-space-1*: *emeasure M* (*space M*) = *1*

lemma *prob-spaceI*[*Pure.intro*!]:
assumes ∗: *emeasure M* (*space M*) = *1*
shows *prob-space M*
by (*simp add*: *assms finite-measureI prob-space-axioms.intro prob-space-def*)

lemma *prob-space-imp-sigma-finite*: *prob-space M* ⟹ *sigma-finite-measure M*
unfolding *prob-space-def finite-measure-def* **by** *simp*

abbreviation (**in** *prob-space*) *events* ≡ *sets M*
abbreviation (**in** *prob-space*) *prob* ≡ *measure M*
abbreviation (**in** *prob-space*) *random-variable M′ X* ≡ *X* ∈ *measurable M M′*
abbreviation (**in** *prob-space*) *expectation* ≡ *integral*^{L} *M*
abbreviation (**in** *prob-space*) *variance X* ≡ *integral*^{L} *M* (λ*x*. (*X x* − *expectation X*)^2)

lemma (**in** *prob-space*) *finite-measure* [*simp*]: *finite-measure M*
by *unfold-locales*

lemma (**in** *prob-space*) *prob-space-distr*:
assumes *f*: *f* ∈ *measurable M M′* **shows** *prob-space* (*distr M M′ f*)
proof (*rule prob-spaceI*)
have *f* −‘ *space M′* ∩ *space M* = *space M* **using** *f* **by** (*auto dest*: *measurable-space*)
with *f* **show** *emeasure* (*distr M M′ f*) (*space* (*distr M M′ f*)) = *1*
by (*auto simp*: *emeasure-distr emeasure-space-1*)
qed

lemma *prob-space-distrD*:
assumes *f*: *f* ∈ *measurable M N* **and** *M*: *prob-space* (*distr M N f*) **shows** *prob-space M*
proof
interpret *M*: *prob-space distr M N f* **by** *fact*
have *f* −‘ *space N* ∩ *space M* = *space M*
using *f*[*THEN measurable-space*] **by** *auto*
then show *emeasure M* (*space M*) = *1*
using *M.emeasure-space-1* **by** (*simp add*: *emeasure-distr*[*OF f*])
qed

lemma (**in** *prob-space*) *prob-space*: *prob* (*space M*) = *1*
by (*simp add*: *emeasure-space-1 measure-eq-emeasure-eq-ennreal*)

lemma (**in** *prob-space*) *prob-le-1*[*simp*, *intro*]: *prob A* ≤ *1*
using *bounded-measure*[*of A*] **by** (*simp add*: *prob-space*)

lemma (**in** *prob-space*) *not-empty*: *space M* ≠ {}
using *prob-space* **by** *auto*

lemma (**in** *prob-space*) *emeasure-eq-1-AE*:
S ∈ *sets M* ⟹ *AE x in M*. *x* ∈ *S* ⟹ *emeasure M S* = *1*
by (*subst emeasure-eq-AE*[**where** *B*=*space M*]) (*auto simp*: *emeasure-space-1*)

lemma (**in** *prob-space*) *emeasure-le-1*: *emeasure M S* ≤ *1*
unfolding *ennreal-1*[*symmetric*] *emeasure-eq-measure* **by** (*subst ennreal-le-iff*) *auto*

lemma (**in** *prob-space*) *emeasure-ge-1-iff*: *emeasure M A* ≥ *1* ⟷ *emeasure M A* = *1*
by (*rule iffI*, *intro antisym emeasure-le-1*) *simp-all*

lemma (**in** *prob-space*) *AE-iff-emeasure-eq-1*:
assumes [*measurable*]: *Measurable.pred M P*
shows (*AE x in M*. *P x*) ⟷ *emeasure M* {*x*∈*space M*. *P x*} = *1*
proof −
have ∗: {*x* ∈ *space M*. ¬ *P x*} = *space M* − {*x*∈*space M*. *P x*}
by *auto*
show *?thesis*
by (*auto simp add*: *ennreal-minus-eq-0* ∗ *emeasure-compl emeasure-space-1 AE-iff-measurable*[*OF - refl*]
intro: *antisym emeasure-le-1*)
qed

lemma (**in** *prob-space*) *measure-le-1*: *emeasure M X* ≤ *1*
using *emeasure-space*[*of M X*] **by** (*simp add*: *emeasure-space-1*)

lemma (**in** *prob-space*) *measure-ge-1-iff*: *measure M A* ≥ *1* ⟷ *measure M A* = *1*
by (*auto intro*!: *antisym*)

lemma (**in** *prob-space*) *AE-I-eq-1*:
assumes *emeasure M* {*x*∈*space M*. *P x*} = *1* {*x*∈*space M*. *P x*} ∈ *sets M*
shows *AE x in M*. *P x*
proof (*rule AE-I*)
show *emeasure M* (*space M* − {*x* ∈ *space M*. *P x*}) = *0*
using *assms emeasure-space-1* **by** (*simp add*: *emeasure-compl*)
qed (*insert assms*, *auto*)

lemma *prob-space-restrict-space*:
S ∈ *sets M* ⟹ *emeasure M S* = *1* ⟹ *prob-space* (*restrict-space M S*)
by (*intro prob-spaceI*)

(*simp add*: *emeasure-restrict-space space-restrict-space*)

lemma (**in** *prob-space*) *prob-compl*:
assumes *A*: $A \in events$
shows $prob\ (space\ M - A) = 1 - prob\ A$
using *finite-measure-compl*[*OF A*] **by** (*simp add*: *prob-space*)

lemma (**in** *prob-space*) *AE-in-set-eq-1*:
assumes *A*[*measurable*]: $A \in events$ **shows** $(AE\ x\ in\ M.\ x \in A) \longleftrightarrow prob\ A = 1$
proof −
have ∗: $\{x \in space\ M.\ x \in A\} = A$
using *A*[*THEN sets.sets-into-space*] **by** *auto*
show *?thesis*
by (*subst AE-iff-emeasure-eq-1*) (*auto simp*: *emeasure-eq-measure* ∗)
qed

lemma (**in** *prob-space*) *AE-False*: $(AE\ x\ in\ M.\ False) \longleftrightarrow False$
proof
assume $AE\ x\ in\ M.\ False$
then have $AE\ x\ in\ M.\ x \in \{\}$ **by** *simp*
then show *False*
by (*subst* (*asm*) *AE-in-set-eq-1*) *auto*
qed *simp*

lemma (**in** *prob-space*) *AE-prob-1*:
assumes $prob\ A = 1$ **shows** $AE\ x\ in\ M.\ x \in A$
proof −
from ‹$prob\ A = 1$› **have** $A \in events$
by (*metis measure-notin-sets zero-neq-one*)
with *AE-in-set-eq-1 assms* **show** *?thesis* **by** *simp*
qed

lemma (**in** *prob-space*) *AE-const*[*simp*]: $(AE\ x\ in\ M.\ P) \longleftrightarrow P$
by (*cases P*) (*auto simp*: *AE-False*)

lemma (**in** *prob-space*) *ae-filter-bot*: $ae\text{-}filter\ M \neq bot$
by (*simp add*: *trivial-limit-def*)

lemma (**in** *prob-space*) *AE-contr*:
assumes *ae*: $AE\ \omega\ in\ M.\ P\ \omega\ AE\ \omega\ in\ M.\ \neg\ P\ \omega$
shows *False*
proof −
from *ae* **have** $AE\ \omega\ in\ M.\ False$ **by** *eventually-elim auto*
then show *False* **by** *auto*
qed

lemma (**in** *prob-space*) *integral-ge-const*:
fixes *c* :: *real*

shows *integrable M f* $\Longrightarrow$ (*AE x in M.* $c \leq f\ x$) $\Longrightarrow$ $c \leq (\int x.\ f\ x\ \partial M)$
using *integral-mono-AE*[*of M* $\lambda x.\ c\ f$] *prob-space* **by** *simp*

lemma (**in** *prob-space*) *integral-le-const*:
fixes c :: *real*
shows *integrable M f* $\Longrightarrow$ (*AE x in M.* $f\ x \leq c$) $\Longrightarrow$ $(\int x.\ f\ x\ \partial M) \leq c$
using *integral-mono-AE*[*of M f* $\lambda x.\ c$] *prob-space* **by** *simp*

lemma (**in** *prob-space*) *nn-integral-ge-const*:
(*AE x in M.* $c \leq f\ x$) $\Longrightarrow$ $c \leq (\int^+ x.\ f\ x\ \partial M)$
using *nn-integral-mono-AE*[*of* $\lambda x.\ c\ f\ M$] *emeasure-space-1*
by (*simp split*: *if-split-asm*)

lemma (**in** *prob-space*) *expectation-less*:
fixes X :: - $\Rightarrow$ *real*
assumes [*simp*]: *integrable M X*
assumes *gt*: *AE x in M.* $X\ x < b$
shows *expectation* $X < b$
proof −
have *expectation* $X <$ *expectation* $(\lambda x.\ b)$
using *gt emeasure-space-1*
by (*intro integral-less-AE-space*) *auto*
then show *?thesis* **using** *prob-space* **by** *simp*
qed

lemma (**in** *prob-space*) *expectation-greater*:
fixes X :: - $\Rightarrow$ *real*
assumes [*simp*]: *integrable M X*
assumes *gt*: *AE x in M.* $a < X\ x$
shows $a <$ *expectation X*
proof −
have *expectation* $(\lambda x.\ a) <$ *expectation X*
using *gt emeasure-space-1*
by (*intro integral-less-AE-space*) *auto*
then show *?thesis* **using** *prob-space* **by** *simp*
qed

lemma (**in** *prob-space*) *jensens-inequality*:
fixes q :: *real* $\Rightarrow$ *real*
assumes X: *integrable M X AE x in M.* $X\ x \in I$
assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I =$ *UNIV*
assumes q: *integrable M* $(\lambda x.\ q\ (X\ x))$ *convex-on I q*
shows q (*expectation X*) $\leq$ *expectation* $(\lambda x.\ q\ (X\ x))$
proof −
let *?F* = $\lambda x.$ *Inf* $((\lambda t.\ (q\ x - q\ t)\ /\ (x - t))\ `\ (\{x<..\} \cap I))$
from $X(2)$ *AE-False* **have** $I \neq \{\}$ **by** *auto*

from I **have** *open I* **by** *auto*

```
  note I
  moreover
  { assume I ⊆ {a <..}
    with X have a < expectation X
      by (intro expectation-greater) auto }
  moreover
  { assume I ⊆ {..< b}
    with X have expectation X < b
      by (intro expectation-less) auto }
  ultimately have expectation X ∈ I
    by (elim disjE)  (auto simp: subset-eq)
  moreover
  { fix y assume y: y ∈ I
    with q(2) ‹open I› have Sup ((λx. q x + ?F x * (y - x)) ` I) = q y
      by (auto intro!: cSup-eq-maximum convex-le-Inf-differential image-eqI [OF -
y] simp: interior-open) }
  ultimately have q (expectation X) = Sup ((λx. q x + ?F x * (expectation X -
x)) ` I)
    by simp
  also have ... ≤ expectation (λw. q (X w))
  proof (rule cSup-least)
    show (λx. q x + ?F x * (expectation X - x)) ` I ≠ {}
      using ‹I ≠ {}› by auto
  next
    fix k assume k ∈ (λx. q x + ?F x * (expectation X - x)) ` I
    then obtain x
      where x: k = q x + (INF t∈{x<..} ∩ I. (q x - q t) / (x - t)) * (expectation
X - x) x ∈ I ..
    have q x + ?F x * (expectation X  - x) = expectation (λw. q x + ?F x * (X
w - x))
      using prob-space by (simp add: X)
    also have ... ≤ expectation (λw. q (X w))
      using ‹x ∈ I› ‹open I› X(2)
    apply (intro integral-mono-AE Bochner-Integration.integrable-add Bochner-Integration.integrable-mult-righ
Bochner-Integration.integrable-diff
              integrable-const X q)
      apply (elim eventually-mono)
      apply (intro convex-le-Inf-differential)
      apply (auto simp: interior-open q)
      done
    finally show k ≤ expectation (λw. q (X w)) using x by auto
  qed
  finally show q (expectation X) ≤ expectation (λx. q (X x)) .
qed

lemma (in prob-space) finite-borel-measurable-integrable:
  assumes f∈ borel-measurable M
  and finite (f`(space M))
  shows integrable M f
```

proof −
have *simple-function M f* **using** *assms* **by** (*simp add*: *simple-function-borel-measurable*)
moreover have *emeasure M* {*y* ∈ *space M*. *f y* ≠ *0*} ≠ ∞ **by** *simp*
ultimately have *Bochner-Integration.simple-bochner-integrable M f*
using *Bochner-Integration.simple-bochner-integrable.simps* **by** *blast*
hence *has-bochner-integral M f* (*Bochner-Integration.simple-bochner-integral M f*)
using *has-bochner-integral-simple-bochner-integrable* **by** *auto*
thus *?thesis* **using** *integrable.simps* **by** *auto*
qed

1.1 Introduce binder for probability

syntax
-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('𝒫'((/- *in* -./ -)'))›)
syntax-consts
-prob == *measure*
translations
𝒫(*x in M*. *P*) => *CONST measure M* {*x* ∈ *CONST space M*. *P*}

print-translation ‹
let
fun to-pattern (*Const* (***const-syntax*** ‹*Pair*›, -) $ *l* $ *r*) =
Syntax.const ***const-syntax*** ‹*Pair*› :: *to-pattern l* @ *to-pattern r*
| *to-pattern* (*t as* (*Const* (***syntax-const*** ‹*-bound*›, -)) $ -) = [*t*]

fun mk-pattern ((*t*, *n*) :: *xs*) = *mk-patterns n xs* |>> *curry list-comb t*
and mk-patterns 0 xs = ([], *xs*)
| *mk-patterns n xs* =
let
val (*t*, *xs'*) = *mk-pattern xs*
val (*ts*, *xs''*) = *mk-patterns* (*n* − *1*) *xs'*
in
(*t* :: *ts*, *xs''*)
end

fun unnest-tuples
(*Const* (***syntax-const*** ‹*-pattern*›, -) $
t1 $
(*t as* (*Const* (***syntax-const*** ‹*-pattern*›, -) $ - $ -)))
= *let*
val (- $ *t2* $ *t3*) = *unnest-tuples t*
in
Syntax.const ***syntax-const*** ‹*-pattern*› $
unnest-tuples t1 $
(*Syntax.const* ***syntax-const*** ‹*-patterns*› $ *t2* $ *t3*)
end
| *unnest-tuples pat* = *pat*

```
  fun tr' ctxt [sig-alg, Const (const-syntax ‹Collect›, -) $ t] =
    let
      val bound-dummyT = Const (syntax-const ‹-bound›, dummyT)

      fun go pattern elem
        (Const (const-syntax ‹conj›, -) $
       (Const (const-syntax ‹Set.member›, -) $ elem' $ (Const (const-syntax ‹space›,
-) $ sig-alg')) $
          u)
        = let
          val - = if sig-alg aconv sig-alg' andalso to-pattern elem' = rev elem then
() else raise Match;
          val (pat, rest) = mk-pattern (rev pattern);
          val - = case rest of [] => () | - => raise Match
        in
          Syntax.const syntax-const ‹-prob› $ unnest-tuples pat $ sig-alg $ u
        end
      | go pattern elem (Abs abs) =
        let
          val (x as (- $ tx), t) = Syntax-Trans.atomic-abs-tr' ctxt abs
        in
          go ((x, 0) :: pattern) (bound-dummyT $ tx :: elem) t
        end
      | go pattern elem (Const (const-syntax ‹case-prod›, -) $ t) =
        go
          ((Syntax.const syntax-const ‹-pattern›, 2) :: pattern)
          (Syntax.const const-syntax ‹Pair› :: elem)
          t
    in
      go [] [] t
    end
  in
    [(const-syntax ‹Sigma-Algebra.measure›, tr')]
  end
›
```

definition
cond-prob M P Q = $\mathcal{P}(\omega$ *in M. P* $\omega \wedge$ *Q* $\omega)$ / $\mathcal{P}(\omega$ *in M. Q* $\omega)$

syntax
-conditional-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹($\mathcal{P}$'(- *in* -. - |/ -'))›)
syntax-consts
-conditional-prob == *cond-prob*
translations
$\mathcal{P}$(*x in M. P* | *Q*) => *CONST cond-prob M* (λ*x. P*) (λ*x. Q*)

lemma (**in** *prob-space*) *AE-E-prob*:
assumes *ae*: *AE x in M. P x*
obtains *S* **where** *S* ⊆ {*x* ∈ *space M. P x*} *S* ∈ *events prob S* = *1*

proof −
from *ae*[*THEN AE-E*] **obtain** *N*
where $\{x \in space\ M.\ \neg\ P\ x\} \subseteq N\ emeasure\ M\ N = 0\ N \in events$ **by** *auto*
then show *thesis*
by (*intro that*[*of space M − N*])
(*auto simp*: *prob-compl prob-space emeasure-eq-measure measure-nonneg*)
qed

lemma (**in** *prob-space*) *prob-neg*: $\{x{\in}space\ M.\ P\ x\} \in events \Longrightarrow \mathcal{P}(x\ in\ M.\ \neg\ P\ x) = 1\ -\ \mathcal{P}(x\ in\ M.\ P\ x)$
by (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *simp add*: *prob-compl*[*symmetric*])

lemma (**in** *prob-space*) *prob-eq-AE*:
$(AE\ x\ in\ M.\ P\ x \longleftrightarrow Q\ x) \Longrightarrow \{x{\in}space\ M.\ P\ x\} \in events \Longrightarrow \{x{\in}space\ M.\ Q\ x\} \in events \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = \mathcal{P}(x\ in\ M.\ Q\ x)$
by (*rule finite-measure-eq-AE*) *auto*

lemma (**in** *prob-space*) *prob-eq-0-AE*:
assumes *not*: $AE\ x\ in\ M.\ \neg\ P\ x$ **shows** $\mathcal{P}(x\ in\ M.\ P\ x) = 0$
proof *cases*
assume $\{x{\in}space\ M.\ P\ x\} \in events$
with *not* **have** $\mathcal{P}(x\ in\ M.\ P\ x) = \mathcal{P}(x\ in\ M.\ False)$
by (*intro prob-eq-AE*) *auto*
then show *?thesis* **by** *simp*
qed (*simp add*: *measure-notin-sets*)

lemma (**in** *prob-space*) *prob-Collect-eq-0*:
$\{x \in space\ M.\ P\ x\} \in sets\ M \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = 0 \longleftrightarrow (AE\ x\ in\ M.\ \neg\ P\ x)$
using *AE-iff-measurable*[*OF - refl, of M* $\lambda x.\ \neg\ P\ x$] **by** (*simp add*: *emeasure-eq-measure measure-nonneg*)

lemma (**in** *prob-space*) *prob-Collect-eq-1*:
$\{x \in space\ M.\ P\ x\} \in sets\ M \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = 1 \longleftrightarrow (AE\ x\ in\ M.\ P\ x)$
using *AE-in-set-eq-1*[*of* $\{x{\in}space\ M.\ P\ x\}$] **by** *simp*

lemma (**in** *prob-space*) *prob-eq-0*:
$A \in sets\ M \Longrightarrow prob\ A = 0 \longleftrightarrow (AE\ x\ in\ M.\ x \notin A)$
using *AE-iff-measurable*[*OF - refl, of M* $\lambda x.\ x \notin A$]
by (*auto simp add*: *emeasure-eq-measure Int-def*[*symmetric*] *measure-nonneg*)

lemma (**in** *prob-space*) *prob-eq-1*:
$A \in sets\ M \Longrightarrow prob\ A = 1 \longleftrightarrow (AE\ x\ in\ M.\ x \in A)$
using *AE-in-set-eq-1*[*of A*] **by** *simp*

lemma (**in** *prob-space*) *prob-sums*:
assumes *P*: $\bigwedge n.\ \{x{\in}space\ M.\ P\ n\ x\} \in events$
assumes *Q*: $\{x{\in}space\ M.\ Q\ x\} \in events$
assumes *ae*: $AE\ x\ in\ M.\ (\forall\ n.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists!n.\ P\ n\ x))$
shows $(\lambda n.\ \mathcal{P}(x\ in\ M.\ P\ n\ x))\ sums\ \mathcal{P}(x\ in\ M.\ Q\ x)$

proof −
from *ae*[*THEN AE-E-prob*] **obtain** *S*
where *S*:
$S \subseteq \{x \in \textit{space } M.\ (\forall n.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists! n.\ P\ n\ x))\}$
$S \in \textit{events}$
$\textit{prob}\ S = 1$
by *auto*
then have *disj*: *disjoint-family* $(\lambda n.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
by (*auto simp*: *disjoint-family-on-def*)
from *S* **have** *ae-S*:
$AE\ x\ \textit{in}\ M.\ x \in \{x \in \textit{space } M.\ Q\ x\} \longleftrightarrow x \in (\bigcup n.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
$\bigwedge n.\ AE\ x\ \textit{in}\ M.\ x \in \{x \in \textit{space } M.\ P\ n\ x\} \longleftrightarrow x \in \{x \in \textit{space } M.\ P\ n\ x\} \cap S$
using *ae* **by** (*auto dest*!: *AE-prob-1*)
from *ae-S* **have** ∗:
$\mathcal{P}(x\ \textit{in}\ M.\ Q\ x) = \textit{prob}\ (\bigcup n.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
using *P Q S* **by** (*intro finite-measure-eq-AE*) *auto*
from *ae-S* **have** ∗∗:
$\bigwedge n.\ \mathcal{P}(x\ \textit{in}\ M.\ P\ n\ x) = \textit{prob}\ (\{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
using *P Q S* **by** (*intro finite-measure-eq-AE*) *auto*
show *?thesis*
unfolding ∗ ∗∗ **using** *S P disj*
by (*intro finite-measure-UNION*) *auto*
qed

lemma (**in** *prob-space*) *prob-sum*:
assumes [*simp*, *intro*]: *finite I*
assumes *P*: $\bigwedge n.\ n \in I \Longrightarrow \{x \in \textit{space } M.\ P\ n\ x\} \in \textit{events}$
assumes *Q*: $\{x \in \textit{space } M.\ Q\ x\} \in \textit{events}$
assumes *ae*: $AE\ x\ \textit{in}\ M.\ (\forall n \in I.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists! n \in I.\ P\ n\ x))$
shows $\mathcal{P}(x\ \textit{in}\ M.\ Q\ x) = (\sum n \in I.\ \mathcal{P}(x\ \textit{in}\ M.\ P\ n\ x))$
proof −
from *ae*[*THEN AE-E-prob*] **obtain** *S*
where *S*:
$S \subseteq \{x \in \textit{space } M.\ (\forall n \in I.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists! n.\ n \in I \wedge P\ n\ x))\}$
$S \in \textit{events}$
$\textit{prob}\ S = 1$
by *auto*
then have *disj*: *disjoint-family-on* $(\lambda n.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)\ I$
by (*auto simp*: *disjoint-family-on-def*)
from *S* **have** *ae-S*:
$AE\ x\ \textit{in}\ M.\ x \in \{x \in \textit{space } M.\ Q\ x\} \longleftrightarrow x \in (\bigcup n \in I.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
$\bigwedge n.\ n \in I \Longrightarrow AE\ x\ \textit{in}\ M.\ x \in \{x \in \textit{space } M.\ P\ n\ x\} \longleftrightarrow x \in \{x \in \textit{space } M.\ P\ n\ x\} \cap S$
using *ae* **by** (*auto dest*!: *AE-prob-1*)
from *ae-S* **have** ∗:
$\mathcal{P}(x\ \textit{in}\ M.\ Q\ x) = \textit{prob}\ (\bigcup n \in I.\ \{x \in \textit{space } M.\ P\ n\ x\} \cap S)$
using *P Q S* **by** (*intro finite-measure-eq-AE*) (*auto intro*!: *sets.Int*)
from *ae-S* **have** ∗∗:

$\bigwedge n.\ n \in I \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ n\ x) = prob\ (\{x{\in}space\ M.\ P\ n\ x\} \cap S)$
using *P Q S* **by** (*intro finite-measure-eq-AE*) *auto*
show *?thesis*
using *S P disj*
by (*auto simp add*: ∗ ∗∗ *simp del*: *UN-simps intro*!: *finite-measure-finite-Union*)
qed

lemma (**in** *prob-space*) *prob-EX-countable*:
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}space\ M.\ P\ i\ x\} \in sets\ M$ **and** *I*: *countable I*
assumes *disj*: *AE x in M.* $\forall i{\in}I.\ \forall j{\in}I.\ P\ i\ x \longrightarrow P\ j\ x \longrightarrow i = j$
shows $\mathcal{P}(x\ in\ M.\ \exists i{\in}I.\ P\ i\ x) = (\int^+ i.\ \mathcal{P}(x\ in\ M.\ P\ i\ x)\ \partial count\text{-}space\ I)$
proof −
let *?N*= $\lambda x.\ \exists!i{\in}I.\ P\ i\ x$
have *ennreal* $(\mathcal{P}(x\ in\ M.\ \exists i{\in}I.\ P\ i\ x)) = \mathcal{P}(x\ in\ M.\ (\exists i{\in}I.\ P\ i\ x \wedge ?N\ x))$
unfolding *ennreal-inj*[*OF measure-nonneg measure-nonneg*]
proof (*rule prob-eq-AE*)
show *AE x in M.* $(\exists i{\in}I.\ P\ i\ x) = (\exists i{\in}I.\ P\ i\ x \wedge ?N\ x)$
using *disj* **by** *eventually-elim blast*
qed (*auto intro*!: *sets.sets-Collect-countable-Ex′ sets.sets-Collect-conj sets.sets-Collect-countable-Ex1′ I sets*)+
also have $\mathcal{P}(x\ in\ M.\ (\exists i{\in}I.\ P\ i\ x \wedge ?N\ x)) = emeasure\ M\ (\bigcup i{\in}I.\ \{x{\in}space\ M.\ P\ i\ x \wedge ?N\ x\})$
unfolding *emeasure-eq-measure* **by** (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *simp*: *measure-nonneg*)
also have $\ldots = (\int^+ i.\ emeasure\ M\ \{x{\in}space\ M.\ P\ i\ x \wedge ?N\ x\}\ \partial count\text{-}space\ I)$
by (*rule emeasure-UN-countable*)
(*auto intro*!: *sets.sets-Collect-countable-Ex′ sets.sets-Collect-conj sets.sets-Collect-countable-Ex1′ I sets*
simp: *disjoint-family-on-def*)
also have $\ldots = (\int^+ i.\ \mathcal{P}(x\ in\ M.\ P\ i\ x)\ \partial count\text{-}space\ I)$
unfolding *emeasure-eq-measure* **using** *disj*
by (*intro nn-integral-cong ennreal-inj*[*THEN iffD2*] *prob-eq-AE*)
(*auto intro*!: *sets.sets-Collect-countable-Ex′ sets.sets-Collect-conj sets.sets-Collect-countable-Ex1′ I sets measure-nonneg*)+
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *cond-prob-eq-AE*:
assumes *P*: *AE x in M.* $Q\ x \longrightarrow P\ x \longleftrightarrow P'\ x$ $\{x{\in}space\ M.\ P\ x\} \in events$ $\{x{\in}space\ M.\ P'\ x\} \in events$
assumes *Q*: *AE x in M.* $Q\ x \longleftrightarrow Q'\ x$ $\{x{\in}space\ M.\ Q\ x\} \in events$ $\{x{\in}space\ M.\ Q'\ x\} \in events$
shows *cond-prob M P Q* = *cond-prob M P′ Q′*
using *P Q*
by (*auto simp*: *cond-prob-def intro*!: *arg-cong2*[**where** *f*=(/)] *prob-eq-AE sets.sets-Collect-conj*)

lemma (**in** *prob-space*) *joint-distribution-Times-le-fst*:
random-variable MX X $\Longrightarrow$ *random-variable MY Y* $\Longrightarrow$ *A* ∈ *sets MX* $\Longrightarrow$ *B* ∈

sets MY
  $\Longrightarrow$ *emeasure* (*distr M* (*MX* $\bigotimes_M$ *MY*) ($\lambda x.$ (*X x*, *Y x*))) (*A* $\times$ *B*) $\leq$ *emeasure* (*distr M MX X*) *A*
 by (*auto simp*: *emeasure-distr measurable-pair-iff comp-def intro*!: *emeasure-mono measurable-sets*)

lemma (**in** *prob-space*) *joint-distribution-Times-le-snd*:
  random-variable MX X $\Longrightarrow$ *random-variable MY Y* $\Longrightarrow$ *A* $\in$ *sets MX* $\Longrightarrow$ *B* $\in$ *sets MY*
  $\Longrightarrow$ *emeasure* (*distr M* (*MX* $\bigotimes_M$ *MY*) ($\lambda x.$ (*X x*, *Y x*))) (*A* $\times$ *B*) $\leq$ *emeasure* (*distr M MY Y*) *B*
 by (*auto simp*: *emeasure-distr measurable-pair-iff comp-def intro*!: *emeasure-mono measurable-sets*)

lemma (**in** *prob-space*) *variance-eq*:
  fixes *X* :: *'a* $\Rightarrow$ *real*
  assumes [*simp*]: *integrable M X*
  assumes [*simp*]: *integrable M* ($\lambda x.$ (*X x*)^2)
  shows *variance X* = *expectation* ($\lambda x.$ (*X x*)^2) $-$ (*expectation X*)^2
  by (*simp add*: *field-simps prob-space power2-diff power2-eq-square*[*symmetric*])

lemma (**in** *prob-space*) *variance-positive*: *0* $\leq$ *variance* (*X*::*'a* $\Rightarrow$ *real*)
  by (*intro integral-nonneg-AE*) (*auto intro*!: *integral-nonneg-AE*)

lemma (**in** *prob-space*) *variance-mean-zero*:
  expectation X = *0* $\Longrightarrow$ *variance X* = *expectation* ($\lambda x.$ (*X x*) ^ *2*)
  by *simp*

theorem (**in** *prob-space*) *Chebyshev-inequality*:
  assumes [*measurable*]: *random-variable borel f*
  assumes *integrable M* ($\lambda x.$ *f x* ^ *2*)
  defines μ $\equiv$ *expectation f*
  assumes *a* > *0*
  shows *prob* {*x*$\in$*space M*. |*f x* $-$ μ| $\geq$ *a*} $\leq$ *variance f* / *a*^{2}
  unfolding μ*-def*
proof (*rule second-moment-method*)
  have *integrable*: *integrable M f*
    using *assms* **by** (*blast dest*: *square-integrable-imp-integrable*)
  show *integrable M* ($\lambda x.$ (*f x* $-$ *expectation f*)^2)
    using *assms integrable* **unfolding** *power2-eq-square ring-distribs*
    by (*intro Bochner-Integration.integrable-diff*) *auto*
qed (*use assms* **in** *auto*)

locale *pair-prob-space* = *pair-sigma-finite M1 M2* + *M1*: *prob-space M1* + *M2*: *prob-space M2* **for** *M1 M2*

sublocale *pair-prob-space* $\subseteq$ *P*?: *prob-space M1* $\bigotimes_M$ *M2*
proof
  show *emeasure* (*M1* $\bigotimes_M$ *M2*) (*space* (*M1* $\bigotimes_M$ *M2*)) = *1*

by (*simp add*: *M2.emeasure-pair-measure-Times M1.emeasure-space-1 M2.emeasure-space-1 space-pair-measure*)
qed

locale *product-prob-space* = *product-sigma-finite M* **for** *M* :: *'i* ⇒ *'a measure* +
fixes *I* :: *'i set*
assumes *prob-space*: ⋀*i*. *prob-space* (*M i*)

sublocale *product-prob-space* ⊆ *M?*: *prob-space M i* **for** *i*
by (*rule prob-space*)

locale *finite-product-prob-space* = *finite-product-sigma-finite M I* + *product-prob-space M I* **for** *M I*

sublocale *finite-product-prob-space* ⊆ *prob-space* Π$_M$ *i*∈*I*. *M i*
proof
show *emeasure* (Π$_M$ *i*∈*I*. *M i*) (*space* (Π$_M$ *i*∈*I*. *M i*)) = *1*
by (*simp add*: *measure-times M.emeasure-space-1 prod.neutral-const space-PiM*)
qed

lemma (**in** *finite-product-prob-space*) *prob-times*:
assumes *X*: ⋀*i*. *i* ∈ *I* ⟹ *X i* ∈ *sets* (*M i*)
shows *prob* (Π$_E$ *i*∈*I*. *X i*) = (∏ *i*∈*I*. *M.prob i* (*X i*))
proof −
have *ennreal* (*measure* (Π$_M$ *i*∈*I*. *M i*) (Π$_E$ *i*∈*I*. *X i*)) = *emeasure* (Π$_M$ *i*∈*I*. *M i*) (Π$_E$ *i*∈*I*. *X i*)
using *X* **by** (*simp add*: *emeasure-eq-measure*)
also have ... = (∏ *i*∈*I*. *emeasure* (*M i*) (*X i*))
using *measure-times X* **by** *simp*
also have ... = *ennreal* (∏ *i*∈*I*. *measure* (*M i*) (*X i*))
using *X* **by** (*simp add*: *M.emeasure-eq-measure prod-ennreal measure-nonneg*)
finally show *?thesis* **by** (*simp add*: *measure-nonneg prod-nonneg*)
qed

lemma *product-prob-spaceI*:
assumes ⋀*i*. *prob-space* (*M i*)
shows *product-prob-space M*
unfolding *product-prob-space-def product-prob-space-axioms-def product-sigma-finite-def*
proof *safe*
fix *i*
interpret *prob-space M i*
by (*rule assms*)
show *sigma-finite-measure* (*M i*) *prob-space* (*M i*)
by *unfold-locales*
qed

1.2 Distributions

definition *distributed* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *ennreal*) ⇒ *bool*
where
distributed M N X f ⟷
distr M N X = density N f ∧ *f* ∈ *borel-measurable N* ∧ *X* ∈ *measurable M N*

lemma
assumes *distributed M N X f*
shows *distributed-distr-eq-density*: *distr M N X = density N f*
and *distributed-measurable*: *X* ∈ *measurable M N*
and *distributed-borel-measurable*: *f* ∈ *borel-measurable N*
using *assms* **by** (*simp-all add*: *distributed-def*)

lemma
assumes *D*: *distributed M N X f*
shows *distributed-measurable'*[*measurable-dest*]:
g ∈ *measurable L M* ⟹ (λ*x*. *X* (*g x*)) ∈ *measurable L N*
and *distributed-borel-measurable'*[*measurable-dest*]:
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
using *distributed-measurable*[*OF D*] *distributed-borel-measurable*[*OF D*]
by *simp-all*

lemma *distributed-real-measurable*:
(⋀*x*. *x* ∈ *space N* ⟹ *0* ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹ *f* ∈ *borel-measurable N*
by (*simp-all add*: *distributed-def*)

lemma *distributed-real-measurable'*:
(⋀*x*. *x* ∈ *space N* ⟹ *0* ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
using *distributed-real-measurable*[*measurable*] **by** *simp*

lemma *joint-distributed-measurable1*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h1* ∈ *measurable N M* ⟹ (λ*x*. *X* (*h1 x*)) ∈ *measurable N S*
by *simp*

lemma *joint-distributed-measurable2*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h2* ∈ *measurable N M* ⟹ (λ*x*. *Y* (*h2 x*)) ∈ *measurable N T*
by *simp*

lemma *distributed-count-space*:
assumes *X*: *distributed M* (*count-space A*) *X P* **and** *a*: *a* ∈ *A* **and** *A*: *finite A*
shows *P a = emeasure M* (*X* −‘ {*a*} ∩ *space M*)
proof −
have *emeasure M* (*X* −‘ {*a*} ∩ *space M*) = *emeasure* (*distr M* (*count-space A*) *X*) {*a*}

using *X a A* **by** (*simp add*: *emeasure-distr*)
also have ... = *emeasure* (*density* (*count-space A*) *P*) {*a*}
using *X* **by** (*simp add*: *distributed-distr-eq-density*)
also have ... = (∫⁺*x*. *P a* ∗ *indicator* {*a*} *x* ∂*count-space A*)
using *X a* **by** (*auto simp add*: *emeasure-density distributed-def indicator-def intro*!: *nn-integral-cong*)
also have ... = *P a*
using *X a* **by** (*subst nn-integral-cmult-indicator*) (*auto simp*: *distributed-def one-ennreal-def*[*symmetric*] *AE-count-space*)
finally show *?thesis* **..**
qed

lemma *distributed-cong-density*:
(*AE x in N*. *f x* = *g x*) ⟹ *g* ∈ *borel-measurable N* ⟹ *f* ∈ *borel-measurable N* ⟹
distributed M N X f ⟷ *distributed M N X g*
by (*auto simp*: *distributed-def intro*!: *density-cong*)

lemma (**in** *prob-space*) *distributed-imp-emeasure-nonzero*:
assumes *X*: *distributed M MX X Px*
shows *emeasure MX* {*x* ∈ *space MX*. *Px x* ≠ *0*} ≠ *0*
proof
note *Px* = *distributed-borel-measurable*[*OF X*]
interpret *X*: *prob-space distr M MX X*
using *distributed-measurable*[*OF X*] **by** (*rule prob-space-distr*)

assume *emeasure MX* {*x* ∈ *space MX*. *Px x* ≠ *0*} = *0*
with *Px* **have** *AE x in MX*. *Px x* = *0*
by (*intro AE-I*[*OF subset-refl*]) (*auto simp*: *borel-measurable-ennreal-iff*)
moreover
from *X.emeasure-space-1* **have** (∫⁺*x*. *Px x* ∂*MX*) = *1*
unfolding *distributed-distr-eq-density*[*OF X*] **using** *Px*
by (*subst* (*asm*) *emeasure-density*)
(*auto simp*: *borel-measurable-ennreal-iff intro*!: *integral-cong cong*: *nn-integral-cong*)
ultimately show *False*
by (*simp add*: *nn-integral-cong-AE*)
qed

lemma *subdensity*:
assumes *T*: *T* ∈ *measurable P Q*
assumes *f*: *distributed M P X f*
assumes *g*: *distributed M Q Y g*
assumes *Y*: *Y* = *T* ∘ *X*
shows *AE x in P*. *g* (*T x*) = *0* ⟶ *f x* = *0*
proof −
have {*x*∈*space Q*. *g x* = *0*} ∈ *null-sets* (*distr M Q* (*T* ∘ *X*))
using *g Y* **by** (*auto simp*: *null-sets-density-iff distributed-def*)
also have *distr M Q* (*T* ∘ *X*) = *distr* (*distr M P X*) *Q T*
using *T f*[*THEN distributed-measurable*] **by** (*rule distr-distr*[*symmetric*])

finally have $T -` \{x{\in}space\ Q.\ g\ x = 0\} \cap space\ P \in null\text{-}sets\ (distr\ M\ P\ X)$
using T **by** (*subst* (*asm*) *null-sets-distr-iff*) *auto*
also have $T -` \{x{\in}space\ Q.\ g\ x = 0\} \cap space\ P = \{x{\in}space\ P.\ g\ (T\ x) = 0\}$
using T **by** (*auto dest*: *measurable-space*)
finally show *?thesis*
using *f g* **by** (*auto simp add*: *null-sets-density-iff distributed-def*)
qed

lemma *subdensity-real*:
fixes $g :: 'a \Rightarrow real$ **and** $f :: 'b \Rightarrow real$
assumes T: $T \in measurable\ P\ Q$
assumes f: *distributed M P X f*
assumes g: *distributed M Q Y g*
assumes Y: $Y = T \circ X$
shows $(AE\ x\ in\ P.\ 0 \le g\ (T\ x)) \Longrightarrow (AE\ x\ in\ P.\ 0 \le f\ x) \Longrightarrow AE\ x\ in\ P.\ g\ (T\ x) = 0 \longrightarrow f\ x = 0$
using *subdensity*[*OF T*, *of M X* $\lambda x.\ ennreal\ (f\ x)$ *Y* $\lambda x.\ ennreal\ (g\ x)$] *assms*
by *auto*

lemma *distributed-emeasure*:
distributed M N X f $\Longrightarrow A \in sets\ N \Longrightarrow emeasure\ M\ (X -` A \cap space\ M) = (\int^+ x.\ f\ x * indicator\ A\ x\ \partial N)$
by (*auto simp*: *distributed-distr-eq-density*[*symmetric*] *emeasure-density*[*symmetric*] *emeasure-distr*)

lemma *distributed-nn-integral*:
distributed M N X f $\Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow (\int^+ x.\ f\ x * g\ x\ \partial N) = (\int^+ x.\ g\ (X\ x)\ \partial M)$
by (*auto simp*: *distributed-distr-eq-density*[*symmetric*] *nn-integral-density*[*symmetric*] *nn-integral-distr*)

lemma *distributed-integral*:
distributed M N X f $\Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow (\bigwedge x.\ x \in space\ N \Longrightarrow 0 \le f\ x) \Longrightarrow$
$(\int x.\ f\ x * g\ x\ \partial N) = (\int x.\ g\ (X\ x)\ \partial M)$
supply *distributed-real-measurable*[*measurable*]
by (*auto simp*: *distributed-distr-eq-density*[*symmetric*] *integral-real-density*[*symmetric*] *integral-distr*)

lemma *distributed-transform-integral*:
assumes *Px*: *distributed M N X Px* $\bigwedge x.\ x \in space\ N \Longrightarrow 0 \le Px\ x$
assumes *distributed M P Y Py* $\bigwedge x.\ x \in space\ P \Longrightarrow 0 \le Py\ x$
assumes Y: $Y = T \circ X$ **and** T: $T \in measurable\ N\ P$ **and** f: $f \in borel\text{-}measurable\ P$
shows $(\int x.\ Py\ x * f\ x\ \partial P) = (\int x.\ Px\ x * f\ (T\ x)\ \partial N)$
proof −
have $(\int x.\ Py\ x * f\ x\ \partial P) = (\int x.\ f\ (Y\ x)\ \partial M)$
by (*rule distributed-integral*) *fact*+
also have $\ldots = (\int x.\ f\ (T\ (X\ x))\ \partial M)$

using *Y* **by** *simp*
also have ... = $(\int x.\ Px\ x * f\ (T\ x)\ \partial N)$
using *measurable-comp*[*OF T f*] *Px* **by** (*intro distributed-integral*[*symmetric*]) (*auto simp*: *comp-def*)
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *distributed-unique*:
assumes *Px*: *distributed M S X Px*
assumes *Py*: *distributed M S X Py*
shows *AE x in S. Px x = Py x*
proof −
interpret *X*: *prob-space distr M S X*
using *Px* **by** (*intro prob-space-distr*) *simp*
have *sigma-finite-measure* (*distr M S X*) **..**
with *sigma-finite-density-unique*[*of Px S Py*] *Px Py*
show *?thesis*
by (*auto simp*: *distributed-def*)
qed

lemma (**in** *prob-space*) *distributed-jointI*:
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *X*[*measurable*]: *X* ∈ *measurable M S* **and** *Y*[*measurable*]: *Y* ∈ *measurable M T*
assumes [*measurable*]: *f* ∈ *borel-measurable* ($S \bigotimes_M T$) **and** *f*: *AE x in S* $\bigotimes_M$ *T. 0* ≤ *f x*
assumes *eq*: $\bigwedge A\ B.\ A \in sets\ S \Longrightarrow B \in sets\ T \Longrightarrow$
emeasure M {*x* ∈ *space M. X x* ∈ *A* ∧ *Y x* ∈ *B*} = $(\int^+ x.\ (\int^+ y.\ f\ (x,\ y) * indicator\ B\ y\ \partial T) * indicator\ A\ x\ \partial S)$
shows *distributed M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) *f*
unfolding *distributed-def*
proof *safe*
interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**

from *ST.sigma-finite-up-in-pair-measure-generator*
obtain *F* :: *nat* ⇒ (′*b* × ′*c*) *set*
where *F*: *range F* ⊆ {*A* × *B* |*A B. A* ∈ *sets S* ∧ *B* ∈ *sets T*} ∧ *incseq F* ∧
$\bigcup$ (*range F*) = *space S* × *space T* ∧ (∀ *i. emeasure* ($S \bigotimes_M T$) (*F i*) ≠ ∞) **..**
let *?E* = {*a* × *b* |*a b. a* ∈ *sets S* ∧ *b* ∈ *sets T*}
let *?P* = $S \bigotimes_M T$
show *distr M ?P* ($\lambda x.\ (X\ x,\ Y\ x)$) = *density ?P f* (**is** *?L* = *?R*)
proof (*rule measure-eqI-generator-eq*[*OF Int-stable-pair-measure-generator*[*of S T*]])
show *?E* ⊆ *Pow* (*space ?P*)
using *sets.space-closed*[*of S*] *sets.space-closed*[*of T*] **by** (*auto simp*: *space-pair-measure*)
show *sets ?L* = *sigma-sets* (*space ?P*) *?E*
by (*simp add*: *sets-pair-measure space-pair-measure*)

```
    then show sets ?R = sigma-sets (space ?P) ?E
      by simp
  next
    interpret L: prob-space ?L
      by (rule prob-space-distr) (auto intro!: measurable-Pair)
    show range F ⊆ ?E (⋃ i. F i) = space ?P ⋀i. emeasure ?L (F i) ≠ ∞
      using F by (auto simp: space-pair-measure)
  next
    fix E assume E ∈ ?E
    then obtain A B where E[simp]: E = A × B
      and A[measurable]: A ∈ sets S and B[measurable]: B ∈ sets T by auto
    have emeasure ?L E = emeasure M {x ∈ space M. X x ∈ A ∧ Y x ∈ B}
      by (auto intro!: arg-cong[where f=emeasure M] simp add: emeasure-distr
measurable-Pair)
    also have ... = (∫⁺x. (∫⁺y. (f (x, y) * indicator B y) * indicator A x ∂T)
∂S)
      using f by (auto simp add: eq nn-integral-multc intro!: nn-integral-cong)
    also have ... = emeasure ?R E
      by (auto simp add: emeasure-density T.nn-integral-fst[symmetric]
            intro!: nn-integral-cong split: split-indicator)
    finally show emeasure ?L E = emeasure ?R E .
  qed
qed (auto simp: f)

lemma (in prob-space) distributed-swap:
  assumes sigma-finite-measure S sigma-finite-measure T
  assumes Pxy: distributed M (S ⨂M T) (λx. (X x, Y x)) Pxy
  shows distributed M (T ⨂M S) (λx. (Y x, X x)) (λ(x, y). Pxy (y, x))
proof −
  interpret S: sigma-finite-measure S by fact
  interpret T: sigma-finite-measure T by fact
  interpret ST: pair-sigma-finite S T ..
  interpret TS: pair-sigma-finite T S ..

  note Pxy[measurable]
  show ?thesis
    apply (subst TS.distr-pair-swap)
    unfolding distributed-def
  proof safe
    let ?D = distr (S ⨂M T) (T ⨂M S) (λ(x, y). (y, x))
    show 1: (λ(x, y). Pxy (y, x)) ∈ borel-measurable ?D
      by auto
    show 2: random-variable (distr (S ⨂M T) (T ⨂M S) (λ(x, y). (y, x))) (λx.
(Y x, X x))
      using Pxy by auto
    { fix A assume A: A ∈ sets (T ⨂M S)
      let ?B = (λ(x, y). (y, x)) −‘ A ∩ space (S ⨂M T)
      from sets.sets-into-space[OF A]
      have emeasure M ((λx. (Y x, X x)) −‘ A ∩ space M) =
```

```
      emeasure M ((λx. (X x, Y x)) -` ?B ∩ space M)
       by (auto intro!: arg-cong2[where f=emeasure] simp: space-pair-measure)
     also have ... = (∫⁺ x. Pxy x * indicator ?B x ∂(S ⨂M T))
       using Pxy A by (intro distributed-emeasure) auto
     finally have emeasure M ((λx. (Y x, X x)) -` A ∩ space M) =
       (∫⁺ x. Pxy x * indicator A (snd x, fst x) ∂(S ⨂M T))
       by (auto intro!: nn-integral-cong split: split-indicator) }
   note * = this
   show distr M ?D (λx. (Y x, X x)) = density ?D (λ(x, y). Pxy (y, x))
     apply (intro measure-eqI)
     apply (simp-all add: emeasure-distr[OF 2] emeasure-density[OF 1])
     apply (subst nn-integral-distr)
     apply (auto intro!: * simp: comp-def split-beta)
     done
 qed
qed

lemma (in prob-space) distr-marginal1:
 assumes sigma-finite-measure S sigma-finite-measure T
 assumes Pxy: distributed M (S ⨂M T) (λx. (X x, Y x)) Pxy
 defines Px ≡ λx. (∫⁺z. Pxy (x, z) ∂T)
 shows distributed M S X Px
 unfolding distributed-def
proof safe
 interpret S: sigma-finite-measure S by fact
 interpret T: sigma-finite-measure T by fact
 interpret ST: pair-sigma-finite S T ..

 note Pxy[measurable]
 show X: X ∈ measurable M S by simp

 show borel: Px ∈ borel-measurable S
   by (auto intro!: T.nn-integral-fst simp: Px-def)

 interpret Pxy: prob-space distr M (S ⨂M T) (λx. (X x, Y x))
   by (intro prob-space-distr) simp

 show distr M S X = density S Px
 proof (rule measure-eqI)
   fix A assume A: A ∈ sets (distr M S X)
   with X measurable-space[of Y M T]
   have emeasure (distr M S X) A = emeasure (distr M (S ⨂M T) (λx. (X x,
Y x))) (A × space T)
     by (auto simp add: emeasure-distr intro!: arg-cong[where f=emeasure M])
   also have ... = emeasure (density (S ⨂M T) Pxy) (A × space T)
     using Pxy by (simp add: distributed-def)
   also have ... = ∫⁺ x. ∫⁺ y. Pxy (x, y) * indicator (A × space T) (x, y) ∂T
∂S
     using A borel Pxy
```

by (*simp add*: *emeasure-density T.nn-integral-fst*[*symmetric*])
also have ... = $\int^+ x.\ Px\ x * indicator\ A\ x\ \partial S$
proof (*rule nn-integral-cong*)
fix x **assume** $x \in space\ S$
moreover have *eq*: $\bigwedge y.\ y \in space\ T \Longrightarrow indicator\ (A \times space\ T)\ (x,\ y) = indicator\ A\ x$
by (*auto simp*: *indicator-def*)
ultimately have $(\int^+ y.\ Pxy\ (x,\ y) * indicator\ (A \times space\ T)\ (x,\ y)\ \partial T) = (\int^+ y.\ Pxy\ (x,\ y)\ \partial T) * indicator\ A\ x$
by (*simp add*: *eq nn-integral-multc cong*: *nn-integral-cong*)
also have $(\int^+ y.\ Pxy\ (x,\ y)\ \partial T) = Px\ x$
by (*simp add*: *Px-def*)
finally show $(\int^+ y.\ Pxy\ (x,\ y) * indicator\ (A \times space\ T)\ (x,\ y)\ \partial T) = Px\ x * indicator\ A\ x$ **.**
qed
finally show *emeasure* (*distr M S X*) A = *emeasure* (*density S Px*) A
using A *borel Pxy* **by** (*simp add*: *emeasure-density*)
qed *simp*
qed

lemma (**in** *prob-space*) *distr-marginal2*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Pxy*: *distributed M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) *Pxy*
shows *distributed M T Y* ($\lambda y.\ (\int^+ x.\ Pxy\ (x,\ y)\ \partial S)$)
using *distr-marginal1*[*OF T S distributed-swap*[*OF S T*]] *Pxy* **by** *simp*

lemma (**in** *prob-space*) *distributed-marginal-eq-joint1*:
assumes *T*: *sigma-finite-measure T*
assumes *S*: *sigma-finite-measure S*
assumes *Px*: *distributed M S X Px*
assumes *Pxy*: *distributed M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) *Pxy*
shows *AE x in S*. *Px x* = $(\int^+ y.\ Pxy\ (x,\ y)\ \partial T)$
using *Px distr-marginal1*[*OF S T Pxy*] **by** (*rule distributed-unique*)

lemma (**in** *prob-space*) *distributed-marginal-eq-joint2*:
assumes *T*: *sigma-finite-measure T*
assumes *S*: *sigma-finite-measure S*
assumes *Py*: *distributed M T Y Py*
assumes *Pxy*: *distributed M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) *Pxy*
shows *AE y in T*. *Py y* = $(\int^+ x.\ Pxy\ (x,\ y)\ \partial S)$
using *Py distr-marginal2*[*OF S T Pxy*] **by** (*rule distributed-unique*)

lemma (**in** *prob-space*) *distributed-joint-indep'*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *X*[*measurable*]: *distributed M S X Px* **and** *Y*[*measurable*]: *distributed M T Y Py*
assumes *indep*: *distr M S X* $\bigotimes_M$ *distr M T Y* = *distr M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$)
shows *distributed M* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) ($\lambda(x,\ y).\ Px\ x * Py\ y$)

unfolding *distributed-def*
proof *safe*
interpret S: *sigma-finite-measure* S **by** *fact*
interpret T: *sigma-finite-measure* T **by** *fact*
interpret ST: *pair-sigma-finite* S T **..**

interpret X: *prob-space density* S Px
unfolding *distributed-distr-eq-density*[*OF* X, *symmetric*]
by (*rule prob-space-distr*) *simp*
have *sf-X*: *sigma-finite-measure* (*density* S Px) **..**

interpret Y: *prob-space density* T Py
unfolding *distributed-distr-eq-density*[*OF* Y, *symmetric*]
by (*rule prob-space-distr*) *simp*
have *sf-Y*: *sigma-finite-measure* (*density* T Py) **..**

show *distr* M ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) = *density* ($S \bigotimes_M T$) ($\lambda(x,\ y).\ Px\ x * Py\ y$)
unfolding *indep*[*symmetric*] *distributed-distr-eq-density*[*OF X*] *distributed-distr-eq-density*[*OF Y*]
using *distributed-borel-measurable*[*OF* X]
using *distributed-borel-measurable*[*OF* Y]
by (*rule pair-measure-density*[*OF - - T sf-Y*])

show *random-variable* ($S \bigotimes_M T$) ($\lambda x.\ (X\ x,\ Y\ x)$) **by** *auto*

show *Pxy*: $(\lambda(x,\ y).\ Px\ x * Py\ y) \in$ *borel-measurable* ($S \bigotimes_M T$) **by** *auto*
qed

lemma *distributed-integrable*:
distributed M N X f $\Longrightarrow$ $g \in$ *borel-measurable* N $\Longrightarrow$ ($\bigwedge x.\ x \in$ *space* $N \Longrightarrow 0 \leq f\ x$) $\Longrightarrow$
integrable N ($\lambda x.\ f\ x * g\ x$) $\longleftrightarrow$ *integrable* M ($\lambda x.\ g\ (X\ x)$)
supply *distributed-real-measurable*[*measurable*]
by (*auto simp*: *distributed-distr-eq-density*[*symmetric*] *integrable-real-density*[*symmetric*] *integrable-distr-eq*)

lemma *distributed-transform-integrable*:
assumes *Px*: *distributed* M N X Px $\bigwedge x.\ x \in$ *space* $N \Longrightarrow 0 \leq Px\ x$
assumes *distributed* M P Y Py $\bigwedge x.\ x \in$ *space* $P \Longrightarrow 0 \leq Py\ x$
assumes Y: $Y = (\lambda x.\ T\ (X\ x))$ **and** T: $T \in$ *measurable* N P **and** f: $f \in$ *borel-measurable* P
shows *integrable* P ($\lambda x.\ Py\ x * f\ x$) $\longleftrightarrow$ *integrable* N ($\lambda x.\ Px\ x * f\ (T\ x)$)
proof −
have *integrable* P ($\lambda x.\ Py\ x * f\ x$) $\longleftrightarrow$ *integrable* M ($\lambda x.\ f\ (Y\ x)$)
by (*rule distributed-integrable*) *fact+*
also have ... $\longleftrightarrow$ *integrable* M ($\lambda x.\ f\ (T\ (X\ x))$)
using Y **by** *simp*
also have ... $\longleftrightarrow$ *integrable* N ($\lambda x.\ Px\ x * f\ (T\ x)$)

using *measurable-comp*[*OF T f*] *Px* **by** (*intro distributed-integrable*[*symmetric*]) (*auto simp*: *comp-def*)
finally show *?thesis* **.**
qed

lemma *distributed-integrable-var*:
fixes $X :: 'a \Rightarrow real$
shows *distributed M lborel X* $(\lambda x.\ ennreal\ (f\ x)) \Longrightarrow (\bigwedge x.\ 0 \le f\ x) \Longrightarrow$
integrable lborel $(\lambda x.\ f\ x * x) \Longrightarrow$ *integrable M X*
using *distributed-integrable*[*of M lborel X f* $\lambda x.\ x$] **by** *simp*

lemma (**in** *prob-space*) *distributed-variance*:
fixes *f*::*real* $\Rightarrow$ *real*
assumes *D*: *distributed M lborel X f* **and** [*simp*]: $\bigwedge x.\ 0 \le f\ x$
shows *variance X* $= (\int x.\ x^2 * f\ (x + expectation\ X)\ \partial lborel)$
proof (*subst distributed-integral*[*OF D*, *symmetric*])
show $(\int x.\ f\ x * (x - expectation\ X)^2\ \partial lborel) = (\int x.\ x^2 * f\ (x + expectation\ X)\ \partial lborel)$
by (*subst lborel-integral-real-affine*[**where** *c=1* **and** *t=expectation X*]) (*auto simp*: *ac-simps*)
qed *simp-all*

lemma (**in** *prob-space*) *variance-affine*:
fixes *f*::*real* $\Rightarrow$ *real*
assumes [*arith*]: $b \neq 0$
assumes *D*[*intro*]: *distributed M lborel X f*
assumes [*simp*]: *prob-space* (*density lborel f*)
assumes *I*[*simp*]: *integrable M X*
assumes *I2*[*simp*]: *integrable M* $(\lambda x.\ (X\ x)^2)$
shows *variance* $(\lambda x.\ a + b * X\ x) = b^2 * variance\ X$
by (*subst variance-eq*)
(*auto simp*: *power2-sum power-mult-distrib prob-space variance-eq right-diff-distrib*)

definition
simple-distributed M X f $\longleftrightarrow$
$(\forall x.\ 0 \le f\ x) \wedge$
distributed M (*count-space* (*X*'*space M*)) *X* $(\lambda x.\ ennreal\ (f\ x)) \wedge$
finite (*X*'*space M*)

lemma *simple-distributed-nonneg*[*dest*]: *simple-distributed M X f* $\Longrightarrow$ $0 \le f\ x$
by (*auto simp*: *simple-distributed-def*)

lemma *simple-distributed*:
simple-distributed M X Px $\Longrightarrow$ *distributed M* (*count-space* (*X*'*space M*)) *X Px*
unfolding *simple-distributed-def* **by** *auto*

lemma *simple-distributed-finite*[*dest*]: *simple-distributed M X P* $\Longrightarrow$ *finite* (*X*'*space M*)
by (*simp add*: *simple-distributed-def*)

lemma (**in** *prob-space*) *distributed-simple-function-superset*:
assumes *X*: *simple-function M X* ⋀*x*. *x* ∈ *X* ‘ *space M* ⟹ *P x* = *measure M* (*X* −‘ {*x*} ∩ *space M*)
assumes *A*: *X*‘*space M* ⊆ *A finite A*
defines *S* ≡ *count-space A* **and** *P′* ≡ (λ*x*. *if x* ∈ *X*‘*space M then P x else 0*)
shows *distributed M S X P′*
unfolding *distributed-def*
proof *safe*
show (λ*x*. *ennreal* (*P′ x*)) ∈ *borel-measurable S* **unfolding** *S-def* **by** *simp*
show *distr M S X* = *density S P′*
proof (*rule measure-eqI-finite*)
show *sets* (*distr M S X*) = *Pow A sets* (*density S P′*) = *Pow A*
using *A* **unfolding** *S-def* **by** *auto*
show *finite A* **by** *fact*
fix *a* **assume** *a*: *a* ∈ *A*
then have *a* ∉ *X*‘*space M* ⟹ *X* −‘ {*a*} ∩ *space M* = {} **by** *auto*
with *A a X* **have** *emeasure* (*distr M S X*) {*a*} = *P′ a*
by (*subst emeasure-distr*)
(*auto simp add*: *S-def P′-def simple-functionD emeasure-eq-measure measurable-count-space-eq2*
intro!: *arg-cong*[**where** *f*=*prob*])
also have ... = (∫⁺*x*. *ennreal* (*P′ a*) ∗ *indicator* {*a*} *x* ∂*S*)
using *A X a*
by (*subst nn-integral-cmult-indicator*)
(*auto simp*: *S-def P′-def simple-distributed-def simple-functionD measure-nonneg*)
also have ... = (∫⁺*x*. *ennreal* (*P′ x*) ∗ *indicator* {*a*} *x* ∂*S*)
by (*auto simp*: *indicator-def intro*!: *nn-integral-cong*)
also have ... = *emeasure* (*density S P′*) {*a*}
using *a A* **by** (*intro emeasure-density*[*symmetric*]) (*auto simp*: *S-def*)
finally show *emeasure* (*distr M S X*) {*a*} = *emeasure* (*density S P′*) {*a*} **.**
qed
show *random-variable S X*
using *X*(*1*) *A* **by** (*auto simp*: *measurable-def simple-functionD S-def*)
qed

lemma (**in** *prob-space*) *simple-distributedI*:
assumes *X*: *simple-function M X*
⋀*x*. *0* ≤ *P x*
⋀*x*. *x* ∈ *X* ‘ *space M* ⟹ *P x* = *measure M* (*X* −‘ {*x*} ∩ *space M*)
shows *simple-distributed M X P*
unfolding *simple-distributed-def*
proof (*safe intro*!: *X*)
have *distributed M* (*count-space* (*X* ‘ *space M*)) *X* (λ*x*. *ennreal* (*if x* ∈ *X*‘*space M then P x else 0*))
(**is** *?A*)
using *simple-functionD*[*OF X*(*1*)] **by** (*intro distributed-simple-function-superset*[*OF X*(*1*,*3*)]) *auto*

also have *?A* $\longleftrightarrow$ *distributed M* (*count-space* (*X* ‘ *space M*)) *X* (λ*x*. *ennreal* (*P x*))
by (*rule distributed-cong-density*) *auto*
finally show ... **.**
qed (*rule simple-functionD*[*OF X*(*1*)])

lemma *simple-distributed-joint-finite*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Px*
shows *finite* (*X* ‘ *space M*) *finite* (*Y* ‘ *space M*)
proof −
have *finite* ((λ*x*. (*X x*, *Y x*)) ‘ *space M*)
using *X* **by** (*auto simp*: *simple-distributed-def simple-functionD*)
then have *finite* (*fst* ‘ (λ*x*. (*X x*, *Y x*)) ‘ *space M*) *finite* (*snd* ‘ (λ*x*. (*X x*, *Y x*)) ‘ *space M*)
by *auto*
then show *fin*: *finite* (*X* ‘ *space M*) *finite* (*Y* ‘ *space M*)
by (*auto simp*: *image-image*)
qed

lemma *simple-distributed-joint2-finite*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*, *Z x*)) *Px*
shows *finite* (*X* ‘ *space M*) *finite* (*Y* ‘ *space M*) *finite* (*Z* ‘ *space M*)
proof −
have *finite* ((λ*x*. (*X x*, *Y x*, *Z x*)) ‘ *space M*)
using *X* **by** (*auto simp*: *simple-distributed-def simple-functionD*)
then have *finite* (*fst* ‘ (λ*x*. (*X x*, *Y x*, *Z x*)) ‘ *space M*)
finite ((*fst* ∘ *snd*) ‘ (λ*x*. (*X x*, *Y x*, *Z x*)) ‘ *space M*)
finite ((*snd* ∘ *snd*) ‘ (λ*x*. (*X x*, *Y x*, *Z x*)) ‘ *space M*)
by *auto*
then show *fin*: *finite* (*X* ‘ *space M*) *finite* (*Y* ‘ *space M*) *finite* (*Z* ‘ *space M*)
by (*auto simp*: *image-image*)
qed

lemma *simple-distributed-simple-function*:
simple-distributed M X Px ⟹ *simple-function M X*
unfolding *simple-distributed-def distributed-def*
by (*auto simp*: *simple-function-def measurable-count-space-eq2*)

lemma *simple-distributed-measure*:
simple-distributed M X P ⟹ *a* ∈ *X*‘*space M* ⟹ *P a* = *measure M* (*X* −‘ {*a*} ∩ *space M*)
using *distributed-count-space*[*of M X*‘*space M X P a*, *symmetric*]
by (*auto simp*: *simple-distributed-def measure-def*)

lemma (**in** *prob-space*) *simple-distributed-joint*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Px*
defines *S* ≡ *count-space* (*X*‘*space M*) ⨂$_M$ *count-space* (*Y*‘*space M*)
defines *P* ≡ (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*))‘*space M then Px x else 0*)
shows *distributed M S* (λ*x*. (*X x*, *Y x*)) *P*

proof −
from *simple-distributed-joint-finite*[*OF X*, *simp*]
have *S-eq*: $S = \mathit{count\text{-}space}\ (X\text{`}\mathit{space}\ M \times Y\text{`}\mathit{space}\ M)$
by (*simp add*: *S-def pair-measure-count-space*)
show *?thesis*
unfolding *S-eq P-def*
proof (*rule distributed-simple-function-superset*)
show *simple-function M* $(\lambda x.\ (X\ x,\ Y\ x))$
using *X* **by** (*rule simple-distributed-simple-function*)
fix *x* **assume** $x \in (\lambda x.\ (X\ x,\ Y\ x))\ `\ \mathit{space}\ M$
from *simple-distributed-measure*[*OF X this*]
show $Px\ x = \mathit{prob}\ ((\lambda x.\ (X\ x,\ Y\ x)) -` \{x\} \cap \mathit{space}\ M)$ **.**
qed *auto*
qed

lemma (**in** *prob-space*) *simple-distributed-joint2*:
assumes *X*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\ Px$
defines $S \equiv \mathit{count\text{-}space}\ (X\text{`}\mathit{space}\ M) \bigotimes_M \mathit{count\text{-}space}\ (Y\text{`}\mathit{space}\ M) \bigotimes_M \mathit{count\text{-}space}\ (Z\text{`}\mathit{space}\ M)$
defines $P \equiv (\lambda x.\ \mathit{if}\ x \in (\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\text{`}\mathit{space}\ M\ \mathit{then}\ Px\ x\ \mathit{else}\ 0)$
shows *distributed M S* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\ P$
proof −
from *simple-distributed-joint2-finite*[*OF X*, *simp*]
have *S-eq*: $S = \mathit{count\text{-}space}\ (X\text{`}\mathit{space}\ M \times Y\text{`}\mathit{space}\ M \times Z\text{`}\mathit{space}\ M)$
by (*simp add*: *S-def pair-measure-count-space*)
show *?thesis*
unfolding *S-eq P-def*
proof (*rule distributed-simple-function-superset*)
show *simple-function M* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$
using *X* **by** (*rule simple-distributed-simple-function*)
fix *x* **assume** $x \in (\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\ `\ \mathit{space}\ M$
from *simple-distributed-measure*[*OF X this*]
show $Px\ x = \mathit{prob}\ ((\lambda x.\ (X\ x,\ Y\ x,\ Z\ x)) -` \{x\} \cap \mathit{space}\ M)$ **.**
qed *auto*
qed

lemma (**in** *prob-space*) *simple-distributed-sum-space*:
assumes *X*: *simple-distributed M X f*
shows $\mathit{sum}\ f\ (X\text{`}\mathit{space}\ M) = 1$
proof −
from *X* **have** $\mathit{sum}\ f\ (X\text{`}\mathit{space}\ M) = \mathit{prob}\ (\bigcup i \in X\text{`}\mathit{space}\ M.\ X -` \{i\} \cap \mathit{space}\ M)$
by (*subst finite-measure-finite-Union*)
(*auto simp add*: *disjoint-family-on-def simple-distributed-measure simple-distributed-simple-function simple-functionD*
intro!: *sum.cong arg-cong*[**where** *f*=*prob*])
also have $\ldots = \mathit{prob}\ (\mathit{space}\ M)$
by (*auto intro*!: *arg-cong*[**where** *f*=*prob*])
finally show *?thesis*
using *emeasure-space-1* **by** (*simp add*: *emeasure-eq-measure*)

qed

lemma (**in** *prob-space*) *distributed-marginal-eq-joint-simple*:
assumes *Px*: *simple-function M X*
assumes *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
assumes *y*: *y* ∈ *Y*'*space M*
shows *Py y* = (∑ *x*∈*X*'*space M*. *if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ' *space M then Pxy* (*x*, *y*) *else 0*)
proof −
note *Px* = *simple-distributedI*[*OF Px measure-nonneg refl*]
have *AE y in count-space* (*Y* ' *space M*). *ennreal* (*Py y*) =
∫⁺ *x*. *ennreal* (*if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ' *space M then Pxy* (*x*, *y*) *else 0*) ∂*count-space* (*X* ' *space M*)
using *sigma-finite-measure-count-space-finite sigma-finite-measure-count-space-finite simple-distributed*[*OF Py*] *simple-distributed-joint*[*OF Pxy*]
by (*rule distributed-marginal-eq-joint2*) (*auto intro*: *Py Px simple-distributed-finite*)
then have *ennreal* (*Py y*) =
(∑ *x*∈*X*'*space M*. *ennreal* (*if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ' *space M then Pxy* (*x*, *y*) *else 0*))
using *y Px*[*THEN simple-distributed-finite*]
by (*auto simp*: *AE-count-space nn-integral-count-space-finite*)
also have ... = (∑ *x*∈*X*'*space M*. *if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ' *space M then Pxy* (*x*, *y*) *else 0*)
using *Pxy* **by** (*intro sum-ennreal*) *auto*
finally show *?thesis*
using *simple-distributed-nonneg*[*OF Py*] *simple-distributed-nonneg*[*OF Pxy*]
by (*subst* (*asm*) *ennreal-inj*) (*auto intro*!: *sum-nonneg*)
qed

lemma *distributedI-real*:
fixes *f* :: *'a* ⇒ *real*
assumes *gen*: *sets M1* = *sigma-sets* (*space M1*) *E* **and** *Int-stable E*
and *A*: *range A* ⊆ *E* (⋃ *i*::*nat*. *A i*) = *space M1* ⋀*i*. *emeasure* (*distr M M1 X*) (*A i*) ≠ ∞
and *X*: *X* ∈ *measurable M M1*
and *f*: *f* ∈ *borel-measurable M1 AE x in M1*. 0 ≤ *f x*
and *eq*: ⋀*A*. *A* ∈ *E* ⟹ *emeasure M* (*X* −' *A* ∩ *space M*) = (∫⁺ *x*. *f x* ∗ *indicator A x* ∂*M1*)
shows *distributed M M1 X f*
unfolding *distributed-def*
proof (*intro conjI*)
show *distr M M1 X* = *density M1 f*
proof (*rule measure-eqI-generator-eq*[**where** *A*=*A*])
{ **fix** *A* **assume** *A*: *A* ∈ *E*
then have *A* ∈ *sigma-sets* (*space M1*) *E* **by** *auto*
then have *A* ∈ *sets M1*
using *gen* **by** *simp*

with *f A eq*[*of A*] *X* **show** *emeasure* (*distr M M1 X*) *A* = *emeasure* (*density M1 f*) *A*
by (*auto simp add*: *emeasure-distr emeasure-density ennreal-indicator intro*!: *nn-integral-cong split*: *split-indicator*) **}**
note *eq-E* = *this*
show *Int-stable E* **by** *fact*
{ fix *e* **assume** *e* ∈ *E*
then have *e* ∈ *sigma-sets* (*space M1*) *E* **by** *auto*
then have *e* ∈ *sets M1* **unfolding** *gen* **.**
then have *e* ⊆ *space M1* **by** (*rule sets.sets-into-space*) **}**
then show *E* ⊆ *Pow* (*space M1*) **by** *auto*
show *sets* (*distr M M1 X*) = *sigma-sets* (*space M1*) *E*
sets (*density M1* (λ*x*. *ennreal* (*f x*))) = *sigma-sets* (*space M1*) *E*
unfolding *gen*[*symmetric*] **by** *auto*
qed *fact*+
qed (*insert X f*, *auto*)

lemma *distributedI-borel-atMost*:
fixes *f* :: *real* ⇒ *real*
assumes [*measurable*]: *X* ∈ *borel-measurable M*
and [*measurable*]: *f* ∈ *borel-measurable borel* **and** *f*[*simp*]: *AE x in lborel*. 0 ≤ *f x*
and *g-eq*: ⋀*a*. (∫⁺*x*. *f x* ∗ *indicator* {..*a*} *x* ∂*lborel*) = *ennreal* (*g a*)
and *M-eq*: ⋀*a*. *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *ennreal* (*g a*)
shows *distributed M lborel X f*
proof (*rule distributedI-real*)
show *sets* (*lborel*::*real measure*) = *sigma-sets* (*space lborel*) (*range atMost*)
by (*simp add*: *borel-eq-atMost*)
show *Int-stable* (*range atMost* :: *real set set*)
by (*auto simp*: *Int-stable-def*)
have *vimage-eq*: ⋀*a*. (*X* −‘ {..*a*} ∩ *space M*) = {*x*∈*space M*. *X x* ≤ *a*} **by** *auto*
define *A* **where** *A i* = {.. *real i*} **for** *i* :: *nat*
then show *range A* ⊆ *range atMost* (⋃ *i*. *A i*) = *space lborel*
⋀*i*. *emeasure* (*distr M lborel X*) (*A i*) ≠ ∞
by (*auto simp*: *real-arch-simple emeasure-distr vimage-eq M-eq*)

fix *A* :: *real set* **assume** *A* ∈ *range atMost*
then obtain *a* **where** *A*: *A* = {..*a*} **by** *auto*
show *emeasure M* (*X* −‘ *A* ∩ *space M*) = (∫⁺*x*. *f x* ∗ *indicator A x* ∂*lborel*)
unfolding *vimage-eq A M-eq g-eq* **..**
qed *auto*

lemma (**in** *prob-space*) *uniform-distributed-params*:
assumes *X*: *distributed M MX X* (λ*x*. *indicator A x* / *measure MX A*)
shows *A* ∈ *sets MX measure MX A* ≠ *0*
proof −
interpret *X*: *prob-space distr M MX X*
using *distributed-measurable*[*OF X*] **by** (*rule prob-space-distr*)

show *measure MX A ≠ 0*
proof
assume *measure MX A = 0*
with *X.emeasure-space-1 X.prob-space distributed-distr-eq-density*[*OF X*]
show *False*
by (*simp add*: *emeasure-density zero-ennreal-def*[*symmetric*])
qed
with *measure-notin-sets*[*of A MX*] **show** *A ∈ sets MX*
by *blast*
qed

lemma *prob-space-uniform-measure*:
assumes *A*: *emeasure M A ≠ 0 emeasure M A ≠ ∞*
shows *prob-space* (*uniform-measure M A*)
proof
show *emeasure* (*uniform-measure M A*) (*space* (*uniform-measure M A*)) = *1*
using *emeasure-uniform-measure*[*OF emeasure-neq-0-sets*[*OF A*(*1*)], *of space M*]
using *sets.sets-into-space*[*OF emeasure-neq-0-sets*[*OF A*(*1*)]] *A*
by (*simp add*: *Int-absorb2 less-top*)
qed

lemma *prob-space-uniform-count-measure*: *finite A ⟹ A ≠ {} ⟹ prob-space* (*uniform-count-measure A*)
by *standard* (*auto simp*: *emeasure-uniform-count-measure space-uniform-count-measure one-ennreal-def*)

lemma (**in** *prob-space*) *measure-uniform-measure-eq-cond-prob*:
assumes [*measurable*]: *Measurable.pred M P Measurable.pred M Q*
shows *𝒫*(*x in uniform-measure M* {*x∈space M*. *Q x*}. *P x*) = *𝒫*(*x in M*. *P x* | *Q x*)
proof *cases*
assume *Q*: *measure M* {*x∈space M*. *Q x*} = *0*
then have ∗: *AE x in M*. ¬ *Q x*
by (*simp add*: *prob-eq-0*)
then have *density M* (*λx*. *indicator* {*x ∈ space M*. *Q x*} *x* / *emeasure M* {*x ∈ space M*. *Q x*}) = *density M* (*λx*. *0*)
by (*intro density-cong*) *auto*
with ∗ **show** *?thesis*
unfolding *uniform-measure-def*
by (*simp add*: *emeasure-density measure-def cond-prob-def emeasure-eq-0-AE*)
next
assume *Q*: *measure M* {*x∈space M*. *Q x*} ≠ *0*
then show *𝒫*(*x in uniform-measure M* {*x ∈ space M*. *Q x*}. *P x*) = *cond-prob M P Q*
by (*subst measure-uniform-measure*)
(*auto simp*: *emeasure-eq-measure cond-prob-def measure-nonneg intro*!: *arg-cong*[**where** *f=prob*])
qed

lemma *prob-space-point-measure*:
finite S $\Longrightarrow$ ($\bigwedge$*s. s* $\in$ *S* $\Longrightarrow$ *0* $\leq$ *p s*) $\Longrightarrow$ ($\sum$ *s*$\in$*S. p s*) = *1* $\Longrightarrow$ *prob-space* (*point-measure S p*)
by (*rule prob-spaceI*) (*simp add*: *space-point-measure emeasure-point-measure-finite*)

lemma (**in** *prob-space*) *distr-pair-fst*: *distr* (*N* $\bigotimes_M$ *M*) *N fst* = *N*
proof (*intro measure-eqI*)
fix *A* **assume** *A*: *A* $\in$ *sets* (*distr* (*N* $\bigotimes_M$ *M*) *N fst*)
from *A* **have** *emeasure* (*distr* (*N* $\bigotimes_M$ *M*) *N fst*) *A* = *emeasure* (*N* $\bigotimes_M$ *M*) (*A* $\times$ *space M*)
by (*auto simp add*: *emeasure-distr space-pair-measure dest*: *sets.sets-into-space intro*!: *arg-cong2*[**where** *f*=*emeasure*])
with *A* **show** *emeasure* (*distr* (*N* $\bigotimes_M$ *M*) *N fst*) *A* = *emeasure N A*
by (*simp add*: *emeasure-pair-measure-Times emeasure-space-1*)
qed *simp*

lemma (**in** *product-prob-space*) *distr-reorder*:
assumes *inj-on t J t* $\in$ *J* $\rightarrow$ *K finite K*
shows *distr* (*PiM K M*) (*Pi*$_M$ *J* (λ*x. M* (*t x*))) ($\lambda\omega$. λ*n*$\in$*J*. ω (*t n*)) = *PiM J* (λ*x. M* (*t x*))
proof (*rule product-sigma-finite.PiM-eqI*)
show *product-sigma-finite* (λ*x. M* (*t x*)) **..**
have *t*‘*J* $\subseteq$ *K* **using** *assms* **by** *auto*
then show [*simp*]: *finite J*
by (*rule finite-imageD*[*OF finite-subset*]) *fact*+
fix *A* **assume** *A*: $\bigwedge$*i. i* $\in$ *J* $\Longrightarrow$ *A i* $\in$ *sets* (*M* (*t i*))
moreover have (($\lambda\omega$. λ*n*$\in$*J*. ω (*t n*)) $-$‘ *Pi*$_E$ *J A* $\cap$ *space* (*Pi*$_M$ *K M*)) =
(Π_E *i*$\in$*K. if i* $\in$ *t*‘*J then A* (*the-inv-into J t i*) *else space* (*M i*))
using *A A*[*THEN sets.sets-into-space*] ‹*t* $\in$ *J* $\rightarrow$ *K*› ‹*inj-on t J*›
by (*subst prod-emb-Pi*[*symmetric*]) (*auto simp*: *space-PiM PiE-iff the-inv-into-f-f prod-emb-def*)
ultimately show *distr* (*Pi*$_M$ *K M*) (*Pi*$_M$ *J* (λ*x. M* (*t x*))) ($\lambda\omega$. λ*n*$\in$*J*. ω (*t n*)) (*Pi*$_E$ *J A*) = ($\prod$ *i*$\in$*J*. *M* (*t i*) (*A i*))
using *assms*
apply (*subst emeasure-distr*)
apply (*auto intro*!: *sets-PiM-I-finite simp*: *Pi-iff*)
apply (*subst emeasure-PiM*)
apply (*auto simp*: *the-inv-into-f-f* ‹*inj-on t J*› *prod.reindex*[*OF* ‹*inj-on t J*›]
if-distrib[**where** *f*=*emeasure* (*M* -)] *prod.If-cases emeasure-space-1 Int-absorb1* ‹*t*‘*J* $\subseteq$ *K*›)
done
qed *simp*

lemma (**in** *product-prob-space*) *distr-restrict*:
J $\subseteq$ *K* $\Longrightarrow$ *finite K* $\Longrightarrow$ (Π_M *i*$\in$*J*. *M i*) = *distr* (Π_M *i*$\in$*K*. *M i*) (Π_M *i*$\in$*J*. *M i*) (λ*f. restrict f J*)
using *distr-reorder*[*of* λ*x. x J K*] **by** (*simp add*: *Pi-iff subset-eq*)

lemma (**in** *product-prob-space*) *emeasure-prod-emb*[*simp*]:
assumes *L*: $J \subseteq L$ *finite L* **and** *X*: $X \in sets\ (Pi_M\ J\ M)$
shows $emeasure\ (Pi_M\ L\ M)\ (prod\text{-}emb\ L\ M\ J\ X) = emeasure\ (Pi_M\ J\ M)\ X$
by (*subst distr-restrict*[*OF L*])
(*simp add*: *prod-emb-def space-PiM emeasure-distr measurable-restrict-subset L X*)

lemma *emeasure-distr-restrict*:
assumes $I \subseteq K$ **and** *Q*[*measurable-cong*]: $sets\ Q = sets\ (PiM\ K\ M)$ **and** *A*[*measurable*]: $A \in sets\ (PiM\ I\ M)$
shows $emeasure\ (distr\ Q\ (PiM\ I\ M)\ (\lambda\omega.\ restrict\ \omega\ I))\ A = emeasure\ Q\ (prod\text{-}emb\ K\ M\ I\ A)$
using ‹$I \subseteq K$› *sets-eq-imp-space-eq*[*OF Q*]
by (*subst emeasure-distr*)
(*auto simp*: *measurable-cong-sets*[*OF Q*] *prod-emb-def space-PiM*[*symmetric*] *intro*!: *measurable-restrict*)

lemma (**in** *prob-space*) *prob-space-completion*: *prob-space* (*completion M*)
by (*rule prob-spaceI*) (*simp add*: *emeasure-space-1*)

lemma *distr-PiM-finite-prob-space*:
assumes *fin*: *finite I*
assumes *product-prob-space M*
assumes *product-prob-space M'*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in measurable\ (M\ i)\ (M'\ i)$
shows $distr\ (PiM\ I\ M)\ (PiM\ I\ M')\ (compose\ I\ f) = PiM\ I\ (\lambda i.\ distr\ (M\ i)\ (M'\ i)\ f)$
proof −
interpret *M*: *product-prob-space M* **by** *fact*
interpret *M'*: *product-prob-space M'* **by** *fact*
define *N* **where** $N = (\lambda i.\ if\ i \in I\ then\ distr\ (M\ i)\ (M'\ i)\ f\ else\ M'\ i)$
have [*intro*]: *prob-space* (*N i*) **for** *i*
by (*auto simp*: *N-def intro*!: *M.M.prob-space-distr M'.prob-space*)

interpret *N*: *product-prob-space N*
by (*intro product-prob-spaceI*) (*auto simp*: *N-def M'.prob-space intro*: *M.M.prob-space-distr*)

have $distr\ (PiM\ I\ M)\ (PiM\ I\ M')\ (compose\ I\ f) = PiM\ I\ N$
proof (*rule N.PiM-eqI*)
have *N-events-eq*: $sets\ (Pi_M\ I\ N) = sets\ (Pi_M\ I\ M')$
unfolding *N-def* **by** (*intro sets-PiM-cong*) *auto*
also have $\ldots = sets\ (distr\ (Pi_M\ I\ M)\ (Pi_M\ I\ M')\ (compose\ I\ f))$
by *simp*
finally show $sets\ (distr\ (Pi_M\ I\ M)\ (Pi_M\ I\ M')\ (compose\ I\ f)) = sets\ (Pi_M\ I\ N)$ **..**

fix *A* **assume** *A*: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in N.M.events\ i$
have $emeasure\ (distr\ (Pi_M\ I\ M)\ (Pi_M\ I\ M')\ (compose\ I\ f))\ (Pi_E\ I\ A) =$
$emeasure\ (Pi_M\ I\ M)\ (compose\ I\ f\ -`\ Pi_E\ I\ A \cap space\ (Pi_M\ I\ M))$

proof (*intro emeasure-distr*)
show *compose I f* ∈ Pi_M *I M* →$_M$ Pi_M *I M′*
unfolding *compose-def* **by** *measurable*
show Pi_E *I A* ∈ *sets* (Pi_M *I M′*)
unfolding *N-events-eq* [*symmetric*] **by** (*intro sets-PiM-I-finite fin A*)
qed
also have *compose I f* −‘ Pi_E *I A* ∩ *space* (Pi_M *I M*) = Pi_E *I* (λ*i*. *f* −‘ *A i* ∩ *space* (*M i*))
using *A* **by** (*auto simp*: *space-PiM PiE-def Pi-def extensional-def N-def compose-def*)
also have *emeasure* (Pi_M *I M*) (Pi_E *I* (λ*i*. *f* −‘ *A i* ∩ *space* (*M i*))) =
(∏ *i*∈*I*. *emeasure* (*M i*) (*f* −‘ *A i* ∩ *space* (*M i*)))
using *A* **by** (*intro M.emeasure-PiM fin*) (*auto simp*: *N-def*)
also have ... = (∏ *i*∈*I*. *emeasure* (*distr* (*M i*) (*M′ i*) *f*) (*A i*))
using *A* **by** (*intro prod.cong emeasure-distr* [*symmetric*]) (*auto simp*: *N-def*)
also have ... = (∏ *i*∈*I*. *emeasure* (*N i*) (*A i*))
unfolding *N-def* **by** (*intro prod.cong*) (*auto simp*: *N-def*)
finally show *emeasure* (*distr* (Pi_M *I M*) (Pi_M *I M′*) (*compose I f*)) (Pi_E *I A*) = ... **.**
qed *fact+*
also have *PiM I N* = *PiM I* (λ*i*. *distr* (*M i*) (*M′ i*) *f*)
by (*intro PiM-cong*) (*auto simp*: *N-def*)
finally show *?thesis* **.**
qed

end

2 Distribution Functions

Shows that the cumulative distribution function (cdf) of a distribution (a measure on the reals) is nondecreasing and right continuous, which tends to 0 and 1 in either direction.

Conversely, every such function is the cdf of a unique distribution. This direction defines the measure in the obvious way on half-open intervals, and then applies the Caratheodory extension theorem.

theory *Distribution-Functions*
imports *Probability-Measure*
begin

lemma *UN-Ioc-eq-UNIV*: (⋃ *n*. { −*real n* <.. *real n*}) = *UNIV*
by *auto*
(*metis le-less-trans minus-minus neg-less-iff-less not-le real-arch-simple of-nat-0-le-iff reals-Archimedean2*)

2.1 Properties of cdf's

definition

cdf :: *real measure* ⇒ *real* ⇒ *real*
where
cdf M ≡ λ*x*. *measure M* {..*x*}

lemma *cdf-def2*: *cdf M x* = *measure M* {..*x*}
by (*simp add*: *cdf-def*)

locale *finite-borel-measure* = *finite-measure M* **for** *M* :: *real measure* +
assumes *M-is-borel*: *sets M* = *sets borel*
begin

lemma *sets-M*[*intro*]: *a* ∈ *sets borel* ⟹ *a* ∈ *sets M*
using *M-is-borel* **by** *auto*

lemma *cdf-diff-eq*:
assumes *x* < *y*
shows *cdf M y* − *cdf M x* = *measure M* {*x*<..*y*}
proof −
from *assms* **have** ∗: {..*x*} ∪ {*x*<..*y*} = {..*y*} **by** *auto*
have *measure M* {..*y*} = *measure M* {..*x*} + *measure M* {*x*<..*y*}
by (*subst finite-measure-Union* [*symmetric*], *auto simp add*: ∗)
thus *?thesis*
unfolding *cdf-def* **by** *auto*
qed

lemma *cdf-nondecreasing*: *x* ≤ *y* ⟹ *cdf M x* ≤ *cdf M y*
unfolding *cdf-def* **by** (*auto intro*!: *finite-measure-mono*)

lemma *borel-UNIV*: *space M* = *UNIV*
by (*metis in-mono sets.sets-into-space space-in-borel top-le M-is-borel*)

lemma *cdf-nonneg*: *cdf M x* ≥ *0*
unfolding *cdf-def* **by** (*rule measure-nonneg*)

lemma *cdf-bounded*: *cdf M x* ≤ *measure M* (*space M*)
unfolding *cdf-def* **by** (*intro bounded-measure*)

lemma *cdf-lim-infty*:
((λ*i*. *cdf M* (*real i*)) ⟶ *measure M* (*space M*))
proof −
have (λ*i*. *cdf M* (*real i*)) ⟶ *measure M* (⋃ *i*::*nat*. {..*real i*})
unfolding *cdf-def* **by** (*rule finite-Lim-measure-incseq*) (*auto simp*: *incseq-def*)
also have (⋃ *i*::*nat*. {..*real i*}) = *space M*
by (*auto simp*: *borel-UNIV intro*: *real-arch-simple*)
finally show *?thesis* **.**
qed

lemma *cdf-lim-at-top*: (*cdf M* ⟶ *measure M* (*space M*)) *at-top*
by (*rule tendsto-at-topI-sequentially-real*)

(*simp-all add*: *mono-def cdf-nondecreasing cdf-lim-infty*)

lemma *cdf-lim-neg-infty*: $((\lambda i.\ cdf\ M\ (-\ real\ i)) \longrightarrow 0)$
proof −
have $(\lambda i.\ cdf\ M\ (-\ real\ i)) \longrightarrow measure\ M\ (\bigcap\ i{::}nat.\ \{..\ -\ real\ i\ \})$
unfolding *cdf-def* **by** (*rule finite-Lim-measure-decseq*) (*auto simp*: *decseq-def*)
also have $(\bigcap\ i{::}nat.\ \{..-\ real\ i\}) = \{\}$
by *auto* (*metis leD le-minus-iff reals-Archimedean2*)
finally show *?thesis*
by *simp*
qed

lemma *cdf-lim-at-bot*: (*cdf M* $\longrightarrow$ *0*) *at-bot*
proof −
have ∗: $((\lambda x :: real.\ -\ cdf\ M\ (-\ x)) \longrightarrow 0)$ *at-top*
by (*intro tendsto-at-topI-sequentially-real monoI*)
(*auto simp*: *cdf-nondecreasing cdf-lim-neg-infty tendsto-minus-cancel-left*[*symmetric*])
from *filterlim-compose* [*OF* ∗, *OF filterlim-uminus-at-top-at-bot*]
show *?thesis*
unfolding *tendsto-minus-cancel-left*[*symmetric*] **by** *simp*
qed

lemma *cdf-is-right-cont*: *continuous* (*at-right a*) (*cdf M*)
unfolding *continuous-within*
proof (*rule tendsto-at-right-sequentially*[**where** *b*=*a* + *1*])
fix *f* :: *nat* ⇒ *real* **and** *x* **assume** *f*: *decseq f f* $\longrightarrow$ *a*
then have $(\lambda n.\ cdf\ M\ (f\ n)) \longrightarrow measure\ M\ (\bigcap i.\ \{..\ f\ i\})$
using ‹*decseq f*› **unfolding** *cdf-def*
by (*intro finite-Lim-measure-decseq*) (*auto simp*: *decseq-def*)
also have $(\bigcap i.\ \{..\ f\ i\}) = \{..\ a\}$
using *decseq-ge*[*OF f*] **by** (*auto intro*: *order-trans LIMSEQ-le-const*[*OF f*(*2*)])
finally show $(\lambda n.\ cdf\ M\ (f\ n)) \longrightarrow cdf\ M\ a$
by (*simp add*: *cdf-def*)
qed *simp*

lemma *cdf-at-left*: (*cdf M* $\longrightarrow$ *measure M* {..<*a*}) (*at-left a*)
proof (*rule tendsto-at-left-sequentially*[*of a* − *1*])
fix *f* :: *nat* ⇒ *real* **and** *x* **assume** *f*: *incseq f f* $\longrightarrow$ *a* $\bigwedge x.\ f\ x < a$ $\bigwedge x.\ a - 1 < f\ x$
then have $(\lambda n.\ cdf\ M\ (f\ n)) \longrightarrow measure\ M\ (\bigcup i.\ \{..\ f\ i\})$
using ‹*incseq f*› **unfolding** *cdf-def*
by (*intro finite-Lim-measure-incseq*) (*auto simp*: *incseq-def*)
also have $(\bigcup i.\ \{..\ f\ i\}) = \{..<a\}$
by (*auto dest*!: *order-tendstoD*(*1*)[*OF f*(*2*)] *eventually-happens'*[*OF sequentially-bot*]
intro: *less-imp-le le-less-trans f*(*3*))
finally show $(\lambda n.\ cdf\ M\ (f\ n)) \longrightarrow measure\ M\ \{..<a\}$
by (*simp add*: *cdf-def*)
qed *auto*

lemma *isCont-cdf*: *isCont* (*cdf M*) $x \longleftrightarrow$ *measure M* $\{x\} = 0$
proof −
have *isCont* (*cdf M*) $x \longleftrightarrow$ *cdf M x* = *measure M* $\{..<x\}$
by (*auto simp*: *continuous-at-split cdf-is-right-cont continuous-within*[**where** *s*={..< -}]
cdf-at-left tendsto-unique[*OF - cdf-at-left*])
also have *cdf M x* = *measure M* $\{..<x\} \longleftrightarrow$ *measure M* $\{x\} = 0$
unfolding *cdf-def ivl-disj-un*(*2*)[*symmetric*]
by (*subst finite-measure-Union*) *auto*
finally show *?thesis* **.**
qed

lemma *countable-atoms*: *countable* $\{x.$ *measure M* $\{x\} > 0\}$
using *countable-support* **unfolding** *zero-less-measure-iff* **.**

end

locale *real-distribution* = *prob-space M* **for** *M* :: *real measure* +
assumes *events-eq-borel* [*simp*, *measurable-cong*]: *sets M* = *sets borel*
begin

lemma *finite-borel-measure-M*: *finite-borel-measure M*
by *standard auto*

sublocale *finite-borel-measure M*
by (*rule finite-borel-measure-M*)

lemma *space-eq-univ* [*simp*]: *space M* = *UNIV*
using *events-eq-borel*[*THEN sets-eq-imp-space-eq*] **by** *simp*

lemma *cdf-bounded-prob*: $\bigwedge x.$ *cdf M x* $\leq$ *1*
by (*subst prob-space* [*symmetric*], *rule cdf-bounded*)

lemma *cdf-lim-infty-prob*: $(\lambda i.$ *cdf M* (*real i*)) $\longrightarrow$ *1*
by (*subst prob-space* [*symmetric*], *rule cdf-lim-infty*)

lemma *cdf-lim-at-top-prob*: (*cdf M* $\longrightarrow$ *1*) *at-top*
by (*subst prob-space* [*symmetric*], *rule cdf-lim-at-top*)

lemma *measurable-finite-borel* [*simp*]:
f $\in$ *borel-measurable borel* $\Longrightarrow$ *f* $\in$ *borel-measurable M*
by (*rule borel-measurable-subalgebra*[**where** *N*=*borel*]) *auto*

end

lemma (**in** *prob-space*) *real-distribution-distr* [*intro*, *simp*]:
random-variable borel X $\Longrightarrow$ *real-distribution* (*distr M borel X*)
unfolding *real-distribution-def real-distribution-axioms-def* **by** (*auto intro*!: *prob-space-distr*)

2.2 Uniqueness

lemma (**in** *finite-borel-measure*) *emeasure-Ioc*:
assumes $a \leq b$ **shows** *emeasure M* $\{a <.. b\} =$ *cdf M b* $-$ *cdf M a*
proof $-$
have $\{a <.. b\} = \{..b\} - \{..a\}$
by *auto*
moreover have $\{..x\} \in$ *sets M* **for** x
using *atMost-borel*[*of x*] *M-is-borel* **by** *auto*
moreover note $\langle a \leq b \rangle$
ultimately show *?thesis*
by (*simp add*: *emeasure-eq-measure finite-measure-Diff cdf-def*)
qed

lemma *cdf-unique'*:
fixes *M1 M2*
assumes *finite-borel-measure M1* **and** *finite-borel-measure M2*
assumes *cdf M1* $=$ *cdf M2*
shows *M1* $=$ *M2*
proof (*rule measure-eqI-generator-eq*[**where** Ω=*UNIV*])
fix *X* **assume** $X \in$ *range* $(\lambda(a, b).\ \{a<..b::real\})$
then obtain *a b* **where** *Xeq*: $X = \{a<..b\}$ **by** *auto*
then show *emeasure M1 X* $=$ *emeasure M2 X*
by (*cases* $a \leq b$)
(*simp-all add*: *assms*(*1*,*2*)[*THEN finite-borel-measure.emeasure-Ioc*] *assms*(*3*))
next
show $(\bigcup i.\ \{-\ real\ (i::nat)<..real\ i\}) = UNIV$
by (*rule UN-Ioc-eq-UNIV*)
qed (*auto simp*: *finite-borel-measure.emeasure-Ioc*[*OF assms*(*1*)]
assms(*1*,*2*)[*THEN finite-borel-measure.M-is-borel*] *borel-sigma-sets-Ioc*
Int-stable-def)

lemma *cdf-unique*:
real-distribution M1 $\Longrightarrow$ *real-distribution M2* $\Longrightarrow$ *cdf M1* $=$ *cdf M2* $\Longrightarrow$ *M1* $=$ *M2*
using *cdf-unique'*[*of M1 M2*] **by** (*simp add*: *real-distribution.finite-borel-measure-M*)

lemma
fixes $F :: real \Rightarrow real$
assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
and *right-cont-F* : $\bigwedge a.$ *continuous* (*at-right a*) *F*
and *lim-F-at-bot* : $(F \longrightarrow 0)$ *at-bot*
and *lim-F-at-top* : $(F \longrightarrow m)$ *at-top*
and *m*: $0 \leq m$
shows *interval-measure-UNIV*: *emeasure* (*interval-measure F*) *UNIV* $= m$
and *finite-borel-measure-interval-measure*: *finite-borel-measure* (*interval-measure F*)
proof $-$
let *?F* $=$ *interval-measure F*
{ have *ennreal* $(m - 0) = (SUP\ i.\ ennreal\ (F\ (real\ i) - F\ (-\ real\ i)))$

by (*intro LIMSEQ-unique*[*OF - LIMSEQ-SUP*] *tendsto-ennrealI tendsto-intros lim-F-at-bot*[*THEN filterlim-compose*] *lim-F-at-top*[*THEN filterlim-compose*] *lim-F-at-bot*[*THEN filterlim-compose*] *filterlim-real-sequentially filterlim-uminus-at-top*[*THEN iffD1*])
(*auto simp*: *incseq-def nondecF intro*!: *diff-mono*)
also have ... = (*SUP i. emeasure ?F* {− *real i*<..*real i*})
by (*subst emeasure-interval-measure-Ioc*) (*simp-all add*: *nondecF right-cont-F*)
also have ... = *emeasure ?F* (⋃ *i*::*nat*. {− *real i*<..*real i*})
by (*rule SUP-emeasure-incseq*) (*auto simp*: *incseq-def*)
also have (⋃ *i*. {− *real* (*i*::*nat*)<..*real i*}) = *space ?F*
by (*simp add*: *UN-Ioc-eq-UNIV*)
finally have *emeasure ?F* (*space ?F*) = *m*
by *simp* **}**
note ∗ = *this*
then show *emeasure* (*interval-measure F*) *UNIV* = *m*
by *simp*

interpret *finite-measure ?F*
proof
show *emeasure ?F* (*space ?F*) ≠ ∞
using ∗ **by** *simp*
qed
show *finite-borel-measure* (*interval-measure F*)
proof qed *simp-all*
qed

lemma *real-distribution-interval-measure*:
fixes *F* :: *real* ⇒ *real*
assumes *nondecF* : ⋀ *x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y* **and**
right-cont-F : ⋀*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* ⟶ *0*) *at-bot* **and**
lim-F-at-top : (*F* ⟶ *1*) *at-top*
shows *real-distribution* (*interval-measure F*)
proof −
let *?F* = *interval-measure F*
interpret *prob-space ?F*
proof qed (*use interval-measure-UNIV*[*OF assms*] **in** *simp*)
show *?thesis*
proof qed *simp-all*
qed

lemma
fixes *F* :: *real* ⇒ *real*
assumes *nondecF* : ⋀ *x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y* **and**
right-cont-F : ⋀*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* ⟶ *0*) *at-bot*
shows *emeasure-interval-measure-Iic*: *emeasure* (*interval-measure F*) {.. *x*} = *F x*

```
    and measure-interval-measure-Iic: measure (interval-measure F) {.. x} = F x
  unfolding cdf-def
proof −
  have F-nonneg[simp]: 0 ≤ F y for y
   using lim-F-at-bot by (rule tendsto-upperbound) (auto simp: eventually-at-bot-linorder
nondecF intro!: exI[of - y])

  have emeasure (interval-measure F) (⋃ i::nat. {−real i <.. x}) = F x − ennreal
0
  proof (intro LIMSEQ-unique[OF Lim-emeasure-incseq])
    have (λi. F x − F (− real i)) ⟶ F x − 0
    by (intro tendsto-intros lim-F-at-bot[THEN filterlim-compose] filterlim-real-sequentially
              filterlim-uminus-at-top[THEN iffD1])
    from tendsto-ennrealI[OF this]
   show (λi. emeasure (interval-measure F) {− real i<..x}) ⟶ F x − ennreal
0
      apply (rule filterlim-cong[THEN iffD1, rotated 3])
       apply simp
      apply simp
      apply (rule eventually-sequentiallyI[where c=nat (ceiling (− x))])
      apply (simp add: emeasure-interval-measure-Ioc right-cont-F nondecF)
      done
  qed (auto simp: incseq-def)
  also have (⋃ i::nat. {−real i <.. x}) = {..x}
    by auto (metis minus-minus neg-less-iff-less reals-Archimedean2)
  finally show emeasure (interval-measure F) {..x} = F x
    by simp
  then show measure (interval-measure F) {..x} = F x
    by (simp add: measure-def)
qed

lemma cdf-interval-measure:
  (⋀ x y. x ≤ y ⟹ F x ≤ F y) ⟹ (⋀a. continuous (at-right a) F) ⟹ (F ⟶
0) at-bot ⟹ cdf (interval-measure F) = F
  by (simp add: cdf-def fun-eq-iff measure-interval-measure-Iic)

end
```

3 Weak Convergence of Functions and Distributions

Properties of weak convergence of functions and measures, including the portmanteau theorem.

```
theory Weak-Convergence
  imports Distribution-Functions
begin
```

4 Weak Convergence of Functions

definition
weak-conv :: $(nat \Rightarrow (real \Rightarrow real)) \Rightarrow (real \Rightarrow real) \Rightarrow bool$
where
weak-conv F-seq F $\equiv \forall x.$ *isCont F x* $\longrightarrow$ $(\lambda n.$ *F-seq n x*$) \longrightarrow$ *F x*

5 Weak Convergence of Distributions

definition
weak-conv-m :: $(nat \Rightarrow real\ measure) \Rightarrow real\ measure \Rightarrow bool$
where
weak-conv-m M-seq M $\equiv$ *weak-conv* $(\lambda n.$ *cdf* $(M\text{-}seq\ n))$ $(cdf\ M)$

6 Skorohod's theorem

locale *right-continuous-mono* =
fixes *f* :: $real \Rightarrow real$ **and** *a b* :: *real*
assumes *cont*: $\bigwedge x.$ *continuous* $(at\text{-}right\ x)$ *f*
assumes *mono*: *mono f*
assumes *bot*: $(f \longrightarrow a)$ *at-bot*
assumes *top*: $(f \longrightarrow b)$ *at-top*
begin

abbreviation *I* :: $real \Rightarrow real$ **where**
$I\ \omega \equiv Inf\ \{x.\ \omega \leq f\ x\}$

lemma *pseudoinverse*: **assumes** $a < \omega\ \omega < b$ **shows** $\omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
proof
let *?F* $= \{x.\ \omega \leq f\ x\}$
obtain *y* **where** $f\ y < \omega$
by (*metis eventually-happens' trivial-limit-at-bot-linorder order-tendstoD*(*2*) *bot* ‹$a < \omega$›)
with *mono* **have** *bdd*: *bdd-below ?F*
by (*auto intro*!: *bdd-belowI*[*of - y*] *elim*: *mono-invE*[*OF - less-le-trans*])

have *ne*: $?F \neq \{\}$
using *order-tendstoD*(*1*)[*OF top* ‹$\omega < b$›]
by (*auto dest*!: *eventually-happens'*[*OF trivial-limit-at-top-linorder*] *intro*: *less-imp-le*)

show $\omega \leq f\ x \Longrightarrow I\ \omega \leq x$
by (*auto intro*!: *cInf-lower bdd*)

{ assume $*$: $I\ \omega \leq x$
have $\omega \leq (INF\ s{\in}\{x.\ \omega \leq f\ x\}.\ f\ s)$
by (*rule cINF-greatest*[*OF ne*]) *auto*
also have $\ldots = f\ (I\ \omega)$
using *continuous-at-Inf-mono*[*OF mono cont ne bdd*] **..**

also have $\ldots \leq f\ x$
using $*$ **by** (*rule monoD*[*OF* ‹*mono f*›])
finally show $\omega \leq f\ x$ **. }**
qed

lemma *pseudoinverse'*: $\forall \omega \in \{a<..<b\}.\ \forall x.\ \omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
by (*intro ballI allI impI pseudoinverse*) *auto*

lemma *mono-I*: *mono-on* $\{a <..< b\}$ *I*
unfolding *mono-on-def* **by** (*metis order.trans order.refl pseudoinverse'*)

end

locale *cdf-distribution* = *real-distribution*
begin

abbreviation $C \equiv$ *cdf M*

sublocale *right-continuous-mono C 0 1*
by *standard*
(*auto intro*: *cdf-nondecreasing cdf-is-right-cont cdf-lim-at-top-prob cdf-lim-at-bot monoI*)

lemma *measurable-C*[*measurable*]: $C \in$ *borel-measurable borel*
by (*intro borel-measurable-mono mono*)

lemma *measurable-CI*[*measurable*]: *I* ∈ *borel-measurable* (*restrict-space borel* {*0*<..<*1*})
by (*intro borel-measurable-mono-on-fnc mono-I*)

lemma *emeasure-distr-I*: *emeasure* (*distr* (*restrict-space lborel* {*0*<..<*1*::*real*}) *borel I*) *UNIV* = *1*
by (*simp add*: *emeasure-distr space-restrict-space emeasure-restrict-space*)

lemma *distr-I-eq-M*: *distr* (*restrict-space lborel* {*0*<..<*1*::*real*}) *borel I* = *M* (**is** *?I* = -)
proof (*intro cdf-unique ext*)
let *?*Ω = *restrict-space lborel* {*0*<..<*1*}::*real measure*
interpret Ω: *prob-space ?*Ω
by (*auto simp add*: *emeasure-restrict-space space-restrict-space intro*!: *prob-spaceI*)
show *real-distribution ?I*
by *auto*

fix *x*
have *cdf ?I x* = *measure lborel* {ω∈{*0*<..<*1*}. ω ≤ *C x*}
by (*subst cdf-def*)
(*auto simp*: *pseudoinverse*[*symmetric*] *measure-distr space-restrict-space measure-restrict-space*
intro!: *arg-cong2*[**where** *f*=*measure*])
also have $\ldots$ = *measure lborel* {*0* <..< *C x*}

using *cdf-bounded-prob*[*of x*] *AE-lborel-singleton*[*of C x*]
by (*auto intro*!: *arg-cong*[**where** *f*=*enn2real*] *emeasure-eq-AE simp*: *measure-def*)
also have ... = *C x*
by (*simp add*: *cdf-nonneg*)
finally show *cdf* (*distr ?Ω borel I*) *x* = *C x* **.**
qed *standard*

end

context
fixes μ :: *nat* ⇒ *real measure*
and *M* :: *real measure*
assumes μ: ⋀*n*. *real-distribution* (μ *n*)
assumes *M*: *real-distribution M*
assumes μ*-to-M*: *weak-conv-m* μ *M*
begin

theorem *Skorohod*:
∃ (Ω :: *real measure*) (*Y-seq* :: *nat* ⇒ *real* ⇒ *real*) (*Y* :: *real* ⇒ *real*).
prob-space Ω ∧
(∀ *n*. *Y-seq n* ∈ *measurable* Ω *borel*) ∧
(∀ *n*. *distr* Ω *borel* (*Y-seq n*) = μ *n*) ∧
Y ∈ *measurable* Ω *lborel* ∧
distr Ω *borel Y* = *M* ∧
(∀ *x* ∈ *space* Ω. (λ*n*. *Y-seq n x*) ⟶ *Y x*)
proof −
interpret μ: *cdf-distribution* μ *n* **for** *n*
unfolding *cdf-distribution-def* **by** (*rule* μ)
interpret *M*: *cdf-distribution M*
unfolding *cdf-distribution-def* **by** (*rule M*)

have *conv*: *measure M* {*x*} = *0* ⟹ (λ*n*. μ.*C n x*) ⟶ *M.C x* **for** *x*
using μ*-to-M M.isCont-cdf* **by** (*auto simp*: *weak-conv-m-def weak-conv-def*)

let *?Ω* = *restrict-space lborel* {*0*<..<*1*} :: *real measure*
have *prob-space ?Ω*
by (*auto simp*: *space-restrict-space emeasure-restrict-space intro*!: *prob-spaceI*)
interpret Ω: *prob-space ?Ω*
by *fact*

have *Y-distr*: *distr ?Ω borel M.I* = *M*
by (*rule M.distr-I-eq-M*)

have *Y-cts-cnv*: (λ*n*. μ.*I n* ω) ⟶ *M.I* ω
if ω: ω ∈ {*0*<..<*1*} *isCont M.I* ω **for** ω :: *real*
proof (*intro limsup-le-liminf-real*)
show *liminf* (λ*n*. μ.*I n* ω) ≥ *M.I* ω
unfolding *le-Liminf-iff*

```
  proof safe
    fix B :: ereal assume B: B < M.I ω
    then show ∀ F n in sequentially. B < μ.I n ω
    proof (cases B)
      case (real r)
      with B have r: r < M.I ω
        by simp
      then obtain x where x: r < x x < M.I ω measure M {x} = 0
        using open-minus-countable[OF M.countable-support, of {r<..<M.I ω}]
by auto
      then have Fx-less: M.C x < ω
        using M.pseudoinverse' ω not-less by blast

      have ∀ F n in sequentially. μ.C n x < ω
        using order-tendstoD(2)[OF conv[OF x(3)] Fx-less] .
      then have ∀ F n in sequentially. x < μ.I n ω
          by eventually-elim (insert ω μ.pseudoinverse[symmetric], simp add:
not-le[symmetric])
      then show ?thesis
        by eventually-elim (insert x(1), simp add: real)
    qed auto
  qed

  have *: limsup (λn. μ.I n ω) ≤ M.I ω'
    if ω': 0 < ω' ω' < 1 ω < ω' for ω' :: real
  proof (rule dense-ge-bounded)
    fix B' assume ereal (M.I ω') < B' B' < ereal (M.I ω' + 1)
    then obtain B where M.I ω' < B and [simp]: B' = ereal B
      by (cases B') auto
    then obtain y where y: M.I ω' < y y < B measure M {y} = 0
      using open-minus-countable[OF M.countable-support, of {M.I ω'<..<B}]
by auto
    then have ω' ≤ M.C (M.I ω')
      using M.pseudoinverse' ω' by (metis greaterThanLessThan-iff order-refl)
    also have ... ≤ M.C y
      using M.mono y unfolding mono-def by auto
    finally have Fy-gt: ω < M.C y
      using ω'(3) by simp

    have ∀ F n in sequentially. ω ≤ μ.C n y
      using order-tendstoD(1)[OF conv[OF y(3)] Fy-gt] by eventually-elim (rule
less-imp-le)
    then have 2: ∀ F n in sequentially. μ.I n ω ≤ ereal y
      by simp (subst μ.pseudoinverse'[rule-format, OF ω(1), symmetric])
    then show limsup (λn. μ.I n ω) ≤ B'
      using ‹y < B›
      by (intro Limsup-bounded[rotated]) (auto intro: le-less-trans elim: eventu-
ally-mono)
  qed simp
```

```
  have **: (M.I ⟶ ereal (M.I ω)) (at-right ω)
    using ω(2) by (auto intro: tendsto-within-subset simp: continuous-at)
  show limsup (λn. μ.I n ω) ≤ M.I ω
    using ω
    by (intro tendsto-lowerbound[OF **])
       (auto intro!: exI[of - 1] * simp: eventually-at-right[of - 1])
 qed

 let ?D = {ω∈{0<..<1}. ¬ isCont M.I ω}
 have D-countable: countable ?D
   using mono-on-ctble-discont[OF M.mono-I] by (simp add: at-within-open[of -
{0 <..< 1}] cong: conj-cong)
 hence D: emeasure ?Ω ?D = 0
   using emeasure-lborel-countable[OF D-countable]
   by (subst emeasure-restrict-space) auto

 define Y' where Y' ω = (if ω ∈ ?D then 0 else M.I ω) for ω
 have Y'-AE: AE ω in ?Ω. Y' ω = M.I ω
   by (rule AE-I [OF - D]) (auto simp: space-restrict-space sets-restrict-space-iff
Y'-def)

 define Y-seq' where Y-seq' n ω = (if ω ∈ ?D then 0 else μ.I n ω) for n ω
 have Y-seq'-AE: ⋀n. AE ω in ?Ω. Y-seq' n ω = μ.I n ω
   by (rule AE-I [OF - D]) (auto simp: space-restrict-space sets-restrict-space-iff
Y-seq'-def)

 have Y'-cnv: ∀ω∈{0<..<1}. (λn. Y-seq' n ω) ⟶ Y' ω
   by (auto simp: Y'-def Y-seq'-def Y-cts-cnv)

 have [simp]: Y-seq' n ∈ borel-measurable ?Ω for n
   by (rule measurable-discrete-difference[of μ.I n - - ?D])
      (insert μ.measurable-CI[of n] D-countable, auto simp: sets-restrict-space
Y-seq'-def)
 moreover have distr ?Ω borel (Y-seq' n) = μ n for n
   using μ.distr-I-eq-M [of n] Y-seq'-AE [of n]
   by (subst distr-cong-AE[where f = Y-seq' n and g = μ.I n], auto)
 moreover have [simp]: Y' ∈ borel-measurable ?Ω
   by (rule measurable-discrete-difference[of M.I - - ?D])
     (insert M.measurable-CI D-countable, auto simp: sets-restrict-space Y'-def)
 moreover have distr ?Ω borel Y' = M
   using M.distr-I-eq-M Y'-AE
   by (subst distr-cong-AE[where f = Y' and g = M.I], auto)
 ultimately have prob-space ?Ω ∧ (∀ n. Y-seq' n ∈ borel-measurable ?Ω) ∧
   (∀ n. distr ?Ω borel (Y-seq' n) = μ n) ∧ Y' ∈ measurable ?Ω lborel ∧ distr ?Ω
borel Y' = M ∧
   (∀ x∈space ?Ω. (λn. Y-seq' n x) ⟶ Y' x)
   using Y'-cnv ‹prob-space ?Ω› by (auto simp: space-restrict-space)
 thus ?thesis by metis
```

qed

The Portmanteau theorem, that is, the equivalence of various definitions of weak convergence.

theorem *weak-conv-imp-bdd-ae-continuous-conv*:
fixes
f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*}
assumes
discont-null: *M* ({*x*. ¬ *isCont f x*}) = *0* **and**
f-bdd: ⋀*x*. *norm* (*f x*) ≤ *B* **and**
[*measurable*]: *f* ∈ *borel-measurable borel*
shows
(λ *n*. *integral*^{L} (*μ n*) *f*) ⟶ *integral*^{L} *M f*
proof −
have *0* ≤ *B*
using *norm-ge-zero f-bdd* **by** (*rule order-trans*)
note *Skorohod*
then obtain *Omega Y-seq Y* **where**
ps-Omega [*simp*]: *prob-space Omega* **and**
Y-seq-measurable [*measurable*]: ⋀*n*. *Y-seq n* ∈ *borel-measurable Omega* **and**
distr-Y-seq: ⋀*n*. *distr Omega borel* (*Y-seq n*) = *μ n* **and**
Y-measurable [*measurable*]: *Y* ∈ *borel-measurable Omega* **and**
distr-Y: *distr Omega borel Y* = *M* **and**
YnY: ⋀*x* :: *real*. *x* ∈ *space Omega* ⟹ (λ*n*. *Y-seq n x*) ⟶ *Y x* **by** *force*
interpret *prob-space Omega* **by** *fact*
have ∗: *emeasure Omega* (*Y* −‘ {*x*. ¬ *isCont f x*} ∩ *space Omega*) = *0*
by (*subst emeasure-distr* [*symmetric*, **where** *N*=*borel*]) (*auto simp*: *distr-Y discont-null*)
have ∗: *AE x in Omega*. (λ*n*. *f* (*Y-seq n x*)) ⟶ *f* (*Y x*)
by (*rule AE-I* [*OF* - ∗]) (*auto intro*: *isCont-tendsto-compose YnY*)
show *?thesis*
by (*auto intro*!: *integral-dominated-convergence*[**where** *w*=λ*x*. *B*]
simp: *f-bdd* ∗ *integral-distr distr-Y-seq* [*symmetric*] *distr-Y* [*symmetric*])
qed

theorem *weak-conv-imp-integral-bdd-continuous-conv*:
fixes *f* :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*}
assumes
⋀*x*. *isCont f x* **and**
⋀*x*. *norm* (*f x*) ≤ *B*
shows
(λ *n*. *integral*^{L} (*μ n*) *f*) ⟶ *integral*^{L} *M f*
using *assms*
by (*intro weak-conv-imp-bdd-ae-continuous-conv*)
(*auto intro*!: *borel-measurable-continuous-onI continuous-at-imp-continuous-on*)

theorem *weak-conv-imp-continuity-set-conv*:
fixes *f* :: *real* ⇒ *real*
assumes [*measurable*]: *A* ∈ *sets borel* **and** *M* (*frontier A*) = *0*

shows $(\lambda n.\ measure\ (\mu\ n)\ A) \longrightarrow measure\ M\ A$
proof −
interpret M: *real-distribution* M **by** *fact*
interpret μ: *real-distribution* $\mu\ n$ **for** n **by** *fact*

have $(\lambda n.\ (\int x.\ indicator\ A\ x\ \partial\mu\ n) :: real) \longrightarrow (\int x.\ indicator\ A\ x\ \partial M)$
by (*intro weak-conv-imp-bdd-ae-continuous-conv*[**where** *B=1*])
(*auto intro*: *assms simp*: *isCont-indicator*)
then show *?thesis*
by *simp*
qed

end

definition
cts-step :: *real* ⇒ *real* ⇒ *real* ⇒ *real*
where
cts-step a b x ≡ *if x* ≤ *a then 1 else if x* ≥ *b then 0 else* (*b* − *x*) / (*b* − *a*)

lemma *cts-step-uniformly-continuous*:
assumes [*arith*]: *a* < *b*
shows *uniformly-continuous-on UNIV* (*cts-step a b*)
unfolding *uniformly-continuous-on-def*
proof *clarsimp*
fix *e* :: *real* **assume** [*arith*]: *0* < *e*
let *?d* = *min* (*e* ∗ (*b* − *a*)) (*b* − *a*)
have *?d* > *0*
by (*auto simp add*: *field-simps*)
moreover have *dist x′ x* < *?d* ⟹ *dist* (*cts-step a b x′*) (*cts-step a b x*) < *e* **for** *x x′*
by (*auto simp*: *dist-real-def divide-simps cts-step-def*)
ultimately show ∃ *d* > *0*. ∀ *x x′*. *dist x′ x* < *d* ⟶ *dist* (*cts-step a b x′*) (*cts-step a b x*) < *e*
by *blast*
qed

lemma (**in** *real-distribution*) *integrable-cts-step*: *a* < *b* ⟹ *integrable M* (*cts-step a b*)
by (*rule integrable-const-bound* [*of - 1*]) (*auto simp*: *cts-step-def*[*abs-def*])

lemma (**in** *real-distribution*) *cdf-cts-step*:
assumes [*arith*]: *x* < *y*
shows *cdf M x* ≤ *integral*^{L} *M* (*cts-step x y*) **and** *integral*^{L} *M* (*cts-step x y*) ≤ *cdf M y*
proof −
have *cdf M x* = *integral*^{L} *M* (*indicator* {..*x*})
by (*simp add*: *cdf-def*)
also have ... ≤ *expectation* (*cts-step x y*)
by (*intro integral-mono integrable-cts-step*)

```
      (auto simp: cts-step-def less-top[symmetric] split: split-indicator)
  finally show cdf M x ≤ expectation (cts-step x y) .
next
  have expectation (cts-step x y) ≤ integral^L M (indicator {..y})
    by (intro integral-mono integrable-cts-step)
       (auto simp: cts-step-def less-top[symmetric] split: split-indicator)
  also have ... = cdf M y
    by (simp add: cdf-def)
  finally show expectation (cts-step x y) ≤ cdf M y .
qed

context
  fixes M-seq :: nat ⇒ real measure
    and M :: real measure
  assumes distr-M-seq [simp]: ⋀n. real-distribution (M-seq n)
  assumes distr-M [simp]: real-distribution M
begin

theorem continuity-set-conv-imp-weak-conv:
  fixes f :: real ⇒ real
  assumes ∗: ⋀A. A ∈ sets borel ⟹ M (frontier A) = 0 ⟹ (λ n. (measure
(M-seq n) A)) ⟶ measure M A
  shows weak-conv-m M-seq M
proof −
  interpret real-distribution M by simp
  show ?thesis
    by (auto intro!: ∗ simp: frontier-real-atMost isCont-cdf emeasure-eq-measure
weak-conv-m-def weak-conv-def cdf-def2)
qed

theorem integral-cts-step-conv-imp-weak-conv:
  assumes integral-conv: ⋀x y. x < y ⟹ (λn. integral^L (M-seq n) (cts-step x y))
⟶ integral^L M (cts-step x y)
  shows weak-conv-m M-seq M
  unfolding weak-conv-m-def weak-conv-def
proof (clarsimp)
  interpret real-distribution M by (rule distr-M)
  fix x assume isCont (cdf M) x
  hence left-cont: continuous (at-left x) (cdf M)
    unfolding continuous-at-split ..
  { fix y :: real assume [arith]: x < y
    have limsup (λn. cdf (M-seq n) x) ≤ limsup (λn. integral^L (M-seq n) (cts-step
x y))
      by (auto intro!: Limsup-mono always-eventually real-distribution.cdf-cts-step)
    also have ... = integral^L M (cts-step x y)
      by (intro lim-imp-Limsup) (auto intro: integral-conv)
    also have ... ≤ cdf M y
      by (simp add: cdf-cts-step)
    finally have limsup (λn. cdf (M-seq n) x) ≤ cdf M y .
```

```
  } note ∗ = this
  { fix y :: real assume [arith]: x > y
    have cdf M y ≤ ereal (integral^L M (cts-step y x))
      by (simp add: cdf-cts-step)
    also have ... = liminf (λn. integral^L (M-seq n) (cts-step y x))
      by (intro lim-imp-Liminf[symmetric]) (auto intro: integral-conv)
    also have ... ≤ liminf (λn. cdf (M-seq n) x)
      by (auto intro!: Liminf-mono always-eventually real-distribution.cdf-cts-step)
    finally have liminf (λn. cdf (M-seq n) x) ≥ cdf M y .
  } note ∗∗ = this

  have limsup (λn. cdf (M-seq n) x) ≤ cdf M x
  proof (rule tendsto-lowerbound)
    show ∀_F i in at-right x. limsup (λxa. ereal (cdf (M-seq xa) x)) ≤ ereal (cdf M
i)
      by (subst eventually-at-right[of - x + 1]) (auto simp: ∗ intro: exI [of - x+1])
  qed (insert cdf-is-right-cont, auto simp: continuous-within)
  moreover have cdf M x ≤ liminf (λn. cdf (M-seq n) x)
  proof (rule tendsto-upperbound)
    show ∀_F i in at-left x. ereal (cdf M i) ≤ liminf (λxa. ereal (cdf (M-seq xa) x))
      by (subst eventually-at-left[of x − 1]) (auto simp: ∗∗ intro: exI [of - x−1])
  qed (insert left-cont, auto simp: continuous-within)
  ultimately show (λn. cdf (M-seq n) x) ⟶ cdf M x
    by (elim limsup-le-liminf-real)
qed

theorem integral-bdd-continuous-conv-imp-weak-conv:
  assumes
    ⋀f. (⋀x. isCont f x) ⟹ (⋀x. abs (f x) ≤ 1) ⟹ (λn. integral^L (M-seq n)
f::real) ⟶ integral^L M f
  shows
    weak-conv-m M-seq M
  apply (rule integral-cts-step-conv-imp-weak-conv [OF assms])
  apply (rule continuous-on-interior)
  apply (rule uniformly-continuous-imp-continuous)
  apply (rule cts-step-uniformly-continuous)
  apply (auto simp: cts-step-def)
  done

end

end
```

7 The Giry monad

```
theory Giry-Monad
  imports Probability-Measure HOL−Library.Monad-Syntax
begin
```

7.1 Sub-probability spaces

locale *subprob-space* = *finite-measure* +
  assumes *emeasure-space-le-1*: *emeasure M* (*space M*) ≤ *1*
  assumes *subprob-not-empty*: *space M* ≠ {}

lemma *subprob-spaceI*[*Pure.intro*!]:
  assumes ∗: *emeasure M* (*space M*) ≤ *1*
  assumes *space M* ≠ {}
  shows *subprob-space M*
proof −
  interpret *finite-measure M*
  proof
    show *emeasure M* (*space M*) ≠ ∞ **using** ∗ **by** (*auto simp*: *top-unique*)
  qed
  show *subprob-space M* **by** *standard fact*+
qed

lemma (**in** *subprob-space*) *emeasure-subprob-space-less-top*: *emeasure M A* ≠ *top*
  by *simp*

lemma *prob-space-imp-subprob-space*:
  prob-space M ⟹ *subprob-space M*
 by (*rule subprob-spaceI*) (*simp-all add*: *prob-space.emeasure-space-1 prob-space.not-empty*)

lemma *subprob-space-imp-sigma-finite*: *subprob-space M* ⟹ *sigma-finite-measure M*
  unfolding *subprob-space-def finite-measure-def* **by** *simp*

sublocale *prob-space* ⊆ *subprob-space*
  by (*rule subprob-spaceI*) (*simp-all add*: *emeasure-space-1 not-empty*)

lemma *subprob-space-sigma* [*simp*]: Ω ≠ {} ⟹ *subprob-space* (*sigma* Ω *X*)
  by(*rule subprob-spaceI*)(*simp-all add*: *emeasure-sigma space-measure-of-conv*)

lemma *subprob-space-null-measure*: *space M* ≠ {} ⟹ *subprob-space* (*null-measure M*)
  by(*simp add*: *null-measure-def*)

lemma (**in** *subprob-space*) *subprob-space-distr*:
  assumes *f*: *f* ∈ *measurable M M'* **and** *space M'* ≠ {} **shows** *subprob-space* (*distr M M' f*)
proof (*rule subprob-spaceI*)
  have *f* −‘ *space M'* ∩ *space M* = *space M* **using** *f* **by** (*auto dest*: *measurable-space*)
  with *f* **show** *emeasure* (*distr M M' f*) (*space* (*distr M M' f*)) ≤ *1*
    by (*auto simp*: *emeasure-distr emeasure-space-le-1*)
  show *space* (*distr M M' f*) ≠ {} **by** (*simp add*: *assms*)
qed

lemma (**in** *subprob-space*) *subprob-emeasure-le-1*: *emeasure M X* $\leq$ *1*
by (*rule order.trans*[*OF emeasure-space emeasure-space-le-1*])

lemma (**in** *subprob-space*) *subprob-measure-le-1*: *measure M X* $\leq$ *1*
using *subprob-emeasure-le-1*[*of X*] **by** (*simp add*: *emeasure-eq-measure*)

lemma (**in** *subprob-space*) *nn-integral-le-const*:
assumes *0* $\leq$ *c AE x in M*. *f x* $\leq$ *c*
shows ($\int^+ x.\ f\ x\ \partial M$) $\leq$ *c*
proof −
have ($\int^+ x.\ f\ x\ \partial M$) $\leq$ ($\int^+ x.\ c\ \partial M$)
by(*rule nn-integral-mono-AE*) *fact*
also have ... $\leq$ *c* $*$ *emeasure M* (*space M*)
using ‹*0* $\leq$ *c*› **by** *simp*
also have ... $\leq$ *c* $*$ *1* **using** *emeasure-space-le-1* ‹*0* $\leq$ *c*› **by**(*rule mult-left-mono*)
finally show *?thesis* **by** *simp*
qed

lemma *emeasure-density-distr-interval*:
fixes *h* :: *real* $\Rightarrow$ *real* **and** *g* :: *real* $\Rightarrow$ *real* **and** *g′* :: *real* $\Rightarrow$ *real*
assumes [*simp*]: *a* $\leq$ *b*
assumes *Mf*[*measurable*]: *f* $\in$ *borel-measurable borel*
assumes *Mg*[*measurable*]: *g* $\in$ *borel-measurable borel*
assumes *Mg′*[*measurable*]: *g′* $\in$ *borel-measurable borel*
assumes *Mh*[*measurable*]: *h* $\in$ *borel-measurable borel*
assumes *prob*: *subprob-space* (*density lborel f*)
assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has-real-derivative g′ x*) (*at x*)
assumes *contg′*: *continuous-on* {*a..b*} *g′*
assumes *mono*: *strict-mono-on* {*a..b*} *g* **and** *inv*: $\bigwedge x.\ h\ x \in \{a..b\} \Longrightarrow g\ (h\ x)$
$= x$
assumes *range*: {*a..b*} $\subseteq$ *range h*
shows *emeasure* (*distr* (*density lborel f*) *lborel h*) {*a..b*} =
emeasure (*density lborel* ($\lambda x.\ f\ (g\ x) * g'\ x$)) {*a..b*}
proof (*cases a* $<$ *b*)
assume *a* $<$ *b*
from *mono* **have** *inj*: *inj-on g* {*a..b*} **by** (*rule strict-mono-on-imp-inj-on*)
from *mono* **have** *mono′*: *mono-on* {*a..b*} *g* **by** (*rule strict-mono-on-imp-mono-on*)
from *mono′ derivg* **have** $\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow g'\ x \geq 0$
by (*rule mono-on-imp-deriv-nonneg*) *auto*
from *contg′ this* **have** *derivg-nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
by (*rule continuous-ge-on-Ioo*) (*simp-all add*: ‹*a* $<$ *b*›)

from *derivg* **have** *contg*: *continuous-on* {*a..b*} *g* **by** (*rule has-real-derivative-imp-continuous-on*)
have *A*: *h* $-$‘ {*a..b*} = {*g a..g b*}
proof (*intro equalityI subsetI*)
fix *x* **assume** *x*: *x* $\in$ *h* $-$‘ {*a..b*}
hence *g* (*h x*) $\in$ {*g a..g b*} **by** (*auto intro*: *mono-onD*[*OF mono′*])
with *inv* **and** *x* **show** *x* $\in$ {*g a..g b*} **by** *simp*

next
fix *y* **assume** *y*: $y \in \{g\ a..g\ b\}$
with *IVT'*[*OF - - - contg, of y*] **obtain** *x* **where** $x \in \{a..b\}$ $y = g\ x$ **by** *auto*
with *range* **and** *inv* **show** $y \in h -` \{a..b\}$ **by** *auto*
qed

have *prob'*: *subprob-space* (*distr* (*density lborel f*) *lborel h*)
by (*rule subprob-space.subprob-space-distr*[*OF prob*]) (*simp-all add*: *Mh*)
have *B*: *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\} =$
$\int^+ x.\ f\ x * indicator\ (h -` \{a..b\})\ x\ \partial lborel$
by (*subst emeasure-distr*) (*simp-all add*: *emeasure-density Mf Mh measurable-sets-borel*[*OF Mh*])
also note *A*
also have *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\} \leq 1$
by (*rule subprob-space.subprob-emeasure-le-1*) (*rule prob'*)
hence *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\} \neq \infty$ **by** (*auto simp*: *top-unique*)
with *assms* **have** $(\int^+ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel) =$
$(\int^+ x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x\ \partial lborel)$
by (*intro nn-integral-substitution-aux*)
(*auto simp*: *derivg-nonneg A B emeasure-density mult.commute* ‹$a < b$›)
also have ... = *emeasure* (*density lborel* ($\lambda x.\ f\ (g\ x) * g'\ x$)) $\{a..b\}$
by (*simp add*: *emeasure-density*)
finally show *?thesis* **.**
next
assume $\neg a < b$
with ‹$a \leq b$› **have** [*simp*]: $b = a$ **by** (*simp add*: *not-less del*: ‹$a \leq b$›)
from *inv* **and** *range* **have** $h -` \{a\} = \{g\ a\}$ **by** *auto*
thus *?thesis* **by** (*simp-all add*: *emeasure-distr emeasure-density measurable-sets-borel*[*OF Mh*])
qed

locale *pair-subprob-space* =
pair-sigma-finite M1 M2 + *M1*: *subprob-space M1* + *M2*: *subprob-space M2* **for** *M1 M2*

sublocale *pair-subprob-space* $\subseteq$ *P?*: *subprob-space M1* $\bigotimes_M$ *M2*
proof
from *mult-le-one*[*OF M1.emeasure-space-le-1 - M2.emeasure-space-le-1*]
show *emeasure* (*M1* $\bigotimes_M$ *M2*) (*space* (*M1* $\bigotimes_M$ *M2*)) ≤ 1
by (*simp add*: *M2.emeasure-pair-measure-Times space-pair-measure*)
from *M1.subprob-not-empty* **and** *M2.subprob-not-empty* **show** *space* (*M1* $\bigotimes_M$ *M2*) $\neq \{\}$
by (*simp add*: *space-pair-measure*)
qed

lemma *subprob-space-null-measure-iff*:
subprob-space (*null-measure M*) $\longleftrightarrow$ *space M* $\neq \{\}$
by (*auto intro*!: *subprob-spaceI dest*: *subprob-space.subprob-not-empty*)

lemma *subprob-space-restrict-space*:
assumes *M*: *subprob-space M*
and *A*: *A* ∩ *space M* ∈ *sets M A* ∩ *space M* ≠ {}
shows *subprob-space* (*restrict-space M A*)
proof(*rule subprob-spaceI*)
have *emeasure* (*restrict-space M A*) (*space* (*restrict-space M A*)) = *emeasure M* (*A* ∩ *space M*)
using *A* **by**(*simp add*: *emeasure-restrict-space space-restrict-space*)
also have ... ≤ *1* **by**(*rule subprob-space.subprob-emeasure-le-1*)(*rule M*)
finally show *emeasure* (*restrict-space M A*) (*space* (*restrict-space M A*)) ≤ *1* .
next
show *space* (*restrict-space M A*) ≠ {}
using *A* **by**(*simp add*: *space-restrict-space*)
qed

definition *subprob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
subprob-algebra K =
(*SUP A* ∈ *sets K*. *vimage-algebra* {*M*. *subprob-space M* ∧ *sets M* = *sets K*} (λ*M*. *emeasure M A*) *borel*)

lemma *space-subprob-algebra*: *space* (*subprob-algebra A*) = {*M*. *subprob-space M* ∧ *sets M* = *sets A*}
by (*auto simp add*: *subprob-algebra-def space-Sup-eq-UN*)

lemma *subprob-algebra-cong*: *sets M* = *sets N* ⟹ *subprob-algebra M* = *subprob-algebra N*
by (*simp add*: *subprob-algebra-def*)

lemma *measurable-emeasure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *emeasure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
by (*auto intro*!: *measurable-Sup1 measurable-vimage-algebra1 simp*: *subprob-algebra-def*)

lemma *measurable-measure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *measure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
unfolding *measure-def* **by** *measurable*

lemma *subprob-measurableD*:
assumes *N*: *N* ∈ *measurable M* (*subprob-algebra S*) **and** *x*: *x* ∈ *space M*
shows *space* (*N x*) = *space S*
and *sets* (*N x*) = *sets S*
and *measurable* (*N x*) *K* = *measurable S K*
and *measurable K* (*N x*) = *measurable K S*
using *measurable-space*[*OF N x*]
by (*auto simp*: *space-subprob-algebra intro*!: *measurable-cong-sets dest*: *sets-eq-imp-space-eq*)

ML ‹

fun subprob-cong thm ctxt = (

```
  let
    val thm' = Thm.transfer' ctxt thm
    val free = thm' |> Thm.concl-of |> HOLogic.dest-Trueprop |> dest-comb |> fst
|>
      dest-comb |> snd |> strip-abs-body |> head-of |> is-Free
  in
   if free then ([], Measurable.add-local-cong (thm' RS @{thm subprob-measurableD(2)})
ctxt)
           else ([], ctxt)
  end
  handle THM - => ([], ctxt) | TERM - => ([], ctxt))

›

setup ‹
  Context.theory-map (Measurable.add-preprocessor subprob-cong subprob-cong)
›

context
  fixes K M N assumes K: K ∈ measurable M (subprob-algebra N)
begin

lemma subprob-space-kernel: a ∈ space M ⟹ subprob-space (K a)
  using measurable-space[OF K] by (simp add: space-subprob-algebra)

lemma sets-kernel: a ∈ space M ⟹ sets (K a) = sets N
  using measurable-space[OF K] by (simp add: space-subprob-algebra)

lemma measurable-emeasure-kernel[measurable]:
    A ∈ sets N ⟹ (λa. emeasure (K a) A) ∈ borel-measurable M
  using measurable-compose[OF K measurable-emeasure-subprob-algebra] .

end

lemma measurable-subprob-algebra:
  (⋀a. a ∈ space M ⟹ subprob-space (K a)) ⟹
  (⋀a. a ∈ space M ⟹ sets (K a) = sets N) ⟹
  (⋀A. A ∈ sets N ⟹ (λa. emeasure (K a) A) ∈ borel-measurable M) ⟹
  K ∈ measurable M (subprob-algebra N)
 by (auto intro!: measurable-Sup2 measurable-vimage-algebra2 simp: subprob-algebra-def)

lemma measurable-submarkov:
  K ∈ measurable M (subprob-algebra M) ⟷
    (∀x∈space M. subprob-space (K x) ∧ sets (K x) = sets M) ∧
    (∀A∈sets M. (λx. emeasure (K x) A) ∈ measurable M borel)
proof
  assume (∀x∈space M. subprob-space (K x) ∧ sets (K x) = sets M) ∧
    (∀A∈sets M. (λx. emeasure (K x) A) ∈ borel-measurable M)
  then show K ∈ measurable M (subprob-algebra M)
```

by (*intro measurable-subprob-algebra*) *auto*
next
assume $K \in$ *measurable* M (*subprob-algebra* M)
then show $(\forall x{\in}$*space* M. *subprob-space* $(K\ x) \wedge$ *sets* $(K\ x) =$ *sets* $M) \wedge$
$(\forall A{\in}$*sets* M. $(\lambda x$. *emeasure* $(K\ x)\ A) \in$ *borel-measurable* $M)$
by (*auto dest*: *subprob-space-kernel sets-kernel*)
qed

lemma *measurable-subprob-algebra-generated*:
assumes *eq*: *sets* $N =$ *sigma-sets* $\Omega\ G$ **and** *Int-stable* $G\ G \subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *subprob-space* $(K\ a)$
assumes *sets*: $\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *sets* $(K\ a) =$ *sets* N
assumes $\bigwedge A$. $A \in G \Longrightarrow (\lambda a$. *emeasure* $(K\ a)\ A) \in$ *borel-measurable* M
assumes Ω: $(\lambda a$. *emeasure* $(K\ a)\ \Omega) \in$ *borel-measurable* M
shows $K \in$ *measurable* M (*subprob-algebra* N)
proof (*rule measurable-subprob-algebra*)
fix a **assume** $a \in$ *space* M **then show** *subprob-space* $(K\ a)$ *sets* $(K\ a) =$ *sets* N
by *fact+*
next
interpret G: *sigma-algebra* Ω *sigma-sets* $\Omega\ G$
using ‹$G \subseteq$ *Pow* Ω› **by** (*rule sigma-algebra-sigma-sets*)
fix A **assume** $A \in$ *sets* N **with** *assms*(*2*,*3*) **show** $(\lambda a$. *emeasure* $(K\ a)\ A) \in$
borel-measurable M
unfolding ‹*sets* $N =$ *sigma-sets* $\Omega\ G$›
proof (*induction rule*: *sigma-sets-induct-disjoint*)
case (*basic* A) **then show** *?case* **by** *fact*
next
case *empty* **then show** *?case* **by** *simp*
next
case (*compl* A)
have $(\lambda a$. *emeasure* $(K\ a)\ (\Omega - A)) \in$ *borel-measurable* $M \longleftrightarrow$
$(\lambda a$. *emeasure* $(K\ a)\ \Omega -$ *emeasure* $(K\ a)\ A) \in$ *borel-measurable* M
using *G.top G.sets-into-space sets eq compl subprob-space.emeasure-subprob-space-less-top*[*OF subsp*]
by (*intro measurable-cong emeasure-Diff*) *auto*
with *compl* Ω **show** *?case*
by *simp*
next
case (*union* F)
moreover have $(\lambda a$. *emeasure* $(K\ a)\ (\bigcup i$. $F\ i)) \in$ *borel-measurable* $M \longleftrightarrow$
$(\lambda a$. $\sum i$. *emeasure* $(K\ a)\ (F\ i)) \in$ *borel-measurable* M
using *sets union eq*
by (*intro measurable-cong suminf-emeasure*[*symmetric*]) *auto*
ultimately show *?case*
by *auto*
qed
qed

lemma *space-subprob-algebra-empty-iff*:

space (*subprob-algebra N*) = {} ⟷ *space N* = {}
proof
have ⋀*x*. *x* ∈ *space N* ⟹ *density N* (λ-. *0*) ∈ *space* (*subprob-algebra N*)
by (*auto simp*: *space-subprob-algebra emeasure-density intro*!: *subprob-spaceI*)
then show *space* (*subprob-algebra N*) = {} ⟹ *space N* = {}
by *auto*
next
assume *space N* = {}
hence *sets N* = {{}} **by** (*simp add*: *space-empty-iff*)
moreover have ⋀*M*. *subprob-space M* ⟹ *sets M* ≠ {{}}
by (*simp add*: *subprob-space.subprob-not-empty space-empty-iff*[*symmetric*])
ultimately show *space* (*subprob-algebra N*) = {} **by** (*auto simp*: *space-subprob-algebra*)
qed

lemma *nn-integral-measurable-subprob-algebra*[*measurable*]:
assumes *f*: *f* ∈ *borel-measurable N*
shows (λ*M*. *integral*^{N} *M f*) ∈ *borel-measurable* (*subprob-algebra N*) (**is** - ∈ *?B*)
using *f*
proof *induct*
case (*cong f g*)
moreover have (λ*M'*. ∫$^+$*M''*. *f M'' ∂M'*) ∈ *?B* ⟷ (λ*M'*. ∫$^+$*M''*. *g M'' ∂M'*) ∈ *?B*
by (*intro measurable-cong nn-integral-cong cong*)
(*auto simp*: *space-subprob-algebra dest*!: *sets-eq-imp-space-eq*)
ultimately show *?case* **by** *simp*
next
case (*set B*)
then have (λ*M'*. ∫$^+$*M''*. *indicator B M'' ∂M'*) ∈ *?B* ⟷ (λ*M'*. *emeasure M' B*) ∈ *?B*
by (*intro measurable-cong nn-integral-indicator*) (*simp add*: *space-subprob-algebra*)
with *set* **show** *?case*
by (*simp add*: *measurable-emeasure-subprob-algebra*)
next
case (*mult f c*)
then have (λ*M'*. ∫$^+$*M''*. *c* ∗ *f M'' ∂M'*) ∈ *?B* ⟷ (λ*M'*. *c* ∗ ∫$^+$*M''*. *f M'' ∂M'*) ∈ *?B*
by (*intro measurable-cong nn-integral-cmult*) (*auto simp add*: *space-subprob-algebra*)
with *mult* **show** *?case*
by *simp*
next
case (*add f g*)
then have (λ*M'*. ∫$^+$*M''*. *f M'' + g M'' ∂M'*) ∈ *?B* ⟷ (λ*M'*. (∫$^+$*M''*. *f M'' ∂M'*) + (∫$^+$*M''*. *g M'' ∂M'*)) ∈ *?B*
by (*intro measurable-cong nn-integral-add*) (*auto simp add*: *space-subprob-algebra*)
with *add* **show** *?case*
by (*simp add*: *ac-simps*)
next
case (*seq F*)
then have (λ*M'*. ∫$^+$*M''*. (*SUP i*. *F i*) *M'' ∂M'*) ∈ *?B* ⟷ (λ*M'*. *SUP i*.

$(\int^+ M''.\ F\ i\ M''\ \partial M')) \in ?B$
unfolding *SUP-apply*
by (*intro measurable-cong nn-integral-monotone-convergence-SUP*) (*auto simp add*: *space-subprob-algebra*)
with *seq* **show** *?case*
by (*simp add*: *ac-simps*)
qed

lemma *measurable-distr*:
assumes [*measurable*]: $f \in$ *measurable M N*
shows $(\lambda M'.$ *distr M' N f*$) \in$ *measurable* (*subprob-algebra M*) (*subprob-algebra N*)
proof (*cases space N* $= \{\}$)
case *False*
show *?thesis*
proof (*rule measurable-subprob-algebra*)
fix *A* **assume** *A*: $A \in$ *sets N*
then have $(\lambda M'.$ *emeasure* (*distr M' N f*) *A*$) \in$ *borel-measurable* (*subprob-algebra M*) $\longleftrightarrow$
$(\lambda M'.$ *emeasure M'* (*f* $-`$ *A* $\cap$ *space M*)$) \in$ *borel-measurable* (*subprob-algebra M*)
by (*intro measurable-cong*)
(*auto simp*: *emeasure-distr space-subprob-algebra*
intro!: *arg-cong2*[**where** *f*=*emeasure*] *sets-eq-imp-space-eq arg-cong2*[**where** *f*=$(\cap)$])
also have ...
using *A* **by** (*intro measurable-emeasure-subprob-algebra*) *simp*
finally show $(\lambda M'.$ *emeasure* (*distr M' N f*) *A*$) \in$ *borel-measurable* (*subprob-algebra M*) **.**
qed (*auto intro*!: *subprob-space.subprob-space-distr simp*: *space-subprob-algebra False cong*: *measurable-cong-sets*)
qed (*use assms* **in** ‹*auto simp*: *measurable-empty-iff space-subprob-algebra-empty-iff*›)

lemma *emeasure-space-subprob-algebra*[*measurable*]:
$(\lambda a.$ *emeasure a* (*space a*)$) \in$ *borel-measurable* (*subprob-algebra N*)
proof−
have $(\lambda a.$ *emeasure a* (*space N*)$) \in$ *borel-measurable* (*subprob-algebra N*) (**is** *?f* $\in$ *?M*)
by (*rule measurable-emeasure-subprob-algebra*) *simp*
also have *?f* $\in$ *?M* $\longleftrightarrow$ $(\lambda a.$ *emeasure a* (*space a*)$) \in$ *?M*
by (*rule measurable-cong*) (*auto simp*: *space-subprob-algebra dest*: *sets-eq-imp-space-eq*)
finally show *?thesis* **.**
qed

lemma *integrable-measurable-subprob-algebra*[*measurable*]:
fixes *f* :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $f \in$ *borel-measurable N*
shows *Measurable.pred* (*subprob-algebra N*) $(\lambda M.$ *integrable M f*$)$
proof (*rule measurable-cong*[*THEN iffD2*])

show $M \in$ *space* (*subprob-algebra N*) $\Longrightarrow$ *integrable M f* $\longleftrightarrow$ ($\int^+ x.$ *norm* (*f x*) ∂M) $< \infty$ **for** *M*
by (*auto simp*: *space-subprob-algebra integrable-iff-bounded*)
qed *measurable*

lemma *integral-measurable-subprob-algebra*[*measurable*]:
fixes *f* :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *f* [*measurable*]: *f* $\in$ *borel-measurable N*
shows ($\lambda M.$ *integral*^{L} *M f*) $\in$ *subprob-algebra N* $\rightarrow_M$ *borel*
proof −
from *borel-measurable-implies-sequence-metric*[*OF f*, *of 0*]
obtain *F* **where** *F*: $\bigwedge i.$ *simple-function N* (*F i*)
$\bigwedge x.$ *x* $\in$ *space N* $\Longrightarrow$ ($\lambda i.$ *F i x*) $\longrightarrow$ *f x*
$\bigwedge i\ x.$ *x* $\in$ *space N* $\Longrightarrow$ *norm* (*F i x*) $\leq$ *2* $*$ *norm* (*f x*)
unfolding *norm-conv-dist* **by** *blast*

have [*measurable*]: *F i* $\in$ *N* $\rightarrow_M$ *count-space UNIV* **for** *i*
using *F*(*1*) **by** (*rule measurable-simple-function*)

define *F′* **where** [*abs-def*]:
F′ M i = (*if integrable M f then integral*^{L} *M* (*F i*) *else 0*) **for** *M i*

have ($\lambda M.$ *F′ M i*) $\in$ *subprob-algebra N* $\rightarrow_M$ *borel* **for** *i*
proof (*rule measurable-cong*[*THEN iffD2*])
fix *M* **assume** *M* $\in$ *space* (*subprob-algebra N*)
then have [*simp*]: *sets M* = *sets N space M* = *space N subprob-space M*
by (*auto simp*: *space-subprob-algebra intro*!: *sets-eq-imp-space-eq*)
interpret *subprob-space M* **by** *fact*
have *F′ M i* = (*if integrable M f then Bochner-Integration.simple-bochner-integral M* (*F i*) *else 0*)
using *F*(*1*)
by (*subst simple-bochner-integrable-eq-integral*)
(*auto simp*: *simple-bochner-integrable.simps simple-function-def F′-def*)
then show *F′ M i* = (*if integrable M f then* $\sum y \in F\ i$ ‘ *space N*. *measure M* {*x*$\in$*space N*. *F i x* = *y*} $*_R$ *y else 0*)
unfolding *simple-bochner-integral-def* **by** *simp*
qed *measurable*
moreover
have *F′ M* $\longrightarrow$ *integral*^{L} *M f* **if** *M*: *M* $\in$ *space* (*subprob-algebra N*) **for** *M*
proof *cases*
from *M* **have** [*simp*]: *sets M* = *sets N space M* = *space N*
by (*auto simp*: *space-subprob-algebra intro*!: *sets-eq-imp-space-eq*)
assume *integrable M f* **then show** *?thesis*
unfolding *F′-def* **using** *F*(*1*)[*THEN borel-measurable-simple-function*] *F*
by (*auto intro*!: *integral-dominated-convergence*[**where** *w*=$\lambda x.$ *2* $*$ *norm* (*f x*)]
cong: *measurable-cong-sets*)
qed (*auto simp*: *F′-def not-integrable-integral-eq*)
ultimately show *?thesis*
by (*rule borel-measurable-LIMSEQ-metric*)

qed

lemma *measurable-pair-measure*:
assumes *f*: $f \in$ *measurable M* (*subprob-algebra N*)
assumes *g*: $g \in$ *measurable M* (*subprob-algebra L*)
shows $(\lambda x.\ f\ x \bigotimes_M g\ x) \in$ *measurable M* (*subprob-algebra* $(N \bigotimes_M L)$)
proof (*rule measurable-subprob-algebra*)
{ fix *x* **assume** $x \in$ *space M*
with *measurable-space*[*OF f*] *measurable-space*[*OF g*]
have *fx*: $f\ x \in$ *space* (*subprob-algebra N*) **and** *gx*: $g\ x \in$ *space* (*subprob-algebra L*)
by *auto*
interpret *F*: *subprob-space f x*
using *fx* **by** (*simp add*: *space-subprob-algebra*)
interpret *G*: *subprob-space g x*
using *gx* **by** (*simp add*: *space-subprob-algebra*)

interpret *pair-subprob-space f x g x* **..**
show *subprob-space* $(f\ x \bigotimes_M g\ x)$ **by** *unfold-locales*
show *sets-eq*: *sets* $(f\ x \bigotimes_M g\ x) =$ *sets* $(N \bigotimes_M L)$
using *fx gx* **by** (*simp add*: *space-subprob-algebra*)

have *1*: $\bigwedge A\ B.\ A \in$ *sets* $N \Longrightarrow B \in$ *sets* $L \Longrightarrow$ *emeasure* $(f\ x \bigotimes_M g\ x)\ (A \times B) =$ *emeasure* $(f\ x)\ A *$ *emeasure* $(g\ x)\ B$
using *fx gx* **by** (*intro G.emeasure-pair-measure-Times*) (*auto simp*: *space-subprob-algebra*)
have *emeasure* $(f\ x \bigotimes_M g\ x)$ (*space* $(f\ x \bigotimes_M g\ x)) =$
emeasure $(f\ x)$ (*space* $(f\ x)) *$ *emeasure* $(g\ x)$ (*space* $(g\ x)$)
by (*subst G.emeasure-pair-measure-Times*[*symmetric*]) (*simp-all add*: *space-pair-measure*)
hence *2*: $\bigwedge A.\ A \in$ *sets* $(N \bigotimes_M L) \Longrightarrow$ *emeasure* $(f\ x \bigotimes_M g\ x)$ (*space N* $\times$ *space L* $- A) =$
$\ldots -$ *emeasure* $(f\ x \bigotimes_M g\ x)\ A$
using *emeasure-compl*[*simplified*, *OF - P.emeasure-finite*]
unfolding *sets-eq*
unfolding *sets-eq-imp-space-eq*[*OF sets-eq*]
by (*simp add*: *space-pair-measure G.emeasure-pair-measure-Times*)
note *1 2 sets-eq* **}**
note *Times* = *this*(*1*) **and** *Compl* = *this*(*2*) **and** *sets-eq* = *this*(*3*)

fix *A* **assume** *A*: $A \in$ *sets* $(N \bigotimes_M L)$
show $(\lambda a.$ *emeasure* $(f\ a \bigotimes_M g\ a)\ A) \in$ *borel-measurable M*
using *Int-stable-pair-measure-generator pair-measure-closed A*
unfolding *sets-pair-measure*
proof (*induct A rule*: *sigma-sets-induct-disjoint*)
case (*basic A*) **then show** *?case*
by (*auto intro*!: *borel-measurable-times-ennreal simp*: *Times cong*: *measurable-cong*)
(*auto intro*!: *measurable-emeasure-kernel f g*)
next

```
    case (compl A)
    then have A: A ∈ sets (N ⨂M L)
      by (auto simp: sets-pair-measure)
    have (λx. emeasure (f x) (space (f x)) * emeasure (g x) (space (g x)) −
                emeasure (f x ⨂M g x) A) ∈ borel-measurable M (is ?f ∈ ?M)
      using compl(2) f g by measurable
    thus ?case by (simp add: Compl A cong: measurable-cong)
  next
    case (union A)
    then have range A ⊆ sets (N ⨂M L) disjoint-family A
      by (auto simp: sets-pair-measure)
    then have (λa. emeasure (f a ⨂M g a) (⋃i. A i)) ∈ borel-measurable M ⟷
      (λa. ∑i. emeasure (f a ⨂M g a) (A i)) ∈ borel-measurable M
      by (intro measurable-cong suminf-emeasure[symmetric])
         (auto simp: sets-eq)
    also have ...
      using union by auto
    finally show ?case .
  qed simp
qed

lemma restrict-space-measurable:
  assumes X: X ≠ {} X ∈ sets K
  assumes N: N ∈ measurable M (subprob-algebra K)
  shows (λx. restrict-space (N x) X) ∈ measurable M (subprob-algebra (restrict-space
K X))
proof (rule measurable-subprob-algebra)
  fix a assume a: a ∈ space M
  from N[THEN measurable-space, OF this]
  have subprob-space (N a) and [simp]: sets (N a) = sets K space (N a) = space
K
    by (auto simp add: space-subprob-algebra dest: sets-eq-imp-space-eq)
  then interpret subprob-space N a
    by simp
  show subprob-space (restrict-space (N a) X)
  proof
    show space (restrict-space (N a) X) ≠ {}
      using X by (auto simp add: space-restrict-space)
    show emeasure (restrict-space (N a) X) (space (restrict-space (N a) X)) ≤ 1
        using X by (simp add: emeasure-restrict-space space-restrict-space sub-
prob-emeasure-le-1)
  qed
  show sets (restrict-space (N a) X) = sets (restrict-space K X)
    by (intro sets-restrict-space-cong) fact
next
  fix A assume A: A ∈ sets (restrict-space K X)
  show (λa. emeasure (restrict-space (N a) X) A) ∈ borel-measurable M
  proof (subst measurable-cong)
    fix a assume a ∈ space M
```

```
    from N[THEN measurable-space, OF this]
    have [simp]: sets (N a) = sets K space (N a) = space K
      by (auto simp add: space-subprob-algebra dest: sets-eq-imp-space-eq)
    show emeasure (restrict-space (N a) X) A = emeasure (N a) (A ∩ X)
    using X A by (subst emeasure-restrict-space) (auto simp add: sets-restrict-space
ac-simps)
  next
    show (λw. emeasure (N w) (A ∩ X)) ∈ borel-measurable M
      using A X
      by (intro measurable-compose[OF N measurable-emeasure-subprob-algebra])
         (auto simp: sets-restrict-space)
  qed
qed
```

7.2 Properties of "return"

```
definition return :: 'a measure ⇒ 'a ⇒ 'a measure where
  return R x = measure-of (space R) (sets R) (λA. indicator A x)

lemma space-return[simp]: space (return M x) = space M
  by (simp add: return-def)

lemma sets-return[simp]: sets (return M x) = sets M
  by (simp add: return-def)

lemma measurable-return1[simp]: measurable (return N x) L = measurable N L
  by (simp cong: measurable-cong-sets)

lemma measurable-return2[simp]: measurable L (return N x) = measurable L N
  by (simp cong: measurable-cong-sets)

lemma return-sets-cong: sets M = sets N ⟹ return M = return N
  by (auto dest: sets-eq-imp-space-eq simp: fun-eq-iff return-def)

lemma return-cong: sets A = sets B ⟹ return A x = return B x
  by (auto simp add: return-def dest: sets-eq-imp-space-eq)

lemma emeasure-return[simp]:
  assumes A ∈ sets M
  shows emeasure (return M x) A = indicator A x
proof (rule emeasure-measure-of[OF return-def])
  show sets M ⊆ Pow (space M) by (rule sets.space-closed)
 show positive (sets (return M x)) (λA. indicator A x) by (simp add: positive-def)
  from assms show A ∈ sets (return M x) unfolding return-def by simp
  show countably-additive (sets (return M x)) (λA. indicator A x)
    by (auto intro!: countably-additiveI suminf-indicator)
qed

lemma prob-space-return: x ∈ space M ⟹ prob-space (return M x)
```

by *rule simp*

lemma *subprob-space-return*: $x \in space\ M \Longrightarrow subprob\text{-}space\ (return\ M\ x)$
by (*intro prob-space-return prob-space-imp-subprob-space*)

lemma *subprob-space-return-ne*:
assumes $space\ M \neq \{\}$ **shows** *subprob-space* (*return M x*)
by (*metis assms emeasure-return indicator-simps*(*2*) *sets.top space-return subprob-spaceI subprob-space-return zero-le*)

lemma *measure-return*: **assumes** *X*: $X \in sets\ M$ **shows** *measure* (*return M x*) *X* = *indicator X x*
unfolding *measure-def emeasure-return*[*OF X*, *of x*] **by** (*simp split*: *split-indicator*)

lemma *AE-return*:
assumes [*simp*]: $x \in space\ M$ **and** [*measurable*]: *Measurable.pred M P*
shows (*AE y in return M x. P y*) $\longleftrightarrow$ *P x*
proof −
have (*AE y in return M x. y* $\notin$ {*x*∈*space M*. ¬ *P x*}) $\longleftrightarrow$ *P x*
by (*subst AE-iff-null-sets*[*symmetric*]) (*simp-all add*: *null-sets-def split*: *split-indicator*)
also have (*AE y in return M x. y* $\notin$ {*x*∈*space M*. ¬ *P x*}) $\longleftrightarrow$ (*AE y in return M x. P y*)
by (*rule AE-cong*) *auto*
finally show *?thesis* **.**
qed

lemma *nn-integral-return*:
assumes $x \in space\ M\ g \in borel\text{-}measurable\ M$
shows $(\int^+ a.\ g\ a\ \partial return\ M\ x) = g\ x$
proof−
interpret *prob-space return M x* **by** (*rule prob-space-return*[*OF* ‹*x* ∈ *space M*›])
have $(\int^+ a.\ g\ a\ \partial return\ M\ x) = (\int^+ a.\ g\ x\ \partial return\ M\ x)$ **using** *assms*
by (*intro nn-integral-cong-AE*) (*auto simp*: *AE-return*)
also have ... = *g x*
using *nn-integral-const*[*of return M x*] *emeasure-space-1* **by** *simp*
finally show *?thesis* **.**
qed

lemma *integral-return*:
fixes *g* :: - ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes $x \in space\ M\ g \in borel\text{-}measurable\ M$
shows $(\int a.\ g\ a\ \partial return\ M\ x) = g\ x$
proof−
interpret *prob-space return M x* **by** (*rule prob-space-return*[*OF* ‹*x* ∈ *space M*›])
have $(\int a.\ g\ a\ \partial return\ M\ x) = (\int a.\ g\ x\ \partial return\ M\ x)$ **using** *assms*
by (*intro integral-cong-AE*) (*auto simp*: *AE-return*)
then show *?thesis*
using *prob-space* **by** *simp*
qed

lemma *return-measurable*[*measurable*]: *return N* ∈ *measurable N* (*subprob-algebra N*)
by (*rule measurable-subprob-algebra*) (*auto simp*: *subprob-space-return*)

lemma *distr-return*:
assumes *f* ∈ *measurable M N* **and** *x* ∈ *space M*
shows *distr* (*return M x*) *N f* = *return N* (*f x*)
using *assms* **by** (*intro measure-eqI*) (*simp-all add*: *indicator-def emeasure-distr*)

lemma *return-restrict-space*:
Ω ∈ *sets M* ⟹ *return* (*restrict-space M* Ω) *x* = *restrict-space* (*return M x*) Ω
by (*auto intro*!: *measure-eqI simp*: *sets-restrict-space emeasure-restrict-space*)

lemma *measurable-distr2*:
assumes *f*[*measurable*]: *case-prod f* ∈ *measurable* (*L* ⨂$_M$ *M*) *N*
assumes *g*[*measurable*]: *g* ∈ *measurable L* (*subprob-algebra M*)
shows (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *measurable L* (*subprob-algebra N*)
proof −
have (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *measurable L* (*subprob-algebra N*)
⟷ (λ*x*. *distr* (*return L x* ⨂$_M$ *g x*) *N* (*case-prod f*)) ∈ *measurable L* (*subprob-algebra N*)
proof (*rule measurable-cong*)
fix *x* **assume** *x*: *x* ∈ *space L*
have *gx*: *g x* ∈ *space* (*subprob-algebra M*)
using *measurable-space*[*OF g x*] **.**
then have [*simp*]: *sets* (*g x*) = *sets M*
by (*simp add*: *space-subprob-algebra*)
then have [*simp*]: *space* (*g x*) = *space M*
by (*rule sets-eq-imp-space-eq*)
let *?R* = *return L x*
from *measurable-compose-Pair1*[*OF x f*] **have** *f-M'*: *f x* ∈ *measurable M N*
by *simp*
interpret *subprob-space g x*
using *gx* **by** (*simp add*: *space-subprob-algebra*)
have *space-pair-M'*[*simp*]: ⋀*X*. *space* (*X* ⨂$_M$ *g x*) = *space* (*X* ⨂$_M$ *M*)
by (*simp add*: *space-pair-measure*)
show *distr* (*g x*) *N* (*f x*) = *distr* (*?R* ⨂$_M$ *g x*) *N* (*case-prod f*) (**is** *?l* = *?r*)
proof (*rule measure-eqI*)
show *sets ?l* = *sets ?r*
by *simp*
next
fix *A* **assume** *A* ∈ *sets ?l*
then have *A*[*measurable*]: *A* ∈ *sets N*
by *simp*
then have *emeasure ?r A* = *emeasure* (*?R* ⨂$_M$ *g x*) ((λ(*x*, *y*). *f x y*) −‘ *A* ∩ *space* (*?R* ⨂$_M$ *g x*))
by (*auto simp add*: *emeasure-distr f-M' cong*: *measurable-cong-sets*)
also have ... = (∫$^+$*M''*. *emeasure* (*g x*) (*f M''* −‘ *A* ∩ *space M*) ∂*?R*)

```
      apply (subst emeasure-pair-measure-alt)
        apply (force simp add: f-M' cong: measurable-cong-sets intro!: measur-
able-sets[OF - A])
      apply (intro nn-integral-cong arg-cong[where f=emeasure (g x)])
      apply (auto simp: space-subprob-algebra space-pair-measure)
      done
    also have ... = emeasure (g x) (f x -` A ∩ space M)
      by (subst nn-integral-return)
         (auto simp: x intro!: measurable-emeasure)
    also have ... = emeasure ?l A
      by (simp add: emeasure-distr f-M' cong: measurable-cong-sets)
    finally show emeasure ?l A = emeasure ?r A ..
  qed
 qed
 also have ...
 proof (intro measurable-compose[OF measurable-pair-measure measurable-distr])
  show return L ∈ L →M subprob-algebra L
    by (rule return-measurable)
 qed measurable
 finally show ?thesis .
qed

lemma nn-integral-measurable-subprob-algebra2:
 assumes f[measurable]: (λ(x, y). f x y) ∈ borel-measurable (M ⨂M N)
 assumes N[measurable]: L ∈ measurable M (subprob-algebra N)
 shows (λx. integral^N (L x) (f x)) ∈ borel-measurable M
proof −
 note nn-integral-measurable-subprob-algebra[measurable]
 note measurable-distr2[measurable]
 have (λx. integral^N (distr (L x) (M ⨂M N) (λy. (x, y))) (λ(x, y). f x y)) ∈
borel-measurable M
   by measurable
 then show (λx. integral^N (L x) (f x)) ∈ borel-measurable M
   by (rule measurable-cong[THEN iffD1, rotated])
      (simp add: nn-integral-distr)
qed

lemma emeasure-measurable-subprob-algebra2:
 assumes A[measurable]: (SIGMA x:space M. A x) ∈ sets (M ⨂M N)
 assumes L[measurable]: L ∈ measurable M (subprob-algebra N)
 shows (λx. emeasure (L x) (A x)) ∈ borel-measurable M
proof −
 { fix x assume x ∈ space M
   then have Pair x -` Sigma (space M) A = A x
     by auto
   with sets-Pair1[OF A, of x] have A x ∈ sets N
     by auto }
 note ** = this
```

have $*$: $\bigwedge x$. *fst* $x \in$ *space* $M \Longrightarrow$ *snd* $x \in A$ (*fst* x) $\longleftrightarrow$ $x \in$ (*SIGMA* x:*space* M. A x)
by (*auto simp*: *fun-eq-iff*)
have *MN*: *Measurable.pred* ($M \bigotimes_M N$) (λw. $w \in$ *Sigma* (*space* M) A)
by *auto*
then have ($\lambda(x, y)$. *indicator* (A x) y::*ennreal*) $\in$ *borel-measurable* ($M \bigotimes_M N$)
apply *measurable*
by (*smt* (*verit*, *best*) *MN measurable-cong mem-Sigma-iff prod.collapse space-pair-measure*)
then have (λx. *integral*^{N} (L x) (*indicator* (A x))) $\in$ *borel-measurable* M
by (*intro nn-integral-measurable-subprob-algebra2*[**where** N=N] L)
then show (λx. *emeasure* (L x) (A x)) $\in$ *borel-measurable* M
by (*smt* (*verit*) $**$ L *measurable-cong-simp nn-integral-indicator sets-kernel*)
qed

lemma *measure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *measure* (L x) (A x)) $\in$ *borel-measurable* M
unfolding *measure-def*
by (*intro borel-measurable-enn2real emeasure-measurable-subprob-algebra2*[*OF assms*])

definition *select-sets* M = (*SOME* N. *sets* M = *sets* (*subprob-algebra* N))

lemma *select-sets1*:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *sets* M = *sets* (*subprob-algebra* (*select-sets* M))
unfolding *select-sets-def* **by** (*rule someI*)

lemma *sets-select-sets*[*simp*]:
assumes *sets*: *sets* M = *sets* (*subprob-algebra* N)
shows *sets* (*select-sets* M) = *sets* N
unfolding *select-sets-def*
proof (*rule someI2*)
show *sets* M = *sets* (*subprob-algebra* N)
by *fact*
next
fix L **assume** *sets* M = *sets* (*subprob-algebra* L)
with *sets* **have** *eq*: *space* (*subprob-algebra* N) = *space* (*subprob-algebra* L)
by (*intro sets-eq-imp-space-eq*) *simp*
show *sets* L = *sets* N
proof *cases*
assume *space* (*subprob-algebra* N) = {}
with *space-subprob-algebra-empty-iff*[*of* N] *space-subprob-algebra-empty-iff*[*of* L]
show *?thesis*
by (*simp add*: *eq space-empty-iff*)
next
assume *space* (*subprob-algebra* N) $\neq$ {}
with *eq* **show** *?thesis*

by (*smt* (*verit*) *equals0I mem-Collect-eq space-subprob-algebra*)
qed
qed

lemma *space-select-sets*[*simp*]:
sets M = *sets* (*subprob-algebra N*) ⟹ *space* (*select-sets M*) = *space N*
by (*intro sets-eq-imp-space-eq sets-select-sets*)

7.3 Join

definition *join* :: *'a measure measure* ⇒ *'a measure* **where**
join M = *measure-of* (*space* (*select-sets M*)) (*sets* (*select-sets M*)) (λ*B*. ∫⁺ *M'*. *emeasure M' B* ∂*M*)

lemma
shows *space-join*[*simp*]: *space* (*join M*) = *space* (*select-sets M*)
and *sets-join*[*simp*]: *sets* (*join M*) = *sets* (*select-sets M*)
by (*simp-all add*: *join-def*)

lemma *emeasure-join*:
assumes *M*[*simp*, *measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*) **and** *A*: *A* ∈ *sets N*
shows *emeasure* (*join M*) *A* = (∫⁺ *M'*. *emeasure M' A* ∂*M*)
proof (*rule emeasure-measure-of*[*OF join-def*])
show *countably-additive* (*sets* (*join M*)) (λ*B*. ∫⁺ *M'*. *emeasure M' B* ∂*M*)
proof (*rule countably-additiveI*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *sets* (*join M*) *disjoint-family A*
have (∑ *i*. ∫⁺ *M'*. *emeasure M'* (*A i*) ∂*M*) = (∫⁺*M'*. (∑ *i*. *emeasure M'* (*A i*)) ∂*M*)
using *A* **by** (*subst nn-integral-suminf*) (*auto simp*: *measurable-emeasure-subprob-algebra*)
also have ... = (∫⁺*M'*. *emeasure M'* (⋃ *i*. *A i*) ∂*M*)
proof (*rule nn-integral-cong*)
fix *M'* **assume** *M'* ∈ *space M*
then show (∑ *i*. *emeasure M'* (*A i*)) = *emeasure M'* (⋃ *i*. *A i*)
using *A sets-eq-imp-space-eq*[*OF M*] **by** (*simp add*: *suminf-emeasure space-subprob-algebra*)
qed
finally show (∑ *i*. ∫⁺*M'*. *emeasure M'* (*A i*) ∂*M*) = (∫⁺*M'*. *emeasure M'* (⋃ *i*. *A i*) ∂*M*) **.**
qed
qed (*auto simp*: *A sets.space-closed positive-def*)

lemma *measurable-join*:
join ∈ *measurable* (*subprob-algebra* (*subprob-algebra N*)) (*subprob-algebra N*)
proof (*cases space N* ≠ {}, *rule measurable-subprob-algebra*)
fix *A* **assume** *A* ∈ *sets N*
let *?B* = *borel-measurable* (*subprob-algebra* (*subprob-algebra N*))
have (λ*M'*. *emeasure* (*join M'*) *A*) ∈ *?B* ⟷ (λ*M'*. (∫⁺ *M''*. *emeasure M''* *A* ∂*M'*)) ∈ *?B*

proof (*rule measurable-cong*)
fix M' **assume** $M' \in$ *space* (*subprob-algebra* (*subprob-algebra* N))
then show *emeasure* (*join* M') $A = (\int^+ M''.$ *emeasure* M'' A $\partial M')$
by (*intro emeasure-join*) (*auto simp*: *space-subprob-algebra* ‹$A \in$*sets* N›)
qed
also have $(\lambda M'.\ \int^+ M''.$ *emeasure* M'' A $\partial M') \in$ *?B*
using *measurable-emeasure-subprob-algebra*[*OF* ‹$A \in$*sets* N›]
by (*rule nn-integral-measurable-subprob-algebra*)
finally show $(\lambda M'.$ *emeasure* (*join* M') $A) \in$ *borel-measurable* (*subprob-algebra* (*subprob-algebra* N)) **.**
next
assume [*simp*]: *space* $N \neq \{\}$
fix M **assume** M: $M \in$ *space* (*subprob-algebra* (*subprob-algebra* N))
then have $(\int^+ M'.$ *emeasure* M' (*space* N) $\partial M) \leq (\int^+ M'.\ 1\ \partial M)$
proof (*intro nn-integral-mono*)
show $\bigwedge x.\ [\![M \in$ *space* (*subprob-algebra* (*subprob-algebra* N)); $x \in$ *space* $M]\!]$
$\Longrightarrow$ *emeasure* x (*space* N) ≤ 1
by (*smt* (*verit*) *mem-Collect-eq sets-eq-imp-space-eq space-subprob-algebra subprob-space.subprob-emeasure-le-1*)
qed
with M **show** *subprob-space* (*join* M)
by (*intro subprob-spaceI*)
(*auto simp*: *emeasure-join space-subprob-algebra M dest*: *subprob-space.emeasure-space-le-1*)
next
assume $\neg$(*space* $N \neq \{\}$)
thus *?thesis* **by** (*simp add*: *measurable-empty-iff space-subprob-algebra-empty-iff*)
qed (*auto simp*: *space-subprob-algebra*)

lemma *nn-integral-join*:
assumes f: $f \in$ *borel-measurable* N
and M[*measurable-cong*]: *sets* M = *sets* (*subprob-algebra* N)
shows $(\int^+ x.\ f\ x\ \partial$*join* $M) = (\int^+ M'.\ \int^+ x.\ f\ x\ \partial M'\ \partial M)$
using f
proof *induct*
case (*cong f g*)
moreover have *integral*^{N} (*join* M) f = *integral*^{N} (*join* M) g
by (*intro nn-integral-cong cong*) (*simp add*: M)
moreover from M **have** $(\int^+ M'.$ *integral*^{N} M' f $\partial M) = (\int^+ M'.$ *integral*^{N} M' g $\partial M)$
by (*intro nn-integral-cong cong*)
(*auto simp add*: *space-subprob-algebra dest*!: *sets-eq-imp-space-eq*)
ultimately show *?case*
by *simp*
next
case (*set A*)
with M **have** $(\int^+ M'.$ *integral*^{N} M' (*indicator* A) $\partial M) = (\int^+ M'.$ *emeasure* M' A $\partial M)$
by (*intro nn-integral-cong nn-integral-indicator*)
(*auto simp*: *space-subprob-algebra dest*!: *sets-eq-imp-space-eq*)

with *set* **show** *?case*
using *M* **by** (*simp add*: *emeasure-join*)
next
case (*mult f c*)
have ($\int^+ M'.\ \int^+ x.\ c * f\ x\ \partial M'\ \partial M) = (\int^+ M'.\ c * \int^+ x.\ f\ x\ \partial M'\ \partial M)$
using *mult M M*[*THEN sets-eq-imp-space-eq*]
by (*intro nn-integral-cong nn-integral-cmult*) (*auto simp add*: *space-subprob-algebra*)
also have ... $= c * (\int^+ M'.\ \int^+ x.\ f\ x\ \partial M'\ \partial M)$
using *nn-integral-measurable-subprob-algebra*[*OF mult*(*2*)]
by (*intro nn-integral-cmult mult*) (*simp add*: *M*)
also have ... $= c * (integral^N\ (join\ M)\ f)$
by (*simp add*: *mult*)
also have ... $= (\int^+ x.\ c * f\ x\ \partial join\ M)$
using *mult*(*2*,*3*) **by** (*intro nn-integral-cmult*[*symmetric*] *mult*) (*simp add*: *M cong*: *measurable-cong-sets*)
finally show *?case* **by** *simp*
next
case (*add f g*)
have ($\int^+ M'.\ \int^+ x.\ f\ x + g\ x\ \partial M'\ \partial M) = (\int^+ M'.\ (\int^+ x.\ f\ x\ \partial M') + (\int^+ x.\ g\ x\ \partial M')\ \partial M)$
using *add M M*[*THEN sets-eq-imp-space-eq*]
by (*intro nn-integral-cong nn-integral-add*) (*auto simp add*: *space-subprob-algebra*)
also have ... $= (\int^+ M'.\ \int^+ x.\ f\ x\ \partial M'\ \partial M) + (\int^+ M'.\ \int^+ x.\ g\ x\ \partial M'\ \partial M)$
using *nn-integral-measurable-subprob-algebra*[*OF add*(*1*)]
using *nn-integral-measurable-subprob-algebra*[*OF add*(*4*)]
by (*intro nn-integral-add add*) (*simp-all add*: *M*)
also have ... $= (integral^N\ (join\ M)\ f) + (integral^N\ (join\ M)\ g)$
by (*simp add*: *add*)
also have ... $= (\int^+ x.\ f\ x + g\ x\ \partial join\ M)$
using *add* **by** (*intro nn-integral-add*[*symmetric*] *add*) (*simp-all add*: *M cong*: *measurable-cong-sets*)
finally show *?case* **by** (*simp add*: *ac-simps*)
next
case (*seq F*)
have ($\int^+ M'.\ \int^+ x.\ (SUP\ i.\ F\ i)\ x\ \partial M'\ \partial M) = (\int^+ M'.\ (SUP\ i.\ \int^+ x.\ F\ i\ x\ \partial M')\ \partial M)$
using *seq M M*[*THEN sets-eq-imp-space-eq*] **unfolding** *SUP-apply*
by (*intro nn-integral-cong nn-integral-monotone-convergence-SUP*)
(*auto simp add*: *space-subprob-algebra*)
also have ... $= (SUP\ i.\ \int^+ M'.\ \int^+ x.\ F\ i\ x\ \partial M'\ \partial M)$
using *nn-integral-measurable-subprob-algebra*[*OF seq*(*1*)] *seq*
by (*intro nn-integral-monotone-convergence-SUP*)
(*simp-all add*: *M incseq-nn-integral incseq-def le-fun-def nn-integral-mono*)
also have ... $= (SUP\ i.\ integral^N\ (join\ M)\ (F\ i))$
by (*simp add*: *seq*)
also have ... $= (\int^+ x.\ (SUP\ i.\ F\ i\ x)\ \partial join\ M)$
using *seq* **by** (*intro nn-integral-monotone-convergence-SUP*[*symmetric*] *seq*)
(*simp-all add*: *M cong*: *measurable-cong-sets*)
finally show *?case* **by** (*simp add*: *ac-simps image-comp*)

qed

lemma *measurable-join1*:
⟦ *f* ∈ *measurable N K*; *sets M* = *sets* (*subprob-algebra N*) ⟧
⟹ *f* ∈ *measurable* (*join M*) *K*
by(*simp add*: *measurable-def*)

lemma
fixes *f* :: - ⇒ *real*
assumes *f-measurable* [*measurable*]: *f* ∈ *borel-measurable N*
and *f-bounded*: ⋀*x*. *x* ∈ *space N* ⟹ |*f x*| ≤ *B*
and *M* [*measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*)
and *fin*: *finite-measure M*
and *M-bounded*: *AE M′ in M*. *emeasure M′* (*space M′*) ≤ *ennreal B′*
shows *integrable-join*: *integrable* (*join M*) *f* (**is** *?integrable*)
and *integral-join*: *integral*^{L} (*join M*) *f* = ∫ *M′*. *integral*^{L} *M′ f* ∂*M* (**is** *?integral*)
proof(*case-tac* [!] *space N* = {})
assume ∗: *space N* = {}
show *?integrable*
using *M* ∗ **by**(*simp add*: *real-integrable-def measurable-def nn-integral-empty*)
have (∫ *M′*. *integral*^{L} *M′ f* ∂*M*) = (∫ *M′*. *0* ∂*M*)
proof(*rule Bochner-Integration.integral-cong*)
fix *M′*
assume *M′* ∈ *space M*
with *sets-eq-imp-space-eq*[*OF M*] **have** *space M′* = *space N*
by(*auto simp add*: *space-subprob-algebra dest*: *sets-eq-imp-space-eq*)
with ∗ **show** (∫ *x*. *f x* ∂*M′*) = *0* **by**(*simp add*: *Bochner-Integration.integral-empty*)
qed *simp*
then show *?integral*
using *M* ∗ **by**(*simp add*: *Bochner-Integration.integral-empty*)
next
assume ∗: *space N* ≠ {}

from ∗ **have** *B* [*simp*]: *0* ≤ *B* **by**(*auto dest*: *f-bounded*)

have [*measurable*]: *f* ∈ *borel-measurable* (*join M*) **using** *f-measurable M*
by(*rule measurable-join1*)

{ fix *f M′*
assume [*measurable*]: *f* ∈ *borel-measurable N*
and *f-bounded*: ⋀*x*. *x* ∈ *space N* ⟹ *f x* ≤ *B*
and *M′* ∈ *space M emeasure M′* (*space M′*) ≤ *ennreal B′*
have *AE x in M′*. *ennreal* (*f x*) ≤ *ennreal B*
proof(*rule AE-I2*)
fix *x*
assume *x* ∈ *space M′*
with ‹*M′* ∈ *space M*› *sets-eq-imp-space-eq*[*OF M*]
have *x* ∈ *space N* **by**(*auto simp add*: *space-subprob-algebra dest*: *sets-eq-imp-space-eq*)
from *f-bounded*[*OF this*] **show** *ennreal* (*f x*) ≤ *ennreal B* **by** *simp*

```
    qed
    then have (∫⁺ x. ennreal (f x) ∂M') ≤ (∫⁺ x. ennreal B ∂M')
      by(rule nn-integral-mono-AE)
    also have ... = ennreal B * emeasure M' (space M') by(simp)
    also have ... ≤ ennreal B * ennreal B' by(rule mult-left-mono)(fact, simp)
    also have ... ≤ ennreal B * ennreal |B'| by(rule mult-left-mono)(simp-all)
    finally have (∫⁺ x. ennreal (f x) ∂M') ≤ ennreal (B * |B'|) by (simp add:
ennreal-mult) }
  note bounded1 = this

  have bounded:
    ⋀f. ⟦ f ∈ borel-measurable N; ⋀x. x ∈ space N ⟹ f x ≤ B ⟧
    ⟹ (∫⁺ x. ennreal (f x) ∂join M) ≠ top
  proof −
    fix f
    assume [measurable]: f ∈ borel-measurable N
      and f-bounded: ⋀x. x ∈ space N ⟹ f x ≤ B
    have (∫⁺ x. ennreal (f x) ∂join M) = (∫⁺ M'. ∫⁺ x. ennreal (f x) ∂M' ∂M)
      by(rule nn-integral-join[OF - M]) simp
    also have ... ≤ ∫⁺ M'. B * |B'| ∂M
      using bounded1[OF ‹f ∈ borel-measurable N› f-bounded]
     by(rule nn-integral-mono-AE[OF AE-mp[OF M-bounded AE-I2], rule-format])
    also have ... = B * |B'| * emeasure M (space M) by simp
    also have ... < ∞
      using finite-measure.finite-emeasure-space[OF fin]
      by(simp add: ennreal-mult-less-top less-top)
    finally show ?thesis f by simp
  qed
  have f-pos: (∫⁺ x. ennreal (f x) ∂join M) ≠ ∞
    and f-neg: (∫⁺ x. ennreal (− f x) ∂join M) ≠ ∞
    using f-bounded by(auto del: notI intro!: bounded simp add: abs-le-iff)

  show ?integrable using f-pos f-neg by(simp add: real-integrable-def)

  note [measurable] = nn-integral-measurable-subprob-algebra

  have int-f: (∫⁺ x. f x ∂join M) = ∫⁺ M'. ∫⁺ x. f x ∂M' ∂M
    by(simp add: nn-integral-join[OF - M])
  have int-mf: (∫⁺ x. − f x ∂join M) = (∫⁺ M'. ∫⁺ x. − f x ∂M' ∂M)
    by(simp add: nn-integral-join[OF - M])

  have pos-finite: AE M' in M. (∫⁺ x. f x ∂M') ≠ ∞
    using AE-space M-bounded
  proof eventually-elim
    fix M' assume M' ∈ space M emeasure M' (space M') ≤ ennreal B'
    then have (∫⁺ x. ennreal (f x) ∂M') ≤ ennreal (B * |B'|)
      using f-measurable by(auto intro!: bounded1 dest: f-bounded)
    then show (∫⁺ x. ennreal (f x) ∂M') ≠ ∞
      by (auto simp: top-unique)
```

qed
hence [*simp*]: $(\int^+ M'.\ ennreal\ (enn2real\ (\int^+ x.\ f\ x\ \partial M'))\ \partial M) = (\int^+ M'.\ \int^+ x.\ f\ x\ \partial M'\ \partial M)$
by (*rule nn-integral-cong-AE*[*OF AE-mp*]) (*simp add*: *less-top*)
from *f-pos* **have** [*simp*]: *integrable M* $(\lambda M'.\ enn2real\ (\int^+ x.\ f\ x\ \partial M'))$
by(*simp add*: *int-f real-integrable-def nn-integral-0-iff-AE*[*THEN iffD2*] *ennreal-neg enn2real-nonneg*)

have *neg-finite*: *AE* M' *in M*. $(\int^+ x.\ -f\ x\ \partial M') \neq \infty$
using *AE-space M-bounded*
proof *eventually-elim*
fix M' **assume** $M' \in space\ M\ emeasure\ M'\ (space\ M') \leq ennreal\ B'$
then have $(\int^+ x.\ ennreal\ (-f\ x)\ \partial M') \leq ennreal\ (B * |B'|)$
using *f-measurable* **by**(*auto intro*!: *bounded1 dest*: *f-bounded*)
then show $(\int^+ x.\ ennreal\ (-f\ x)\ \partial M') \neq \infty$
by (*auto simp*: *top-unique*)
qed
hence [*simp*]: $(\int^+ M'.\ ennreal\ (enn2real\ (\int^+ x.\ -f\ x\ \partial M'))\ \partial M) = (\int^+ M'.\ \int^+ x.\ -f\ x\ \partial M'\ \partial M)$
by (*rule nn-integral-cong-AE*[*OF AE-mp*]) (*simp add*: *less-top*)
from *f-neg* **have** [*simp*]: *integrable M* $(\lambda M'.\ enn2real\ (\int^+ x.\ -f\ x\ \partial M'))$
by(*simp add*: *int-mf real-integrable-def nn-integral-0-iff-AE*[*THEN iffD2*] *ennreal-neg enn2real-nonneg*)

have $(\int x.\ f\ x\ \partial join\ M) = enn2real\ (\int^+ N.\ \int^+ x.\ f\ x\ \partial N\ \partial M) - enn2real\ (\int^+ N.\ \int^+ x.\ -f\ x\ \partial N\ \partial M)$
unfolding *real-lebesgue-integral-def*[*OF* ‹*?integrable*›] **by** (*simp add*: *nn-integral-join*[*OF - M*])
also have $\ldots = (\int N.\ enn2real\ (\int^+ x.\ f\ x\ \partial N)\ \partial M) - (\int N.\ enn2real\ (\int^+ x.\ -f\ x\ \partial N)\ \partial M)$
using *pos-finite neg-finite* **by** (*subst* (*1 2*) *integral-eq-nn-integral*) (*auto simp*: *enn2real-nonneg*)
also have $\ldots = (\int N.\ enn2real\ (\int^+ x.\ f\ x\ \partial N) - enn2real\ (\int^+ x.\ -f\ x\ \partial N)\ \partial M)$
by *simp*
also have $\ldots = \int M'.\ \int x.\ f\ x\ \partial M'\ \partial M$
proof (*rule integral-cong-AE*)
show *AE x in M*.
$enn2real\ (\int^+ x.\ ennreal\ (f\ x)\ \partial x) - enn2real\ (\int^+ x.\ ennreal\ (-f\ x)\ \partial x) = integral^L\ x\ f$
using *AE-space M-bounded*
proof *eventually-elim*
fix M' **assume** $M' \in space\ M\ emeasure\ M'\ (space\ M') \leq B'$
then interpret *subprob-space* M'
by (*auto simp*: *M*[*THEN sets-eq-imp-space-eq*] *space-subprob-algebra*)

from ‹$M' \in space\ M$› *sets-eq-imp-space-eq*[*OF M*]
have [*measurable-cong*]: *sets* M' = *sets N* **by**(*simp add*: *space-subprob-algebra*)
hence [*simp*]: *space* M' = *space N* **by**(*rule sets-eq-imp-space-eq*)

```
    have integrable M' f
      by(rule integrable-const-bound[where B=B])(auto simp add: f-bounded)
    then show enn2real (∫⁺ x. f x ∂M') - enn2real (∫⁺ x. - f x ∂M') = ∫ x.
f x ∂M'
      by(simp add: real-lebesgue-integral-def)
  qed
 qed simp-all
 finally show ?integral by simp
qed

lemma join-assoc:
 assumes M[measurable-cong]: sets M = sets (subprob-algebra (subprob-algebra
N))
 shows join (distr M (subprob-algebra N) join) = join (join M)
proof (rule measure-eqI)
 fix A assume A ∈ sets (join (distr M (subprob-algebra N) join))
 then have A: A ∈ sets N by simp
 show emeasure (join (distr M (subprob-algebra N) join)) A = emeasure (join
(join M)) A
   using measurable-join[of N]
  by (auto simp: M A nn-integral-distr emeasure-join measurable-emeasure-subprob-algebra
            sets-eq-imp-space-eq[OF M] space-subprob-algebra nn-integral-join[OF
- M]
          intro!: nn-integral-cong emeasure-join)
qed (simp add: M)

lemma join-return:
 assumes sets M = sets N and subprob-space M
 shows join (return (subprob-algebra N) M) = M
 by (rule measure-eqI)
    (simp-all add: emeasure-join space-subprob-algebra
                 measurable-emeasure-subprob-algebra nn-integral-return assms)

lemma join-return':
 assumes sets N = sets M
 shows join (distr M (subprob-algebra N) (return N)) = M
proof (rule measure-eqI)
 fix A
 have return N ∈ measurable M (subprob-algebra N)
   using assms by auto
 moreover
 assume A ∈ sets (join (distr M (subprob-algebra N) (return N)))
 ultimately show emeasure (join (distr M (subprob-algebra N) (return N))) A
= emeasure M A
  by (simp add: emeasure-join nn-integral-distr measurable-emeasure-subprob-algebra
assms)
qed (simp add: assms)

lemma join-distr-distr:
```

fixes $f :: 'a \Rightarrow 'b$ **and** $M :: 'a\ measure\ measure$ **and** $N :: 'b\ measure$
assumes $sets\ M = sets\ (subprob\text{-}algebra\ R)$ **and** $f \in measurable\ R\ N$
shows $join\ (distr\ M\ (subprob\text{-}algebra\ N)\ (\lambda M.\ distr\ M\ N\ f)) = distr\ (join\ M)\ N\ f$ (**is** *?r = ?l*)
proof (*rule measure-eqI*)
fix A **assume** $A \in sets\ ?r$
hence *A-in-N*: $A \in sets\ N$ **by** *simp*

from *assms* **have** $f \in measurable\ (join\ M)\ N$
by (*simp cong*: *measurable-cong-sets*)
moreover from *assms* **and** *A-in-N* **have** $f -` A \cap space\ R \in sets\ R$
by (*intro measurable-sets*) *simp-all*
ultimately have $emeasure\ (distr\ (join\ M)\ N\ f)\ A = \int^+ M'.\ emeasure\ M'\ (f -` A \cap space\ R)\ \partial M$
by (*simp-all add*: *A-in-N emeasure-distr emeasure-join assms*)

also have $... = \int^+ x.\ emeasure\ (distr\ x\ N\ f)\ A\ \partial M$ **using** *A-in-N*
proof (*intro nn-integral-cong*, *subst emeasure-distr*)
fix M' **assume** $M' \in space\ M$
from *assms* **have** $space\ M = space\ (subprob\text{-}algebra\ R)$
using *sets-eq-imp-space-eq* **by** *blast*
with ‹$M' \in space\ M$› **have** [*simp*]: $sets\ M' = sets\ R$ **using** *space-subprob-algebra* **by** *blast*
show $f \in measurable\ M'\ N$ **by** (*simp cong*: *measurable-cong-sets add*: *assms*)
have $space\ M' = space\ R$ **by** (*rule sets-eq-imp-space-eq*) *simp*
thus $emeasure\ M'\ (f -` A \cap space\ R) = emeasure\ M'\ (f -` A \cap space\ M')$ **by** *simp*
qed

also have $(\lambda M.\ distr\ M\ N\ f) \in measurable\ M\ (subprob\text{-}algebra\ N)$
by (*simp cong*: *measurable-cong-sets add*: *assms measurable-distr*)
hence $(\int^+ x.\ emeasure\ (distr\ x\ N\ f)\ A\ \partial M) =$
$emeasure\ (join\ (distr\ M\ (subprob\text{-}algebra\ N)\ (\lambda M.\ distr\ M\ N\ f)))\ A$
by (*simp-all add*: *emeasure-join assms A-in-N nn-integral-distr measurable-emeasure-subprob-algebra*)
finally show $emeasure\ ?r\ A = emeasure\ ?l\ A$ **..**
qed *simp*

definition $bind :: 'a\ measure \Rightarrow ('a \Rightarrow 'b\ measure) \Rightarrow 'b\ measure$ **where**
$bind\ M\ f = (if\ space\ M = \{\}\ then\ count\text{-}space\ \{\}\ else$
$join\ (distr\ M\ (subprob\text{-}algebra\ (f\ (SOME\ x.\ x \in space\ M)))\ f))$

adhoc-overloading *Monad-Syntax.bind* $\rightleftharpoons$ *bind*

lemma *bind-empty*:
$space\ M = \{\} \Longrightarrow bind\ M\ f = count\text{-}space\ \{\}$
by (*simp add*: *bind-def*)

lemma *bind-nonempty*:

space M ≠ {} ⟹ bind M f = join (distr M (subprob-algebra (f (SOME x. x ∈ space M))) f)
by (*simp add*: *bind-def*)

lemma *sets-bind-empty*: *sets M = {} ⟹ sets (bind M f) = {{}}*
by *auto*

lemma *space-bind-empty*: *space M = {} ⟹ space (bind M f) = {}*
by (*simp add*: *bind-def*)

lemma *sets-bind*[*simp*, *measurable-cong*]:
assumes *f*: *⋀x. x ∈ space M ⟹ sets (f x) = sets N* **and** *M*: *space M ≠ {}*
shows *sets (bind M f) = sets N*
using *f* [*of SOME x. x ∈ space M*] **by** (*simp add*: *bind-nonempty M some-in-eq*)

lemma *space-bind*[*simp*]:
assumes *⋀x. x ∈ space M ⟹ sets (f x) = sets N* **and** *space M ≠ {}*
shows *space (bind M f) = space N*
using *assms* **by** (*intro sets-eq-imp-space-eq sets-bind*)

lemma *bind-cong-All*:
assumes *∀ x ∈ space M. f x = g x*
shows *bind M f = bind M g*
proof (*cases space M = {}*)
assume *space M ≠ {}*
hence (*SOME x. x ∈ space M*) *∈ space M* **by** (*rule-tac someI-ex*) *blast*
with *assms* **have** *f* (*SOME x. x ∈ space M*) *= g* (*SOME x. x ∈ space M*) **by** *blast*
with ‹*space M ≠ {}*› **and** *assms* **show** *?thesis* **by** (*simp add*: *bind-nonempty cong*: *distr-cong*)
qed (*simp add*: *bind-empty*)

lemma *bind-cong*:
M = N ⟹ (⋀x. x ∈ space M ⟹ f x = g x) ⟹ bind M f = bind N g
using *bind-cong-All*[*of M f g*] **by** *auto*

lemma *bind-nonempty′*:
assumes *f ∈ measurable M* (*subprob-algebra N*) *x ∈ space M*
shows *bind M f = join* (*distr M* (*subprob-algebra N*) *f*)
proof −
have *join* (*distr M* (*subprob-algebra* (*f* (*SOME x. x ∈ space M*))) *f*) *= join* (*distr M* (*subprob-algebra N*) *f*)
by (*metis assms someI-ex subprob-algebra-cong subprob-measurableD*(*2*))
with *assms* **show** *?thesis*
by (*metis bind-nonempty empty-iff*)
qed

lemma *bind-nonempty″*:
assumes *f ∈ measurable M* (*subprob-algebra N*) *space M ≠ {}*

shows *bind M f = join (distr M (subprob-algebra N) f)*
using *assms* **by** (*auto intro*: *bind-nonempty'*)

lemma *emeasure-bind*:
⟦*space M ≠ {}*; *f ∈ measurable M (subprob-algebra N)*;*X ∈ sets N*⟧
⟹ *emeasure (M ⤜ f) X = ∫⁺x. emeasure (f x) X ∂M*
by (*simp add*: *bind-nonempty'' emeasure-join nn-integral-distr measurable-emeasure-subprob-algebra*)

lemma *nn-integral-bind*:
assumes *f*: *f ∈ borel-measurable B*
assumes *N*: *N ∈ measurable M (subprob-algebra B)*
shows *(∫⁺x. f x ∂(M ⤜ N)) = (∫⁺x. ∫⁺y. f y ∂N x ∂M)*
proof *cases*
assume *M*: *space M ≠ {}* **show** *?thesis*
unfolding *bind-nonempty''*[*OF N M*] *nn-integral-join*[*OF f sets-distr*]
by (*rule nn-integral-distr*[*OF N*])
(*simp add*: *f nn-integral-measurable-subprob-algebra*)
qed (*simp add*: *bind-empty space-empty*[*of M*] *nn-integral-count-space*)

lemma *AE-bind*:
assumes *N*[*measurable*]: *N ∈ measurable M (subprob-algebra B)*
assumes *P*[*measurable*]: *Measurable.pred B P*
shows *(AE x in M ⤜ N. P x) ⟷ (AE x in M. AE y in N x. P y)*
proof *cases*
assume *M*: *space M = {}* **show** *?thesis*
unfolding *bind-empty*[*OF M*] **unfolding** *space-empty*[*OF M*] **by** (*simp add*: *AE-count-space*)
next
assume *M*: *space M ≠ {}*
note *sets-kernel*[*OF N*, *simp*]
have ∗: *(∫⁺x. indicator {x. ¬ P x} x ∂(M ⤜ N)) = (∫⁺x. indicator {x∈space B. ¬ P x} x ∂(M ⤜ N))*
by (*intro nn-integral-cong*) (*simp add*: *space-bind*[*OF - M*] *split*: *split-indicator*)

have *(AE x in M ⤜ N. P x) ⟷ (∫⁺ x. integral^N (N x) (indicator {x ∈ space B. ¬ P x}) ∂M) = 0*
by (*simp add*: *AE-iff-nn-integral sets-bind*[*OF - M*] *space-bind*[*OF - M*] ∗ *nn-integral-bind*[**where** *B=B*]
del: *nn-integral-indicator*)
also have *... = (AE x in M. integral^N (N x) (indicator {x ∈ space B. ¬ P x}) = 0)*
proof (*rule nn-integral-0-iff-AE*)
show *(λx. integral^N (N x) (indicator {x ∈ space B. ¬ P x})) ∈ borel-measurable M*
apply (*rule measurable-compose*[*OF N nn-integral-measurable-subprob-algebra*])
by *measurable*
qed
also have *... = (AE x in M. AE y in N x. P y)*
apply (*intro eventually-subst AE-I2*)

by (*auto simp add*: *subprob-measurableD*(*1*)[*OF N*] *intro*!: *AE-iff-measurable*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *measurable-bind'*:
assumes *M1*: $f \in$ *measurable* M (*subprob-algebra* N) **and**
M2: *case-prod* $g \in$ *measurable* ($M \bigotimes_M N$) (*subprob-algebra* R)
shows (λx. *bind* ($f\ x$) ($g\ x$)) $\in$ *measurable* M (*subprob-algebra* R)
proof (*subst measurable-cong*)
fix x **assume** *x-in-M*: $x \in$ *space* M
with *assms* **have** *space* ($f\ x$) $\neq$ {}
by (*blast dest*: *subprob-space-kernel subprob-space.subprob-not-empty*)
moreover from *M2 x-in-M* **have** $g\ x \in$ *measurable* ($f\ x$) (*subprob-algebra* R)
by (*subst measurable-cong-sets*[*OF sets-kernel*[*OF M1 x-in-M*] *refl*])
(*auto dest*: *measurable-Pair2*)
ultimately show *bind* ($f\ x$) ($g\ x$) = *join* (*distr* ($f\ x$) (*subprob-algebra* R) ($g\ x$))
by (*simp-all add*: *bind-nonempty''*)
next
show (λw. *join* (*distr* ($f\ w$) (*subprob-algebra* R) ($g\ w$))) $\in$ *measurable* M (*subprob-algebra* R)
apply (*rule measurable-compose*[*OF* - *measurable-join*])
apply (*rule measurable-distr2*[*OF M2 M1*])
done
qed

lemma *measurable-bind*[*measurable* (*raw*)]:
assumes *M1*: $f \in$ *measurable* M (*subprob-algebra* N) **and**
M2: (λx. g (*fst* x) (*snd* x)) $\in$ *measurable* ($M \bigotimes_M N$) (*subprob-algebra* R)
shows (λx. *bind* ($f\ x$) ($g\ x$)) $\in$ *measurable* M (*subprob-algebra* R)
using *assms* **by** (*auto intro*: *measurable-bind' simp*: *measurable-split-conv*)

lemma *measurable-bind2*:
assumes $f \in$ *measurable* M (*subprob-algebra* N) **and** $g \in$ *measurable* N (*subprob-algebra* R)
shows (λx. *bind* ($f\ x$) g) $\in$ *measurable* M (*subprob-algebra* R)
using *assms* **by** (*intro measurable-bind' measurable-const*) *auto*

lemma *subprob-space-bind*:
assumes *subprob-space* M $f \in$ *measurable* M (*subprob-algebra* N)
shows *subprob-space* ($M \gg\!= f$)
proof (*rule subprob-space-kernel*[*of* λx. $x \gg\!= f$])
show (λx. $x \gg\!= f$) $\in$ *measurable* (*subprob-algebra* M) (*subprob-algebra* N)
by (*rule measurable-bind*, *rule measurable-ident-sets*, *rule refl*,
rule measurable-compose[*OF measurable-snd assms*(*2*)])
from *assms*(*1*) **show** $M \in$ *space* (*subprob-algebra* M)
by (*simp add*: *space-subprob-algebra*)
qed

lemma

fixes $f :: - \Rightarrow real$
assumes *f-measurable* [*measurable*]: $f \in borel\text{-}measurable\ K$
and *f-bounded*: $\bigwedge x.\ x \in space\ K \Longrightarrow |f\ x| \leq B$
and N [*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ K)$
and *fin*: *finite-measure M*
and *M-bounded*: $AE\ x\ in\ M.\ emeasure\ (N\ x)\ (space\ (N\ x)) \leq ennreal\ B'$
shows *integrable-bind*: *integrable* (*bind M N*) *f* (**is** *?integrable*)
and *integral-bind*: $integral^L\ (bind\ M\ N)\ f = \int\ x.\ integral^L\ (N\ x)\ f\ \partial M$ (**is** *?integral*)
proof(*case-tac* [!] $space\ M = \{\}$)
assume [*simp*]: $space\ M \neq \{\}$
interpret *finite-measure M* **by**(*rule fin*)

have *integrable* (*join* (*distr M* (*subprob-algebra K*) *N*)) *f*
using *f-measurable f-bounded*
by(*rule integrable-join*[**where** *B'*=*B'*])(*simp-all add*: *finite-measure-distr AE-distr-iff M-bounded*)
then show *?integrable* **by**(*simp add*: *bind-nonempty''*[**where** *N*=*K*])

have $integral^L\ (join\ (distr\ M\ (subprob\text{-}algebra\ K)\ N))\ f = \int\ M'.\ integral^L\ M'\ f\ \partial distr\ M\ (subprob\text{-}algebra\ K)\ N$
using *f-measurable f-bounded*
by(*rule integral-join*[**where** *B'*=*B'*])(*simp-all add*: *finite-measure-distr AE-distr-iff M-bounded*)
also have $\ldots = \int\ x.\ integral^L\ (N\ x)\ f\ \partial M$
by(*rule integral-distr*)(*simp-all add*: *integral-measurable-subprob-algebra*[*OF* -])
finally show *?integral* **by**(*simp add*: *bind-nonempty''*[**where** *N*=*K*])
qed(*simp-all add*: *bind-def integrable-count-space lebesgue-integral-count-space-finite Bochner-Integration.integral-empty*)

lemma (**in** *prob-space*) *prob-space-bind*:
assumes *ae*: $AE\ x\ in\ M.\ prob\text{-}space\ (N\ x)$
and *N*[*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ S)$
shows *prob-space* $(M \gg\!= N)$
proof
have $emeasure\ (M \gg\!= N)\ (space\ (M \gg\!= N)) = (\int^+ x.\ emeasure\ (N\ x)\ (space\ (N\ x))\ \partial M)$
by (*subst emeasure-bind*[**where** *N*=*S*])
(*auto simp*: *not-empty space-bind*[*OF sets-kernel*] *subprob-measurableD*[*OF N*] *intro*!: *nn-integral-cong*)
also have $\ldots = (\int^+ x.\ 1\ \partial M)$
using *ae* **by** (*intro nn-integral-cong-AE*, *eventually-elim*) (*rule prob-space.emeasure-space-1*)
finally show $emeasure\ (M \gg\!= N)\ (space\ (M \gg\!= N)) = 1$
by (*simp add*: *emeasure-space-1*)
qed

lemma (**in** *subprob-space*) *bind-in-space*:
$A \in measurable\ M\ (subprob\text{-}algebra\ N) \Longrightarrow (M \gg\!= A) \in space\ (subprob\text{-}algebra\ N)$

by (*auto simp add*: *space-subprob-algebra subprob-not-empty sets-kernel intro*!: *subprob-space-bind*)
unfold-locales

lemma (**in** *subprob-space*) *measure-bind*:
assumes *f*: $f \in$ *measurable M* (*subprob-algebra N*) **and** *X*: $X \in$ *sets N*
shows *measure* ($M \gg\!= f$) $X = \int x.$ *measure* ($f\ x$) $X\ \partial M$
proof −
interpret *Mf*: *subprob-space M* $\gg\!= f$
by (*rule subprob-space-bind*[*OF - f*]) *unfold-locales*

{ fix x **assume** $x \in$ *space M*
from *f*[*THEN measurable-space*, *OF this*] **interpret** *subprob-space f x*
by (*simp add*: *space-subprob-algebra*)
have *emeasure* ($f\ x$) $X =$ *ennreal* (*measure* ($f\ x$) X) *measure* ($f\ x$) $X \leq 1$
by (*auto simp*: *emeasure-eq-measure subprob-measure-le-1*) **}**
note *this*[*simp*]

have *emeasure* ($M \gg\!= f$) $X = \int^+ x.$ *emeasure* ($f\ x$) $X\ \partial M$
using *subprob-not-empty f X* **by** (*rule emeasure-bind*)
also have $\ldots = \int^+ x.$ *ennreal* (*measure* ($f\ x$) X) ∂M
by (*intro nn-integral-cong*) *simp*
also have $\ldots = \int x.$ *measure* ($f\ x$) $X\ \partial M$
by (*intro nn-integral-eq-integral integrable-const-bound*[**where** *B=1*]
measure-measurable-subprob-algebra2[*OF - f*] *pair-measureI X*)
(*auto simp*: *measure-nonneg*)
finally show *?thesis*
by (*simp add*: *Mf.emeasure-eq-measure measure-nonneg integral-nonneg*)
qed

lemma *emeasure-bind-const*:
space M $\neq \{\} \Longrightarrow X \in$ *sets N* $\Longrightarrow$ *subprob-space N* $\Longrightarrow$
emeasure ($M \gg\!= (\lambda x.\ N)$) $X =$ *emeasure N X* $*$ *emeasure M* (*space M*)
by (*simp add*: *bind-nonempty emeasure-join nn-integral-distr*
space-subprob-algebra measurable-emeasure-subprob-algebra)

lemma *emeasure-bind-const′*:
assumes *subprob-space M subprob-space N*
shows *emeasure* ($M \gg\!= (\lambda x.\ N)$) $X =$ *emeasure N X* $*$ *emeasure M* (*space M*)
using *assms*
proof (*case-tac* $X \in$ *sets N*)
fix X **assume** $X \in$ *sets N*
thus *emeasure* ($M \gg\!= (\lambda x.\ N)$) $X =$ *emeasure N X* $*$ *emeasure M* (*space M*)
using *assms*
by (*subst emeasure-bind-const*)
(*simp-all add*: *subprob-space.subprob-not-empty subprob-space.emeasure-space-le-1*)
next
fix X **assume** $X \notin$ *sets N*
with *assms* **show** *emeasure* ($M \gg\!= (\lambda x.\ N)$) $X =$ *emeasure N X* $*$ *emeasure M*

(*space M*)
  by (*simp add*: *sets-bind*[*of - - N*] *subprob-space.subprob-not-empty*
      space-subprob-algebra emeasure-notin-sets)
qed

lemma *emeasure-bind-const-prob-space*:
  assumes *prob-space M subprob-space N*
  shows *emeasure* (*M* ⤜ (λ*x*. *N*)) *X* = *emeasure N X*
  using *assms* **by** (*simp add*: *emeasure-bind-const′ prob-space-imp-subprob-space*
      prob-space.emeasure-space-1)

lemma *bind-return*:
  assumes *f* ∈ *measurable M* (*subprob-algebra N*) **and** *x* ∈ *space M*
  shows *bind* (*return M x*) *f* = *f x*
  using *sets-kernel*[*OF assms*] *assms*
  by (*simp-all add*: *distr-return join-return subprob-space-kernel bind-nonempty′*
      cong: *subprob-algebra-cong*)

lemma *bind-return′*:
  shows *bind M* (*return M*) = *M*
  by (*cases space M* = {})
    (*simp-all add*: *bind-empty space-empty*[*symmetric*] *bind-nonempty join-return′*
      cong: *subprob-algebra-cong*)

lemma *distr-bind*:
  assumes *N*: *N* ∈ *measurable M* (*subprob-algebra K*) *space M* ≠ {}
  assumes *f*: *f* ∈ *measurable K R*
  shows *distr* (*M* ⤜ *N*) *R f* = (*M* ⤜ (λ*x*. *distr* (*N x*) *R f*))
proof −
  have *distr* (*join* (*distr M* (*subprob-algebra K*) *N*)) *R f* =
      join (*distr M* (*subprob-algebra R*) (λ*x*. *distr* (*N x*) *R f*))
   by (*simp add*: *assms distr-distr*[*OF measurable-distr*] *comp-def flip*: *join-distr-distr*)
  with *assms* **show** *?thesis*
    unfolding *bind-nonempty′′*[*OF N*]
    by (*smt* (*verit*) *bind-nonempty sets-distr subprob-algebra-cong*)
qed

lemma *bind-distr*:
  assumes *f*[*measurable*]: *f* ∈ *measurable M X*
  assumes *N*[*measurable*]: *N* ∈ *measurable X* (*subprob-algebra K*) **and** *space M* ≠
{}
  shows (*distr M X f* ⤜ *N*) = (*M* ⤜ (λ*x*. *N* (*f x*)))
proof −
  have *space X* ≠ {} *space M* ≠ {}
    using ‹*space M* ≠ {}› *f*[*THEN measurable-space*] **by** *auto*
  then show *?thesis*
    by (*simp add*: *bind-nonempty′′*[**where** *N*=*K*] *distr-distr comp-def*)
qed

lemma *bind-count-space-singleton*:
assumes *subprob-space* (*f x*)
shows *count-space* {*x*} ⤜ *f* = *f x*
proof−
have *A*: ⋀*A*. *A* ⊆ {*x*} ⟹ *A* = {} ∨ *A* = {*x*} **by** *auto*
have *count-space* {*x*} = *return* (*count-space* {*x*}) *x*
by (*intro measure-eqI*) (*auto dest*: *A*)
also have ... ⤜ *f* = *f x*
by (*subst bind-return*[*of* - - *f x*]) (*auto simp*: *space-subprob-algebra assms*)
finally show *?thesis* **.**
qed

lemma *restrict-space-bind*:
assumes *N*: *N* ∈ *measurable M* (*subprob-algebra K*)
assumes *space M* ≠ {}
assumes *X*[*simp*]: *X* ∈ *sets K X* ≠ {}
shows *restrict-space* (*bind M N*) *X* = *bind M* (λ*x*. *restrict-space* (*N x*) *X*)
proof (*rule measure-eqI*)
note *N-sets* = *sets-bind*[*OF sets-kernel*[*OF N*] *assms*(*2*), *simp*]
note *N-space* = *sets-eq-imp-space-eq*[*OF N-sets*, *simp*]
show *sets* (*restrict-space* (*bind M N*) *X*) = *sets* (*bind M* (λ*x*. *restrict-space* (*N x*) *X*))
by (*simp add*: *sets-restrict-space assms*(*2*) *sets-bind*[*OF sets-kernel*[*OF restrict-space-measurable*[*OF assms*(*4*,*3*,*1*)]]])
fix *A* **assume** *A* ∈ *sets* (*restrict-space* (*M* ⤜ *N*) *X*)
with *X* **have** *A*: *A* ∈ *sets K A* ⊆ *X*
by (*auto simp*: *sets-restrict-space*)
then have *emeasure* (*restrict-space* (*M* ⤜ *N*) *X*) *A* = *emeasure* (*M* ⤜ *N*) *A*
by (*simp add*: *emeasure-restrict-space*)
also have ... = ∫⁺ *x*. *emeasure* (*N x*) *A* ∂*M*
by (*metis* ‹*A* ∈ *sets K*› *N* ‹*space M* ≠ {}› *emeasure-bind*)
also have ... = ∫⁺ *x*. *emeasure* (*restrict-space* (*N x*) *X*) *A* ∂*M*
using *A assms* **by** (*smt* (*verit*, *best*) *emeasure-restrict-space nn-integral-cong sets.Int-space-eq2 subprob-measurableD*(*2*))
also have ... = *emeasure* (*M* ⤜ (λ*x*. *restrict-space* (*N x*) *X*)) *A*
using *A assms*
apply (*subst emeasure-bind*[*OF* - *restrict-space-measurable*])
apply (*auto simp*: *sets-restrict-space*)
done
finally show *emeasure* (*restrict-space* (*M* ⤜ *N*) *X*) *A* = *emeasure* (*M* ⤜ (λ*x*. *restrict-space* (*N x*) *X*)) *A* **.**
qed

lemma *bind-restrict-space*:
assumes *A*: *A* ∩ *space M* ≠ {} *A* ∩ *space M* ∈ *sets M*
and *f*: *f* ∈ *measurable* (*restrict-space M A*) (*subprob-algebra N*)
shows *restrict-space M A* ⤜ *f* = *M* ⤜ (λ*x*. *if x* ∈ *A then f x else null-measure* (*f* (*SOME x*. *x* ∈ *A* ∧ *x* ∈ *space M*)))
(**is** *?lhs* = *?rhs* **is** - = *M* ⤜ *?f*)

proof −
let *?P* = λ*x*. *x* ∈ *A* ∧ *x* ∈ *space M*
let *?x* = *Eps ?P*
let *?c* = *null-measure* (*f ?x*)
from *A* **have** *?P ?x* **by**−(*rule someI-ex*, *blast*)
hence *?x* ∈ *space* (*restrict-space M A*) **by**(*simp add*: *space-restrict-space*)
with *f* **have** *f ?x* ∈ *space* (*subprob-algebra N*) **by**(*rule measurable-space*)
hence *sps*: *subprob-space* (*f ?x*)
and *sets*: *sets* (*f ?x*) = *sets N*
by(*simp-all add*: *space-subprob-algebra*)
have *space* (*f ?x*) ≠ {} **using** *sps* **by**(*rule subprob-space.subprob-not-empty*)
moreover have *sets ?c* = *sets N* **by**(*simp add*: *null-measure-def*)(*simp add*: *sets*)
ultimately have *c*: *?c* ∈ *space* (*subprob-algebra N*)
by(*simp add*: *space-subprob-algebra subprob-space-null-measure*)
from *f A c* **have** *f'*: *?f* ∈ *measurable M* (*subprob-algebra N*)
by(*simp add*: *measurable-restrict-space-iff*)

from *A* **have** [*simp*]: *space M* ≠ {} **by** *blast*

have *?lhs* = *join* (*distr* (*restrict-space M A*) (*subprob-algebra N*) *f*)
using *assms* **by**(*simp add*: *space-restrict-space bind-nonempty''*)
also have ... = *join* (*distr M* (*subprob-algebra N*) *?f*)
by(*rule measure-eqI*)(*auto simp add*: *emeasure-join nn-integral-distr nn-integral-restrict-space f f' A intro*: *nn-integral-cong*)
also have ... = *?rhs* **using** *f'* **by**(*simp add*: *bind-nonempty''*)
finally show *?thesis* .
qed

lemma *bind-const'*: ⟦*prob-space M*; *subprob-space N*⟧ ⟹ *M* ⤜ (λ*x*. *N*) = *N*
by (*intro measure-eqI*)
(*simp-all add*: *space-subprob-algebra prob-space.not-empty emeasure-bind-const-prob-space*)

lemma *bind-return-distr*:
assumes *space M* ≠ {} *f* ∈ *measurable M N*
shows *bind M* (*return N* ∘ *f*) = *distr M N f*
proof −
have *bind M* (*return N* ∘ *f*)
= *join* (*distr M* (*subprob-algebra* (*return N* (*f* (*SOME x*. *x* ∈ *space M*)))) (*return N* ∘ *f*))
by (*simp add*: *Giry-Monad.bind-def assms*)
also have ... = *join* (*distr M* (*subprob-algebra N*) (*return N* ∘ *f*))
by (*metis sets-return subprob-algebra-cong*)
also have ... = *distr M N f*
by (*metis assms*(2) *distr-distr join-return' return-measurable sets-distr*)
finally show *?thesis* .
qed

lemma *bind-return-distr'*:

space M ≠ {} ⟹ f ∈ measurable M N ⟹ bind M (λx. return N (f x)) = distr M N f
using *bind-return-distr*[*of M f N*] **by** (*simp add*: *comp-def*)

lemma *bind-assoc*:
fixes *f* :: *'a ⇒ 'b measure* **and** *g* :: *'b ⇒ 'c measure*
assumes *M1*: *f ∈ measurable M* (*subprob-algebra N*) **and** *M2*: *g ∈ measurable N* (*subprob-algebra R*)
shows *bind* (*bind M f*) *g* = *bind M* (λx. *bind* (*f x*) *g*)
proof (*cases space M* = {})
assume [*simp*]: *space M ≠ {}*
from *assms* **have** [*simp*]: *space N ≠ {} space R ≠ {}*
by (*auto simp*: *measurable-empty-iff space-subprob-algebra-empty-iff*)
from *assms* **have** *sets-fx*[*simp*]: ⋀x. *x ∈ space M ⟹ sets* (*f x*) = *sets N*
by (*simp add*: *sets-kernel*)
have *ex-in*: ⋀*A*. *A ≠ {} ⟹ ∃ x. x ∈ A* **by** *blast*
note *sets-some*[*simp*] = *sets-kernel*[*OF M1 someI-ex*[*OF ex-in*[*OF ‹space M ≠ {}›*]]]
sets-kernel[*OF M2 someI-ex*[*OF ex-in*[*OF ‹space N ≠ {}›*]]]
note *space-some*[*simp*] = *sets-eq-imp-space-eq*[*OF this*(*1*)] *sets-eq-imp-space-eq*[*OF this*(*2*)]

have ∗: (λx. *distr x* (*subprob-algebra R*) *g*) ∘ *f ∈ M* →*M subprob-algebra* (*subprob-algebra R*)
using *M1 M2 measurable-comp measurable-distr* **by** *blast*
have *bind M* (λx. *bind* (*f x*) *g*) =
join (*distr M* (*subprob-algebra R*) (*join* ∘ (λx. (*distr x* (*subprob-algebra R*) *g*)) ∘ *f*))
by (*simp add*: *sets-eq-imp-space-eq*[*OF sets-fx*] *bind-nonempty o-def*
cong: *subprob-algebra-cong distr-cong*)
also have *distr M* (*subprob-algebra R*) (*join* ∘ (λx. (*distr x* (*subprob-algebra R*) *g*)) ∘ *f*) =
distr (*distr* (*distr M* (*subprob-algebra N*) *f*)
(*subprob-algebra* (*subprob-algebra R*))
(λx. *distr x* (*subprob-algebra R*) *g*))
(*subprob-algebra R*) *join*
by (*simp add*: *distr-distr M1 M2 measurable-distr measurable-join fun.map-comp* ∗)
also have *join* ... = *bind* (*bind M f*) *g*
by (*simp add*: *join-assoc join-distr-distr M2 bind-nonempty cong*: *subprob-algebra-cong*)
finally show *?thesis* **..**
qed (*simp add*: *bind-empty*)

lemma *double-bind-assoc*:
assumes *Mg*: *g ∈ measurable N* (*subprob-algebra N'*)
assumes *Mf*: *f ∈ measurable M* (*subprob-algebra M'*)
assumes *Mh*: *case-prod h ∈ measurable* (*M* ⨂*M M'*) *N*
shows *do* {*x* ← *M*; *y* ← *f x*; *g* (*h x y*)} = *do* {*x* ← *M*; *y* ← *f x*; *return N* (*h x*

y)} ⤜ *g*
proof −
have *do* {*x* ← *M*; *y* ← *f x*; *return N* (*h x y*)} ⤜ *g* =
do {*x* ← *M*; *do* {*y* ← *f x*; *return N* (*h x y*)} ⤜ *g*}
using *Mh* **by** (*auto intro*!: *bind-assoc measurable-bind'*[*OF Mf*] *Mf Mg*
measurable-compose[*OF - return-measurable*] *simp*: *measurable-split-conv*)
also from *Mh* **have** ⋀*x*. *x* ∈ *space M* ⟹ *h x* ∈ *measurable M' N* **by** *measurable*
hence *do* {*x* ← *M*; *do* {*y* ← *f x*; *return N* (*h x y*)} ⤜ *g*} =
do {*x* ← *M*; *y* ← *f x*; *return N* (*h x y*) ⤜ *g*}
apply (*intro ballI bind-cong refl bind-assoc*)
apply (*subst measurable-cong-sets*[*OF sets-kernel*[*OF Mf*] *refl*], *simp*)
apply (*rule measurable-compose*[*OF - return-measurable*], *auto intro*: *Mg*)
done
also have ⋀*x*. *x* ∈ *space M* ⟹ *space* (*f x*) = *space M'*
by (*intro sets-eq-imp-space-eq sets-kernel*[*OF Mf*])
with *measurable-space*[*OF Mh*]
have *do* {*x* ← *M*; *y* ← *f x*; *return N* (*h x y*) ⤜ *g*} = *do* {*x* ← *M*; *y* ← *f x*; *g* (*h x y*)}
by (*intro ballI bind-cong bind-return*[*OF Mg*]) (*auto simp*: *space-pair-measure*)
finally show *?thesis* **..**
qed

lemma (**in** *prob-space*) *M-in-subprob*[*measurable* (*raw*)]: *M* ∈ *space* (*subprob-algebra M*)
by (*simp add*: *space-subprob-algebra*) *unfold-locales*

lemma (**in** *pair-prob-space*) *pair-measure-eq-bind*:
(*M1* ⨂$_M$ *M2*) = (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *return* (*M1* ⨂$_M$ *M2*) (*x*, *y*))))
proof (*rule measure-eqI*)
have *ps-M2*: *prob-space M2* **by** *unfold-locales*
note *return-measurable*[*measurable*]
show *sets* (*M1* ⨂$_M$ *M2*) = *sets* (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *return* (*M1* ⨂$_M$ *M2*) (*x*, *y*))))
by (*simp-all add*: *M1.not-empty M2.not-empty*)
fix *A* **assume** [*measurable*]: *A* ∈ *sets* (*M1* ⨂$_M$ *M2*)
show *emeasure* (*M1* ⨂$_M$ *M2*) *A* = *emeasure* (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *return* (*M1* ⨂$_M$ *M2*) (*x*, *y*)))) *A*
by (*auto simp*: *M2.emeasure-pair-measure M1.not-empty M2.not-empty emeasure-bind*[**where** *N*=*M1* ⨂$_M$ *M2*]
intro!: *nn-integral-cong*)
qed

lemma (**in** *pair-prob-space*) *bind-rotate*:
assumes *C*[*measurable*]: (λ(*x*, *y*). *C x y*) ∈ *measurable* (*M1* ⨂$_M$ *M2*) (*subprob-algebra N*)
shows (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *C x y*))) = (*M2* ⤜ (λ*y*. *M1* ⤜ (λ*x*. *C x y*)))
proof −

interpret *swap*: *pair-prob-space M2 M1* **by** *unfold-locales*
note *measurable-bind*[**where** *N=M2*, *measurable*]
note *measurable-bind*[**where** *N=M1*, *measurable*]
have [*simp*]: *M1* ∈ *space* (*subprob-algebra M1*) *M2* ∈ *space* (*subprob-algebra M2*)
by (*auto simp*: *space-subprob-algebra*) *unfold-locales*
have (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *C x y*))) =
(*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *return* (*M1* ⨂$_M$ *M2*) (*x*, *y*)))) ⤜ (λ(*x*, *y*). *C x y*)
by (*auto intro*!: *bind-cong simp*: *bind-return*[**where** *N=N*] *space-pair-measure bind-assoc*[**where** *N=M1* ⨂$_M$ *M2* **and** *R=N*])
also have ... = (*distr* (*M2* ⨂$_M$ *M1*) (*M1* ⨂$_M$ *M2*) (λ(*x*, *y*). (*y*, *x*))) ⤜ (λ(*x*, *y*). *C x y*)
unfolding *pair-measure-eq-bind*[*symmetric*] *distr-pair-swap*[*symmetric*] **..**
also have ... = (*M2* ⤜ (λ*x*. *M1* ⤜ (λ*y*. *return* (*M2* ⨂$_M$ *M1*) (*x*, *y*)))) ⤜ (λ(*y*, *x*). *C x y*)
unfolding *swap.pair-measure-eq-bind*[*symmetric*]
by (*auto simp add*: *space-pair-measure M1.not-empty M2.not-empty bind-distr*[*OF - C*] *intro*!: *bind-cong*)
also have ... = (*M2* ⤜ (λ*y*. *M1* ⤜ (λ*x*. *C x y*)))
by (*auto intro*!: *bind-cong simp*: *bind-return*[**where** *N=N*] *space-pair-measure bind-assoc*[**where** *N=M2* ⨂$_M$ *M1* **and** *R=N*])
finally show *?thesis* **.**
qed

lemma *bind-return''*: *sets M* = *sets N* ⟹ *M* ⤜ *return N* = *M*
by (*cases space M* = {})
(*simp-all add*: *bind-empty space-empty*[*symmetric*] *bind-nonempty join-return'*
cong: *subprob-algebra-cong*)

lemma (**in** *prob-space*) *distr-const*[*simp*]:
c ∈ *space N* ⟹ *distr M N* (λ*x*. *c*) = *return N c*
by (*rule measure-eqI*) (*auto simp*: *emeasure-distr emeasure-space-1*)

lemma *return-count-space-eq-density*:
return (*count-space M*) *x* = *density* (*count-space M*) (*indicator* {*x*})
by (*rule measure-eqI*)
(*auto simp*: *indicator-inter-arith*[*symmetric*] *emeasure-density split*: *split-indicator*)

lemma *null-measure-in-space-subprob-algebra* [*simp*]:
null-measure M ∈ *space* (*subprob-algebra M*) ⟷ *space M* ≠ {}
by(*simp add*: *space-subprob-algebra subprob-space-null-measure-iff*)

7.4 Giry monad on probability spaces

definition *prob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
prob-algebra K = *restrict-space* (*subprob-algebra K*) {*M*. *prob-space M*}

lemma *space-prob-algebra*: *space* (*prob-algebra M*) = {*N*. *sets N* = *sets M* ∧ *prob-space N*}
unfolding *prob-algebra-def* **by** (*auto simp*: *space-subprob-algebra space-restrict-space*

prob-space-imp-subprob-space)

lemma *measurable-measure-prob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *Sigma-Algebra.measure M a*) ∈ *prob-algebra A* →$_M$ *borel*
unfolding *prob-algebra-def* **by** (*intro measurable-restrict-space1 measurable-measure-subprob-algebra*)

lemma *measurable-prob-algebraD*:
f ∈ *N* →$_M$ *prob-algebra M* ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M*
unfolding *prob-algebra-def measurable-restrict-space2-iff* **by** *auto*

lemma *measure-measurable-prob-algebra2*:
Sigma (*space M*) *A* ∈ *sets* (*M* ⨂$_M$ *N*) ⟹ *L* ∈ *M* →$_M$ *prob-algebra N* ⟹
(λ*x*. *Sigma-Algebra.measure* (*L x*) (*A x*)) ∈ *borel-measurable M*
using *measure-measurable-subprob-algebra2*[*of M A N L*] **by** (*auto intro*: *measurable-prob-algebraD*)

lemma *measurable-prob-algebraI*:
(⋀*x*. *x* ∈ *space N* ⟹ *prob-space* (*f x*)) ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M* ⟹
f ∈ *N* →$_M$ *prob-algebra M*
unfolding *prob-algebra-def* **by** (*intro measurable-restrict-space2*) *auto*

lemma *measurable-distr-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *N*
shows (λ*M'*. *distr M' N f*) ∈ *prob-algebra M* →$_M$ *prob-algebra N*
unfolding *prob-algebra-def measurable-restrict-space2-iff*
proof (*intro conjI measurable-restrict-space1 measurable-distr f*)
show (λ*M'*. *distr M' N f*) ∈ *space* (*restrict-space* (*subprob-algebra M*) (*Collect prob-space*)) → *Collect prob-space*
using *f* **by** (*auto simp*: *space-restrict-space space-subprob-algebra intro*!: *prob-space.prob-space-distr*)
qed

lemma *measurable-return-prob-space*[*measurable*]: *return N* ∈ *N* →$_M$ *prob-algebra N*
by (*rule measurable-prob-algebraI*) (*auto simp*: *prob-space-return*)

lemma *measurable-distr-prob-space2*[*measurable* (*raw*)]:
assumes *f*: *g* ∈ *L* →$_M$ *prob-algebra M* (λ(*x*, *y*). *f x y*) ∈ *L* ⨂$_M$ *M* →$_M$ *N*
shows (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *L* →$_M$ *prob-algebra N*
unfolding *prob-algebra-def measurable-restrict-space2-iff*
proof (*intro conjI measurable-restrict-space1 measurable-distr2*[**where** *M*=*M*] *f measurable-prob-algebraD*)
show (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *space L* → *Collect prob-space*
using *f subprob-measurableD*[*OF measurable-prob-algebraD*[*OF f*(*1*)]]
by (*auto simp*: *measurable-restrict-space2-iff prob-algebra-def*
intro!: *prob-space.prob-space-distr*)
qed

lemma *measurable-bind-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *prob-algebra N* **and** *g*: *g* ∈ *N* →$_M$ *prob-algebra R*

shows $(\lambda x.\ bind\ (f\ x)\ g) \in M \to_M prob\text{-}algebra\ R$
unfolding *prob-algebra-def measurable-restrict-space2-iff*
proof (*intro conjI measurable-restrict-space1 measurable-bind2*[**where** $N{=}N$] *f g measurable-prob-algebraD*)
show $(\lambda x.\ f\ x \gg\!= g) \in space\ M \to Collect\ prob\text{-}space$
using *g f subprob-measurableD*[*OF measurable-prob-algebraD*[*OF f*]]
by (*auto simp*: *measurable-restrict-space2-iff prob-algebra-def*
intro!: *prob-space.prob-space-bind*[**where** $S{=}R$] *AE-I2*)
qed

lemma *measurable-bind-prob-space2*[*measurable* (*raw*)]:
assumes *f*: $f \in M \to_M prob\text{-}algebra\ N$ **and** *g*: $(\lambda(x,\ y).\ g\ x\ y) \in (M \bigotimes_M N) \to_M prob\text{-}algebra\ R$
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in M \to_M prob\text{-}algebra\ R$
unfolding *prob-algebra-def measurable-restrict-space2-iff*
proof (*intro conjI measurable-restrict-space1 measurable-bind*[**where** $N{=}N$] *f g measurable-prob-algebraD*)
show $(\lambda x.\ f\ x \gg\!= g\ x) \in space\ M \to Collect\ prob\text{-}space$
using *g f subprob-measurableD*[*OF measurable-prob-algebraD*[*OF f*]]
using *measurable-space*[*OF g*]
by (*auto simp*: *measurable-restrict-space2-iff prob-algebra-def space-pair-measure Pi-iff*
intro!: *prob-space.prob-space-bind*[**where** $S{=}R$] *AE-I2*)
qed (*use g* **in** *simp*)

lemma *measurable-prob-algebra-generated*:
assumes *eq*: *sets* N = *sigma-sets* Ω G **and** *Int-stable* G $G \subseteq Pow\ \Omega$
assumes *subsp*: $\bigwedge a.\ a \in space\ M \Longrightarrow prob\text{-}space\ (K\ a)$
assumes *sets*: $\bigwedge a.\ a \in space\ M \Longrightarrow sets\ (K\ a) = sets\ N$
assumes $\bigwedge A.\ A \in G \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in borel\text{-}measurable\ M$
shows $K \in measurable\ M\ (prob\text{-}algebra\ N)$
unfolding *measurable-restrict-space2-iff prob-algebra-def*
proof
show $K \in M \to_M subprob\text{-}algebra\ N$
proof (*rule measurable-subprob-algebra-generated*[*OF assms*(*1*,*2*,*3*) - *assms*(*5*,*6*)])
fix *a* **assume** $a \in space\ M$ **then show** $subprob\text{-}space\ (K\ a)$
using *subsp*[*of a*] **by** (*intro prob-space-imp-subprob-space*)
next
have $(\lambda a.\ emeasure\ (K\ a)\ \Omega) \in borel\text{-}measurable\ M \longleftrightarrow (\lambda a.\ 1{::}ennreal) \in borel\text{-}measurable\ M$
using *sets-eq-imp-space-eq*[*of sigma* Ω G N] ‹$G \subseteq Pow\ \Omega$› *eq sets-eq-imp-space-eq*[*OF sets*]
prob-space.emeasure-space-1[*OF subsp*]
by (*intro measurable-cong*) *auto*
then show $(\lambda a.\ emeasure\ (K\ a)\ \Omega) \in borel\text{-}measurable\ M$ **by** *simp*
qed
qed (*use subsp* **in** *auto*)

lemma *in-space-prob-algebra*:
x ∈ *space* (*prob-algebra M*) ⟹ *emeasure x* (*space M*) = *1*
unfolding *prob-algebra-def space-restrict-space space-subprob-algebra*
by (*auto dest*!: *prob-space.emeasure-space-1 sets-eq-imp-space-eq*)

lemma *prob-space-pair*:
assumes *prob-space M prob-space N* **shows** *prob-space* (*M* ⨂$_M$ *N*)
by (*metis assms measurable-fst prob-space.distr-pair-fst prob-space-distrD*)

lemma *measurable-pair-prob*[*measurable*]:
f ∈ *M* →$_M$ *prob-algebra N* ⟹ *g* ∈ *M* →$_M$ *prob-algebra L* ⟹ (λ*x*. *f x* ⨂$_M$ *g x*) ∈ *M* →$_M$ *prob-algebra* (*N* ⨂$_M$ *L*)
unfolding *prob-algebra-def measurable-restrict-space2-iff*
by (*auto intro*!: *measurable-pair-measure prob-space-pair*)

lemma *emeasure-bind-prob-algebra*:
assumes *A*: *A* ∈ *space* (*prob-algebra N*)
assumes *B*: *B* ∈ *N* →$_M$ *prob-algebra L*
assumes *X*: *X* ∈ *sets L*
shows *emeasure* (*bind A B*) *X* = (∫$^+$*x*. *emeasure* (*B x*) *X* ∂*A*)
using *A B*
by (*intro emeasure-bind*[*OF - - X*])
(*auto simp*: *space-prob-algebra measurable-prob-algebraD cong*: *measurable-cong-sets intro*!: *prob-space.not-empty*)

lemma *prob-space-bind′*:
assumes *A*: *A* ∈ *space* (*prob-algebra M*) **and** *B*: *B* ∈ *M* →$_M$ *prob-algebra N* **shows** *prob-space* (*A* ⤜ *B*)
using *measurable-bind-prob-space*[*OF measurable-const*, *OF A B*, *THEN measurable-space*, *of undefined count-space UNIV*]
by (*simp add*: *space-prob-algebra*)

lemma *sets-bind′*:
assumes *A*: *A* ∈ *space* (*prob-algebra M*) **and** *B*: *B* ∈ *M* →$_M$ *prob-algebra N* **shows** *sets* (*A* ⤜ *B*) = *sets N*
using *measurable-bind-prob-space*[*OF measurable-const*, *OF A B*, *THEN measurable-space*, *of undefined count-space UNIV*]
by (*simp add*: *space-prob-algebra*)

lemma *bind-cong-AE′*:
assumes *M*: *M* ∈ *space* (*prob-algebra L*)
and *f*: *f* ∈ *L* →$_M$ *prob-algebra N* **and** *g*: *g* ∈ *L* →$_M$ *prob-algebra N*
and *ae*: *AE x in M*. *f x* = *g x*
shows *bind M f* = *bind M g*
proof (*rule measure-eqI*)
show *sets* (*M* ⤜ *f*) = *sets* (*M* ⤜ *g*)
unfolding *sets-bind′*[*OF M f*] *sets-bind′*[*OF M g*] **..**
show *A* ∈ *sets* (*M* ⤜ *f*) ⟹ *emeasure* (*M* ⤜ *f*) *A* = *emeasure* (*M* ⤜ *g*) *A* **for** *A*

unfolding *sets-bind'*[*OF M f*]
using *emeasure-bind-prob-algebra*[*OF M f*, *of A*] *emeasure-bind-prob-algebra*[*OF M g*, *of A*] *ae*
by (*auto intro*: *nn-integral-cong-AE*)
qed

lemma *density-discrete*:
countable A $\Longrightarrow$ *sets N* = *Set.Pow A* $\Longrightarrow$ $(\bigwedge x.\ f\ x \geq 0) \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x = emeasure\ N\ \{x\}) \Longrightarrow$
density (*count-space A*) *f* = *N*
by (*rule measure-eqI-countable*[*of* - *A*]) (*auto simp*: *emeasure-density*)

lemma *distr-density-discrete*:
fixes *f'*
assumes *countable A*
assumes *f'* $\in$ *borel-measurable M*
assumes *g* $\in$ *measurable M* (*count-space A*)
defines *f* $\equiv \lambda x.\ \int^+ t.$ (*if g t* = *x then 1 else 0*) $*$ *f' t* ∂M
assumes $\bigwedge x.\ \ x \in space\ M \Longrightarrow g\ x \in A$
shows *density* (*count-space A*) $(\lambda x.\ f\ x)$ = *distr* (*density M f'*) (*count-space A*) *g*
proof (*rule density-discrete*)
fix *x* **assume** *x*: *x* $\in$ *A*
have *f x* = $\int^+ t.$ *indicator* (*g* $-`$ $\{x\}$ $\cap$ *space M*) *t* $*$ *f' t* ∂M (**is** - = *?I*)
unfolding *f-def*
by (*intro nn-integral-cong*) (*simp split*: *split-indicator*)
also from *x* **have** *in-sets*: *g* $-`$ $\{x\}$ $\cap$ *space M* $\in$ *sets M*
by (*intro measurable-sets*[*OF assms*(*3*)]) *simp*
have *?I* = *emeasure* (*density M f'*) (*g* $-`$ $\{x\}$ $\cap$ *space M*) **unfolding** *f-def*
by (*subst emeasure-density*[*OF assms*(*2*) *in-sets*], *subst mult.commute*) (*rule refl*)
also from *assms*(*3*) *x* **have** ... = *emeasure* (*distr* (*density M f'*) (*count-space A*) *g*) $\{x\}$
by (*subst emeasure-distr*) *simp-all*
finally show *f x* = *emeasure* (*distr* (*density M f'*) (*count-space A*) *g*) $\{x\}$.
qed (*use assms* **in** *auto*)

lemma *bind-cong-AE*:
assumes *M* = *N*
assumes *f*: *f* $\in$ *measurable N* (*subprob-algebra B*)
assumes *g*: *g* $\in$ *measurable N* (*subprob-algebra B*)
assumes *ae*: *AE x in N*. *f x* = *g x*
shows *bind M f* = *bind N g*
proof *cases*
assume *space N* = {} **then show** *?thesis*
using ‹*M* = *N*› **by** (*simp add*: *bind-empty*)
next
assume *space N* $\neq$ {}
show *?thesis* **unfolding** ‹*M* = *N*›
proof (*rule measure-eqI*)

```
    have ∗: sets (N ⤜ f) = sets B
      using sets-bind[OF sets-kernel[OF f] ‹space N ≠ {}›] by simp
    then show sets (N ⤜ f) = sets (N ⤜ g)
      using sets-bind[OF sets-kernel[OF g] ‹space N ≠ {}›] by auto
    fix A assume A ∈ sets (N ⤜ f)
    then have A ∈ sets B
      unfolding ∗ .
    with ae f g ‹space N ≠ {}› show emeasure (N ⤜ f) A = emeasure (N ⤜
g) A
    by (subst (1 2) emeasure-bind[where N=B]) (auto intro!: nn-integral-cong-AE)
  qed
qed

lemma bind-cong-simp: M = N ⟹ (⋀x. x∈space M =simp=> f x = g x) ⟹
bind M f = bind N g
  by (auto simp: simp-implies-def intro!: bind-cong)

lemma sets-bind-measurable:
  assumes f: f ∈ measurable M (subprob-algebra B)
  assumes M: space M ≠ {}
  shows sets (M ⤜ f) = sets B
  using M by (intro sets-bind[OF sets-kernel[OF f]]) auto

lemma space-bind-measurable:
  assumes f: f ∈ measurable M (subprob-algebra B)
  assumes M: space M ≠ {}
  shows space (M ⤜ f) = space B
  using M by (intro space-bind[OF sets-kernel[OF f]]) auto

lemma bind-distr-return:
  f ∈ M →M N ⟹ g ∈ N →M L ⟹ space M ≠ {} ⟹
    distr M N f ⤜ (λx. return L (g x)) = distr M L (λx. g (f x))
  by (subst bind-distr[OF - measurable-compose[OF - return-measurable]])
     (auto intro!: bind-return-distr′)

lemma (in prob-space) AE-eq-constD:
  assumes AE x in M. x = y
  shows   M = return M y y ∈ space M
proof −
  have AE x in M. x ∈ space M
    by auto
  with assms have AE x in M. y ∈ space M
    by eventually-elim auto
  thus y ∈ space M
    by simp
  show M = return M y
  proof (rule measure-eqI)
    fix X assume X: X ∈ sets M
    have AE x in M. (x ∈ X) = (x ∈ (if y ∈ X then space M else {}))
```

using *assms* **by** *eventually-elim* (*use X* ‹*y* ∈ *space M*› **in** *auto*)
hence *emeasure M X* = *emeasure M* (*if y* ∈ *X then space M else* {})
using *X* **by** (*intro emeasure-eq-AE*) *auto*
also have ... = *emeasure* (*return M y*) *X*
using *X* **by** (*auto simp*: *emeasure-space-1*)
finally show *emeasure M X* = ...
qed *auto*
qed

end

8 Projective Family

theory *Projective-Family*
imports *Giry-Monad*
begin

lemma *vimage-restrict-preserve-mono*:
assumes *J*: *J* ⊆ *I*
and *sets*: *A* ⊆ (Π$_E$ *i*∈*J*. *S i*) *B* ⊆ (Π$_E$ *i*∈*J*. *S i*) **and** *ne*: (Π$_E$ *i*∈*I*. *S i*) ≠ {}
and *eq*: (λ*x*. *restrict x J*) −‘ *A* ∩ (Π$_E$ *i*∈*I*. *S i*) ⊆ (λ*x*. *restrict x J*) −‘ *B* ∩ (Π$_E$ *i*∈*I*. *S i*)
shows *A* ⊆ *B*
proof (*intro subsetI*)
fix *x* **assume** *x* ∈ *A*
from *ne* **obtain** *y* **where** *y*: ⋀*i*. *i* ∈ *I* ⟹ *y i* ∈ *S i* **by** *auto*
have *J* ∩ (*I* − *J*) = {} **by** *auto*
show *x* ∈ *B*
proof *cases*
assume *x*: *x* ∈ (Π$_E$ *i*∈*J*. *S i*)
have *merge J* (*I* − *J*) (*x*,*y*) ∈ (λ*x*. *restrict x J*) −‘ *A* ∩ (Π$_E$ *i*∈*I*. *S i*)
using *y x* ‹*J* ⊆ *I*› *PiE-cancel-merge*[*of J I* − *J x y S*] ‹*x*∈*A*›
by (*auto simp del*: *PiE-cancel-merge simp add*: *Un-absorb1*)
also have ... ⊆ (λ*x*. *restrict x J*) −‘ *B* ∩ (Π$_E$ *i*∈*I*. *S i*) **by** *fact*
finally show *x* ∈ *B*
using *y x* ‹*J* ⊆ *I*› *PiE-cancel-merge*[*of J I* − *J x y S*] ‹*x*∈*A*› *eq*
by (*auto simp del*: *PiE-cancel-merge simp add*: *Un-absorb1*)
qed (*insert* ‹*x*∈*A*› *sets*, *auto*)
qed

locale *projective-family* =
fixes *I* :: *'i set* **and** *P* :: *'i set* ⇒ (*'i* ⇒ *'a*) *measure* **and** *M* :: *'i* ⇒ *'a measure*
assumes *P*: ⋀*J H*. *J* ⊆ *H* ⟹ *finite H* ⟹ *H* ⊆ *I* ⟹ *P J* = *distr* (*P H*) (*PiM J M*) (λ*f*. *restrict f J*)
assumes *prob-space-P*: ⋀*J*. *finite J* ⟹ *J* ⊆ *I* ⟹ *prob-space* (*P J*)
begin

lemma *sets-P*: *finite J* $\Longrightarrow$ $J \subseteq I$ $\Longrightarrow$ *sets* (*P J*) = *sets* (*PiM J M*)
by (*subst P*[*of J J*]) *simp-all*

lemma *space-P*: *finite J* $\Longrightarrow$ $J \subseteq I$ $\Longrightarrow$ *space* (*P J*) = *space* (*PiM J M*)
using *sets-P* **by** (*rule sets-eq-imp-space-eq*)

lemma *not-empty-M*: $i \in I$ $\Longrightarrow$ *space* (*M i*) $\neq$ {}
using *prob-space-P*[*THEN prob-space.not-empty*] **by** (*auto simp*: *space-PiM space-P*)

lemma *not-empty*: *space* (*PiM I M*) $\neq$ {}
by (*simp add*: *not-empty-M*)

abbreviation
emb L K $\equiv$ *prod-emb L M K*

lemma *emb-preserve-mono*:
assumes $J \subseteq L$ $L \subseteq I$ **and** *sets*: $X \in$ *sets* (Pi_M *J M*) $Y \in$ *sets* (Pi_M *J M*)
assumes *emb L J X* $\subseteq$ *emb L J Y*
shows $X \subseteq Y$
proof (*rule vimage-restrict-preserve-mono*)
show $X \subseteq (\Pi_E\ i \in J.\ space\ (M\ i))$ $Y \subseteq (\Pi_E\ i \in J.\ space\ (M\ i))$
using *sets*[*THEN sets.sets-into-space*] **by** (*auto simp*: *space-PiM*)
show $(\Pi_E\ i \in L.\ space\ (M\ i)) \neq \{\}$
using ‹$L \subseteq I$› **by** (*auto simp add*: *not-empty-M space-PiM*[*symmetric*])
show $(\lambda x.\ restrict\ x\ J) -` X \cap (\Pi_E\ i \in L.\ space\ (M\ i)) \subseteq (\lambda x.\ restrict\ x\ J) -` Y \cap (\Pi_E\ i \in L.\ space\ (M\ i))$
using ‹*prod-emb L M J X* $\subseteq$ *prod-emb L M J Y*› **by** (*simp add*: *prod-emb-def*)
qed *fact*

lemma *emb-injective*:
assumes *L*: $J \subseteq L$ $L \subseteq I$ **and** *X*: $X \in$ *sets* (Pi_M *J M*) **and** *Y*: $Y \in$ *sets* (Pi_M *J M*)
shows *emb L J X* = *emb L J Y* $\Longrightarrow$ $X = Y$
by (*intro antisym emb-preserve-mono*[*OF L X Y*] *emb-preserve-mono*[*OF L Y X*]) *auto*

lemma *emeasure-P*: $J \subseteq K$ $\Longrightarrow$ *finite K* $\Longrightarrow$ $K \subseteq I$ $\Longrightarrow$ $X \in$ *sets* (*PiM J M*) $\Longrightarrow$ *P K* (*emb K J X*) = *P J X*
by (*auto intro*!: *emeasure-distr-restrict*[*symmetric*] *simp*: *P sets-P*)

inductive-set *generator* :: ('*i* $\Rightarrow$ '*a*) *set set* **where**
finite J $\Longrightarrow$ $J \subseteq I$ $\Longrightarrow$ $X \in$ *sets* (Pi_M *J M*) $\Longrightarrow$ *emb I J X* $\in$ *generator*

lemma *algebra-generator*: *algebra* (*space* (*PiM I M*)) *generator*
unfolding *algebra-iff-Int*
proof (*safe elim*!: *generator.cases*)
fix *J X* **assume** *J*: *finite J* $J \subseteq I$ **and** *X*: $X \in$ *sets* (*PiM J M*)

from *X*[*THEN sets.sets-into-space*] *J* **show** $x \in$ *emb I J X* $\Longrightarrow$ $x \in$ *space* (*PiM*

I M) **for** *x*
by (*auto simp*: *prod-emb-def space-PiM*)

have *emb I J* (*space* (*PiM J M*) − *X*) ∈ *generator*
by (*intro generator.intros J sets.Diff sets.top X*)
with *J* **show** *space* (*Pi*$_M$ *I M*) − *emb I J X* ∈ *generator*
by (*simp add*: *space-PiM prod-emb-PiE*)

fix *K Y* **assume** *K*: *finite K K* ⊆ *I* **and** *Y*: *Y* ∈ *sets* (*PiM K M*)
have *emb I* (*J* ∪ *K*) (*emb* (*J* ∪ *K*) *J X*) ∩ *emb I* (*J* ∪ *K*) (*emb* (*J* ∪ *K*) *K Y*) ∈ *generator*
unfolding *prod-emb-Int*[*symmetric*]
by (*intro generator.intros sets.Int measurable-prod-emb*) (*auto intro*!: *J K X Y*)
with *J K X Y* **show** *emb I J X* ∩ *emb I K Y* ∈ *generator*
by *simp*
qed (*force simp*: *generator.simps prod-emb-empty*[*symmetric*])

interpretation *generator*: *algebra space* (*PiM I M*) *generator*
by (*rule algebra-generator*)

lemma *sets-PiM-generator*: *sets* (*PiM I M*) = *sigma-sets* (*space* (*PiM I M*)) *generator*
proof (*intro antisym sets.sigma-sets-subset*)
show *sets* (*PiM I M*) ⊆ *sigma-sets* (*space* (*PiM I M*)) *generator*
unfolding *sets-PiM-single space-PiM*[*symmetric*]
proof (*intro sigma-sets-mono*′, *safe*)
fix *i A* **assume** *i* ∈ *I* **and** *A*: *A* ∈ *sets* (*M i*)
then have {*f* ∈ *space* (*Pi*$_M$ *I M*). *f i* ∈ *A*} = *emb I* {*i*} (Π$_E$ *j*∈{*i*}. *A*)
by (*auto simp*: *prod-emb-def space-PiM restrict-def Pi-iff PiE-iff extensional-def*)
with ‹*i*∈*I*› *A* **show** {*f* ∈ *space* (*Pi*$_M$ *I M*). *f i* ∈ *A*} ∈ *generator*
by (*auto intro*!: *generator.intros sets-PiM-I-finite*)
qed
qed (*auto elim*!: *generator.cases*)

definition *mu-G* (‹*μG*›) **where**
μG A = (*THE x*. ∀ *J*⊆*I*. *finite J* ⟶ (∀ *X*∈*sets* (*Pi*$_M$ *J M*). *A* = *emb I J X* ⟶ *x* = *emeasure* (*P J*) *X*))

definition *lim* :: (′*i* ⇒ ′*a*) *measure* **where**
lim = *extend-measure* (*space* (*PiM I M*)) *generator* (λ*x*. *x*) *μG*

lemma *space-lim*[*simp*]: *space lim* = *space* (*PiM I M*)
using *generator.space-closed*
unfolding *lim-def* **by** (*intro space-extend-measure*) *simp*

lemma *sets-lim*[*simp*, *measurable*]: *sets lim* = *sets* (*PiM I M*)
using *generator.space-closed* **by** (*simp add*: *lim-def sets-PiM-generator sets-extend-measure*)

lemma *mu-G-spec*:
assumes *J*: *finite J* $J \subseteq I$ $X \in$ *sets* (Pi_M *J M*)
shows μG (*emb I J X*) = *emeasure* (*P J*) *X*
unfolding *mu-G-def*
proof (*intro the-equality allI impI ballI*)
fix *K Y* **assume** *K*: *finite K* $K \subseteq I$ $Y \in$ *sets* (Pi_M *K M*)
and [*simp*]: *emb I J X* = *emb I K Y*
have *emeasure* (*P K*) *Y* = *emeasure* (*P* ($K \cup J$)) (*emb* ($K \cup J$) *K Y*)
using *K J* **by** (*simp add*: *emeasure-P*)
also have *emb* ($K \cup J$) *K Y* = *emb* ($K \cup J$) *J X*
using *K J* **by** (*simp add*: *emb-injective*[*of K* $\cup$ *J I*])
also have *emeasure* (*P* ($K \cup J$)) (*emb* ($K \cup J$) *J X*) = *emeasure* (*P J*) *X*
using *K J* **by** (*subst emeasure-P*) *simp-all*
finally show *emeasure* (*P J*) *X* = *emeasure* (*P K*) *Y* **..**
qed (*insert J*, *force*)

lemma *positive-mu-G*: *positive generator* μG
proof −
show *?thesis*
proof (*safe intro*!: *positive-def*[*THEN iffD2*])
obtain *J* **where** *finite J* $J \subseteq I$ **by** *auto*
then have μG (*emb I J* {}) = *0*
by (*subst mu-G-spec*) *auto*
then show μG {} = *0* **by** *simp*
qed
qed

lemma *additive-mu-G*: *additive generator* μG
proof (*safe intro*!: *additive-def*[*THEN iffD2*] *elim*!: *generator.cases*)
fix *J X K Y* **assume** *J*: *finite J* $J \subseteq I$ $X \in$ *sets* (*PiM J M*)
and *K*: *finite K* $K \subseteq I$ $Y \in$ *sets* (*PiM K M*)
and *emb I J X* $\cap$ *emb I K Y* = {}
then have *JK-disj*: *emb* ($J \cup K$) *J X* $\cap$ *emb* ($J \cup K$) *K Y* = {}
by (*intro emb-injective*[*of J* $\cup$ *K I* - {}]) (*auto simp*: *sets.Int prod-emb-Int*)
have μG (*emb I J X* $\cup$ *emb I K Y*) = μG (*emb I* ($J \cup K$) (*emb* ($J \cup K$) *J X* $\cup$ *emb* ($J \cup K$) *K Y*))
using *J K* **by** *simp*
also have ... = *emeasure* (*P* ($J \cup K$)) (*emb* ($J \cup K$) *J X* $\cup$ *emb* ($J \cup K$) *K Y*)
using *J K* **by** (*simp add*: *mu-G-spec sets.Un del*: *prod-emb-Un*)
also have ... = μG (*emb I J X*) + μG (*emb I K Y*)
using *J K JK-disj* **by** (*simp add*: *plus-emeasure*[*symmetric*] *mu-G-spec emeasure-P sets-P*)
finally show μG (*emb I J X* $\cup$ *emb I K Y*) = μG (*emb I J X*) + μG (*emb I K Y*) **.**
qed

lemma *emeasure-lim*:
assumes *JX*: *finite J* $J \subseteq I$ $X \in$ *sets* (*PiM J M*)

assumes *cont*: $\bigwedge J\ X.\ (\bigwedge i.\ J\ i \subseteq I) \Longrightarrow$ *incseq* $J \Longrightarrow (\bigwedge i.$ *finite* $(J\ i)) \Longrightarrow (\bigwedge i.$ $X\ i \in$ *sets* $(PiM\ (J\ i)\ M)) \Longrightarrow$
decseq $(\lambda i.$ *emb* $I\ (J\ i)\ (X\ i)) \Longrightarrow 0 <$ $(INF\ i.\ P\ (J\ i)\ (X\ i)) \Longrightarrow (\bigcap i.$ *emb* I $(J\ i)\ (X\ i)) \neq \{\}$
shows *emeasure lim* $(emb\ I\ J\ X) = P\ J\ X$
proof −
have $\exists \mu.\ (\forall s \in$ *generator*. $\mu\ s = \mu G\ s) \wedge$
measure-space $(space\ (PiM\ I\ M))$ $(sigma\text{-}sets\ (space\ (PiM\ I\ M))\ generator)\ \mu$
proof (*rule generator.caratheodory-empty-continuous*[*OF positive-mu-G additive-mu-G*])
show $\bigwedge A.\ A \in$ *generator* $\Longrightarrow \mu G\ A \neq \infty$
proof (*clarsimp elim*!: *generator.cases simp*: *mu-G-spec del*: *notI*)
fix J **assume** *finite* $J\ J \subseteq I$
then interpret *prob-space* $P\ J$ **by** (*rule prob-space-P*)
show $\bigwedge X.\ X \in sets\ (Pi_M\ J\ M) \Longrightarrow$ *emeasure* $(P\ J)\ X \neq top$
by *simp*
qed
next
fix A **assume** *range* $A \subseteq$ *generator* **and** *decseq* $A\ (\bigcap i.\ A\ i) = \{\}$
then have $\forall i.\ \exists J\ X.\ A\ i = emb\ I\ J\ X \wedge$ *finite* $J \wedge J \subseteq I \wedge X \in sets\ (PiM\ J\ M)$
unfolding *image-subset-iff generator.simps* **by** *blast*
then obtain $J\ X$ **where** A: $\bigwedge i.\ A\ i = emb\ I\ (J\ i)\ (X\ i)$
and $*$: $\bigwedge i.$ *finite* $(J\ i)\ \bigwedge i.\ J\ i \subseteq I\ \bigwedge i.\ X\ i \in sets\ (PiM\ (J\ i)\ M)$
by *metis*
have $(INF\ i.\ P\ (J\ i)\ (X\ i)) = 0$
proof (*rule ccontr*)
assume *INF-P*: $(INF\ i.\ P\ (J\ i)\ (X\ i)) \neq 0$
have $(\bigcap i.\ emb\ I\ (\bigcup n{\leq}i.\ J\ n)\ (emb\ (\bigcup n{\leq}i.\ J\ n)\ (J\ i)\ (X\ i))) \neq \{\}$
proof (*rule cont*)
show *decseq* $(\lambda i.\ emb\ I\ (\bigcup n{\leq}i.\ J\ n)\ (emb\ (\bigcup n{\leq}i.\ J\ n)\ (J\ i)\ (X\ i)))$
using $*$ ‹*decseq A*› **by** (*subst prod-emb-trans*) (*auto simp*: *A*[*abs-def*])
show $0 <$ $(INF\ i.\ P\ (\bigcup n{\leq}i.\ J\ n)\ (emb\ (\bigcup n{\leq}i.\ J\ n)\ (J\ i)\ (X\ i)))$
using $*$ *INF-P* **by** (*subst emeasure-P*) (*auto simp*: *less-le intro*!: *INF-greatest*)
show *incseq* $(\lambda i.\ \bigcup n{\leq}i.\ J\ n)$
by (*force simp*: *incseq-def*)
qed (*insert* $*$, *auto*)
with ‹$(\bigcap i.\ A\ i) = \{\}$› $*$ **show** *False*
by (*subst* (*asm*) *prod-emb-trans*) (*auto simp*: *A*[*abs-def*])
qed
moreover have $(\lambda i.\ P\ (J\ i)\ (X\ i)) \longrightarrow (INF\ i.\ P\ (J\ i)\ (X\ i))$
proof (*intro LIMSEQ-INF antimonoI*)
fix $x\ y ::$ *nat* **assume** $x \leq y$
have $P\ (J\ y \cup J\ x)\ (emb\ (J\ y \cup J\ x)\ (J\ y)\ (X\ y)) \leq P\ (J\ y \cup J\ x)\ (emb\ (J\ y \cup J\ x)\ (J\ x)\ (X\ x))$
using ‹*decseq A*›[*THEN decseqD*, *OF* ‹$x{\leq}y$›] $*$
by (*auto simp*: *A sets-P del*: *subsetI intro*!: *emeasure-mono* ‹$x \leq y$›
emb-preserve-mono[*of* $J\ y \cup J\ x\ I$, **where** X=$emb\ (J\ y \cup J\ x)\ (J\ y)$

```
(X y)])
      then show P (J y) (X y) ≤ P (J x) (X x)
        using ∗ by (simp add: emeasure-P)
    qed
    ultimately show (λi. μG (A i)) ⟶ 0
      by (auto simp: A[abs-def] mu-G-spec ∗)
  qed
  then obtain μ where eq: ∀ s∈generator. μ s = μG s
    and ms: measure-space (space (PiM I M)) (sets (PiM I M)) μ
    by (metis sets-PiM-generator)
  show ?thesis
  proof (subst emeasure-extend-measure[OF lim-def])
    show A ∈ generator ⟹ μ A = μG A for A
      using eq by simp
    show positive (sets lim) μ countably-additive (sets lim) μ
      using ms by (auto simp add: measure-space-def)
    show (λx. x) ' generator ⊆ Pow (space (Pi_M I M))
      using generator.space-closed by simp
    show emb I J X ∈ generator μG (emb I J X) = emeasure (P J) X
      using JX by (auto intro: generator.intros simp: mu-G-spec)
  qed
qed

end

sublocale product-prob-space ⊆ projective-family I λJ. PiM J M M
  unfolding projective-family-def
proof (intro conjI allI impI distr-restrict)
  show ⋀J. finite J ⟹ prob-space (Pi_M J M)
    by (intro prob-spaceI) (simp add: space-PiM emeasure-PiM emeasure-space-1)
qed auto
```

Proof due to Ionescu Tulcea.

```
locale Ionescu-Tulcea =
  fixes P :: nat ⇒ (nat ⇒ 'a) ⇒ 'a measure and M :: nat ⇒ 'a measure
  assumes P[measurable]: ⋀i. P i ∈ measurable (PiM {0..<i} M) (subprob-algebra
(M i))
  assumes prob-space-P: ⋀i x. x ∈ space (PiM {0..<i} M) ⟹ prob-space (P i
x)
begin

lemma non-empty[simp]: space (M i) ≠ {}
proof (induction i rule: less-induct)
  case (less i)
  then obtain x where ⋀j. j < i ⟹ x j ∈ space (M j)
    unfolding ex-in-conv[symmetric] by metis
  then have ∗: restrict x {0..<i} ∈ space (PiM {0..<i} M)
    by (auto simp: space-PiM PiE-iff)
  then interpret prob-space P i (restrict x {0..<i})
```

by (*rule prob-space-P*)
show *?case*
using *not-empty subprob-measurableD(1)*[*OF P*, *OF* ∗] **by** *simp*
qed

lemma *space-PiM-not-empty*[*simp*]: *space* (*PiM UNIV M*) ≠ {}
unfolding *space-PiM-empty-iff* **by** *auto*

lemma *space-P*: *x* ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *space* (*P n x*) = *space* (*M n*)
by (*simp add*: *P*[*THEN subprob-measurableD(1)*])

lemma *sets-P*[*measurable-cong*]: *x* ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *sets* (*P n x*) = *sets* (*M n*)
by (*simp add*: *P*[*THEN subprob-measurableD(2)*])

definition *eP* :: *nat* ⇒ (*nat* ⇒ *'a*) ⇒ (*nat* ⇒ *'a*) *measure* **where**
eP n ω = *distr* (*P n ω*) (*PiM* {*0*..<*Suc n*} *M*) (*fun-upd ω n*)

lemma *measurable-eP*[*measurable*]:
eP n ∈ *measurable* (*PiM* {*0*..< *n*} *M*) (*subprob-algebra* (*PiM* {*0*..<*Suc n*} *M*))
by (*auto simp*: *eP-def*[*abs-def*] *measurable-split-conv*
intro!: *measurable-fun-upd*[**where** *J*={*0*..<*n*}] *measurable-distr2*[*OF* - *P*])

lemma *space-eP*:
x ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *space* (*eP n x*) = *space* (*PiM* {*0*..<*Suc n*} *M*)
by (*simp add*: *eP-def*)

lemma *sets-eP*[*measurable*]:
x ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *sets* (*eP n x*) = *sets* (*PiM* {*0*..<*Suc n*} *M*)
by (*simp add*: *eP-def*)

lemma *prob-space-eP*: *x* ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *prob-space* (*eP n x*)
unfolding *eP-def*
by (*intro prob-space.prob-space-distr prob-space-P measurable-fun-upd*[**where** *J*={*0*..<*n*}])
auto

lemma *nn-integral-eP*:
ω ∈ *space* (*PiM* {*0*..<*n*} *M*) ⟹ *f* ∈ *borel-measurable* (*PiM* {*0*..<*Suc n*} *M*)
⟹
(∫⁺*x*. *f x* ∂*eP n ω*) = (∫⁺*x*. *f* (*fun-upd ω n x*) ∂*P n ω*)
unfolding *eP-def*
by (*subst nn-integral-distr*) (*auto intro*!: *measurable-fun-upd*[**where** *J*={*0*..<*n*}]
simp: *space-PiM PiE-iff*)

lemma *emeasure-eP*:
assumes *ω*[*simp*]: *ω* ∈ *space* (*PiM* {*0*..<*n*} *M*) **and** *A*[*measurable*]: *A* ∈ *sets*
(*PiM* {*0*..<*Suc n*} *M*)
shows *eP n ω A* = *P n ω* ((λ*x*. *fun-upd ω n x*) −‘ *A* ∩ *space* (*M n*))
using *nn-integral-eP*[*of ω n indicator A*]

apply (*simp add*: *sets-eP nn-integral-indicator*[*symmetric*] *sets-P del*: *nn-integral-indicator*)
apply (*subst nn-integral-indicator*[*symmetric*])
using *measurable-sets*[*OF measurable-fun-upd*[*OF - measurable-const*[*OF* ω] *measurable-id*] *A*, *of n*]
apply (*auto simp add*: *sets-P atLeastLessThanSuc space-P simp del*: *nn-integral-indicator intro*!: *nn-integral-cong split*: *split-indicator*)
done

primrec $C :: nat \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$C\ n\ 0\ \omega = return\ (PiM\ \{0..<n\}\ M)\ \omega$
| $C\ n\ (Suc\ m)\ \omega = C\ n\ m\ \omega \gg\!= eP\ (n + m)$

lemma *measurable-C*[*measurable*]:
$C\ n\ m \in measurable\ (PiM\ \{0..<n\}\ M)\ (subprob\text{-}algebra\ (PiM\ \{0..<n + m\}\ M))$
by (*induction m*) *auto*

lemma *space-C*:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (C\ n\ m\ x) = space\ (PiM\ \{0..<n + m\}\ M)$
by (*simp add*: *measurable-C*[*THEN subprob-measurableD*(*1*)])

lemma *sets-C*[*measurable-cong*]:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (C\ n\ m\ x) = sets\ (PiM\ \{0..<n + m\}\ M)$
by (*simp add*: *measurable-C*[*THEN subprob-measurableD*(*2*)])

lemma *prob-space-C*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow prob\text{-}space\ (C\ n\ m\ x)$
proof (*induction m*)
case (*Suc m*) **then show** *?case*
by (*auto intro*!: *prob-space.prob-space-bind*[**where** *S*=$PiM\ \{0..<Suc\ (n + m)\}$ *M*]
simp: *space-C prob-space-eP*)
qed (*auto intro*!: *prob-space-return simp*: *space-PiM*)

lemma *split-C*:
assumes ω: $\omega \in space\ (PiM\ \{0..<n\}\ M)$ **shows** $(C\ n\ m\ \omega \gg\!= C\ (n + m)\ l) = C\ n\ (m + l)\ \omega$
proof (*induction l*)
case *0*
with ω **show** *?case*
by (*simp add*: *bind-return-distr'* *prob-space-C*[*THEN prob-space.not-empty*]
distr-cong[*OF refl sets-C*[*symmetric*, *OF* ω]])
next
case (*Suc l*) **with** ω **show** *?case*
by (*simp add*: *bind-assoc*[*symmetric*, *OF - measurable-eP*]) (*simp add*: *ac-simps*)
qed

lemma *nn-integral-C*:
assumes $m \le m'$ **and** *f*[*measurable*]: $f \in borel\text{-}measurable\ (PiM\ \{0..<n+m\}$

M)
and *nonneg*: $\bigwedge x.\ x \in space\ (PiM\ \{0..<n+m\}\ M) \Longrightarrow 0 \leq f\ x$
and *x*: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $(\int^+ x.\ f\ x\ \partial C\ n\ m\ x) = (\int^+ x.\ f\ (restrict\ x\ \{0..<n+m\})\ \partial C\ n\ m'\ x)$
using ‹$m \leq m'$›
proof (*induction rule*: *dec-induct*)
case (*step i*)
let *?E* = $\lambda x.\ f\ (restrict\ x\ \{0..<n + m\})$ **and** *?C* = $\lambda i\ f.\ \int^+ x.\ f\ x\ \partial C\ n\ i\ x$
from ‹$m \leq i$› *x* **have** *?C i ?E* = *?C* (*Suc i*) *?E*
by (*auto simp*: *nn-integral-bind*[**where** *B*=*PiM* $\{0\ ..<\ Suc\ (n + i)\}$ *M*] *space-C nn-integral-eP*
intro!: *nn-integral-cong*)
(*simp add*: *space-PiM PiE-iff nonneg prob-space.emeasure-space-1*[*OF prob-space-P*])
with *step* **show** *?case* **by** (*simp del*: *restrict-apply*)
qed (*auto simp*: *space-PiM space-C*[*OF x*] *simp del*: *restrict-apply intro*!: *nn-integral-cong*)

lemma *emeasure-C*:
assumes $m \leq m'$ **and** *A*[*measurable*]: $A \in sets\ (PiM\ \{0..<n+m\}\ M)$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $emeasure\ (C\ n\ m'\ x)\ (prod\text{-}emb\ \{0..<n\ +\ m'\}\ M\ \{0..<n+m\}\ A) = emeasure\ (C\ n\ m\ x)\ A$
using *assms*
by (*subst* (*1 2*) *nn-integral-indicator*[*symmetric*])
(*auto intro*!: *nn-integral-cong split*: *split-indicator simp del*: *nn-integral-indicator*
simp: *nn-integral-C*[*of m m' - n*] *prod-emb-iff space-PiM PiE-iff sets-C space-C*)

lemma *distr-C*:
assumes $m \leq m'$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $C\ n\ m\ x = distr\ (C\ n\ m'\ x)\ (PiM\ \{0..<n+m\}\ M)\ (\lambda x.\ restrict\ x\ \{0..<n+m\})$
proof (*rule measure-eqI*)
fix *A* **assume** $A \in sets\ (C\ n\ m\ x)$
with ‹$m \leq m'$› **show** $emeasure\ (C\ n\ m\ x)\ A = emeasure\ (distr\ (C\ n\ m'\ x)\ (Pi_M\ \{0..<n\ +\ m\}\ M)\ (\lambda x.\ restrict\ x\ \{0..<n\ +\ m\}))\ A$
by (*subst emeasure-C*[*symmetric*, *OF* ‹$m \leq m'$›]) (*auto intro*!: *emeasure-distr-restrict*[*symmetric*] *simp*: *sets-C*)
qed (*simp add*: *sets-C*)

definition *up-to* :: *nat set* ⇒ *nat* **where**
$up\text{-}to\ J = (LEAST\ n.\ \forall i{\geq}n.\ i \notin J)$

lemma *up-to-less*: *finite* $J \Longrightarrow i \in J \Longrightarrow i < up\text{-}to\ J$
unfolding *up-to-def*
by (*rule LeastI2*[*of - Suc* (*Max J*)]) (*auto simp*: *Suc-le-eq not-le*[*symmetric*])

lemma *up-to-iff*: *finite* $J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow (\forall i{\in}J.\ i < n)$
proof *safe*

show *finite J* $\Longrightarrow$ *up-to J* $\leq$ *n* $\Longrightarrow$ *i* $\in$ *J* $\Longrightarrow$ *i* $<$ *n* **for** *i*
using *up-to-less*[*of J i*] **by** *auto*
qed (*auto simp*: *up-to-def intro*!: *Least-le*)

lemma *up-to-iff-Ico*: *finite J* $\Longrightarrow$ *up-to J* $\leq$ *n* $\longleftrightarrow$ *J* $\subseteq$ {*0..<n*}
by (*auto simp*: *up-to-iff*)

lemma *up-to*: *finite J* $\Longrightarrow$ *J* $\subseteq$ {*0..<* *up-to J*}
by (*auto simp*: *up-to-less*)

lemma *up-to-mono*: *J* $\subseteq$ *H* $\Longrightarrow$ *finite H* $\Longrightarrow$ *up-to J* $\leq$ *up-to H*
by (*auto simp add*: *up-to-iff finite-subset up-to-less*)

definition *CI* :: *nat set* $\Rightarrow$ (*nat* $\Rightarrow$ *'a*) *measure* **where**
CI J = *distr* (*C 0* (*up-to J*) (λ*x*. *undefined*)) (*PiM J M*) (λ*f*. *restrict f J*)

sublocale *PF*: *projective-family UNIV CI*
unfolding *projective-family-def*
proof *safe*
show *finite J* $\Longrightarrow$ *prob-space* (*CI J*) **for** *J*
using *up-to*[*of J*] **unfolding** *CI-def*
by (*intro prob-space.prob-space-distr prob-space-C measurable-restrict*) *auto*
note *measurable-cong-sets*[*OF sets-C*, *simp*]
have [*simp*]: *J* $\subseteq$ *H* $\Longrightarrow$ (λ*f*. *restrict f J*) $\in$ *measurable* (*Pi*$_M$ *H M*) (*Pi*$_M$ *J M*)
for *H J*
by (*auto intro*!: *measurable-restrict*)

show *J* $\subseteq$ *H* $\Longrightarrow$ *finite H* $\Longrightarrow$ *CI J* = *distr* (*CI H*) (*Pi*$_M$ *J M*) (λ*f*. *restrict f J*) **for** *J H*
by (*simp add*: *CI-def distr-C*[*OF up-to-mono*[*of J H*]] *up-to up-to-mono distr-distr comp-def*
inf.absorb2 finite-subset)
qed

lemma *emeasure-CI'*:
finite J $\Longrightarrow$ *X* $\in$ *sets* (*PiM J M*) $\Longrightarrow$ *CI J X* = *C 0* (*up-to J*) (λ-. *undefined*) (*PF.emb* {*0..<up-to J*} *J X*)
unfolding *CI-def* **using** *up-to*[*of J*] **by** (*rule emeasure-distr-restrict*) (*auto simp*: *sets-C*)

lemma *emeasure-CI*:
J $\subseteq$ {*0..<n*} $\Longrightarrow$ *X* $\in$ *sets* (*PiM J M*) $\Longrightarrow$ *CI J X* = *C 0 n* (λ-. *undefined*) (*PF.emb* {*0..<n*} *J X*)
apply (*subst emeasure-CI'*, *simp-all add*: *finite-subset*)
apply (*subst emeasure-C*[*symmetric*, *of up-to J n*])
apply (*auto simp*: *finite-subset up-to-iff-Ico up-to-less*)
apply (*subst prod-emb-trans*)
apply (*auto simp*: *up-to-less finite-subset up-to-iff-Ico*)
done

lemma *lim*:
assumes *J*: *finite J* **and** *X*: $X \in sets\ (PiM\ J\ M)$
shows *emeasure PF.lim* (*PF.emb UNIV J X*) = *emeasure* (*CI J*) *X*
proof (*rule PF.emeasure-lim*[*OF J subset-UNIV X*])
fix *J X'* **assume** *J*[*simp*]: $\bigwedge i.$ *finite* (*J i*) **and** *X'*[*measurable*]: $\bigwedge i.\ X'\ i \in sets\ (Pi_M\ (J\ i)\ M)$
and *dec*: *decseq* ($\lambda i.$ *PF.emb UNIV* (*J i*) (*X' i*))
define *X* **where** *X n* =
($\bigcap i \in \{i.\ J\ i \subseteq \{0..<n\}\}.$ *PF.emb* $\{0..<n\}$ (*J i*) (*X' i*)) $\cap$ *space* (*PiM* $\{0..<n\}$ *M*) **for** *n*

have *sets-X*[*measurable*]: *X n* $\in$ *sets* (*PiM* $\{0..<n\}$ *M*) **for** *n*
by (*cases* $\{i.\ J\ i \subseteq \{0..<n\}\} = \{\}$)
(*simp-all add*: *X-def*, *auto intro*!: *sets.countable-INT' sets.Int*)

have *dec-X*: $n \leq m \Longrightarrow X\ m \subseteq PF.emb\ \{0..<m\}\ \{0..<n\}\ (X\ n)$ **for** *n m*
unfolding *X-def* **using** *ivl-subset*[*of 0 n 0 m*]
by (*cases* $\{i.\ J\ i \subseteq \{0..<n\}\} = \{\}$)
(*auto simp add*: *prod-emb-Int prod-emb-PiE space-PiM simp del*: *ivl-subset*)

have *dec-X'*: *PF.emb* $\{0..<n\}$ (*J j*) (*X' j*) $\subseteq$ *PF.emb* $\{0..<n\}$ (*J i*) (*X' i*)
if [*simp*]: *J i* $\subseteq \{0..<n\}$ *J j* $\subseteq \{0..<n\}$ $i \leq j$ **for** *n i j*
by (*rule PF.emb-preserve-mono*[*of* $\{0..<n\}$ *UNIV*]) (*auto del*: *subsetI intro*: *dec*[*THEN antimonoD*])

assume $0 <$ (*INF i. CI* (*J i*) (*X' i*))
also have ... $\leq$ (*INF i. C 0 i* ($\lambda x.$ *undefined*) (*X i*))
proof (*intro INF-greatest*)
fix *n*
interpret *C*: *prob-space C 0 n* ($\lambda x.$ *undefined*)
by (*rule prob-space-C*) *simp*
show (*INF i. CI* (*J i*) (*X' i*)) $\leq$ *C 0 n* ($\lambda x.$ *undefined*) (*X n*)
proof *cases*
assume $\{i.\ J\ i \subseteq \{0..<n\}\} = \{\}$ **with** *C.emeasure-space-1* **show** *?thesis*
by (*auto simp add*: *X-def space-C intro*!: *INF-lower2*[*of 0*] *prob-space.measure-le-1 PF.prob-space-P*)
next
assume $*$: $\{i.\ J\ i \subseteq \{0..<n\}\} \neq \{\}$
have (*INF i. CI* (*J i*) (*X' i*)) $\leq$
(*INF* $i \in \{i.\ J\ i \subseteq \{0..<n\}\}.$ *C 0 n* (λ-. *undefined*) (*PF.emb* $\{0..<n\}$ (*J i*) (*X' i*)))
by (*intro INF-superset-mono*) (*auto simp*: *emeasure-CI*)
also have ... = *C 0 n* (λ-. *undefined*) ($\bigcap i \in \{i.\ J\ i \subseteq \{0..<n\}\}.$ (*PF.emb* $\{0..<n\}$ (*J i*) (*X' i*)))
using $*$ **by** (*intro emeasure-INT-decseq-subset*[*symmetric*]) (*auto intro*!: *dec-X' del*: *subsetI simp*: *sets-C*)
also have ... = *C 0 n* (λ-. *undefined*) (*X n*)
using $*$ **by** (*auto simp add*: *X-def INT-extend-simps*)

```
    finally show (INF i. CI (J i) (X' i)) ≤ C 0 n (λ-. undefined) (X n) .
  qed
 qed
 finally have pos: 0 < (INF i. C 0 i (λx. undefined) (X i)) .
 from less-INF-D[OF this, of 0] have X 0 ≠ {}
  by auto

 { fix ω n assume ω: ω ∈ space (PiM {0..<n} M)
  let ?C = λi. emeasure (C n i ω) (X (n + i))
  let ?C' = λi x. emeasure (C (Suc n) i (fun-upd ω n x)) (X (Suc n + i))
  have M: ⋀i. ?C' i ∈ borel-measurable (P n ω)
   using ω[measurable, simp] measurable-fun-upd[where J={0..<n}] by measurable auto

  assume 0 < (INF i. ?C i)
  also have ... ≤ (INF i. emeasure (C n (1 + i) ω) (X (n + (1 + i))))
   by (intro INF-greatest INF-lower) auto
  also have ... = (INF i. ∫⁺x. ?C' i x ∂P n ω)
   using ω measurable-C[of Suc n]
   apply (intro INF-cong refl)
   apply (subst split-C[symmetric, OF ω])
   apply (subst emeasure-bind[OF - - sets-X])
   apply (simp-all del: C.simps add: space-C)
   apply measurable
   apply simp
   apply (simp add: bind-return[OF measurable-eP] nn-integral-eP)
   done
  also have ... = (∫⁺x. (INF i. ?C' i x) ∂P n ω)
  proof (rule nn-integral-monotone-convergence-INF-AE[symmetric])
   have (∫⁺x. ?C' 0 x ∂P n ω) ≤ (∫⁺x. 1 ∂P n ω)
    by (intro nn-integral-mono) (auto split: split-indicator)
   also have ... < ∞
    using prob-space-P[OF ω, THEN prob-space.emeasure-space-1] by simp
   finally show (∫⁺x. ?C' 0 x ∂P n ω) < ∞ .
  next
   show AE x in P n ω. ?C' (Suc i) x ≤ ?C' i x for i
   proof (rule AE-I2)
    fix x assume x ∈ space (P n ω)
    with ω have ω': fun-upd ω n x ∈ space (PiM {0..<Suc n} M)
     by (auto simp: space-P[OF ω] space-PiM PiE-iff extensional-def)
    with ω show ?C' (Suc i) x ≤ ?C' i x
     apply (subst emeasure-C[symmetric, of i Suc i])
     apply (auto intro!: emeasure-mono[OF dec-X] del: subsetI
                 simp: sets-C space-P)
     apply (subst sets-bind[OF sets-eP])
     apply (simp-all add: space-C space-P)
     done
   qed
  qed fact
```

```
    finally have (∫⁺x. (INF i. ?C' i x) ∂P n ω) ≠ 0
      by simp
    with M have ∃ F x in ae-filter (P n ω). 0 < (INF i. ?C' i x)
      by (subst (asm) nn-integral-0-iff-AE)
         (auto intro!: borel-measurable-INF simp: Filter.not-eventually not-le
zero-less-iff-neq-zero)
    then have ∃ F x in ae-filter (P n ω). x ∈ space (M n) ∧ 0 < (INF i. ?C' i x)
      by (rule frequently-mp[rotated]) (auto simp: space-P ω)
    then obtain x where x ∈ space (M n) 0 < (INF i. ?C' i x)
      by (auto dest: frequently-ex)
    from this(2)[THEN less-INF-D, of 0] this(2)
    have ∃ x. fun-upd ω n x ∈ X (Suc n) ∧ 0 < (INF i. ?C' i x)
      by (intro exI[of - x]) (simp split: split-indicator-asm) }
  note step = this

  let ?ω = λω n x. (restrict ω {0..<n})(n := x)
  let ?L = λω n r. INF i. emeasure (C (Suc n) i (?ω ω n r)) (X (Suc n + i))
  have *: (⋀i. i < n ⟹ ?ω ω i (ω i) ∈ X (Suc i)) ⟹
    restrict ω {0..<n} ∈ space (Pi_M {0..<n} M) for ω n
    using sets.sets-into-space[OF sets-X, of n]
    by (cases n) (auto simp: atLeastLessThanSuc restrict-def[of - {}])
  have ∃ω. ∀ n. ?ω ω n (ω n) ∈ X (Suc n) ∧ 0 < ?L ω n (ω n)
  proof (rule dependent-wellorder-choice)
    fix n ω assume IH: ⋀i. i < n ⟹ ?ω ω i (ω i) ∈ X (Suc i) ∧ 0 < ?L ω i (ω
i)
    show ∃ r. ?ω ω n r ∈ X (Suc n) ∧ 0 < ?L ω n r
    proof (rule step)
      show restrict ω {0..<n} ∈ space (Pi_M {0..<n} M)
        using IH[THEN conjunct1] by (rule *)
      show 0 < (INF i. emeasure (C n i (restrict ω {0..<n})) (X (n + i)))
      proof (cases n)
        case 0 with pos show ?thesis
          by (simp add: CI-def restrict-def)
      next
        case (Suc i) then show ?thesis
          using IH[of i, THEN conjunct2] by (simp add: atLeastLessThanSuc)
      qed
    qed
  qed (simp cong: restrict-cong)
  then obtain ω where ω: ⋀n. ?ω ω n (ω n) ∈ X (Suc n)
    by auto
  from this[THEN *] have ω-space: ω ∈ space (PiM UNIV M)
    by (auto simp: space-PiM PiE-iff Ball-def)
  have *: ω ∈ PF.emb UNIV {0..<n} (X n) for n
  proof (cases n)
    case 0 with ω-space ‹X 0 ≠ {}› sets.sets-into-space[OF sets-X, of 0] show
?thesis
      by (auto simp add: space-PiM prod-emb-def restrict-def PiE-iff)
  next
```

```
    case (Suc i) then show ?thesis
     using ω[of i] ω-space by (auto simp: prod-emb-def space-PiM PiE-iff atLeast-
LessThanSuc)
  qed
  have **: {i. J i ⊆ {0..<up-to (J n)}} ≠ {} for n
    by (auto intro!: exI[of - n] up-to J)
  have ω ∈ PF.emb UNIV (J n) (X' n) for n
   using *[of up-to (J n)] up-to[of J n] by (simp add: X-def prod-emb-Int prod-emb-INT[OF
**])
  then show (⋂ i. PF.emb UNIV (J i) (X' i)) ≠ {}
    by auto
qed

lemma distr-lim: assumes J[simp]: finite J shows distr PF.lim (PiM J M) (λx.
restrict x J) = CI J
  apply (rule measure-eqI)
  apply (simp add: CI-def)
 apply (simp add: emeasure-distr measurable-cong-sets[OF PF.sets-lim] lim[symmetric]
prod-emb-def space-PiM)
  done

end

lemma (in product-prob-space) emeasure-lim-emb:
  assumes *: finite J J ⊆ I X ∈ sets (PiM J M)
  shows emeasure lim (emb I J X) = emeasure (Pi_M J M) X
proof (rule emeasure-lim[OF *], goal-cases)
  case (1 J X)

  have ∃ Q. (∀ i. sets Q = PiM (⋃ i. J i) M ∧ distr Q (PiM (J i) M) (λx. restrict
x (J i)) = Pi_M (J i) M)
  proof cases
    assume finite (⋃ i. J i)
    then have distr (PiM (⋃ i. J i) M) (Pi_M (J i) M) (λx. restrict x (J i)) =
Pi_M (J i) M for i
      by (intro distr-restrict[symmetric]) auto
    then show ?thesis
      by auto
  next
    assume inf: infinite (⋃ i. J i)
    moreover have count: countable (⋃ i. J i)
      using 1(3) by (auto intro: countable-finite)
    define f where f = from-nat-into (⋃ i. J i)
    define t where t = to-nat-on (⋃ i. J i)
    have ft[simp]: x ∈ J i ⟹ f (t x) = x for x i
     unfolding f-def t-def using inf count by (intro from-nat-into-to-nat-on) auto
    have tf[simp]: t (f i) = i for i
      unfolding t-def f-def by (intro to-nat-on-from-nat-into-infinite inf count)
    have inj-t: inj-on t (⋃ i. J i)
```

using *count* **by** (*auto simp*: *t-def*)
then have *inj-t-J*: *inj-on t* (*J i*) **for** *i*
by (*rule inj-on-subset*) *auto*
interpret *IT*: *Ionescu-Tulcea* $\lambda i\ \omega.\ M\ (f\ i)\ \lambda i.\ M\ (f\ i)$
by *standard auto*
interpret *Mf*: *product-prob-space* $\lambda x.\ M\ (f\ x)$ *UNIV*
by *standard*
have *C-eq-PiM*: *IT.C 0 n* (*λ-. undefined*) = *PiM* {*0..<n*} ($\lambda x.\ M\ (f\ x)$) **for** *n*
proof (*induction n*)
case *0* **then show** *?case*
by (*auto simp*: *PiM-empty intro*!: *measure-eqI dest*!: *subset-singletonD*)
next
case (*Suc n*) **then show** *?case*
apply (*auto intro*!: *measure-eqI simp*: *sets-bind*[*OF IT.sets-eP*] *emeasure-bind*[*OF - IT.measurable-eP*])
apply (*auto simp*: *Mf.product-nn-integral-insert nn-integral-indicator*[*symmetric*] *atLeastLessThanSuc IT.emeasure-eP space-PiM*
split: *split-indicator simp del*: *nn-integral-indicator intro*!: *nn-integral-cong*)
done
qed
have *CI-eq-PiM*: *IT.CI X* = *PiM X* ($\lambda x.\ M\ (f\ x)$) **if** *X*: *finite X* **for** *X*
by (*auto simp*: *IT.up-to-less X IT.CI-def C-eq-PiM intro*!: *Mf.distr-restrict*[*symmetric*])

let *?Q* = *distr IT.PF.lim* (*PiM* ($\bigcup i.\ J\ i$) *M*) ($\lambda\omega.\ \lambda x{\in}\bigcup i.\ J\ i.\ \omega\ (t\ x)$)
{ fix *i*
have *distr ?Q* ($Pi_M\ (J\ i)\ M$) ($\lambda x.$ *restrict x* (*J i*)) =
distr IT.PF.lim ($Pi_M\ (J\ i)\ M$) (($\lambda\omega.\ \lambda n{\in}J\ i.\ \omega\ (t\ n)$) ∘ ($\lambda\omega.$ *restrict ω* (*t‘J i*)))
proof (*subst distr-distr*)
have ($\lambda\omega.\ \omega\ (t\ x)$) ∈ *measurable* ($Pi_M$ *UNIV* ($\lambda x.\ M\ (f\ x)$)) (*M x*) **if** *x*: *x* ∈ *J i* **for** *x i*
using *measurable-component-singleton*[*of t x UNIV* $\lambda x.\ M\ (f\ x)$] **unfolding** *ft*[*OF x*] **by** *simp*
then show ($\lambda\omega.\ \lambda x{\in}\bigcup i.\ J\ i.\ \omega\ (t\ x)$) ∈ *measurable IT.PF.lim* (Pi_M ($\bigcup$(*J ‘ UNIV*)) *M*)
by (*auto intro*!: *measurable-restrict simp*: *measurable-cong-sets*[*OF IT.PF.sets-lim refl*])
qed (*auto intro*!: *distr-cong measurable-restrict measurable-component-singleton*)
also have ... = *distr* (*distr IT.PF.lim* (*PiM* (*t‘J i*) ($\lambda x.\ M\ (f\ x)$))) ($\lambda\omega.$ *restrict ω* (*t‘J i*))) ($Pi_M\ (J\ i)\ M$) ($\lambda\omega.\ \lambda n{\in}J\ i.\ \omega\ (t\ n)$)
proof (*intro distr-distr*[*symmetric*])
have ($\lambda\omega.\ \omega\ (t\ x)$) ∈ *measurable* ($Pi_M$ (*t‘J i*) ($\lambda x.\ M\ (f\ x)$)) (*M x*) **if** *x*: *x* ∈ *J i* **for** *x*
using *measurable-component-singleton*[*of t x t‘J i* $\lambda x.\ M\ (f\ x)$] *x* **unfolding** *ft*[*OF x*] **by** *auto*
then show ($\lambda\omega.\ \lambda n{\in}J\ i.\ \omega\ (t\ n)$) ∈ *measurable* ($Pi_M$ (*t ‘ J i*) ($\lambda x.\ M\ (f\ x)$)) ($Pi_M\ (J\ i)\ M$)
by (*auto intro*!: *measurable-restrict*)

qed (*auto intro*!: *measurable-restrict simp*: *measurable-cong-sets*[*OF IT.PF.sets-lim refl*])
also have ... = *distr* (*PiM* (*t*'*J i*) (λ*x*. *M* (*f x*))) (*Pi*$_M$ (*J i*) *M*) (λω. λ*n*∈*J i*. ω (*t n*))
using ‹*finite* (*J i*)› **by** (*subst IT.distr-lim*) (*auto simp*: *CI-eq-PiM*)
also have ... = *Pi*$_M$ (*J i*) *M*
using *Mf.distr-reorder*[*of t J i*] **by** (*simp add*: *1 inj-t-J cong*: *PiM-cong*)
finally have *distr ?Q* (*Pi*$_M$ (*J i*) *M*) (λ*x*. *restrict x* (*J i*)) = *Pi*$_M$ (*J i*) *M* **. }**
then show ∃ *Q*. ∀ *i*. *sets Q* = *PiM* (⋃ *i*. *J i*) *M* ∧ *distr Q* (*Pi*$_M$ (*J i*) *M*) (λ*x*. *restrict x* (*J i*)) = *Pi*$_M$ (*J i*) *M*
by (*intro exI*[*of* - *?Q*]) *auto*
qed
then obtain *Q* **where** *sets-Q*: *sets Q* = *PiM* (⋃ *i*. *J i*) *M*
and *Q*: ⋀*i*. *distr Q* (*PiM* (*J i*) *M*) (λ*x*. *restrict x* (*J i*)) = *Pi*$_M$ (*J i*) *M* **by** *blast*

from *1* **interpret** *Q*: *prob-space Q*
by (*intro prob-space-distrD*[*of* λ*x*. *restrict x* (*J 0*) *Q PiM* (*J 0*) *M*])
(*auto simp*: *Q measurable-cong-sets*[*OF sets-Q*]
intro!: *prob-space-P measurable-restrict measurable-component-singleton*)

have *0* < (*INF i*. *emeasure* (*Pi*$_M$ (*J i*) *M*) (*X i*)) **by** *fact*
also have ... = (*INF i*. *emeasure Q* (*emb* (⋃ *i*. *J i*) (*J i*) (*X i*)))
by (*simp add*: *emeasure-distr-restrict*[*OF* - *sets-Q 1*(*4*), *symmetric*] *SUP-upper Q*)
also have ... = *emeasure Q* (⋂ *i*. *emb* (⋃ *i*. *J i*) (*J i*) (*X i*))
proof (*rule INF-emeasure-decseq*)
from *1* **show** *decseq* (λ*n*. *emb* (⋃ *i*. *J i*) (*J n*) (*X n*))
by (*intro antimonoI emb-preserve-mono*[**where** *X*=*emb* (⋃ *i*. *J i*) (*J n*) (*X n*) **and** *L*=*I* **and** *J*=⋃ *i*. *J i* **for** *n*]
measurable-prod-emb)
(*auto simp*: *SUP-least SUP-upper antimono-def*)
qed (*insert 1*, *auto simp*: *sets-Q*)
finally have (⋂ *i*. *emb* (⋃ *i*. *J i*) (*J i*) (*X i*)) ≠ {}
by *auto*
moreover have (⋂ *i*. *emb I* (*J i*) (*X i*)) = {} ⟹ (⋂ *i*. *emb* (⋃ *i*. *J i*) (*J i*) (*X i*)) = {}
using *1* **by** (*intro emb-injective*[*of* ⋃ *i*. *J i I* - {}] *sets.countable-INT*) (*auto simp*: *SUP-least SUP-upper*)
ultimately show *?case* **by** *auto*
qed

end

9 Infinite Product Measure

theory *Infinite-Product-Measure*
imports *Probability-Measure Projective-Family*
begin

lemma (**in** *product-prob-space*) *distr-PiM-restrict-finite*:
assumes *finite J* $J \subseteq I$
shows *distr* (*PiM I M*) (*PiM J M*) (λx. *restrict x J*) = *PiM J M*
proof (*rule PiM-eqI*)
fix *X* **assume** *X*: $\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
{ fix *J X* **assume** *J*: $J \neq \{\} \vee I = \{\}$ *finite J* $J \subseteq I$ **and** *X*: $\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
have *emeasure* (*PiM I M*) (*emb I J* (Pi_E *J X*)) = ($\prod i \in J.\ M\ i\ (X\ i)$)
proof (*subst emeasure-extend-measure-Pair*[*OF PiM-def*, **where** μ'=*lim*], *goal-cases*)
case *1* **then show** *?case*
by (*simp add*: *M.emeasure-space-1 emeasure-PiM Pi-iff sets-PiM-I-finite emeasure-lim-emb*)
next
case (*2 J X*)
then have *emb I J* (Pi_E *J X*) ∈ *sets* (*PiM I M*)
by (*intro measurable-prod-emb sets-PiM-I-finite*) *auto*
from *this*[*THEN sets.sets-into-space*] **show** *?case*
by (*simp add*: *space-PiM*)
qed (*insert assms J X*, *simp-all del*: *sets-lim*
add: *M.emeasure-space-1 sets-lim*[*symmetric*] *emeasure-countably-additive emeasure-positive*) **}**
note $* = this$

have *emeasure* (*PiM I M*) (*emb I J* (Pi_E *J X*)) = ($\prod i \in J.\ M\ i\ (X\ i)$)
proof (*cases* $J \neq \{\} \vee I = \{\}$)
case *False*
then obtain *i* **where** *i*: $J = \{\}$ $i \in I$ **by** *auto*
then have *emb I* $\{\}$ $\{\lambda x.\ undefined\}$ = *emb I* $\{i\}$ ($\Pi_E\ i \in \{i\}.\ space\ (M\ i)$)
by (*auto simp*: *space-PiM prod-emb-def*)
with *i* **show** *?thesis*
by (*simp add*: $*$ *M.emeasure-space-1*)
next
case *True*
then show *?thesis*
by (*simp add*: $*$[*OF - assms X*])
qed
with *assms* **show** *emeasure* (*distr* (Pi_M *I M*) (Pi_M *J M*) ($\lambda x.$ *restrict x J*)) (Pi_E *J X*) = ($\prod i \in J.$ *emeasure* (*M i*) (*X i*))
by (*subst emeasure-distr-restrict*[*OF - refl*]) (*auto intro*!: *sets-PiM-I-finite X*)
qed (*insert assms*, *auto*)

lemma (**in** *product-prob-space*) *emeasure-PiM-emb'*:
$J \subseteq I \Longrightarrow$ *finite J* $\Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow$ *emeasure* (Pi_M *I M*) (*emb I J X*) = *PiM J M X*
by (*subst distr-PiM-restrict-finite*[*symmetric*, *of J*])
(*auto intro*!: *emeasure-distr-restrict*[*symmetric*])

lemma (**in** *product-prob-space*) *emeasure-PiM-emb*:

$J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in$ *sets* $(M\ i)) \Longrightarrow$
emeasure $(Pi_M\ I\ M)$ $(emb\ I\ J\ (Pi_E\ J\ X)) = (\prod\ i{\in}J.$ *emeasure* $(M\ i)\ (X\ i))$
by (*subst emeasure-PiM-emb'*) (*auto intro*!: *emeasure-PiM*)

sublocale *product-prob-space* $\subseteq$ *P?*: *prob-space* $Pi_M\ I\ M$
proof
have $*$: *emb* $I\ \{\}\ \{\lambda x.\ undefined\} =$ *space* $(PiM\ I\ M)$
by (*auto simp*: *prod-emb-def space-PiM*)
show *emeasure* $(Pi_M\ I\ M)$ $(space\ (Pi_M\ I\ M)) = 1$
using *emeasure-PiM-emb*[*of* $\{\}$ λ-. $\{\}$] **by** (*simp add*: $*$)
qed

lemma *prob-space-PiM*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$ **shows** *prob-space* $(PiM\ I\ M)$
proof −
let *?M* $= \lambda i.$ *if* $i \in I$ *then* $M\ i$ *else count-space* $\{undefined\}$
interpret *M'*: *prob-space ?M i* **for** *i*
using *M* **by** (*cases* $i \in I$) (*auto intro*!: *prob-spaceI*)
interpret *product-prob-space ?M I*
by *unfold-locales*
have *prob-space* $(\Pi_M\ i{\in}I.$ *?M i*)
by *unfold-locales*
also have $(\Pi_M\ i{\in}I.$ *?M i*$) = (\Pi_M\ i{\in}I.\ M\ i)$
by (*intro PiM-cong*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect*:
assumes *X*: $J \subseteq I$ *finite* J $\bigwedge i.\ i \in J \Longrightarrow X\ i \in$ *sets* $(M\ i)$
shows *emeasure* $(Pi_M\ I\ M)$ $\{x{\in}space\ (Pi_M\ I\ M).\ \forall\, i{\in}J.\ x\ i \in X\ i\} = (\prod\ i{\in}J.$ *emeasure* $(M\ i)\ (X\ i))$
proof −
have $\{x{\in}space\ (Pi_M\ I\ M).\ \forall\, i{\in}J.\ x\ i \in X\ i\} = emb\ I\ J\ (Pi_E\ J\ X)$
unfolding *prod-emb-def* **using** *assms* **by** (*auto simp*: *space-PiM Pi-iff*)
with *emeasure-PiM-emb*[*OF assms*] **show** *?thesis* **by** *simp*
qed

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect-single*:
assumes *X*: $i \in I$ $A \in$ *sets* $(M\ i)$
shows *emeasure* $(Pi_M\ I\ M)$ $\{x{\in}space\ (Pi_M\ I\ M).\ x\ i \in A\} =$ *emeasure* $(M\ i)\ A$
using *emeasure-PiM-Collect*[*of* $\{i\}$ $\lambda i.\ A$] *assms*
by *simp*

lemma (**in** *product-prob-space*) *measure-PiM-emb*:
assumes $J \subseteq I$ *finite* J $\bigwedge i.\ i \in J \Longrightarrow X\ i \in$ *sets* $(M\ i)$
shows *measure* $(PiM\ I\ M)$ $(emb\ I\ J\ (Pi_E\ J\ X)) = (\prod\ i{\in}J.$ *measure* $(M\ i)\ (X\ i))$
using *emeasure-PiM-emb*[*OF assms*]
unfolding *emeasure-eq-measure M.emeasure-eq-measure*

by (*simp add*: *prod-ennreal measure-nonneg prod-nonneg*)

lemma *sets-Collect-single'*:
$i \in I \Longrightarrow \{x \in space\ (M\ i).\ P\ x\} \in sets\ (M\ i) \Longrightarrow \{x \in space\ (PiM\ I\ M).\ P\ (x\ i)\} \in sets\ (PiM\ I\ M)$
by *auto*

lemma (**in** *finite-product-prob-space*) *finite-measure-PiM-emb*:
$(\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow measure\ (PiM\ I\ M)\ (Pi_E\ I\ A) = (\prod i \in I.\ measure\ (M\ i)\ (A\ i))$
by (*rule prob-times*)

lemma (**in** *product-prob-space*) *PiM-component*:
assumes $i \in I$
shows $distr\ (PiM\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
proof (*rule measure-eqI*[*symmetric*])
fix A **assume** $A \in sets\ (M\ i)$
moreover have $((\lambda\omega.\ \omega\ i) -` A \cap space\ (PiM\ I\ M)) = \{x \in space\ (PiM\ I\ M).\ x\ i \in A\}$
by *auto*
ultimately show $emeasure\ (M\ i)\ A = emeasure\ (distr\ (PiM\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i))\ A$
by (*auto simp*: ‹*i*∈*I*› *emeasure-distr measurable-component-singleton emeasure-PiM-Collect-single*)
qed *simp*

lemma (**in** *product-prob-space*) *PiM-eq*:
assumes *M'*: $sets\ M' = sets\ (PiM\ I\ M)$
assumes *eq*: $\bigwedge J\ F.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge j.\ j \in J \Longrightarrow F\ j \in sets\ (M\ j)) \Longrightarrow emeasure\ M'\ (prod\text{-}emb\ I\ M\ J\ (\Pi_E\ j \in J.\ F\ j)) = (\prod j \in J.\ emeasure\ (M\ j)\ (F\ j))$
shows $M' = (PiM\ I\ M)$
proof (*rule measure-eqI-PiM-infinite*[*symmetric*, *OF refl M'*])
show $finite\text{-}measure\ (Pi_M\ I\ M)$
by *standard* (*simp add*: *P.emeasure-space-1*)
qed (*simp add*: *eq emeasure-PiM-emb*)

lemma (**in** *product-prob-space*) *AE-component*: $i \in I \Longrightarrow AE\ x\ in\ M\ i.\ P\ x \Longrightarrow AE\ x\ in\ PiM\ I\ M.\ P\ (x\ i)$
apply (*rule AE-distrD*[*of* $\lambda\omega.\ \omega\ i\ PiM\ I\ M\ M\ i\ P$])
apply *simp*
apply (*subst PiM-component*)
apply *simp-all*
done

lemma *emeasure-PiM-emb*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
assumes *J*: $J \subseteq I\ finite\ J$ **and** *A*: $\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)$
shows $emeasure\ (Pi_M\ I\ M)\ (prod\text{-}emb\ I\ M\ J\ (Pi_E\ J\ A)) = (\prod i \in J.\ emeasure$

(*M i*) (*A i*))
proof −
let *?M* = λ*i*. *if i* ∈ *I then M i else count-space* {*undefined*}
interpret *M'*: *prob-space ?M i* **for** *i*
using *M* **by** (*cases i* ∈ *I*) (*auto intro*!: *prob-spaceI*)
interpret *P*: *product-prob-space ?M I*
by *unfold-locales*
have *emeasure* (*Pi*$_M$ *I M*) (*prod-emb I M J* (*Pi*$_E$ *J A*)) = *emeasure* (*Pi*$_M$ *I ?M*) (*P.emb I J* (*Pi*$_E$ *J A*))
by (*auto simp*: *prod-emb-def PiE-iff intro*!: *arg-cong2*[**where** *f*=*emeasure*] *PiM-cong*)
also have ... = (∏ *i*∈*J*. *emeasure* (*M i*) (*A i*))
using *J A* **by** (*subst P.emeasure-PiM-emb*[*OF J*]) (*auto intro*!: *prod.cong*)
finally show *?thesis* **.**
qed

lemma *distr-pair-PiM-eq-PiM*:
fixes *i'* :: *'i* **and** *I* :: *'i set* **and** *M* :: *'i* ⇒ *'a measure*
assumes *M*: ⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M i*) *prob-space* (*M i'*)
shows *distr* (*M i'* ⨂$_M$ (Π$_M$ *i*∈*I*. *M i*)) (Π$_M$ *i*∈*insert i' I*. *M i*) (λ(*x*, *X*). *X*(*i'* := *x*)) =
(Π$_M$ *i*∈*insert i' I*. *M i*) (**is** *?L* = -)
proof (*rule measure-eqI-PiM-infinite*[*symmetric*, *OF refl*])
interpret *M'*: *prob-space M i'* **by** *fact*
interpret *I*: *prob-space* (Π$_M$ *i*∈*I*. *M i*)
using *M* **by** (*intro prob-space-PiM*) *auto*
interpret *I'*: *prob-space* (Π$_M$ *i*∈*insert i' I*. *M i*)
using *M* **by** (*intro prob-space-PiM*) *auto*
show *finite-measure* (Π$_M$ *i*∈*insert i' I*. *M i*)
by *unfold-locales*
fix *J A* **assume** *J*: *finite J J* ⊆ *insert i' I* **and** *A*: ⋀*i*. *i* ∈ *J* ⟹ *A i* ∈ *sets* (*M i*)
let *?X* = *prod-emb* (*insert i' I*) *M J* (*Pi*$_E$ *J A*)
have *Pi*$_M$ (*insert i' I*) *M ?X* = (∏ *i*∈*J*. *M i* (*A i*))
using *M J A* **by** (*intro emeasure-PiM-emb*) *auto*
also have ... = *M i'* (*if i'* ∈ *J then* (*A i'*) *else space* (*M i'*)) ∗ (∏ *i*∈*J*−{*i'*}. *M i* (*A i*))
using *prod.insert-remove*[*of J* λ*i*. *M i* (*A i*) *i'*] *J M'.emeasure-space-1*
by (*cases i'* ∈ *J*) (*auto simp*: *insert-absorb*)
also have (∏ *i*∈*J*−{*i'*}. *M i* (*A i*)) = *Pi*$_M$ *I M* (*prod-emb I M* (*J* − {*i'*}) (*Pi*$_E$ (*J* − {*i'*}) *A*))
using *M J A* **by** (*intro emeasure-PiM-emb*[*symmetric*]) *auto*
also have *M i'* (*if i'* ∈ *J then* (*A i'*) *else space* (*M i'*)) ∗ ... =
(*M i'* ⨂$_M$ *Pi*$_M$ *I M*) ((*if i'* ∈ *J then* (*A i'*) *else space* (*M i'*)) × *prod-emb I M* (*J* − {*i'*}) (*Pi*$_E$ (*J* − {*i'*}) *A*))
using *J A* **by** (*intro I.emeasure-pair-measure-Times*[*symmetric*] *sets-PiM-I*) *auto*
also have ((*if i'* ∈ *J then* (*A i'*) *else space* (*M i'*)) × *prod-emb I M* (*J* − {*i'*}) (*Pi*$_E$ (*J* − {*i'*}) *A*)) =

$(\lambda(x,\ X).\ X(i' := x))\ -`\ ?X \cap space\ (M\ i' \bigotimes_M\ Pi_M\ I\ M)$
using *A*[*of i'*, *THEN sets.sets-into-space*] **unfolding** *set-eq-iff*
by (*simp add*: *prod-emb-def space-pair-measure space-PiM PiE-fun-upd ac-simps cong*: *conj-cong*)
(*auto simp add*: *Pi-iff Ball-def all-conj-distrib*)
finally show $Pi_M\ (insert\ i'\ I)\ M\ ?X = ?L\ ?X$
using *J A* **by** (*simp add*: *emeasure-distr*)
qed *simp*

lemma *distr-PiM-reindex*:
assumes *M*: $\bigwedge i.\ i \in K \Longrightarrow prob\text{-}space\ (M\ i)$
assumes *f*: $inj\text{-}on\ f\ I\ f \in I \to K$
shows $distr\ (Pi_M\ K\ M)\ (\Pi_M\ i{\in}I.\ M\ (f\ i))\ (\lambda\omega.\ \lambda n{\in}I.\ \omega\ (f\ n)) = (\Pi_M\ i{\in}I.\ M\ (f\ i))$
(**is** *distr ?K ?I ?t = ?I*)
proof (*rule measure-eqI-PiM-infinite*[*symmetric*, *OF refl*])
interpret *prob-space ?I*
using *f M* **by** (*intro prob-space-PiM*) *auto*
show *finite-measure ?I*
by *unfold-locales*
fix *A J* **assume** *J*: *finite J* $J \subseteq I$ **and** *A*: $\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ (f\ i))$
have [*simp*]: $i \in J \Longrightarrow the\text{-}inv\text{-}into\ I\ f\ (f\ i) = i$ **for** *i*
using *J f* **by** (*intro the-inv-into-f-f*) *auto*
have *?I* $(prod\text{-}emb\ I\ (\lambda i.\ M\ (f\ i))\ J\ (Pi_E\ J\ A)) = (\prod j{\in}J.\ M\ (f\ j)\ (A\ j))$
using *f J A* **by** (*intro emeasure-PiM-emb M*) *auto*
also have $\ldots = (\prod j{\in}f`J.\ M\ j\ (A\ (the\text{-}inv\text{-}into\ I\ f\ j)))$
using *f J* **by** (*subst prod.reindex*) (*auto intro*!: *prod.cong intro*: *inj-on-subset simp*: *the-inv-into-f-f*)
also have $\ldots = ?K\ (prod\text{-}emb\ K\ M\ (f`J)\ (\Pi_E\ j{\in}f`J.\ A\ (the\text{-}inv\text{-}into\ I\ f\ j)))$
using *f J A* **by** (*intro emeasure-PiM-emb*[*symmetric*] *M*) (*auto simp*: *the-inv-into-f-f*)
also have $prod\text{-}emb\ K\ M\ (f`J)\ (\Pi_E\ j{\in}f`J.\ A\ (the\text{-}inv\text{-}into\ I\ f\ j)) = ?t\ -`\ prod\text{-}emb\ I\ (\lambda i.\ M\ (f\ i))\ J\ (Pi_E\ J\ A) \cap space\ ?K$
using *f J A* **by** (*auto simp*: *prod-emb-def space-PiM Pi-iff PiE-iff Int-absorb1*)
also have $?K\ \ldots = distr\ ?K\ ?I\ ?t\ (prod\text{-}emb\ I\ (\lambda i.\ M\ (f\ i))\ J\ (Pi_E\ J\ A))$
using *f J A* **by** (*intro emeasure-distr*[*symmetric*] *sets-PiM-I*) (*auto simp*: *Pi-iff*)
finally show $?I\ (prod\text{-}emb\ I\ (\lambda i.\ M\ (f\ i))\ J\ (Pi_E\ J\ A)) = distr\ ?K\ ?I\ ?t\ (prod\text{-}emb\ I\ (\lambda i.\ M\ (f\ i))\ J\ (Pi_E\ J\ A))$ **.**
qed *simp*

lemma *distr-PiM-component*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
assumes $i \in I$
shows $distr\ (Pi_M\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
proof −
have $*$: $(\lambda\omega.\ \omega\ i)\ -`\ A \cap space\ (Pi_M\ I\ M) = prod\text{-}emb\ I\ M\ \{i\}\ (\Pi_E\ i'{\in}\{i\}.\ A)$ **for** *A*
by (*auto simp*: *prod-emb-def space-PiM*)
show *?thesis*
apply (*intro measure-eqI*)

apply (*auto simp add*: *emeasure-distr* ‹*i*∈*I*› ∗ *emeasure-PiM-emb M*)
apply (*subst emeasure-PiM-emb*)
apply (*simp-all add*: *M* ‹*i*∈*I*›)
done
qed

lemma *AE-PiM-component*:
(⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M i*)) ⟹ *i* ∈ *I* ⟹ *AE x in M i*. *P x* ⟹ *AE x in PiM I M*. *P* (*x i*)
using *AE-distrD*[*of* λ*x*. *x i PiM I M M i*]
by (*subst* (*asm*) *distr-PiM-component*[*of I - i*]) (*auto intro*: *AE-distrD*[*of* λ*x*. *x i - - P*])

lemma *decseq-emb-PiE*:
incseq J ⟹ *decseq* (λ*i*. *prod-emb I M* (*J i*) (Π$_E$ *j*∈*J i*. *X j*))
by (*fastforce simp*: *decseq-def prod-emb-def incseq-def Pi-iff*)

9.1 Sequence space

definition *comb-seq* :: *nat* ⇒ (*nat* ⇒ *'a*) ⇒ (*nat* ⇒ *'a*) ⇒ (*nat* ⇒ *'a*) **where**
comb-seq i ω ω' j = (*if j* < *i then ω j else ω'* (*j* − *i*))

lemma *split-comb-seq*: *P* (*comb-seq i ω ω' j*) ⟷ (*j* < *i* ⟶ *P* (*ω j*)) ∧ (∀ *k*. *j* = *i* + *k* ⟶ *P* (*ω' k*))
by (*auto simp*: *comb-seq-def not-less*)

lemma *split-comb-seq-asm*: *P* (*comb-seq i ω ω' j*) ⟷ ¬ ((*j* < *i* ∧ ¬ *P* (*ω j*)) ∨ (∃ *k*. *j* = *i* + *k* ∧ ¬ *P* (*ω' k*)))
by (*auto simp*: *comb-seq-def*)

lemma *measurable-comb-seq*:
(λ(*ω*, *ω'*). *comb-seq i ω ω'*) ∈ *measurable* ((Π$_M$ *i*∈*UNIV*. *M*) ⨂$_M$ (Π$_M$ *i*∈*UNIV*. *M*)) (Π$_M$ *i*∈*UNIV*. *M*)
proof (*rule measurable-PiM-single*)
show (λ(*ω*, *ω'*). *comb-seq i ω ω'*) ∈ *space* ((Π$_M$ *i*∈*UNIV*. *M*) ⨂$_M$ (Π$_M$ *i*∈*UNIV*. *M*)) → (*UNIV* →$_E$ *space M*)
by (*auto simp*: *space-pair-measure space-PiM PiE-iff split*: *split-comb-seq*)
fix *j* :: *nat* **and** *A* **assume** *A*: *A* ∈ *sets M*
then have ∗: {*ω* ∈ *space* ((Π$_M$ *i*∈*UNIV*. *M*) ⨂$_M$ (Π$_M$ *i*∈*UNIV*. *M*)). *case-prod* (*comb-seq i*) *ω j* ∈ *A*} =
(*if j* < *i then* {*ω* ∈ *space* (Π$_M$ *i*∈*UNIV*. *M*). *ω j* ∈ *A*} × *space* (Π$_M$ *i*∈*UNIV*. *M*)
else space (Π$_M$ *i*∈*UNIV*. *M*) × {*ω* ∈ *space* (Π$_M$ *i*∈*UNIV*. *M*). *ω* (*j* − *i*) ∈ *A*})
by (*auto simp*: *space-PiM space-pair-measure comb-seq-def dest*: *sets.sets-into-space*)
show {*ω* ∈ *space* ((Π$_M$ *i*∈*UNIV*. *M*) ⨂$_M$ (Π$_M$ *i*∈*UNIV*. *M*)). *case-prod* (*comb-seq i*) *ω j* ∈ *A*} ∈ *sets* ((Π$_M$ *i*∈*UNIV*. *M*) ⨂$_M$ (Π$_M$ *i*∈*UNIV*. *M*))
unfolding ∗ **by** (*auto simp*: *A intro*!: *sets-Collect-single*)
qed

lemma *measurable-comb-seq'*[*measurable* (*raw*)]:
assumes *f*: $f \in$ *measurable* N (Π_M $i \in$ *UNIV*. M) **and** *g*: $g \in$ *measurable* N (Π_M $i \in$ *UNIV*. M)
shows (λx. *comb-seq* i (f x) (g x)) $\in$ *measurable* N (Π_M $i \in$ *UNIV*. M)
using *measurable-compose*[*OF measurable-Pair*[*OF f g*] *measurable-comb-seq*] **by** *simp*

lemma *comb-seq-0*: *comb-seq* 0 ω ω' = ω'
by (*auto simp add*: *comb-seq-def*)

lemma *comb-seq-Suc*: *comb-seq* (*Suc* n) ω ω' = *comb-seq* n ω (*case-nat* (ω n) ω')
by (*auto simp add*: *comb-seq-def not-less less-Suc-eq le-imp-diff-is-add intro*!: *ext split*: *nat.split*)

lemma *comb-seq-Suc-0*[*simp*]: *comb-seq* (*Suc* 0) ω = *case-nat* (ω 0)
by (*intro ext*) (*simp add*: *comb-seq-Suc comb-seq-0*)

lemma *comb-seq-less*: $i < n \Longrightarrow$ *comb-seq* n ω ω' i = ω i
by (*auto split*: *split-comb-seq*)

lemma *comb-seq-add*: *comb-seq* n ω ω' (i + n) = ω' i
by (*auto split*: *nat.split split-comb-seq*)

lemma *case-nat-comb-seq*: *case-nat* s' (*comb-seq* n ω ω') (i + n) = *case-nat* (*case-nat* s' ω n) ω' i
by (*auto split*: *nat.split split-comb-seq*)

lemma *case-nat-comb-seq'*:
case-nat s (*comb-seq* i ω ω') = *comb-seq* (*Suc* i) (*case-nat* s ω) ω'
by (*auto split*: *split-comb-seq nat.split*)

locale *sequence-space* = *product-prob-space* λi. M *UNIV* :: *nat set* **for** M
begin

abbreviation $S \equiv \Pi_M$ $i \in$ *UNIV*::*nat set*. M

lemma *infprod-in-sets*[*intro*]:
fixes E :: *nat* $\Rightarrow$ *'a set* **assumes** E: $\bigwedge i$. E $i \in$ *sets* M
shows *Pi UNIV E* $\in$ *sets S*
proof −
have *Pi UNIV E* = ($\bigcap i$. *emb UNIV* {..i} (Π_E $j \in$ {..i}. E j))
using *E E*[*THEN sets.sets-into-space*]
by (*auto simp*: *prod-emb-def Pi-iff extensional-def*)
with *E* **show** *?thesis* **by** *auto*
qed

lemma *measure-PiM-countable*:
fixes E :: *nat* $\Rightarrow$ *'a set* **assumes** E: $\bigwedge i$. E $i \in$ *sets* M

shows $(\lambda n.\ \prod i{\leq}n.\ measure\ M\ (E\ i)) \longrightarrow measure\ S\ (Pi\ UNIV\ E)$
proof −
let *?E* = $\lambda n.\ emb\ UNIV\ \{..n\}\ (Pi_E\ \{..\ n\}\ E)$
have $\bigwedge n.\ (\prod i{\leq}n.\ measure\ M\ (E\ i)) = measure\ S\ (?E\ n)$
using *E* **by** (*simp add*: *measure-PiM-emb*)
moreover have $Pi\ UNIV\ E = (\bigcap n.\ ?E\ n)$
using *E E*[*THEN sets.sets-into-space*]
by (*auto simp*: *prod-emb-def extensional-def Pi-iff*)
moreover have *range ?E* ⊆ *sets S*
using *E* **by** *auto*
moreover have *decseq ?E*
by (*auto simp*: *prod-emb-def Pi-iff decseq-def*)
ultimately show *?thesis*
by (*simp add*: *finite-Lim-measure-decseq*)
qed

lemma *nat-eq-diff-eq*:
fixes *a b c* :: *nat*
shows $c \leq b \Longrightarrow a = b - c \longleftrightarrow a + c = b$
by *auto*

lemma *PiM-comb-seq*:
$distr\ (S \bigotimes_M S)\ S\ (\lambda(\omega, \omega').\ comb\text{-}seq\ i\ \omega\ \omega') = S$ (**is** *?D* = -)
proof (*rule PiM-eq*)
let *?I* = *UNIV*::*nat set* **and** *?M* = $\lambda n.\ M$
let *distr - - ?f* = *?D*

fix *J E* **assume** *J*: *finite J J* ⊆ *?I* $\bigwedge j.\ j \in J \Longrightarrow E\ j \in sets\ M$
let *?X* = *prod-emb ?I ?M J* $(\Pi_E\ j{\in}J.\ E\ j)$
have $\bigwedge j\ x.\ j \in J \Longrightarrow x \in E\ j \Longrightarrow x \in space\ M$
using *J*(*3*)[*THEN sets.sets-into-space*] **by** (*auto simp*: *space-PiM Pi-iff subset-eq*)
with *J* **have** *?f* −‘ *?X* ∩ *space* $(S \bigotimes_M S) =$
$(prod\text{-}emb\ ?I\ ?M\ (J \cap \{..<i\})\ (\Pi_E\ j{\in}J \cap \{..<i\}.\ E\ j)) \times$
$(prod\text{-}emb\ ?I\ ?M\ (((+)\ i)\ -‘\ J)\ (\Pi_E\ j{\in}((+)\ i)\ -‘\ J.\ E\ (i + j)))$ (**is** - = *?E* × *?F*)
by (*auto simp*: *space-pair-measure space-PiM prod-emb-def all-conj-distrib PiE-iff split*: *split-comb-seq split-comb-seq-asm*)
then have *emeasure ?D ?X* = *emeasure* $(S \bigotimes_M S)$ (*?E* × *?F*)
by (*subst emeasure-distr*[*OF measurable-comb-seq*])
(*auto intro*!: *sets-PiM-I simp*: *split-beta' J*)
also have ... = *emeasure S ?E* ∗ *emeasure S ?F*
using *J* **by** (*intro P.emeasure-pair-measure-Times*) (*auto intro*!: *sets-PiM-I finite-vimageI simp*: *inj-on-def*)
also have *emeasure S ?F* = $(\prod j{\in}((+)\ i)\ -‘\ J.\ emeasure\ M\ (E\ (i + j)))$
using *J* **by** (*intro emeasure-PiM-emb*) (*simp-all add*: *finite-vimageI inj-on-def*)
also have ... = $(\prod j{\in}J - (J \cap \{..<i\}).\ emeasure\ M\ (E\ j))$
by (*rule prod.reindex-cong* [*of* $\lambda x.\ x - i$])
(*auto simp*: *image-iff ac-simps nat-eq-diff-eq cong*: *conj-cong intro*!: *inj-onI*)

also have *emeasure S ?E* = (∏ *j*∈*J* ∩ {..<*i*}. *emeasure M* (*E j*))
using *J* **by** (*intro emeasure-PiM-emb*) *simp-all*
also have (∏ *j*∈*J* ∩ {..<*i*}. *emeasure M* (*E j*)) * (∏ *j*∈*J* − (*J* ∩ {..<*i*}). *emeasure M* (*E j*)) = (∏ *j*∈*J*. *emeasure M* (*E j*))
by (*subst mult.commute*) (*auto simp*: *J prod.subset-diff*[*symmetric*])
finally show *emeasure ?D ?X* = (∏ *j*∈*J*. *emeasure M* (*E j*)) .
qed *simp-all*

lemma *PiM-iter*:
distr (*M* ⨂$_M$ *S*) *S* (λ(*s*, *ω*). *case-nat s ω*) = *S* (**is** *?D* = -)
proof (*rule PiM-eq*)
let *?I* = *UNIV*::*nat set* **and** *?M* = λ*n*. *M*
let *distr* - - *?f* = *?D*

fix *J E* **assume** *J*: *finite J J* ⊆ *?I* ⋀*j*. *j* ∈ *J* ⟹ *E j* ∈ *sets M*
let *?X* = *prod-emb ?I ?M J* (Π$_E$ *j*∈*J*. *E j*)
have ⋀*j x*. *j* ∈ *J* ⟹ *x* ∈ *E j* ⟹ *x* ∈ *space M*
using *J*(*3*)[*THEN sets.sets-into-space*] **by** (*auto simp*: *space-PiM Pi-iff subset-eq*)
with *J* **have** *?f* −‘ *?X* ∩ *space* (*M* ⨂$_M$ *S*) = (*if 0* ∈ *J then E 0 else space M*) ×
(*prod-emb ?I ?M* (*Suc* −‘ *J*) (Π$_E$ *j*∈*Suc* −‘ *J*. *E* (*Suc j*))) (**is** - = *?E* × *?F*)
by (*auto simp*: *space-pair-measure space-PiM PiE-iff prod-emb-def all-conj-distrib split*: *nat.split nat.split-asm*)
then have *emeasure ?D ?X* = *emeasure* (*M* ⨂$_M$ *S*) (*?E* × *?F*)
by (*subst emeasure-distr*)
(*auto intro*!: *sets-PiM-I simp*: *split-beta*′ *J*)
also have ... = *emeasure M ?E* * *emeasure S ?F*
using *J* **by** (*intro P.emeasure-pair-measure-Times*) (*auto intro*!: *sets-PiM-I finite-vimageI*)
also have *emeasure S ?F* = (∏ *j*∈*Suc* −‘ *J*. *emeasure M* (*E* (*Suc j*)))
using *J* **by** (*intro emeasure-PiM-emb*) (*simp-all add*: *finite-vimageI*)
also have ... = (∏ *j*∈*J* − {*0*}. *emeasure M* (*E j*))
by (*rule prod.reindex-cong* [*of* λ*x*. *x* − *1*])
(*auto simp*: *image-iff nat-eq-diff-eq ac-simps cong*: *conj-cong intro*!: *inj-onI*)
also have *emeasure M ?E* * (∏ *j*∈*J* − {*0*}. *emeasure M* (*E j*)) = (∏ *j*∈*J*. *emeasure M* (*E j*))
by (*auto simp*: *M.emeasure-space-1 prod.remove J*)
finally show *emeasure ?D ?X* = (∏ *j*∈*J*. *emeasure M* (*E j*)) .
qed *simp-all*

end

lemma *PiM-return*:
assumes *finite I*
assumes [*measurable*]: ⋀*i*. *i* ∈ *I* ⟹ {*a i*} ∈ *sets* (*M i*)
shows *PiM I* (λ*i*. *return* (*M i*) (*a i*)) = *return* (*PiM I M*) (*restrict a I*)
proof −
have [*simp*]: *a i* ∈ *space* (*M i*) **if** *i* ∈ *I* **for** *i*

using *assms(2)*[*OF that*] **by** (*meson insert-subset sets.sets-into-space*)
interpret *prob-space PiM I* (λ*i*. *return* (*M i*) (*a i*))
by (*intro prob-space-PiM prob-space-return*) *auto*
have *AE x in PiM I* (λ*i*. *return* (*M i*) (*a i*)). ∀ *i*∈*I*. *x i* = *restrict a I i*
by (*intro eventually-ball-finite ballI AE-PiM-component prob-space-return assms*)
(*auto simp*: *AE-return*)
moreover have *AE x in PiM I* (λ*i*. *return* (*M i*) (*a i*)). *x* ∈ *space* (*PiM I* (λ*i*. *return* (*M i*) (*a i*)))
by *simp*
ultimately have *AE x in PiM I* (λ*i*. *return* (*M i*) (*a i*)). *x* = *restrict a I*
by *eventually-elim* (*auto simp*: *fun-eq-iff space-PiM*)
hence Pi_M *I* (λ*i*. *return* (*M i*) (*a i*)) = *return* (Pi_M *I* (λ*i*. *return* (*M i*) (*a i*))) (*restrict a I*)
by (*rule AE-eq-constD*)
also have ... = *return* (*PiM I M*) (*restrict a I*)
by (*intro return-cong sets-PiM-cong*) *auto*
finally show *?thesis* **.**
qed

lemma *distr-PiM-finite-prob-space'*:
assumes *fin*: *finite I*
assumes ⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M i*)
assumes ⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M' i*)
assumes [*measurable*]: ⋀*i*. *i* ∈ *I* ⟹ *f* ∈ *measurable* (*M i*) (*M' i*)
shows *distr* (*PiM I M*) (*PiM I M'*) (*compose I f*) = *PiM I* (λ*i*. *distr* (*M i*) (*M' i*) *f*)
proof −
define *N* **where** *N* = (λ*i*. *if i* ∈ *I then M i else return* (*count-space UNIV*) *undefined*)
define *N'* **where** *N'* = (λ*i*. *if i* ∈ *I then M' i else return* (*count-space UNIV*) *undefined*)
have [*simp*]: *PiM I N* = *PiM I M PiM I N'* = *PiM I M'*
by (*intro PiM-cong*; *simp add*: *N-def N'-def*)+

have *distr* (*PiM I N*) (*PiM I N'*) (*compose I f*) = *PiM I* (λ*i*. *distr* (*N i*) (*N' i*) *f*)
proof (*rule distr-PiM-finite-prob-space*)
show *product-prob-space N*
by (*rule product-prob-spaceI*) (*auto simp*: *N-def intro*!: *prob-space-return assms*)
show *product-prob-space N'*
by (*rule product-prob-spaceI*) (*auto simp*: *N'-def intro*!: *prob-space-return assms*)
qed (*auto simp*: *N-def N'-def fin*)
also have Pi_M *I* (λ*i*. *distr* (*N i*) (*N' i*) *f*) = Pi_M *I* (λ*i*. *distr* (*M i*) (*M' i*) *f*)
by (*intro PiM-cong*) (*simp-all add*: *N-def N'-def*)
finally show *?thesis* **by** *simp*
qed

end

10 Independent families of events, event sets, and random variables

theory *Independent-Family*
imports *Infinite-Product-Measure*
begin

definition (**in** *prob-space*)
indep-sets F I $\longleftrightarrow$ $(\forall\, i{\in}I.\ F\ i \subseteq \mathit{events}) \wedge$
$(\forall\, J{\subseteq}I.\ J \neq \{\} \longrightarrow \mathit{finite}\ J \longrightarrow (\forall\, A{\in}\mathit{Pi}\ J\ F.\ \mathit{prob}\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ \mathit{prob}\ (A\ j))))$

definition (**in** *prob-space*)
indep-set A B $\longleftrightarrow$ *indep-sets* (*case-bool A B*) *UNIV*

definition (**in** *prob-space*)
indep-events-def-alt: *indep-events A I* $\longleftrightarrow$ *indep-sets* $(\lambda i.\ \{A\ i\})\ I$

lemma (**in** *prob-space*) *indep-events-def*:
indep-events A I $\longleftrightarrow$ $(A`I \subseteq \mathit{events}) \wedge$
$(\forall\, J{\subseteq}I.\ J \neq \{\} \longrightarrow \mathit{finite}\ J \longrightarrow \mathit{prob}\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ \mathit{prob}\ (A\ j)))$
unfolding *indep-events-def-alt indep-sets-def*
apply (*simp add*: *Ball-def Pi-iff image-subset-iff-funcset*)
apply (*intro conj-cong refl arg-cong*[**where** *f*=*All*] *ext imp-cong*)
apply *auto*
done

lemma (**in** *prob-space*) *indep-eventsI*:
$(\bigwedge i.\ i \in I \Longrightarrow F\ i \in \mathit{sets}\ M) \Longrightarrow (\bigwedge J.\ J \subseteq I \Longrightarrow \mathit{finite}\ J \Longrightarrow J \neq \{\} \Longrightarrow \mathit{prob}\ (\bigcap i{\in}J.\ F\ i) = (\prod i{\in}J.\ \mathit{prob}\ (F\ i))) \Longrightarrow \mathit{indep\text{-}events}\ F\ I$
by (*auto simp*: *indep-events-def*)

definition (**in** *prob-space*)
indep-event A B $\longleftrightarrow$ *indep-events* (*case-bool A B*) *UNIV*

lemma (**in** *prob-space*) *indep-sets-cong*:
$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i = G\ i) \Longrightarrow \mathit{indep\text{-}sets}\ F\ I \longleftrightarrow \mathit{indep\text{-}sets}\ G\ J$
by (*simp add*: *indep-sets-def*, *intro conj-cong all-cong imp-cong ball-cong*) *blast*+

lemma (**in** *prob-space*) *indep-events-finite-index-events*:
indep-events F I $\longleftrightarrow$ $(\forall\, J{\subseteq}I.\ J \neq \{\} \longrightarrow \mathit{finite}\ J \longrightarrow \mathit{indep\text{-}events}\ F\ J)$
by (*auto simp*: *indep-events-def*)

lemma (**in** *prob-space*) *indep-sets-finite-index-sets*:
indep-sets F I $\longleftrightarrow$ $(\forall\, J{\subseteq}I.\ J \neq \{\} \longrightarrow \mathit{finite}\ J \longrightarrow \mathit{indep\text{-}sets}\ F\ J)$
proof (*intro iffI allI impI*)
assume $*$: $\forall\, J{\subseteq}I.\ J \neq \{\} \longrightarrow \mathit{finite}\ J \longrightarrow \mathit{indep\text{-}sets}\ F\ J$
show *indep-sets F I* **unfolding** *indep-sets-def*
proof (*intro conjI ballI allI impI*)

fix i **assume** $i \in I$
with $*$[*THEN spec*, *of* $\{i\}$] **show** $F\ i \subseteq events$
by (*auto simp*: *indep-sets-def*)
qed (*insert* $*$, *auto simp*: *indep-sets-def*)
qed (*auto simp*: *indep-sets-def*)

lemma (**in** *prob-space*) *indep-sets-mono-index*:
$J \subseteq I \Longrightarrow$ *indep-sets* $F\ I \Longrightarrow$ *indep-sets* $F\ J$
unfolding *indep-sets-def* **by** *auto*

lemma (**in** *prob-space*) *indep-sets-mono-sets*:
assumes *indep*: *indep-sets* $F\ I$
assumes *mono*: $\bigwedge i.\ i \in I \Longrightarrow G\ i \subseteq F\ i$
shows *indep-sets* $G\ I$
proof −
have $(\forall i \in I.\ F\ i \subseteq events) \Longrightarrow (\forall i \in I.\ G\ i \subseteq events)$
using *mono* **by** *auto*
moreover have $\bigwedge A\ J.\ J \subseteq I \Longrightarrow A \in (\Pi\ j \in J.\ G\ j) \Longrightarrow A \in (\Pi\ j \in J.\ F\ j)$
using *mono* **by** (*auto simp*: *Pi-iff*)
ultimately show *?thesis*
using *indep* **by** (*auto simp*: *indep-sets-def*)
qed

lemma (**in** *prob-space*) *indep-sets-mono*:
assumes *indep*: *indep-sets* $F\ I$
assumes *mono*: $J \subseteq I\ \bigwedge i.\ i \in J \Longrightarrow G\ i \subseteq F\ i$
shows *indep-sets* $G\ J$
apply (*rule indep-sets-mono-sets*)
apply (*rule indep-sets-mono-index*)
apply (*fact* +)
done

lemma (**in** *prob-space*) *indep-setsI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq events$
and $\bigwedge A\ J.\ J \neq \{\} \Longrightarrow J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow (\forall j \in J.\ A\ j \in F\ j) \Longrightarrow$ *prob* $(\bigcap j \in J.\ A\ j) = (\prod j \in J.\ prob\ (A\ j))$
shows *indep-sets* $F\ I$
using *assms* **unfolding** *indep-sets-def* **by** (*auto simp*: *Pi-iff*)

lemma (**in** *prob-space*) *indep-setsD*:
assumes *indep-sets* $F\ I$ **and** $J \subseteq I\ J \neq \{\}$ *finite* $J\ \forall j \in J.\ A\ j \in F\ j$
shows *prob* $(\bigcap j \in J.\ A\ j) = (\prod j \in J.\ prob\ (A\ j))$
using *assms* **unfolding** *indep-sets-def* **by** *auto*

lemma (**in** *prob-space*) *indep-setI*:
assumes *ev*: $A \subseteq events\ B \subseteq events$
and *indep*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow prob\ (a \cap b) = prob\ a * prob\ b$
shows *indep-set* $A\ B$
unfolding *indep-set-def*

proof (*rule indep-setsI*)
fix *F J* **assume** $J \neq \{\}$ $J \subseteq$ *UNIV*
and *F*: $\forall j{\in}J.$ *F j* $\in$ (*case j of True* $\Rightarrow$ *A* | *False* $\Rightarrow$ *B*)
have $J \in$ *Pow UNIV* **by** *auto*
with *F* ‹$J \neq \{\}$› *indep*[*of F True F False*]
show *prob* $(\bigcap j{\in}J.$ *F j*$) = (\prod j{\in}J.$ *prob* (*F j*))
unfolding *UNIV-bool Pow-insert* **by** (*auto simp*: *ac-simps*)
qed (*auto split*: *bool.split simp*: *ev*)

lemma (**in** *prob-space*) *indep-setD*:
assumes *indep*: *indep-set A B* **and** *ev*: $a \in A$ $b \in B$
shows *prob* $(a \cap b) =$ *prob a* $*$ *prob b*
using *indep*[*unfolded indep-set-def*, *THEN indep-setsD*, *of UNIV case-bool a b*] *ev*
by (*simp add*: *ac-simps UNIV-bool*)

lemma (**in** *prob-space*)
assumes *indep*: *indep-set A B*
shows *indep-setD-ev1*: $A \subseteq$ *events*
and *indep-setD-ev2*: $B \subseteq$ *events*
using *indep* **unfolding** *indep-set-def indep-sets-def UNIV-bool* **by** *auto*

lemma (**in** *prob-space*) *indep-sets-Dynkin*:
assumes *indep*: *indep-sets F I*
shows *indep-sets* ($\lambda i.$ *Dynkin* (*space M*) (*F i*)) *I*
(**is** *indep-sets ?F I*)
proof (*subst indep-sets-finite-index-sets*, *intro allI impI ballI*)
fix *J* **assume** *finite J J* $\subseteq$ *I J* $\neq \{\}$
with *indep* **have** *indep-sets F J*
by (*subst* (*asm*) *indep-sets-finite-index-sets*) *auto*
{ fix *J K* **assume** *indep-sets F K*
let *?G* $= \lambda S\ i.$ *if i* $\in$ *S then ?F i else F i*
assume *finite J J* $\subseteq$ *K*
then have *indep-sets* (*?G J*) *K*
proof *induct*
case (*insert j J*)
moreover define *G* **where** *G* $=$ *?G J*
ultimately have *G*: *indep-sets G K* $\bigwedge i.$ $i \in K \Longrightarrow$ *G i* $\subseteq$ *events* **and** $j \in K$
by (*auto simp*: *indep-sets-def*)
let *?D* $= \{E{\in}$*events*. *indep-sets* (*G*($j := \{E\}$)) *K* $\}$
{ fix *X* **assume** *X*: $X \in$ *events*
assume *indep*: $\bigwedge J\ A.$ $J \neq \{\} \Longrightarrow J \subseteq K \Longrightarrow$ *finite J* $\Longrightarrow j \notin J \Longrightarrow (\forall i{\in}J.$ *A i* $\in$ *G i*)
$\Longrightarrow$ *prob* $((\bigcap i{\in}J.$ *A i*$) \cap X) =$ *prob X* $*$ $(\prod i{\in}J.$ *prob* (*A i*))
have *indep-sets* (*G*($j := \{X\}$)) *K*
proof (*rule indep-setsI*)
fix *i* **assume** $i \in K$ **then show** (*G*($j{:=}\{X\}$)) *i* $\subseteq$ *events*
using *G X* **by** *auto*
next

```
      fix A J assume J: J ≠ {} J ⊆ K finite J ∀i∈J. A i ∈ (G(j := {X})) i
      show prob (⋂j∈J. A j) = (∏j∈J. prob (A j))
      proof cases
        assume j ∈ J
        with J have A j = X by auto
        show ?thesis
        proof cases
          assume J = {j} then show ?thesis by simp
        next
          assume J ≠ {j}
          have prob (⋂i∈J. A i) = prob ((⋂i∈J−{j}. A i) ∩ X)
            using ‹j ∈ J› ‹A j = X› by (auto intro!: arg-cong[where f=prob]
split: if-split-asm)
          also have ... = prob X * (∏i∈J−{j}. prob (A i))
          proof (rule indep)
            show J − {j} ≠ {} J − {j} ⊆ K finite (J − {j}) j ∉ J − {j}
              using J ‹J ≠ {j}› ‹j ∈ J› by auto
            show ∀i∈J − {j}. A i ∈ G i
              using J by auto
          qed
          also have ... = prob (A j) * (∏i∈J−{j}. prob (A i))
            using ‹A j = X› by simp
          also have ... = (∏i∈J. prob (A i))
            unfolding prod.insert-remove[OF ‹finite J›, symmetric, of λi. prob
(A i)]
            using ‹j ∈ J› by (simp add: insert-absorb)
          finally show ?thesis .
        qed
      next
        assume j ∉ J
        with J have ∀i∈J. A i ∈ G i by (auto split: if-split-asm)
        with J show ?thesis
          by (intro indep-setsD[OF G(1)]) auto
      qed
    qed }
    note indep-sets-insert = this
    have Dynkin-system (space M) ?D
    proof (rule Dynkin-systemI', simp-all cong del: indep-sets-cong, safe)
      show indep-sets (G(j := {{}})) K
        by (rule indep-sets-insert) auto
    next
      fix X assume X: X ∈ events and G': indep-sets (G(j := {X})) K
      show indep-sets (G(j := {space M − X})) K
      proof (rule indep-sets-insert)
        fix J A assume J: J ≠ {} J ⊆ K finite J j ∉ J and A: ∀i∈J. A i ∈ G i
        then have A-sets: ⋀i. i∈J ⟹ A i ∈ events
          using G by auto
        have prob ((⋂j∈J. A j) ∩ (space M − X)) =
            prob ((⋂j∈J. A j) − (⋂i∈insert j J. (A(j := X)) i))
```

using *A-sets sets.sets-into-space*[*of* - *M*] *X* ‹*J* ≠ {}›
by (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *split*: *if-split-asm*)
also have ... = *prob* (⋂*j*∈*J*. *A j*) − *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *X*)) *i*)
using *J* ‹*J* ≠ {}› ‹*j* ∉ *J*› *A-sets X sets.sets-into-space*
by (*auto intro*!: *finite-measure-Diff sets.finite-INT split*: *if-split-asm*)
finally have *prob* ((⋂*j*∈*J*. *A j*) ∩ (*space M* − *X*)) =
prob (⋂*j*∈*J*. *A j*) − *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *X*)) *i*) **.**
moreover {
have *prob* (⋂*j*∈*J*. *A j*) = (∏*j*∈*J*. *prob* (*A j*))
using *J A* ‹*finite J*› **by** (*intro indep-setsD*[*OF G*(*1*)]) *auto*
then have *prob* (⋂*j*∈*J*. *A j*) = *prob* (*space M*) ∗ (∏*i*∈*J*. *prob* (*A i*))
using *prob-space* **by** *simp* **}**
moreover {
have *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *X*)) *i*) = (∏*i*∈*insert j J*. *prob* ((*A*(*j* := *X*)) *i*))
using *J A* ‹*j* ∈ *K*› **by** (*intro indep-setsD*[*OF G'*]) *auto*
then have *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *X*)) *i*) = *prob X* ∗ (∏*i*∈*J*. *prob* (*A i*))
using ‹*finite J*› ‹*j* ∉ *J*› **by** (*auto intro*!: *prod.cong*) **}**
ultimately have *prob* ((⋂*j*∈*J*. *A j*) ∩ (*space M* − *X*)) = (*prob* (*space M*) − *prob X*) ∗ (∏*i*∈*J*. *prob* (*A i*))
by (*simp add*: *field-simps*)
also have ... = *prob* (*space M* − *X*) ∗ (∏*i*∈*J*. *prob* (*A i*))
using *X A* **by** (*simp add*: *finite-measure-compl*)
finally show *prob* ((⋂*j*∈*J*. *A j*) ∩ (*space M* − *X*)) = *prob* (*space M* − *X*) ∗ (∏*i*∈*J*. *prob* (*A i*)) **.**
qed (*insert X*, *auto*)
next
fix *F* :: *nat* ⇒ *'a set* **assume** *disj*: *disjoint-family F* **and** *range F* ⊆ *?D*
then have *F*: ⋀*i*. *F i* ∈ *events* ⋀*i*. *indep-sets* (*G*(*j*:={*F i*})) *K* **by** *auto*
show *indep-sets* (*G*(*j* := {⋃*k*. *F k*})) *K*
proof (*rule indep-sets-insert*)
fix *J A* **assume** *J*: *j* ∉ *J J* ≠ {} *J* ⊆ *K finite J* **and** *A*: ∀*i*∈*J*. *A i* ∈ *G i*
then have *A-sets*: ⋀*i*. *i*∈*J* ⟹ *A i* ∈ *events*
using *G* **by** *auto*
have *prob* ((⋂*j*∈*J*. *A j*) ∩ (⋃*k*. *F k*)) = *prob* (⋃*k*. (⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*))
using ‹*J* ≠ {}› ‹*j* ∉ *J*› ‹*j* ∈ *K*› **by** (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *split*: *if-split-asm*)
moreover have (λ*k*. *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*)) *sums prob* (⋃*k*. (⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*))
proof (*rule finite-measure-UNION*)
show *disjoint-family* (λ*k*. ⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*)
using *disj* **by** (*rule disjoint-family-on-bisimulation*) *auto*
show *range* (λ*k*. ⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*) ⊆ *events*
using *A-sets F* ‹*finite J*› ‹*J* ≠ {}› ‹*j* ∉ *J*› **by** (*auto intro*!: *sets.Int*)
qed
moreover { fix *k*
from *J A* ‹*j* ∈ *K*› **have** *prob* (⋂*i*∈*insert j J*. (*A*(*j* := *F k*)) *i*) = *prob*

```
(F k) * (∏ i∈J. prob (A i))
          by (subst indep-setsD[OF F(2)]) (auto intro!: prod.cong split: if-split-asm)
           also have ... = prob (F k) * prob (⋂ i∈J. A i)
             using J A ‹j ∈ K› by (subst indep-setsD[OF G(1)]) auto
           finally have prob (⋂ i∈insert j J. (A(j := F k)) i) = prob (F k) * prob
(⋂ i∈J. A i) . }
        ultimately have (λk. prob (F k) * prob (⋂ i∈J. A i)) sums (prob ((⋂ j∈J.
A j) ∩ (⋃ k. F k)))
          by simp
         moreover
          have (λk. prob (F k) * prob (⋂ i∈J. A i)) sums (prob (⋃ k. F k) * prob
(⋂ i∈J. A i))
           using disj F(1) by (intro finite-measure-UNION sums-mult2) auto
         then have (λk. prob (F k) * prob (⋂ i∈J. A i)) sums (prob (⋃ k. F k) *
(∏ i∈J. prob (A i)))
           using J A ‹j ∈ K› by (subst indep-setsD[OF G(1), symmetric]) auto
         ultimately
         show prob ((⋂ j∈J. A j) ∩ (⋃ k. F k)) = prob (⋃ k. F k) * (∏ j∈J. prob
(A j))
           by (auto dest!: sums-unique)
       qed (insert F, auto)
     qed (insert sets.sets-into-space, auto)
      then have mono: Dynkin (space M) (G j) ⊆ {E ∈ events. indep-sets (G(j
:= {E})) K}
     proof (rule Dynkin-system.Dynkin-subset, safe)
       fix X assume X ∈ G j
       then show X ∈ events using G ‹j ∈ K› by auto
       from ‹indep-sets G K›
       show indep-sets (G(j := {X})) K
         by (rule indep-sets-mono-sets) (insert ‹X ∈ G j›, auto)
     qed
     have indep-sets (G(j:=?D)) K
     proof (rule indep-setsI)
       fix i assume i ∈ K then show (G(j := ?D)) i ⊆ events
         using G(2) by auto
     next
      fix A J assume J: J≠{} J ⊆ K finite J and A: ∀ i∈J. A i ∈ (G(j := ?D))
i
       show prob (⋂ j∈J. A j) = (∏ j∈J. prob (A j))
       proof cases
         assume j ∈ J
         with A have indep: indep-sets (G(j := {A j})) K by auto
         from J A show ?thesis
           by (intro indep-setsD[OF indep]) auto
       next
         assume j ∉ J
         with J A have ∀ i∈J. A i ∈ G i by (auto split: if-split-asm)
         with J show ?thesis
           by (intro indep-setsD[OF G(1)]) auto
```

```
      qed
    qed
    then have indep-sets (G(j := Dynkin (space M) (G j))) K
      by (rule indep-sets-mono-sets) (insert mono, auto)
    then show ?case
      by (rule indep-sets-mono-sets) (insert ‹j ∈ K› ‹j ∉ J›, auto simp: G-def)
  qed (insert ‹indep-sets F K›, simp) }
 from this[OF ‹indep-sets F J› ‹finite J› subset-refl]
 show indep-sets ?F J
  by (rule indep-sets-mono-sets) auto
qed

lemma (in prob-space) indep-sets-sigma:
 assumes indep: indep-sets F I
 assumes stable: ⋀i. i ∈ I ⟹ Int-stable (F i)
 shows indep-sets (λi. sigma-sets (space M) (F i)) I
proof −
 from indep-sets-Dynkin[OF indep]
 show ?thesis
 proof (rule indep-sets-mono-sets, subst sigma-eq-Dynkin, simp-all add: stable)
  fix i assume i ∈ I
  with indep have F i ⊆ events by (auto simp: indep-sets-def)
  with sets.sets-into-space show F i ⊆ Pow (space M) by auto
 qed
qed

lemma (in prob-space) indep-sets-sigma-sets-iff:
 assumes ⋀i. i ∈ I ⟹ Int-stable (F i)
 shows indep-sets (λi. sigma-sets (space M) (F i)) I ⟷ indep-sets F I
proof
 assume indep-sets F I then show indep-sets (λi. sigma-sets (space M) (F i)) I
  by (rule indep-sets-sigma) fact
next
 assume indep-sets (λi. sigma-sets (space M) (F i)) I then show indep-sets F I
  by (rule indep-sets-mono-sets) (intro subsetI sigma-sets.Basic)
qed

definition (in prob-space)
 indep-vars-def2: indep-vars M' X I ⟷
  (∀ i∈I. random-variable (M' i) (X i)) ∧
  indep-sets (λi. { X i −‘ A ∩ space M | A. A ∈ sets (M' i)}) I

definition (in prob-space)
 indep-var Ma A Mb B ⟷ indep-vars (case-bool Ma Mb) (case-bool A B) UNIV

lemma (in prob-space) indep-vars-def:
 indep-vars M' X I ⟷
  (∀ i∈I. random-variable (M' i) (X i)) ∧
  indep-sets (λi. sigma-sets (space M) { X i −‘ A ∩ space M | A. A ∈ sets (M'
```

i)}) *I*
unfolding *indep-vars-def2*
apply (*rule conj-cong*[*OF refl*])
apply (*rule indep-sets-sigma-sets-iff*[*symmetric*])
apply (*auto simp*: *Int-stable-def*)
apply (*rule-tac x*=*A* ∩ *Aa* **in** *exI*)
apply *auto*
done

lemma (**in** *prob-space*) *indep-var-eq*:
indep-var S X T Y ⟷
(*random-variable S X* ∧ *random-variable T Y*) ∧
indep-set
(*sigma-sets* (*space M*) { *X* −‘ *A* ∩ *space M* | *A*. *A* ∈ *sets S*})
(*sigma-sets* (*space M*) { *Y* −‘ *A* ∩ *space M* | *A*. *A* ∈ *sets T*})
unfolding *indep-var-def indep-vars-def indep-set-def UNIV-bool*
by (*intro arg-cong2*[**where** *f*=(∧)] *arg-cong2*[**where** *f*=*indep-sets*] *ext*)
(*auto split*: *bool.split*)

lemma (**in** *prob-space*) *indep-sets2-eq*:
indep-set A B ⟷ *A* ⊆ *events* ∧ *B* ⊆ *events* ∧ (∀ *a*∈*A*. ∀ *b*∈*B*. *prob* (*a* ∩ *b*) = *prob a* ∗ *prob b*)
unfolding *indep-set-def*
proof (*intro iffI ballI conjI*)
assume *indep*: *indep-sets* (*case-bool A B*) *UNIV*
{ fix *a b* **assume** *a* ∈ *A b* ∈ *B*
with *indep-setsD*[*OF indep*, *of UNIV case-bool a b*]
show *prob* (*a* ∩ *b*) = *prob a* ∗ *prob b*
unfolding *UNIV-bool* **by** (*simp add*: *ac-simps*) **}**
from *indep* **show** *A* ⊆ *events B* ⊆ *events*
unfolding *indep-sets-def UNIV-bool* **by** *auto*
next
assume ∗: *A* ⊆ *events* ∧ *B* ⊆ *events* ∧ (∀ *a*∈*A*. ∀ *b*∈*B*. *prob* (*a* ∩ *b*) = *prob a* ∗ *prob b*)
show *indep-sets* (*case-bool A B*) *UNIV*
proof (*rule indep-setsI*)
fix *i* **show** (*case i of True* ⇒ *A* | *False* ⇒ *B*) ⊆ *events*
using ∗ **by** (*auto split*: *bool.split*)
next
fix *J X* **assume** *J* ≠ {} *J* ⊆ *UNIV* **and** *X*: ∀ *j*∈*J*. *X j* ∈ (*case j of True* ⇒ *A* | *False* ⇒ *B*)
then have *J* = {*True*} ∨ *J* = {*False*} ∨ *J* = {*True*,*False*}
by (*auto simp*: *UNIV-bool*)
then show *prob* (⋂ *j*∈*J*. *X j*) = (∏ *j*∈*J*. *prob* (*X j*))
using *X* ∗ **by** *auto*
qed
qed

lemma (**in** *prob-space*) *indep-set-sigma-sets*:

assumes *indep-set A B*
assumes *A*: *Int-stable A* **and** *B*: *Int-stable B*
shows *indep-set* (*sigma-sets* (*space M*) *A*) (*sigma-sets* (*space M*) *B*)
proof −
have *indep-sets* (λ*i*. *sigma-sets* (*space M*) (*case i of True* ⇒ *A* | *False* ⇒ *B*)) *UNIV*
proof (*rule indep-sets-sigma*)
show *indep-sets* (*case-bool A B*) *UNIV*
by (*rule* ‹*indep-set A B*›[*unfolded indep-set-def*])
fix *i* **show** *Int-stable* (*case i of True* ⇒ *A* | *False* ⇒ *B*)
using *A B* **by** (*cases i*) *auto*
qed
then show *?thesis*
unfolding *indep-set-def*
by (*rule indep-sets-mono-sets*) (*auto split*: *bool.split*)
qed

lemma (**in** *prob-space*) *indep-eventsI-indep-vars*:
assumes *indep*: *indep-vars N X I*
assumes *P*: ⋀*i*. *i* ∈ *I* ⟹ {*x*∈*space* (*N i*). *P i x*} ∈ *sets* (*N i*)
shows *indep-events* (λ*i*. {*x*∈*space M*. *P i* (*X i x*)}) *I*
proof −
have *indep-sets* (λ*i*. {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*N i*)}) *I*
using *indep* **unfolding** *indep-vars-def2* **by** *auto*
then show *?thesis*
unfolding *indep-events-def-alt*
proof (*rule indep-sets-mono-sets*)
fix *i* **assume** *i* ∈ *I*
then have {{*x* ∈ *space M*. *P i* (*X i x*)}} = {*X i* −‘ {*x*∈*space* (*N i*). *P i x*} ∩ *space M*}
using *indep* **by** (*auto simp*: *indep-vars-def dest*: *measurable-space*)
also have ... ⊆ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*N i*)}
using *P*[*OF* ‹*i* ∈ *I*›] **by** *blast*
finally show {{*x* ∈ *space M*. *P i* (*X i x*)}} ⊆ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*N i*)} **.**
qed
qed

lemma (**in** *prob-space*) *indep-sets-collect-sigma*:
fixes *I* :: *'j* ⇒ *'i set* **and** *J* :: *'j set* **and** *E* :: *'i* ⇒ *'a set set*
assumes *indep*: *indep-sets E* (⋃*j*∈*J*. *I j*)
assumes *Int-stable*: ⋀*i j*. *j* ∈ *J* ⟹ *i* ∈ *I j* ⟹ *Int-stable* (*E i*)
assumes *disjoint*: *disjoint-family-on I J*
shows *indep-sets* (λ*j*. *sigma-sets* (*space M*) (⋃*i*∈*I j*. *E i*)) *J*
proof −
let *?E* = λ*j*. {⋂*k*∈*K*. *E' k*| *E' K*. *finite K* ∧ *K* ≠ {} ∧ *K* ⊆ *I j* ∧ (∀*k*∈*K*. *E' k* ∈ *E k*) }

from *indep* **have** *E*: ⋀*j i*. *j* ∈ *J* ⟹ *i* ∈ *I j* ⟹ *E i* ⊆ *events*

```
    unfolding indep-sets-def by auto
  { fix j
    let ?S = sigma-sets (space M) (⋃ i∈I j. E i)
    assume j ∈ J
    from E[OF this] interpret S: sigma-algebra space M ?S
      using sets.sets-into-space[of - M] by (intro sigma-algebra-sigma-sets) auto

    have sigma-sets (space M) (⋃ i∈I j. E i) = sigma-sets (space M) (?E j)
    proof (rule sigma-sets-eqI)
      fix A assume A ∈ (⋃ i∈I j. E i)
      then obtain i where i ∈ I j A ∈ E i ..
      then show A ∈ sigma-sets (space M) (?E j)
        by (auto intro!: sigma-sets.intros(2−) exI[of - {i}] exI[of - λi. A])
    next
      fix A assume A ∈ ?E j
      then obtain E' K where finite K K ≠ {} K ⊆ I j ⋀k. k ∈ K ⟹ E' k ∈
E k
        and A: A = (⋂ k∈K. E' k)
        by auto
      then have A ∈ ?S unfolding A
        by (safe intro!: S.finite-INT) auto
      then show A ∈ sigma-sets (space M) (⋃ i∈I j. E i)
        by simp
    qed }
  moreover have indep-sets (λj. sigma-sets (space M) (?E j)) J
  proof (rule indep-sets-sigma)
    show indep-sets ?E J
    proof (intro indep-setsI)
    fix j assume j ∈ J with E show ?E j ⊆ events by (force intro!: sets.finite-INT)
    next
      fix K A assume K: K ≠ {} K ⊆ J finite K
        and ∀ j∈K. A j ∈ ?E j
      then have ∀ j∈K. ∃ E' L. A j = (⋂ l∈L. E' l) ∧ finite L ∧ L ≠ {} ∧ L ⊆ I
j ∧ (∀ l∈L. E' l ∈ E l)
        by simp
      from bchoice[OF this] obtain E'
        where ∀ x∈K. ∃ L. A x = ⋂ (E' x ' L) ∧ finite L ∧ L ≠ {} ∧ L ⊆ I x ∧
(∀ l∈L. E' x l ∈ E l)
        ..
      from bchoice[OF this] obtain L
        where A: ⋀j. j∈K ⟹ A j = (⋂ l∈L j. E' j l)
        and L: ⋀j. j∈K ⟹ finite (L j) ⋀j. j∈K ⟹ L j ≠ {} ⋀j. j∈K ⟹ L j
⊆ I j
        and E': ⋀j l. j∈K ⟹ l ∈ L j ⟹ E' j l ∈ E l
        by auto
      { fix k l j assume k ∈ K j ∈ K l ∈ L j l ∈ L k
        have k = j
        proof (rule ccontr)
          assume k ≠ j
```

```
        with disjoint ‹K ⊆ J› ‹k ∈ K› ‹j ∈ K› have I k ∩ I j = {}
          unfolding disjoint-family-on-def by auto
        with L(2,3)[OF ‹j ∈ K›] L(2,3)[OF ‹k ∈ K›]
        show False using ‹l ∈ L k› ‹l ∈ L j› by auto
      qed }
    note L-inj = this

    define k where k l = (SOME k. k ∈ K ∧ l ∈ L k) for l
    { fix x j l assume ∗: j ∈ K l ∈ L j
      have k l = j unfolding k-def
      proof (rule some-equality)
        fix k assume k ∈ K ∧ l ∈ L k
        with ∗ L-inj show k = j by auto
      qed (insert ∗, simp) }
    note k-simp[simp] = this
    let ?E' = λl. E' (k l) l
    have prob (⋂j∈K. A j) = prob (⋂ l∈(⋃ k∈K. L k). ?E' l)
      by (auto simp: A intro!: arg-cong[where f=prob])
    also have ... = (∏ l∈(⋃ k∈K. L k). prob (?E' l))
      using L K E' by (intro indep-setsD[OF indep]) (simp-all add: UN-mono)
    also have ... = (∏ j∈K. ∏ l∈L j. prob (E' j l))
      using K L L-inj by (subst prod.UNION-disjoint) auto
    also have ... = (∏ j∈K. prob (A j))
     using K L E' by (auto simp add: A intro!: prod.cong indep-setsD[OF indep,
symmetric]) blast
    finally show prob (⋂j∈K. A j) = (∏ j∈K. prob (A j)) .
  qed
 next
  fix j assume j ∈ J
  show Int-stable (?E j)
  proof (rule Int-stableI)
    fix a assume a ∈ ?E j then obtain Ka Ea
      where a: a = (⋂ k∈Ka. Ea k) finite Ka Ka ≠ {} Ka ⊆ I j ⋀k. k∈Ka ⟹
Ea k ∈ E k by auto
    fix b assume b ∈ ?E j then obtain Kb Eb
      where b: b = (⋂ k∈Kb. Eb k) finite Kb Kb ≠ {} Kb ⊆ I j ⋀k. k∈Kb ⟹
Eb k ∈ E k by auto
    let ?f = λk. (if k ∈ Ka ∩ Kb then Ea k ∩ Eb k else if k ∈ Kb then Eb k else
if k ∈ Ka then Ea k else {})
    have Ka ∪ Kb = (Ka ∩ Kb) ∪ (Kb − Ka) ∪ (Ka − Kb)
      by blast
    moreover have (⋂ x∈Ka ∩ Kb. Ea x ∩ Eb x) ∩
    (⋂ x∈Kb − Ka. Eb x) ∩ (⋂ x∈Ka − Kb. Ea x) = (⋂ k∈Ka. Ea k) ∩ (⋂ k∈Kb.
Eb k)
      by auto
    ultimately have (⋂ k∈Ka ∪ Kb. ?f k) = (⋂ k∈Ka. Ea k) ∩ (⋂ k∈Kb. Eb k)
(is ?lhs = ?rhs)
      by (simp only: image-Un Inter-Un-distrib) simp
    then have a ∩ b = (⋂ k∈Ka ∪ Kb. ?f k)
```

```
      by (simp only: a(1) b(1))
    with a b ‹j ∈ J› Int-stableD[OF Int-stable] show a ∩ b ∈ ?E j
      by (intro CollectI exI[of - Ka ∪ Kb] exI[of - ?f]) auto
  qed
 qed
 ultimately show ?thesis
  by (simp cong: indep-sets-cong)
qed

lemma (in prob-space) indep-vars-restrict:
  assumes ind: indep-vars M' X I and K: ⋀j. j ∈ L ⟹ K j ⊆ I and J:
disjoint-family-on K L
 shows indep-vars (λj. PiM (K j) M') (λj ω. restrict (λi. X i ω) (K j)) L
 unfolding indep-vars-def
proof safe
 fix j assume j ∈ L then show random-variable (Pi_M (K j) M') (λω. λi∈K j.
X i ω)
  using K ind by (auto simp: indep-vars-def intro!: measurable-restrict)
next
 have X: ⋀i. i ∈ I ⟹ X i ∈ measurable M (M' i)
  using ind by (auto simp: indep-vars-def)
 let ?proj = λj S. {(λω. λi∈K j. X i ω) -` A ∩ space M |A. A ∈ S}
 let ?UN = λj. sigma-sets (space M) (⋃i∈K j. { X i -` A ∩ space M| A. A ∈
sets (M' i) })
 show indep-sets (λi. sigma-sets (space M) (?proj i (sets (Pi_M (K i) M')))) L
 proof (rule indep-sets-mono-sets)
  fix j assume j: j ∈ L
  have sigma-sets (space M) (?proj j (sets (Pi_M (K j) M'))) =
   sigma-sets (space M) (sigma-sets (space M) (?proj j (prod-algebra (K j) M')))
    using j K X[THEN measurable-space] unfolding sets-PiM
    by (subst sigma-sets-vimage-commute) (auto simp add: Pi-iff)
  also have ... = sigma-sets (space M) (?proj j (prod-algebra (K j) M'))
    by (rule sigma-sets-sigma-sets-eq) auto
  also have ... ⊆ ?UN j
  proof (rule sigma-sets-mono, safe del: disjE elim!: prod-algebraE)
    fix J E assume J: finite J J ≠ {} ∨ K j = {}  J ⊆ K j and E: ∀i. i ∈ J
⟶ E i ∈ sets (M' i)
    show (λω. λi∈K j. X i ω) -` prod-emb (K j) M' J (Pi_E J E) ∩ space M ∈
?UN j
    proof cases
     assume K j = {} with J show ?thesis
      by (auto simp add: sigma-sets-empty-eq prod-emb-def)
    next
     assume K j ≠ {} with J have J ≠ {}
      by auto
     { interpret sigma-algebra space M ?UN j
        by (rule sigma-algebra-sigma-sets) auto
      have ⋀A. (⋀i. i ∈ J ⟹ A i ∈ ?UN j) ⟹ ⋂(A ` J) ∈ ?UN j
       using ‹finite J› ‹J ≠ {}› by (rule finite-INT) blast }
```

```
      note INT = this

      from ‹J ≠ {}› J K E[rule-format, THEN sets.sets-into-space] j
      have (λω. λi∈K j. X i ω) −‘ prod-emb (K j) M′ J (Pi_E J E) ∩ space M
        = (⋂i∈J. X i −‘ E i ∩ space M)
        apply (subst prod-emb-PiE[OF - ])
        apply auto []
        apply auto []
        apply (auto simp add: Pi-iff intro!: X[THEN measurable-space])
        apply (erule-tac x=i in ballE)
        apply auto
        done
      also have ... ∈ ?UN j
        apply (rule INT)
        apply (rule sigma-sets.Basic)
        using ‹J ⊆ K j› E
        apply auto
        done
      finally show ?thesis .
    qed
  qed
  finally show sigma-sets (space M) (?proj j (sets (Pi_M (K j) M′))) ⊆ ?UN j .
 next
  show indep-sets ?UN L
  proof (rule indep-sets-collect-sigma)
    show indep-sets (λi. {X i −‘ A ∩ space M |A. A ∈ sets (M′ i)}) (⋃j∈L. K
j)
    proof (rule indep-sets-mono-index)
      show indep-sets (λi. {X i −‘ A ∩ space M |A. A ∈ sets (M′ i)}) I
        using ind unfolding indep-vars-def2 by auto
      show (⋃l∈L. K l) ⊆ I
        using K by auto
    qed
  next
    fix l i assume l ∈ L i ∈ K l
    show Int-stable {X i −‘ A ∩ space M |A. A ∈ sets (M′ i)}
      apply (auto simp: Int-stable-def)
      apply (rule-tac x=A ∩ Aa in exI)
      apply auto
      done
  qed fact
 qed
qed

lemma (in prob-space) indep-var-restrict:
  assumes ind: indep-vars M′ X I and AB: A ∩ B = {} A ⊆ I B ⊆ I
  shows indep-var (PiM A M′) (λω. restrict (λi. X i ω) A) (PiM B M′) (λω.
restrict (λi. X i ω) B)
proof −
```

have $*$:
case-bool (Pi_M *A M'*) (Pi_M *B M'*) = (λb. *PiM* (*case-bool A B b*) *M'*)
case-bool ($\lambda\omega$. $\lambda i{\in}A$. *X i* ω) ($\lambda\omega$. $\lambda i{\in}B$. *X i* ω) = ($\lambda b\ \omega$. $\lambda i{\in}$*case-bool A B b*. *X i* ω)
by (*simp-all add*: *fun-eq-iff split*: *bool.split*)
show *?thesis*
unfolding *indep-var-def* $*$ **using** *AB*
by (*intro indep-vars-restrict*[*OF ind*]) (*auto simp*: *disjoint-family-on-def split*: *bool.split*)
qed

lemma (**in** *prob-space*) *indep-vars-subset*:
assumes *indep-vars M' X I J* $\subseteq$ *I*
shows *indep-vars M' X J*
using *assms* **unfolding** *indep-vars-def indep-sets-def*
by *auto*

lemma (**in** *prob-space*) *indep-vars-cong*:
I = *J* $\Longrightarrow$ ($\bigwedge i$. *i* $\in$ *I* $\Longrightarrow$ *X i* = *Y i*) $\Longrightarrow$ ($\bigwedge i$. *i* $\in$ *I* $\Longrightarrow$ *M' i* = *N' i*) $\Longrightarrow$ *indep-vars M' X I* $\longleftrightarrow$ *indep-vars N' Y J*
unfolding *indep-vars-def2* **by** (*intro conj-cong indep-sets-cong*) *auto*

definition (**in** *prob-space*) *tail-events* **where**
tail-events A = ($\bigcap n$. *sigma-sets* (*space M*) ($\bigcup$ (*A* ' {*n*..})))

lemma (**in** *prob-space*) *tail-events-sets*:
assumes *A*: $\bigwedge i{::}nat$. *A i* $\subseteq$ *events*
shows *tail-events A* $\subseteq$ *events*
proof
fix *X* **assume** *X*: *X* $\in$ *tail-events A*
let *?A* = ($\bigcap n$. *sigma-sets* (*space M*) ($\bigcup$ (*A* ' {*n*..})))
from *X* **have** $\bigwedge n{::}nat$. *X* $\in$ *sigma-sets* (*space M*) ($\bigcup$ (*A* ' {*n*..})) **by** (*auto simp*: *tail-events-def*)
from *this*[*of 0*] **have** *X* $\in$ *sigma-sets* (*space M*) ($\bigcup$(*A* ' *UNIV*)) **by** *simp*
then show *X* $\in$ *events*
by *induct* (*insert A*, *auto*)
qed

lemma (**in** *prob-space*) *sigma-algebra-tail-events*:
assumes $\bigwedge i{::}nat$. *sigma-algebra* (*space M*) (*A i*)
shows *sigma-algebra* (*space M*) (*tail-events A*)
unfolding *tail-events-def*
proof (*simp add*: *sigma-algebra-iff2*, *safe*)
let *?A* = ($\bigcap n$. *sigma-sets* (*space M*) ($\bigcup$ (*A* ' {*n*..})))
interpret *A*: *sigma-algebra space M A i* **for** *i* **by** *fact*
{ fix *X x* **assume** *X* $\in$ *?A x* $\in$ *X*
then have $\bigwedge n$. *X* $\in$ *sigma-sets* (*space M*) ($\bigcup$ (*A* ' {*n*..})) **by** *auto*
from *this*[*of 0*] **have** *X* $\in$ *sigma-sets* (*space M*) ($\bigcup$(*A* ' *UNIV*)) **by** *simp*
then have *X* $\subseteq$ *space M*

```
      by induct (insert A.sets-into-space, auto)
    with ‹x ∈ X› show x ∈ space M by auto }
  { fix F :: nat ⇒ 'a set and n assume range F ⊆ ?A
    then show (⋃(F ` UNIV)) ∈ sigma-sets (space M) (⋃ (A ` {n..}))
      by (intro sigma-sets.Union) auto }
qed (auto intro!: sigma-sets.Compl sigma-sets.Empty)

lemma (in prob-space) kolmogorov-0-1-law:
  fixes A :: nat ⇒ 'a set set
  assumes ⋀i::nat. sigma-algebra (space M) (A i)
  assumes indep: indep-sets A UNIV
  and X: X ∈ tail-events A
  shows prob X = 0 ∨ prob X = 1
proof −
  have A: ⋀i. A i ⊆ events
    using indep unfolding indep-sets-def by simp

  let ?D = {D ∈ events. prob (X ∩ D) = prob X * prob D}
  interpret A: sigma-algebra space M A i for i by fact
  interpret T: sigma-algebra space M tail-events A
    by (rule sigma-algebra-tail-events) fact
  have X ⊆ space M using T.space-closed X by auto

  have X-in: X ∈ events
    using tail-events-sets A X by auto

  interpret D: Dynkin-system space M ?D
  proof (rule Dynkin-systemI)
    fix D assume D ∈ ?D then show D ⊆ space M
      using sets.sets-into-space by auto
  next
    show space M ∈ ?D
      using prob-space ‹X ⊆ space M› by (simp add: Int-absorb2)
  next
    fix A assume A: A ∈ ?D
    have prob (X ∩ (space M − A)) = prob (X − (X ∩ A))
      using ‹X ⊆ space M› by (auto intro!: arg-cong[where f=prob])
    also have ... = prob X − prob (X ∩ A)
      using X-in A by (intro finite-measure-Diff) auto
    also have ... = prob X * prob (space M) − prob X * prob A
      using A prob-space by auto
    also have ... = prob X * prob (space M − A)
      using X-in A sets.sets-into-space
      by (subst finite-measure-Diff) (auto simp: field-simps)
    finally show space M − A ∈ ?D
      using A ‹X ⊆ space M› by auto
  next
    fix F :: nat ⇒ 'a set assume dis: disjoint-family F and range F ⊆ ?D
    then have F: range F ⊆ events ⋀i. prob (X ∩ F i) = prob X * prob (F i)
```

```
    by auto
  have (λi. prob (X ∩ F i)) sums prob (⋃ i. X ∩ F i)
  proof (rule finite-measure-UNION)
    show range (λi. X ∩ F i) ⊆ events
      using F X-in by auto
    show disjoint-family (λi. X ∩ F i)
      using dis by (rule disjoint-family-on-bisimulation) auto
  qed
  with F have (λi. prob X * prob (F i)) sums prob (X ∩ (⋃ i. F i))
    by simp
  moreover have (λi. prob X * prob (F i)) sums (prob X * prob (⋃ i. F i))
    by (intro sums-mult finite-measure-UNION F dis)
  ultimately have prob (X ∩ (⋃ i. F i)) = prob X * prob (⋃ i. F i)
    by (auto dest!: sums-unique)
  with F show (⋃ i. F i) ∈ ?D
    by auto
qed

{ fix n
  have indep-sets (λb. sigma-sets (space M) (⋃ m∈case-bool {..n} {Suc n..} b.
A m)) UNIV
  proof (rule indep-sets-collect-sigma)
    have *: (⋃ b. case b of True ⇒ {..n} | False ⇒ {Suc n..}) = UNIV (is ?U
= -)
      by (simp split: bool.split add: set-eq-iff) (metis not-less-eq-eq)
    with indep show indep-sets A ?U by simp
    show disjoint-family (case-bool {..n} {Suc n..})
      unfolding disjoint-family-on-def by (auto split: bool.split)
    fix m
    show Int-stable (A m)
      unfolding Int-stable-def using A.Int by auto
  qed
  also have (λb. sigma-sets (space M) (⋃ m∈case-bool {..n} {Suc n..} b. A m))
=
      case-bool (sigma-sets (space M) (⋃ m∈{..n}. A m)) (sigma-sets (space M)
(⋃ m∈{Suc n..}. A m))
    by (auto intro!: ext split: bool.split)
  finally have indep: indep-set (sigma-sets (space M) (⋃ m∈{..n}. A m)) (sigma-sets
(space M) (⋃ m∈{Suc n..}. A m))
    unfolding indep-set-def by simp

  have sigma-sets (space M) (⋃ m∈{..n}. A m) ⊆ ?D
  proof (simp add: subset-eq, rule)
    fix D assume D: D ∈ sigma-sets (space M) (⋃ m∈{..n}. A m)
    have X ∈ sigma-sets (space M) (⋃ m∈{Suc n..}. A m)
      using X unfolding tail-events-def by simp
    from indep-setD[OF indep D this] indep-setD-ev1[OF indep] D
    show D ∈ events ∧ prob (X ∩ D) = prob X * prob D
      by (auto simp add: ac-simps)
```

```
    qed }
  then have (⋃ n. sigma-sets (space M) (⋃ m∈{..n}. A m)) ⊆ ?D (is ?A ⊆ -)
    by auto

  note ‹X ∈ tail-events A›
  also {
    have ⋀n. sigma-sets (space M) (⋃ i∈{n..}. A i) ⊆ sigma-sets (space M) ?A
      by (intro sigma-sets-subseteq UN-mono) auto
   then have tail-events A ⊆ sigma-sets (space M) ?A
      unfolding tail-events-def by auto }
  also have sigma-sets (space M) ?A = Dynkin (space M) ?A
  proof (rule sigma-eq-Dynkin)
    { fix B n assume B ∈ sigma-sets (space M) (⋃ m∈{..n}. A m)
      then have B ⊆ space M
        by induct (insert A sets.sets-into-space[of - M], auto) }
    then show ?A ⊆ Pow (space M) by auto
    show Int-stable ?A
    proof (rule Int-stableI)
      fix a b assume a ∈ ?A b ∈ ?A then obtain n m
        where a: n ∈ UNIV a ∈ sigma-sets (space M) (⋃ (A ' {..n}))
          and b: m ∈ UNIV b ∈ sigma-sets (space M) (⋃ (A ' {..m})) by auto
      interpret Amn: sigma-algebra space M sigma-sets (space M) (⋃ i∈{..max m
n}. A i)
        using A sets.sets-into-space[of - M] by (intro sigma-algebra-sigma-sets) auto
      have sigma-sets (space M) (⋃ i∈{..n}. A i) ⊆ sigma-sets (space M) (⋃ i∈{..max
m n}. A i)
        by (intro sigma-sets-subseteq UN-mono) auto
      with a have a ∈ sigma-sets (space M) (⋃ i∈{..max m n}. A i) by auto
      moreover
      have sigma-sets (space M) (⋃ i∈{..m}. A i) ⊆ sigma-sets (space M) (⋃ i∈{..max
m n}. A i)
        by (intro sigma-sets-subseteq UN-mono) auto
      with b have b ∈ sigma-sets (space M) (⋃ i∈{..max m n}. A i) by auto
      ultimately have a ∩ b ∈ sigma-sets (space M) (⋃ i∈{..max m n}. A i)
        using Amn.Int[of a b] by simp
      then show a ∩ b ∈ (⋃ n. sigma-sets (space M) (⋃ i∈{..n}. A i)) by auto
    qed
  qed
  also have Dynkin (space M) ?A ⊆ ?D
    using ‹?A ⊆ ?D› by (auto intro!: D.Dynkin-subset)
  finally show ?thesis by auto
qed

lemma (in prob-space) borel-0-1-law:
  fixes F :: nat ⇒ 'a set
  assumes F2: indep-events F UNIV
  shows prob (⋂ n. ⋃ m∈{n..}. F m) = 0 ∨ prob (⋂ n. ⋃ m∈{n..}. F m) = 1
proof (rule kolmogorov-0-1-law[of λi. sigma-sets (space M) { F i }])
  have F1: range F ⊆ events
```

```
    using F2 by (simp add: indep-events-def subset-eq)
  { fix i show sigma-algebra (space M) (sigma-sets (space M) {F i})
      using sigma-algebra-sigma-sets[of {F i} space M] F1 sets.sets-into-space
      by auto }
  show indep-sets (λi. sigma-sets (space M) {F i}) UNIV
  proof (rule indep-sets-sigma)
    show indep-sets (λi. {F i}) UNIV
      unfolding indep-events-def-alt[symmetric] by fact
    fix i show Int-stable {F i}
      unfolding Int-stable-def by simp
  qed
  let ?Q = λn. ⋃ i∈{n..}. F i
  show (⋂ n. ⋃ m∈{n..}. F m) ∈ tail-events (λi. sigma-sets (space M) {F i})
    unfolding tail-events-def
  proof
    fix j
   interpret S: sigma-algebra space M sigma-sets (space M) (⋃ i∈{j..}. sigma-sets
(space M) {F i})
      using order-trans[OF F1 sets.space-closed]
     by (intro sigma-algebra-sigma-sets) (simp add: sigma-sets-singleton subset-eq)
    have (⋂ n. ?Q n) = (⋂ n∈{j..}. ?Q n)
      by (intro decseq-SucI INT-decseq-offset UN-mono) auto
    also have ... ∈ sigma-sets (space M) (⋃ i∈{j..}. sigma-sets (space M) {F i})
      using order-trans[OF F1 sets.space-closed]
      by (safe intro!: S.countable-INT S.countable-UN)
        (auto simp: sigma-sets-singleton intro!: sigma-sets.Basic bexI)
    finally show (⋂ n. ?Q n) ∈ sigma-sets (space M) (⋃ i∈{j..}. sigma-sets (space
M) {F i})
      by simp
  qed
qed

lemma (in prob-space) borel-0-1-law-AE:
  fixes P :: nat ⇒ 'a ⇒ bool
  assumes indep-events (λm. {x∈space M. P m x}) UNIV (is indep-events ?P -)
  shows (AE x in M. infinite {m. P m x}) ∨ (AE x in M. finite {m. P m x})
proof −
  have [measurable]: ⋀m. {x∈space M. P m x} ∈ sets M
    using assms by (auto simp: indep-events-def)
  have ∗: (⋂ n. ⋃ m∈{n..}. {x ∈ space M. P m x}) ∈ events
    by simp
  from assms have prob (⋂ n. ⋃ m∈{n..}. ?P m) = 0 ∨ prob (⋂ n. ⋃ m∈{n..}.
?P m) = 1
    by (rule borel-0-1-law)
  also have prob (⋂ n. ⋃ m∈{n..}. ?P m) = 1 ⟷ (AE x in M. infinite {m. P
m x})
    using ∗ by (simp add: prob-eq-1)
      (simp add: Bex-def infinite-nat-iff-unbounded-le)
  also have prob (⋂ n. ⋃ m∈{n..}. ?P m) = 0 ⟷ (AE x in M. finite {m. P m
```

$x\}$)
using $*$ **by** (*simp add*: *prob-eq-0*)
(*auto simp add*: *Ball-def finite-nat-iff-bounded not-less* [*symmetric*])
finally show *?thesis*
by *blast*
qed

lemma (**in** *prob-space*) *indep-sets-finite*:
assumes *I*: $I \neq \{\}$ *finite I*
and *F*: $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq$ *events* $\bigwedge i.\ i \in I \Longrightarrow$ *space* $M \in F\ i$
shows *indep-sets F I* $\longleftrightarrow$ $(\forall A{\in}Pi\ I\ F.$ *prob* $(\bigcap j{\in}I.\ A\ j) = (\prod j{\in}I.$ *prob* $(A\ j)))$
proof
assume $*$: *indep-sets F I*
from *I* **show** $\forall A{\in}Pi\ I\ F.$ *prob* $(\bigcap j{\in}I.\ A\ j) = (\prod j{\in}I.$ *prob* $(A\ j))$
by (*intro indep-setsD*[*OF* $*$] *ballI*) *auto*
next
assume *indep*: $\forall A{\in}Pi\ I\ F.$ *prob* $(\bigcap j{\in}I.\ A\ j) = (\prod j{\in}I.$ *prob* $(A\ j))$
show *indep-sets F I*
proof (*rule indep-setsI*[*OF F*(*1*)])
fix *A J* **assume** *J*: $J \neq \{\}$ $J \subseteq I$ *finite J*
assume *A*: $\forall j{\in}J.\ A\ j \in F\ j$
let *?A* $= \lambda j.$ *if* $j \in J$ *then A j else space M*
have *prob* $(\bigcap j{\in}I.$ *?A j*$) =$ *prob* $(\bigcap j{\in}J.\ A\ j)$
using *subset-trans*[*OF F*(*1*) *sets.space-closed*] *J A*
by (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *split*: *if-split-asm*) *blast*
also
from *A F* **have** $(\lambda j.$ *if* $j \in J$ *then A j else space M*$) \in Pi\ I\ F$ (**is** *?A* $\in$ -)
by (*auto split*: *if-split-asm*)
with *indep* **have** *prob* $(\bigcap j{\in}I.$ *?A j*$) = (\prod j{\in}I.$ *prob* $($*?A j*$))$
by *auto*
also have $\ldots = (\prod j{\in}J.$ *prob* $(A\ j))$
unfolding *if-distrib prod.If-cases*[*OF* ‹*finite I*›]
using *prob-space* ‹$J \subseteq I$› **by** (*simp add*: *Int-absorb1 prod.neutral-const*)
finally show *prob* $(\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.$ *prob* $(A\ j))$ **..**
qed
qed

lemma (**in** *prob-space*) *indep-vars-finite*:
fixes *I* :: *'i set*
assumes *I*: $I \neq \{\}$ *finite I*
and *M'*: $\bigwedge i.\ i \in I \Longrightarrow$ *sets* $(M'\ i) =$ *sigma-sets* $($*space* $(M'\ i))\ (E\ i)$
and *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)\ (X\ i)$
and *Int-stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(E\ i)$
and *space*: $\bigwedge i.\ i \in I \Longrightarrow$ *space* $(M'\ i) \in E\ i$ **and** *closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq$ *Pow* $($*space* $(M'\ i))$
shows *indep-vars M' X I* $\longrightarrow$
$(\forall A{\in}(\Pi\ i{\in}I.\ E\ i).$ *prob* $(\bigcap j{\in}I.\ X\ j -` A\ j \cap$ *space* $M) = (\prod j{\in}I.$ *prob* $(X\ j -` A\ j \cap$ *space* $M)))$

proof −
from *rv* **have** *X*: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in space\ M \to space\ (M'\ i)$
unfolding *measurable-def* **by** *simp*

{ fix *i* **assume** *i*∈*I*
from *closed*[*OF* ‹*i* ∈ *I*›]
have *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*M*′ *i*)}
= *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
unfolding *sigma-sets-vimage-commute*[*OF X*, *OF* ‹*i* ∈ *I*›, *symmetric*] *M*′[*OF* ‹*i* ∈ *I*›]
by (*subst sigma-sets-sigma-sets-eq*) *auto* **}**
note *sigma-sets-X* = *this*

{ fix *i* **assume** *i*∈*I*
have *Int-stable* {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
proof (*rule Int-stableI*)
fix *a* **assume** *a* ∈ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
then obtain *A* **where** *a* = *X i* −‘ *A* ∩ *space M A* ∈ *E i* **by** *auto*
moreover
fix *b* **assume** *b* ∈ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
then obtain *B* **where** *b* = *X i* −‘ *B* ∩ *space M B* ∈ *E i* **by** *auto*
moreover
have (*X i* −‘ *A* ∩ *space M*) ∩ (*X i* −‘ *B* ∩ *space M*) = *X i* −‘ (*A* ∩ *B*) ∩ *space M* **by** *auto*
moreover note *Int-stable*[*OF* ‹*i* ∈ *I*›]
ultimately
show *a* ∩ *b* ∈ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
by (*auto simp del*: *vimage-Int intro*!: *exI*[*of* - *A* ∩ *B*] *dest*: *Int-stableD*)
qed **}**
note *indep-sets-X* = *indep-sets-sigma-sets-iff*[*OF this*]

{ fix *i* **assume** *i* ∈ *I*
{ fix *A* **assume** *A* ∈ *E i*
with *M*′[*OF* ‹*i* ∈ *I*›] **have** *A* ∈ *sets* (*M*′ *i*) **by** *auto*
moreover
from *rv*[*OF* ‹*i*∈*I*›] **have** *X i* ∈ *measurable M* (*M*′ *i*) **by** *auto*
ultimately
have *X i* −‘ *A* ∩ *space M* ∈ *sets M* **by** (*auto intro*: *measurable-sets*) **}**
with *X*[*OF* ‹*i*∈*I*›] *space*[*OF* ‹*i*∈*I*›]
have {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*} ⊆ *events*
space M ∈ {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}
by (*auto intro*!: *exI*[*of* - *space* (*M*′ *i*)]) **}**
note *indep-sets-finite-X* = *indep-sets-finite*[*OF I this*]

have (∀ *A*∈Π *i*∈*I*. {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *E i*}. *prob* (⋂ (*A* ‘ *I*)) = (∏ *j*∈*I*. *prob* (*A j*))) =
(∀ *A*∈Π *i*∈*I*. *E i*. *prob* ((⋂ *j*∈*I*. *X j* −‘ *A j*) ∩ *space M*) = (∏ *x*∈*I*. *prob* (*X x* −‘ *A x* ∩ *space M*)))
(**is** *?L* = *?R*)

proof *safe*
fix A **assume** *?L* **and** A: $A \in (\Pi\ i{\in}I.\ E\ i)$
from ‹*?L*›[*THEN bspec*, *of* $\lambda i.\ X\ i -` A\ i \cap$ *space* M] A ‹$I \neq \{\}$›
show *prob* $((\bigcap j{\in}I.\ X\ j -` A\ j) \cap$ *space* $M) = (\prod x{\in}I.$ *prob* $(X\ x -` A\ x \cap$ *space* $M))$
by (*auto simp add*: *Pi-iff*)
next
fix A **assume** *?R* **and** A: $A \in (\Pi\ i{\in}I.\ \{X\ i -` A \cap$ *space* $M\ |A.\ A \in E\ i\})$
from A **have** $\forall\, i{\in}I.\ \exists B.\ A\ i = X\ i -` B \cap$ *space* $M \wedge B \in E\ i$ **by** *auto*
from *bchoice*[*OF this*] **obtain** B **where** B: $\forall\, i{\in}I.\ A\ i = X\ i -` B\ i \cap$ *space* M $B \in (\Pi\ i{\in}I.\ E\ i)$ **by** *auto*
from ‹*?R*›[*THEN bspec*, *OF* $B(2)$] $B(1)$ ‹$I \neq \{\}$›
show *prob* $(\bigcap (A\ `\ I)) = (\prod j{\in}I.$ *prob* $(A\ j))$
by *simp*
qed
then show *?thesis* **using** ‹$I \neq \{\}$›
by (*simp add*: *rv indep-vars-def indep-sets-X sigma-sets-X indep-sets-finite-X cong*: *indep-sets-cong*)
qed

lemma (**in** *prob-space*) *indep-vars-compose*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i.\ Y\ i \circ X\ i)\ I$
unfolding *indep-vars-def*
proof
from *rv* ‹*indep-vars* $M'\ X\ I$›
show $\forall\, i{\in}I.$ *random-variable* $(N\ i)\ (Y\ i \circ X\ i)$
by (*auto simp*: *indep-vars-def*)

have *indep-sets* $(\lambda i.$ *sigma-sets* $($*space* $M)\ \{X\ i -` A \cap$ *space* $M\ |A.\ A \in$ *sets* $(M'\ i)\})\ I$
using ‹*indep-vars* $M'\ X\ I$› **by** (*simp add*: *indep-vars-def*)
then show *indep-sets* $(\lambda i.$ *sigma-sets* $($*space* $M)\ \{(Y\ i \circ X\ i) -` A \cap$ *space* $M\ |A.\ A \in$ *sets* $(N\ i)\})\ I$
proof (*rule indep-sets-mono-sets*)
fix i **assume** $i \in I$
with ‹*indep-vars* $M'\ X\ I$› **have** X: $X\ i \in$ *space* $M \to$ *space* $(M'\ i)$
unfolding *indep-vars-def measurable-def* **by** *auto*
{ fix A **assume** $A \in$ *sets* $(N\ i)$
then have $\exists B.\ (Y\ i \circ X\ i) -` A \cap$ *space* $M = X\ i -` B \cap$ *space* $M \wedge B \in$ *sets* $(M'\ i)$
by (*intro exI*[*of* - $Y\ i -` A \cap$ *space* $(M'\ i)$])
(*auto simp*: *vimage-comp intro*!: *measurable-sets rv* ‹$i \in I$› *funcset-mem*[*OF X*]) **}**
then show *sigma-sets* $($*space* $M)\ \{(Y\ i \circ X\ i) -` A \cap$ *space* $M\ |A.\ A \in$ *sets* $(N\ i)\} \subseteq$
sigma-sets $($*space* $M)\ \{X\ i -` A \cap$ *space* $M\ |A.\ A \in$ *sets* $(M'\ i)\}$
by (*intro sigma-sets-subseteq*) (*auto simp*: *vimage-comp*)

```
  qed
qed

lemma (in prob-space) indep-vars-compose2:
  assumes indep-vars M' X I
  assumes rv: ⋀i. i ∈ I ⟹ Y i ∈ measurable (M' i) (N i)
  shows indep-vars N (λi x. Y i (X i x)) I
  using indep-vars-compose [OF assms] by (simp add: comp-def)

lemma (in prob-space) indep-var-compose:
  assumes indep-var M1 X1 M2 X2 Y1 ∈ measurable M1 N1 Y2 ∈ measurable
M2 N2
  shows indep-var N1 (Y1 ∘ X1) N2 (Y2 ∘ X2)
proof −
  have indep-vars (case-bool N1 N2) (λb. case-bool Y1 Y2 b ∘ case-bool X1 X2 b)
UNIV
    using assms
    by (intro indep-vars-compose[where M'=case-bool M1 M2])
       (auto simp: indep-var-def split: bool.split)
  also have (λb. case-bool Y1 Y2 b ∘ case-bool X1 X2 b) = case-bool (Y1 ∘ X1)
(Y2 ∘ X2)
    by (simp add: fun-eq-iff split: bool.split)
  finally show ?thesis
    unfolding indep-var-def .
qed

lemma (in prob-space) indep-vars-Min:
  fixes X :: 'i ⇒ 'a ⇒ real
  assumes I: finite I i ∉ I and indep: indep-vars (λ-. borel) X (insert i I)
  shows indep-var borel (X i) borel (λω. Min ((λi. X i ω)'I))
proof −
  have indep-var
    borel ((λf. f i) ∘ (λω. restrict (λi. X i ω) {i}))
    borel ((λf. Min (f'I)) ∘ (λω. restrict (λi. X i ω) I))
   using I by (intro indep-var-compose[OF indep-var-restrict[OF indep]] borel-measurable-Min)
auto
  also have ((λf. f i) ∘ (λω. restrict (λi. X i ω) {i})) = X i
    by auto
  also have ((λf. Min (f'I)) ∘ (λω. restrict (λi. X i ω) I)) = (λω. Min ((λi. X i
ω)'I))
    by (auto cong: rev-conj-cong)
  finally show ?thesis
    unfolding indep-var-def .
qed

lemma (in prob-space) indep-vars-sum:
  fixes X :: 'i ⇒ 'a ⇒ real
  assumes I: finite I i ∉ I and indep: indep-vars (λ-. borel) X (insert i I)
  shows indep-var borel (X i) borel (λω. ∑i∈I. X i ω)
```

proof −
have *indep-var*
borel ((λ*f*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) {*i*}))
borel ((λ*f*. ∑ *i*∈*I*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) *I*))
using *I* **by** (*intro indep-var-compose*[*OF indep-var-restrict*[*OF indep*]]) *auto*
also have ((λ*f*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) {*i*})) = *X i*
by *auto*
also have ((λ*f*. ∑ *i*∈*I*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) *I*)) = (λω. ∑ *i*∈*I*. *X i* ω)
by (*auto cong*: *rev-conj-cong*)
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *indep-vars-prod*:
fixes *X* :: *'i* ⇒ *'a* ⇒ *real*
assumes *I*: *finite I i* ∉ *I* **and** *indep*: *indep-vars* (λ-. *borel*) *X* (*insert i I*)
shows *indep-var borel* (*X i*) *borel* (λω. ∏ *i*∈*I*. *X i* ω)
proof −
have *indep-var*
borel ((λ*f*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) {*i*}))
borel ((λ*f*. ∏ *i*∈*I*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) *I*))
using *I* **by** (*intro indep-var-compose*[*OF indep-var-restrict*[*OF indep*]]) *auto*
also have ((λ*f*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) {*i*})) = *X i*
by *auto*
also have ((λ*f*. ∏ *i*∈*I*. *f i*) ∘ (λω. *restrict* (λ*i*. *X i* ω) *I*)) = (λω. ∏ *i*∈*I*. *X i* ω)
by (*auto cong*: *rev-conj-cong*)
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *indep-varsD-finite*:
assumes *X*: *indep-vars M' X I*
assumes *I*: *I* ≠ {} *finite I* ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M' i*)
shows *prob* (⋂ *i*∈*I*. *X i* −‘ *A i* ∩ *space M*) = (∏ *i*∈*I*. *prob* (*X i* −‘ *A i* ∩ *space M*))
proof (*rule indep-setsD*)
show *indep-sets* (λ*i*. *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*M' i*)}) *I*
using *X* **by** (*auto simp*: *indep-vars-def*)
show *I* ⊆ *I I* ≠ {} *finite I* **using** *I* **by** *auto*
show ∀ *i*∈*I*. *X i* −‘ *A i* ∩ *space M* ∈ *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*M' i*)}
using *I* **by** *auto*
qed

lemma (**in** *prob-space*) *indep-varsD*:
assumes *X*: *indep-vars M' X I*
assumes *I*: *J* ≠ {} *finite J J* ⊆ *I* ⋀*i*. *i* ∈ *J* ⟹ *A i* ∈ *sets* (*M' i*)
shows *prob* (⋂ *i*∈*J*. *X i* −‘ *A i* ∩ *space M*) = (∏ *i*∈*J*. *prob* (*X i* −‘ *A i* ∩ *space M*))
proof (*rule indep-setsD*)

show *indep-sets* (λ*i*. *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*M′ i*)}) *I*
using *X* **by** (*auto simp*: *indep-vars-def*)
show ∀ *i*∈*J*. *X i* −‘ *A i* ∩ *space M* ∈ *sigma-sets* (*space M*) {*X i* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets* (*M′ i*)}
using *I* **by** *auto*
qed *fact+*

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM*:
fixes *I* :: *'i set* **and** *X* :: *'i* ⇒ *'a* ⇒ *'b*
assumes *I* ≠ {}
assumes *rv*: ⋀*i*. *random-variable* (*M′ i*) (*X i*)
shows *indep-vars M′ X I* ⟷
distr M (Π$_M$ *i*∈*I*. *M′ i*) (λ*x*. λ*i*∈*I*. *X i x*) = (Π$_M$ *i*∈*I*. *distr M* (*M′ i*) (*X i*))
proof −
let *?P* = Π$_M$ *i*∈*I*. *M′ i*
let *?X* = λ*x*. λ*i*∈*I*. *X i x*
let *?D* = *distr M ?P ?X*
have *X*: *random-variable ?P ?X* **by** (*intro measurable-restrict rv*)
interpret *D*: *prob-space ?D* **by** (*intro prob-space-distr X*)

let *?D′* = λ*i*. *distr M* (*M′ i*) (*X i*)
let *?P′* = Π$_M$ *i*∈*I*. *distr M* (*M′ i*) (*X i*)
interpret *D′*: *prob-space ?D′ i* **for** *i* **by** (*intro prob-space-distr rv*)
interpret *P*: *product-prob-space ?D′ I* **..**

show *?thesis*
proof
assume *indep-vars M′ X I*
show *?D* = *?P′*
proof (*rule measure-eqI-generator-eq*)
show *Int-stable* (*prod-algebra I M′*)
by (*rule Int-stable-prod-algebra*)
show *prod-algebra I M′* ⊆ *Pow* (*space ?P*)
using *prod-algebra-sets-into-space* **by** (*simp add*: *space-PiM*)
show *sets ?D* = *sigma-sets* (*space ?P*) (*prod-algebra I M′*)
by (*simp add*: *sets-PiM space-PiM*)
show *sets ?P′* = *sigma-sets* (*space ?P*) (*prod-algebra I M′*)
by (*simp add*: *sets-PiM space-PiM cong*: *prod-algebra-cong*)
let *?A* = λ*i*. Π$_E$ *i*∈*I*. *space* (*M′ i*)
show *range ?A* ⊆ *prod-algebra I M′* (⋃ *i*. *?A i*) = *space* (*Pi*$_M$ *I M′*)
by (*auto simp*: *space-PiM intro*!: *space-in-prod-algebra cong*: *prod-algebra-cong*)
{ fix *i* **show** *emeasure ?D* (Π$_E$ *i*∈*I*. *space* (*M′ i*)) ≠ ∞ **by** *auto* **}**
next
fix *E* **assume** *E*: *E* ∈ *prod-algebra I M′*
from *prod-algebraE*[*OF E*] **obtain** *J Y*
where *J*:
E = *prod-emb I M′ J* (*Pi*$_E$ *J Y*)
finite J

$J \neq \{\} \vee I = \{\}$
$J \subseteq I$
$\bigwedge i.\ i \in J \Longrightarrow Y\ i \in sets\ (M'\ i)$
by *auto*
from E **have** $E \in sets\ ?P$ **by** (*auto simp*: *sets-PiM*)
then have *emeasure ?D E* = *emeasure M* (*?X* −‘ $E \cap space\ M$)
by (*simp add*: *emeasure-distr X*)
also have *?X* −‘ $E \cap space\ M = (\bigcap i \in J.\ X\ i$ −‘ $Y\ i \cap space\ M)$
using J ‹$I \neq \{\}$› *measurable-space*[*OF rv*] **by** (*auto simp*: *prod-emb-def PiE-iff split*: *if-split-asm*)
also have *emeasure M* $(\bigcap i \in J.\ X\ i$ −‘ $Y\ i \cap space\ M) = (\prod\ i \in J.\ emeasure\ M\ (X\ i$ −‘ $Y\ i \cap space\ M))$
using ‹*indep-vars M′ X I*› J ‹$I \neq \{\}$› **using** *indep-varsD*[*of M′ X I J*]
by (*auto simp*: *emeasure-eq-measure prod-ennreal measure-nonneg prod-nonneg*)
also have ... = $(\prod\ i \in J.\ emeasure\ (?D'\ i)\ (Y\ i))$
using *rv J* **by** (*simp add*: *emeasure-distr*)
also have ... = *emeasure ?P′ E*
using *P.emeasure-PiM-emb*[*of J Y*] J **by** (*simp add*: *prod-emb-def*)
finally show *emeasure ?D E* = *emeasure ?P′ E* **.**
qed
next
assume *?D* = *?P′*
show *indep-vars M′ X I* **unfolding** *indep-vars-def*
proof (*intro conjI indep-setsI ballI rv*)
fix i **show** *sigma-sets* (*space M*) $\{X\ i$ −‘ $A \cap space\ M \mid A.\ A \in sets\ (M'\ i)\} \subseteq events$
by (*auto intro*!: *sets.sigma-sets-subset measurable-sets rv*)
next
fix $J\ Y'$ **assume** J: $J \neq \{\}\ J \subseteq I$ *finite J*
assume Y': $\forall j \in J.\ Y'\ j \in sigma\text{-}sets\ (space\ M)\ \{X\ j$ −‘ $A \cap space\ M \mid A.\ A \in sets\ (M'\ j)\}$
have $\forall j \in J.\ \exists Y.\ Y'\ j = X\ j$ −‘ $Y \cap space\ M \wedge Y \in sets\ (M'\ j)$
proof
fix j **assume** $j \in J$
from Y'[*rule-format*, *OF this*] *rv*[*of j*]
show $\exists Y.\ Y'\ j = X\ j$ −‘ $Y \cap space\ M \wedge Y \in sets\ (M'\ j)$
by (*subst* (*asm*) *sigma-sets-vimage-commute*[*symmetric*, *of - - space* $(M'\ j)$])
(*auto dest*: *measurable-space simp*: *sets.sigma-sets-eq*)
qed
from *bchoice*[*OF this*] **obtain** Y **where**
Y: $\bigwedge j.\ j \in J \Longrightarrow Y'\ j = X\ j$ −‘ $Y\ j \cap space\ M\ \bigwedge j.\ j \in J \Longrightarrow Y\ j \in sets\ (M'\ j)$ **by** *auto*
let *?E* = *prod-emb I M′ J* $(Pi_E\ J\ Y)$
from Y **have** $(\bigcap j \in J.\ Y'\ j) =$ *?X* −‘ *?E* ∩ *space M*
using J ‹$I \neq \{\}$› *measurable-space*[*OF rv*] **by** (*auto simp*: *prod-emb-def PiE-iff split*: *if-split-asm*)
then have *emeasure M* $(\bigcap j \in J.\ Y'\ j) =$ *emeasure M* (*?X* −‘ *?E* ∩ *space M*)
by *simp*

```
    also have ... = emeasure ?D ?E
      using Y J by (intro emeasure-distr[symmetric] X sets-PiM-I) auto
    also have ... = emeasure ?P' ?E
      using ‹?D = ?P'› by simp
    also have ... = (∏ i∈J. emeasure (?D' i) (Y i))
      using P.emeasure-PiM-emb[of J Y] J Y by (simp add: prod-emb-def)
    also have ... = (∏ i∈J. emeasure M (Y' i))
      using rv J Y by (simp add: emeasure-distr)
    finally have emeasure M (⋂j∈J. Y' j) = (∏ i∈J. emeasure M (Y' i)) .
    then show prob (⋂j∈J. Y' j) = (∏ i∈J. prob (Y' i))
    by (auto simp: emeasure-eq-measure prod-ennreal measure-nonneg prod-nonneg)
  qed
 qed
qed

lemma (in prob-space) indep-vars-iff-distr-eq-PiM':
 fixes I :: 'i set and X :: 'i ⇒ 'a ⇒ 'b
 assumes I ≠ {}
 assumes rv: ⋀i. i ∈ I ⟹ random-variable (M' i) (X i)
 shows indep-vars M' X I ⟷
        distr M (Π_M i∈I. M' i) (λx. λi∈I. X i x) = (Π_M i∈I. distr M (M' i)
(X i))
proof −
 from assms obtain j where j: j ∈ I
   by auto
 define N' where N' = (λi. if i ∈ I then M' i else M' j)
 define Y where Y = (λi. if i ∈ I then X i else X j)
 have rv: random-variable (N' i) (Y i) for i
   using j by (auto simp: N'-def Y-def intro: assms)

 have indep-vars M' X I = indep-vars N' Y I
   by (intro indep-vars-cong) (auto simp: N'-def Y-def)
 also have ... ⟷ distr M (Π_M i∈I. N' i) (λx. λi∈I. Y i x) = (Π_M i∈I. distr
M (N' i) (Y i))
   by (intro indep-vars-iff-distr-eq-PiM rv assms)
 also have (Π_M i∈I. N' i) = (Π_M i∈I. M' i)
   by (intro PiM-cong) (simp-all add: N'-def)
 also have (λx. λi∈I. Y i x) = (λx. λi∈I. X i x)
   by (simp-all add: Y-def fun-eq-iff)
 also have (Π_M i∈I. distr M (N' i) (Y i)) = (Π_M i∈I. distr M (M' i) (X i))
   by (intro PiM-cong distr-cong) (simp-all add: N'-def Y-def)
 finally show ?thesis .
qed

lemma (in prob-space) indep-varD:
 assumes indep: indep-var Ma A Mb B
 assumes sets: Xa ∈ sets Ma Xb ∈ sets Mb
 shows prob ((λx. (A x, B x)) −‘ (Xa × Xb) ∩ space M) =
   prob (A −‘ Xa ∩ space M) ∗ prob (B −‘ Xb ∩ space M)
```

proof −
have *prob* ((λ*x*. (*A x*, *B x*)) −‘ (*Xa* × *Xb*) ∩ *space M*) =
prob (⋂ *i*∈*UNIV*. (*case-bool A B i* −‘ *case-bool Xa Xb i* ∩ *space M*))
by (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *simp*: *UNIV-bool*)
also have ... = (∏ *i*∈*UNIV*. *prob* (*case-bool A B i* −‘ *case-bool Xa Xb i* ∩ *space M*))
using *indep* **unfolding** *indep-var-def*
by (*rule indep-varsD*) (*auto split*: *bool.split intro*: *sets*)
also have ... = *prob* (*A* −‘ *Xa* ∩ *space M*) ∗ *prob* (*B* −‘ *Xb* ∩ *space M*)
unfolding *UNIV-bool* **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *prob-indep-random-variable*:
assumes *ind*[*simp*]: *indep-var N X N Y*
assumes [*simp*]: *A* ∈ *sets N B* ∈ *sets N*
shows $\mathcal{P}$(*x in M*. *X x* ∈ *A* ∧ *Y x* ∈ *B*) = $\mathcal{P}$(*x in M*. *X x* ∈ *A*) ∗ $\mathcal{P}$(*x in M*. *Y x* ∈ *B*)
proof−
have $\mathcal{P}$(*x in M*. (*X x*)∈*A* ∧ (*Y x*)∈ *B*) = *prob* ((λ*x*. (*X x*, *Y x*)) −‘ (*A* × *B*) ∩ *space M*)
by (*auto intro*!: *arg-cong*[**where** *f*= *prob*])
also have ...= *prob* (*X* −‘ *A* ∩ *space M*) ∗ *prob* (*Y* −‘ *B* ∩ *space M*)
by (*auto intro*!: *indep-varD*[**where** *Ma*=*N* **and** *Mb*=*N*])
also have ... = $\mathcal{P}$(*x in M*. *X x* ∈ *A*) ∗ $\mathcal{P}$(*x in M*. *Y x* ∈ *B*)
by (*auto intro*!: *arg-cong2*[**where** *f*= (∗)] *arg-cong*[**where** *f*= *prob*])
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*)
assumes *indep-var S X T Y*
shows *indep-var-rv1*: *random-variable S X*
and *indep-var-rv2*: *random-variable T Y*
proof −
have ∀ *i*∈*UNIV*. *random-variable* (*case-bool S T i*) (*case-bool X Y i*)
using *assms* **unfolding** *indep-var-def indep-vars-def* **by** *auto*
then show *random-variable S X random-variable T Y*
unfolding *UNIV-bool* **by** *auto*
qed

lemma (**in** *prob-space*) *indep-var-distribution-eq*:
indep-var S X T Y ⟷ *random-variable S X* ∧ *random-variable T Y* ∧
distr M S X ⨂$_M$ *distr M T Y* = *distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) (**is** - ⟷ - ∧ - ∧ *?S* ⨂$_M$ *?T* = *?J*)
proof *safe*
assume *indep-var S X T Y*
then show *rvs*: *random-variable S X random-variable T Y*
by (*blast dest*: *indep-var-rv1 indep-var-rv2*)+
then have *XY*: *random-variable* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*))

by (*rule measurable-Pair*)

interpret *X*: *prob-space ?S* **by** (*rule prob-space-distr*) *fact*
interpret *Y*: *prob-space ?T* **by** (*rule prob-space-distr*) *fact*
interpret *XY*: *pair-prob-space ?S ?T* **..**
show *?S* $\bigotimes_M$ *?T* = *?J*
proof (*rule pair-measure-eqI*)
show *sigma-finite-measure ?S* **..**
show *sigma-finite-measure ?T* **..**

fix *A B* **assume** *A*: *A* ∈ *sets ?S* **and** *B*: *B* ∈ *sets ?T*
have *emeasure ?J* (*A* × *B*) = *emeasure M* ((λ*x*. (*X x*, *Y x*)) −‘ (*A* × *B*) ∩ *space M*)
using *A B* **by** (*intro emeasure-distr*[*OF XY*]) *auto*
also have ... = *emeasure M* (*X* −‘ *A* ∩ *space M*) ∗ *emeasure M* (*Y* −‘ *B* ∩ *space M*)
using *indep-varD*[*OF* ‹*indep-var S X T Y*›, *of A B*] *A B*
by (*simp add*: *emeasure-eq-measure measure-nonneg ennreal-mult*)
also have ... = *emeasure ?S A* ∗ *emeasure ?T B*
using *rvs A B* **by** (*simp add*: *emeasure-distr*)
finally show *emeasure ?S A* ∗ *emeasure ?T B* = *emeasure ?J* (*A* × *B*) **by** *simp*
qed *simp*
next
assume *rvs*: *random-variable S X random-variable T Y*
then have *XY*: *random-variable* (*S* $\bigotimes_M$ *T*) (λ*x*. (*X x*, *Y x*))
by (*rule measurable-Pair*)

let *?S* = *distr M S X* **and** *?T* = *distr M T Y*
interpret *X*: *prob-space ?S* **by** (*rule prob-space-distr*) *fact*
interpret *Y*: *prob-space ?T* **by** (*rule prob-space-distr*) *fact*
interpret *XY*: *pair-prob-space ?S ?T* **..**

assume *?S* $\bigotimes_M$ *?T* = *?J*

{ fix *S* **and** *X*
have *Int-stable* {*X* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets S*}
proof (*safe intro*!: *Int-stableI*)
fix *A B* **assume** *A* ∈ *sets S B* ∈ *sets S*
then show ∃ *C*. (*X* −‘ *A* ∩ *space M*) ∩ (*X* −‘ *B* ∩ *space M*) = (*X* −‘ *C* ∩ *space M*) ∧ *C* ∈ *sets S*
by (*intro exI*[*of* - *A* ∩ *B*]) *auto*
qed }
note *Int-stable* = *this*

show *indep-var S X T Y* **unfolding** *indep-var-eq*
proof (*intro conjI indep-set-sigma-sets Int-stable rvs*)
show *indep-set* {*X* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets S*} {*Y* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets T*}

proof (*safe intro*!: *indep-setI*)
{ fix A **assume** $A \in$ *sets* S **then show** $X -` A \cap$ *space* $M \in$ *sets* M
using ‹$X \in$ *measurable* M S› **by** (*auto intro*: *measurable-sets*) **}**
{ fix A **assume** $A \in$ *sets* T **then show** $Y -` A \cap$ *space* $M \in$ *sets* M
using ‹$Y \in$ *measurable* M T› **by** (*auto intro*: *measurable-sets*) **}**
next
fix A B **assume** *ab*: $A \in$ *sets* S $B \in$ *sets* T
then have *prob* $((X -` A \cap$ *space* $M) \cap (Y -` B \cap$ *space* $M)) =$ *emeasure* *?J* $(A \times B)$
using XY **by** (*auto simp add*: *emeasure-distr emeasure-eq-measure measure-nonneg intro*!: *arg-cong*[**where** f=*prob*])
also have $\ldots =$ *emeasure* $(?S \bigotimes_M ?T)$ $(A \times B)$
unfolding ‹$?S \bigotimes_M ?T = ?J$› **..**
also have $\ldots =$ *emeasure* *?S* $A *$ *emeasure* *?T* B
using *ab* **by** (*simp add*: *Y.emeasure-pair-measure-Times*)
finally show *prob* $((X -` A \cap$ *space* $M) \cap (Y -` B \cap$ *space* $M)) =$
prob $(X -` A \cap$ *space* $M) *$ *prob* $(Y -` B \cap$ *space* $M)$
using *rvs ab* **by** (*simp add*: *emeasure-eq-measure emeasure-distr measure-nonneg ennreal-mult*[*symmetric*])
qed
qed
qed

lemma (**in** *prob-space*) *distributed-joint-indep*:
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes X: *distributed* M S X *Px* **and** Y: *distributed* M T Y *Py*
assumes *indep*: *indep-var* S X T Y
shows *distributed* M $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $(\lambda(x,\ y).\ Px\ x * Py\ y)$
using *indep-var-distribution-eq*[*of* S X T Y] *indep*
by (*intro distributed-joint-indep'*[*OF* S T X Y]) *auto*

lemma (**in** *prob-space*) *indep-vars-nn-integral*:
assumes I: *finite* I *indep-vars* $(\lambda\text{-}.\ borel)$ X I $\bigwedge i\ \omega.\ i \in I \Longrightarrow 0 \le X\ i\ \omega$
shows $(\int^+ \omega.\ (\prod i \in I.\ X\ i\ \omega)\ \partial M) = (\prod i \in I.\ \int^+ \omega.\ X\ i\ \omega\ \partial M)$
proof *cases*
assume $I \neq \{\}$
define Y **where** [*abs-def*]: $Y\ i\ \omega = (\textit{if } i \in I \textit{ then } X\ i\ \omega \textit{ else } 0)$ **for** $i\ \omega$
{ fix i **have** $i \in I \Longrightarrow$ *random-variable borel* $(X\ i)$
using $I(2)$ **by** (*cases* $i \in I$) (*auto simp*: *indep-vars-def*) **}**
note *rv-X* = *this*

{ fix i **have** *random-variable borel* $(Y\ i)$
using $I(2)$ **by** (*cases* $i \in I$) (*auto simp*: *Y-def rv-X*) **}**
note *rv-Y* = *this*[*measurable*]

interpret Y: *prob-space distr* M *borel* $(Y\ i)$ **for** i
using $I(2)$ **by** (*cases* $i \in I$) (*auto intro*!: *prob-space-distr simp*: *indep-vars-def prob-space-return*)
interpret *product-sigma-finite* $\lambda i.$ *distr* M *borel* $(Y\ i)$

..

have *indep-Y*: *indep-vars* (*λi. borel*) *Y I*
by (*rule indep-vars-cong*[*THEN iffD1*, *OF - - - I*(*2*)]) (*auto simp*: *Y-def*)

have $(\int^+\omega.\ (\prod i{\in}I.\ X\ i\ \omega)\ \partial M) = (\int^+\omega.\ (\prod i{\in}I.\ Y\ i\ \omega)\ \partial M)$
using *I*(*3*) **by** (*auto intro*!: *nn-integral-cong prod.cong simp add*: *Y-def max-def*)
also have ... = $(\int^+\omega.\ (\prod i{\in}I.\ \omega\ i)\ \partial distr\ M\ (Pi_M\ I\ (\lambda i.\ borel))\ (\lambda x.\ \lambda i{\in}I.\ Y\ i\ x))$
by (*subst nn-integral-distr*) *auto*
also have ... = $(\int^+\omega.\ (\prod i{\in}I.\ \omega\ i)\ \partial Pi_M\ I\ (\lambda i.\ distr\ M\ borel\ (Y\ i)))$
unfolding *indep-vars-iff-distr-eq-PiM*[*THEN iffD1*, *OF* ‹*I* ≠ {}› *rv-Y indep-Y*]
..
also have ... = $(\prod i{\in}I.\ (\int^+\omega.\ \omega\ \partial distr\ M\ borel\ (Y\ i)))$
by (*rule product-nn-integral-prod*) (*auto intro*: ‹*finite I*›)
also have ... = $(\prod i{\in}I.\ \int^+\omega.\ X\ i\ \omega\ \partial M)$
by (*intro prod.cong nn-integral-cong*) (*auto simp*: *nn-integral-distr Y-def rv-X*)
finally show *?thesis* **.**
qed (*simp add*: *emeasure-space-1*)

lemma (**in** *prob-space*)
fixes *X* :: *'i* ⇒ *'a* ⇒ *'b*::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *I*: *finite I indep-vars* (*λ-. borel*) *X I* $\bigwedge i.\ i \in I \Longrightarrow integrable\ M\ (X\ i)$
shows *indep-vars-lebesgue-integral*: $(\int\omega.\ (\prod i{\in}I.\ X\ i\ \omega)\ \partial M) = (\prod i{\in}I.\ \int\omega.\ X\ i\ \omega\ \partial M)$ (**is** *?eq*)
and *indep-vars-integrable*: *integrable M* $(\lambda\omega.\ (\prod i{\in}I.\ X\ i\ \omega))$ (**is** *?int*)
proof (*induct rule*: *case-split*)
assume *I* ≠ {}
define *Y* **where** [*abs-def*]: *Y i ω* = (*if i* ∈ *I then X i ω else 0*) **for** *i ω*
{ fix *i* **have** *i* ∈ *I* ⟹ *random-variable borel* (*X i*)
using *I*(*2*) **by** (*cases i*∈*I*) (*auto simp*: *indep-vars-def*) **}**
note *rv-X* = *this*[*measurable*]

{ fix *i* **have** *random-variable borel* (*Y i*)
using *I*(*2*) **by** (*cases i*∈*I*) (*auto simp*: *Y-def rv-X*) **}**
note *rv-Y* = *this*[*measurable*]

{ fix *i* **have** *integrable M* (*Y i*)
using *I*(*3*) **by** (*cases i*∈*I*) (*auto simp*: *Y-def*) **}**
note *int-Y* = *this*

interpret *Y*: *prob-space distr M borel* (*Y i*) **for** *i*
using *I*(*2*) **by** (*cases i* ∈ *I*) (*auto intro*!: *prob-space-distr simp*: *indep-vars-def prob-space-return*)
interpret *product-sigma-finite λi. distr M borel* (*Y i*)
..

have *indep-Y*: *indep-vars* (*λi. borel*) *Y I*
by (*rule indep-vars-cong*[*THEN iffD1*, *OF - - - I*(*2*)]) (*auto simp*: *Y-def*)

```
  have (∫ ω. (∏ i∈I. X i ω) ∂M) = (∫ ω. (∏ i∈I. Y i ω) ∂M)
    using I(3) by (simp add: Y-def)
  also have ... = (∫ ω. (∏ i∈I. ω i) ∂distr M (Pi_M I (λi. borel)) (λx. λi∈I. Y i
x))
    by (subst integral-distr) auto
  also have ... = (∫ ω. (∏ i∈I. ω i) ∂Pi_M I (λi. distr M borel (Y i)))
   unfolding indep-vars-iff-distr-eq-PiM[THEN iffD1, OF ‹I ≠ {}› rv-Y indep-Y]
..
  also have ... = (∏ i∈I. (∫ ω. ω ∂distr M borel (Y i)))
    by (rule product-integral-prod) (auto intro: ‹finite I› simp: integrable-distr-eq
int-Y)
  also have ... = (∏ i∈I. ∫ ω. X i ω ∂M)
    by (intro prod.cong integral-cong)
       (auto simp: integral-distr Y-def rv-X)
  finally show ?eq .

  have integrable (distr M (Pi_M I (λi. borel)) (λx. λi∈I. Y i x)) (λω. (∏ i∈I. ω
i))
   unfolding indep-vars-iff-distr-eq-PiM[THEN iffD1, OF ‹I ≠ {}› rv-Y indep-Y]
    by (intro product-integrable-prod[OF ‹finite I›])
       (simp add: integrable-distr-eq int-Y)
  then show ?int
    by (simp add: integrable-distr-eq Y-def)
qed (simp-all add: prob-space)

lemma (in prob-space)
  fixes X1 X2 :: 'a ⇒ 'b::{real-normed-field, banach, second-countable-topology}
  assumes indep-var borel X1 borel X2 integrable M X1 integrable M X2
  shows indep-var-lebesgue-integral: (∫ ω. X1 ω * X2 ω ∂M) = (∫ ω. X1 ω ∂M)
* (∫ ω. X2 ω ∂M) (is ?eq)
    and indep-var-integrable: integrable M (λω. X1 ω * X2 ω) (is ?int)
unfolding indep-var-def
proof −
  have *: (λω. X1 ω * X2 ω) = (λω. ∏ i∈UNIV. (case-bool X1 X2 i ω))
    by (simp add: UNIV-bool mult.commute)
  have **: (λ -. borel) = case-bool borel borel
    by (rule ext, metis (full-types) bool.simps(3) bool.simps(4))
  show ?eq
    apply (subst *)
    apply (subst indep-vars-lebesgue-integral)
    apply (auto)
    apply (subst **, subst indep-var-def [symmetric], rule assms)
    apply (simp split: bool.split add: assms)
    by (simp add: UNIV-bool mult.commute)
  show ?int
    apply (subst *)
    apply (rule indep-vars-integrable)
    apply auto
```

apply (*subst* ∗∗, *subst indep-var-def* [*symmetric*], *rule assms*)
by (*simp split*: *bool.split add*: *assms*)
qed

end

11 Convolution Measure

theory *Convolution*
imports *Independent-Family*
begin

lemma (**in** *finite-measure*) *sigma-finite-measure*: *sigma-finite-measure M*
..

definition *convolution* :: ('*a* :: *ordered-euclidean-space*) *measure* ⇒ '*a measure* ⇒ '*a measure* (**infix** ‹⋆› *50*) **where**
convolution M N = *distr* (*M* ⨂$_M$ *N*) *borel* (λ(*x*, *y*). *x* + *y*)

lemma
shows *space-convolution*[*simp*]: *space* (*convolution M N*) = *space borel*
and *sets-convolution*[*simp*]: *sets* (*convolution M N*) = *sets borel*
and *measurable-convolution1*[*simp*]: *measurable A* (*convolution M N*) = *measurable A borel*
and *measurable-convolution2*[*simp*]: *measurable* (*convolution M N*) *B* = *measurable borel B*
by (*simp-all add*: *convolution-def*)

lemma *nn-integral-convolution*:
assumes *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*measurable*]: *f* ∈ *borel-measurable borel*
shows (∫⁺*x*. *f x* ∂*convolution M N*) = (∫⁺*x*. ∫⁺*y*. *f* (*x* + *y*) ∂*N* ∂*M*)
proof −
interpret *M*: *finite-measure M* **by** *fact*
interpret *N*: *finite-measure N* **by** *fact*
interpret *pair-sigma-finite M N* **..**
show *?thesis*
unfolding *convolution-def*
by (*simp add*: *nn-integral-distr N*.*nn-integral-fst*[*symmetric*])
qed

lemma *convolution-emeasure*:
assumes *A* ∈ *sets borel finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*simp*]: *space M* = *space N space N* = *space borel*
shows *emeasure* (*M* ⋆ *N*) *A* = ∫⁺*x*. (*emeasure N* {*a*. *a* + *x* ∈ *A*}) ∂*M*
using *assms* **by** (*auto intro*!: *nn-integral-cong simp del*: *nn-integral-indicator simp*: *nn-integral-convolution*

nn-integral-indicator [*symmetric*] *ac-simps split*:*split-indicator*)

lemma *convolution-emeasure'*:
assumes [*simp*]:$A \in$ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows *emeasure* (*M* ⋆ *N*) *A* = $\int^+ x.\ \int^+ y.$ (*indicator A* (*x* + *y*)) *∂N ∂M*
by (*auto simp del*: *nn-integral-indicator simp*: *nn-integral-convolution*
nn-integral-indicator[*symmetric*] *borel-measurable-indicator*)

lemma *convolution-finite*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
shows *finite-measure* (*M* ⋆ *N*)
unfolding *convolution-def*
by (*intro finite-measure-pair-measure finite-measure.finite-measure-distr*) *auto*

lemma *convolution-emeasure-3*:
assumes [*simp*, *measurable*]: *A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* (*L* ⋆ (*M* ⋆ *N*)) *A* = $\int^+ x.\ \int^+ y.\ \int^+ z.$ *indicator A* (*x* + *y* + *z*) *∂N ∂M ∂L*
apply (*subst nn-integral-indicator*[*symmetric*], *simp*)
apply (*subst nn-integral-convolution*,
auto intro!: *borel-measurable-indicator borel-measurable-indicator' convolution-finite*)+
by (*rule nn-integral-cong*)+ (*auto simp*: *semigroup-add-class.add.assoc*)

lemma *convolution-emeasure-3'*:
assumes [*simp*, *measurable*]:*A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* ((*L* ⋆ *M*) ⋆ *N*) *A* = $\int^+ x.\ \int^+ y.\ \int^+ z.$ *indicator A* (*x* + *y* + *z*) *∂N ∂M ∂L*
apply (*subst nn-integral-indicator*[*symmetric*], *simp*)+
apply (*subst nn-integral-convolution*)
apply (*simp-all add*: *convolution-finite*)
apply (*subst nn-integral-convolution*)
apply (*simp-all add*: *finite-measure.sigma-finite-measure sigma-finite-measure.borel-measurable-nn-integral*)
done

lemma *convolution-commutative*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows (*M* ⋆ *N*) = (*N* ⋆ *M*)
proof (*rule measure-eqI*)
interpret *M*: *finite-measure M* **by** *fact*

interpret N: *finite-measure N* **by** *fact*
interpret *pair-sigma-finite M N* **..**

show *sets* $(M \star N) =$ *sets* $(N \star M)$ **by** *simp*

fix A **assume** $A \in$ *sets* $(M \star N)$
then have *1*[*measurable*]:$A \in$ *sets borel* **by** *simp*
have *emeasure* $(M \star N)$ $A = \int^+ x.$ $\int^+ y.$ *indicator* A $(x + y)$ ∂N ∂M **by** (*auto intro*!: *convolution-emeasure'*)
also have ... $= \int^+ x.$ $\int^+ y.$ $(\lambda(x,y).$ *indicator* A $(x + y))$ (x, y) ∂N ∂M **by** (*auto intro*!: *nn-integral-cong*)
also have ... $= \int^+ y.$ $\int^+ x.$ $(\lambda(x,y).$ *indicator* A $(x + y))$ (x, y) ∂M ∂N **by** (*rule Fubini*[*symmetric*]) *simp*
also have ... = *emeasure* $(N \star M)$ A **by** (*auto intro*!: *nn-integral-cong simp*: *add.commute convolution-emeasure'*)
finally show *emeasure* $(M \star N)$ $A =$ *emeasure* $(N \star M)$ A **by** *simp*
qed

lemma *convolution-associative*:
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N = sets borel sets M = sets borel sets L = sets borel*
shows $(L \star (M \star N)) = ((L \star M) \star N)$
by (*auto intro*!: *measure-eqI simp*: *convolution-emeasure-3 convolution-emeasure-3'*)

lemma (**in** *prob-space*) *sum-indep-random-variable*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable borel X*
assumes [*simp*, *measurable*]: *random-variable borel Y*
shows *distr M borel* $(\lambda x.$ X $x +$ Y $x)$ = *convolution* (*distr M borel X*) (*distr M borel Y*)
using *ind* **unfolding** *indep-var-distribution-eq convolution-def*
by (*auto simp*: *distr-distr intro*!:*arg-cong*[**where** $f =$ *distr M borel*])

lemma (**in** *prob-space*) *sum-indep-random-variable-lborel*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable lborel X*
assumes [*simp*, *measurable*]:*random-variable lborel Y*
shows *distr M lborel* $(\lambda x.$ X $x +$ Y $x)$ = *convolution* (*distr M lborel X*) (*distr M lborel Y*)
using *ind* **unfolding** *indep-var-distribution-eq convolution-def*
by (*auto simp*: *distr-distr o-def intro*!: *arg-cong*[**where** $f =$ *distr M borel*] *cong*: *distr-cong*)

lemma *convolution-density*:
fixes f g :: *real* $\Rightarrow$ *ennreal*
assumes [*measurable*]: $f \in$ *borel-measurable borel* $g \in$ *borel-measurable borel*
assumes [*simp*]:*finite-measure* (*density lborel f*) *finite-measure* (*density lborel g*)
shows *density lborel f* $\star$ *density lborel g* = *density lborel* $(\lambda x.$ $\int^+ y.$ f $(x - y) *$ g y ∂*lborel*)

```
    (is ?l = ?r)
proof (intro measure-eqI)
  fix A assume A ∈ sets ?l
  then have [measurable]: A ∈ sets borel
    by simp

  have (∫⁺x. f x * (∫⁺y. g y * indicator A (x + y) ∂lborel) ∂lborel) =
    (∫⁺x. (∫⁺y. g y * (f x * indicator A (x + y)) ∂lborel) ∂lborel)
  proof (intro nn-integral-cong-AE, eventually-elim)
    fix x
    have f x * (∫⁺ y. g y * indicator A (x + y) ∂lborel) =
      (∫⁺ y. f x * (g y * indicator A (x + y)) ∂lborel)
      by (intro nn-integral-cmult[symmetric]) auto
    then show f x * (∫⁺ y. g y * indicator A (x + y) ∂lborel) =
      (∫⁺ y. g y * (f x * indicator A (x + y)) ∂lborel)
      by (simp add: ac-simps)
  qed
  also have ... = (∫⁺y. (∫⁺x. g y * (f x * indicator A (x + y)) ∂lborel) ∂lborel)
    by (intro lborel-pair.Fubini') simp
  also have ... = (∫⁺y. (∫⁺x. f (x − y) * g y * indicator A x ∂lborel) ∂lborel)
  proof (intro nn-integral-cong-AE, eventually-elim)
    fix y
    have (∫⁺x. g y * (f x * indicator A (x + y)) ∂lborel) =
      g y * (∫⁺x. f x * indicator A (x + y) ∂lborel)
      by (intro nn-integral-cmult) auto
    also have ... = g y * (∫⁺x. f (x − y) * indicator A x ∂lborel)
      by (subst nn-integral-real-affine[where c=1 and t=−y])
        (auto simp add: one-ennreal-def[symmetric])
    also have ... = (∫⁺x. g y * (f (x − y) * indicator A x) ∂lborel)
      by (intro nn-integral-cmult[symmetric]) auto
    finally show (∫⁺ x. g y * (f x * indicator A (x + y)) ∂lborel) =
      (∫⁺ x. f (x − y) * g y * indicator A x ∂lborel)
      by (simp add: ac-simps)
  qed
  also have ... = (∫⁺x. (∫⁺y. f (x − y) * g y * indicator A x ∂lborel) ∂lborel)
    by (intro lborel-pair.Fubini') simp
  finally show emeasure ?l A = emeasure ?r A
    by (auto simp: convolution-emeasure' nn-integral-density emeasure-density
      nn-integral-multc)
qed simp

lemma (in prob-space) distributed-finite-measure-density:
  distributed M N X f ⟹ finite-measure (density N f)
  using finite-measure-distr[of X N] distributed-distr-eq-density[of M N X f] by
simp


lemma (in prob-space) distributed-convolution:
  fixes f :: real ⇒ -
```

fixes $g :: real \Rightarrow -$
assumes *indep*: *indep-var borel X borel Y*
assumes *X*: *distributed M lborel X f*
assumes *Y*: *distributed M lborel Y g*
shows *distributed M lborel* $(\lambda x.\ X\ x + Y\ x)$ $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
unfolding *distributed-def*
proof *safe*
have *fg*[*measurable*]: $f \in$ *borel-measurable borel* $g \in$ *borel-measurable borel*
using *distributed-borel-measurable*[*OF X*] *distributed-borel-measurable*[*OF Y*]
by *simp-all*

show $(\lambda x.\ \int^+ xa.\ f\ (x - xa) * g\ xa\ \partial lborel) \in$ *borel-measurable lborel*
by *measurable*

have *distr M borel* $(\lambda x.\ X\ x + Y\ x) =$ (*distr M borel X* $\star$ *distr M borel Y*)
using *distributed-measurable*[*OF X*] *distributed-measurable*[*OF Y*]
by (*intro sum-indep-random-variable*) (*auto simp*: *indep*)
also have $\ldots =$ (*density lborel f* $\star$ *density lborel g*)
using *distributed-distr-eq-density*[*OF X*] *distributed-distr-eq-density*[*OF Y*]
by (*simp cong*: *distr-cong*)
also have $\ldots =$ *density lborel* $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
proof (*rule convolution-density*)
show *finite-measure* (*density lborel f*)
using *X* **by** (*rule distributed-finite-measure-density*)
show *finite-measure* (*density lborel g*)
using *Y* **by** (*rule distributed-finite-measure-density*)
qed *fact*+
finally show *distr M lborel* $(\lambda x.\ X\ x + Y\ x) =$ *density lborel* $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
by (*simp cong*: *distr-cong*)
show *random-variable lborel* $(\lambda x.\ X\ x + Y\ x)$
using *distributed-measurable*[*OF X*] *distributed-measurable*[*OF Y*] **by** *simp*
qed

lemma *prob-space-convolution-density*:
fixes $f :: real \Rightarrow -$
fixes $g :: real \Rightarrow -$
assumes [*measurable*]: $f \in$ *borel-measurable borel*
assumes [*measurable*]: $g \in$ *borel-measurable borel*
assumes *gt-0*[*simp*]: $\bigwedge x.\ 0 \le f\ x\ \bigwedge x.\ 0 \le g\ x$
assumes *prob-space* (*density lborel f*) (**is** *prob-space ?F*)
assumes *prob-space* (*density lborel g*) (**is** *prob-space ?G*)
shows *prob-space* (*density lborel* $(\lambda x. \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$) (**is** *prob-space ?D*)
proof (*subst convolution-density*[*symmetric*])
interpret *F*: *prob-space ?F* **by** *fact*
show *finite-measure ?F* **by** *unfold-locales*
interpret *G*: *prob-space ?G* **by** *fact*
show *finite-measure ?G* **by** *unfold-locales*

interpret *FG*: *pair-prob-space ?F ?G* **..**

show *prob-space* (*density lborel f* ⋆ *density lborel g*)
unfolding *convolution-def* **by** (*rule FG.prob-space-distr*) *simp*
qed *simp-all*

end

12 Information theory

theory *Information*
imports
Independent-Family
begin

12.1 Information theory

locale *information-space* = *prob-space* +
fixes *b* :: *real* **assumes** *b-gt-1*: *1* < *b*

Introduce some simplification rules for logarithm of base *b*.

lemmas *log-simps* = *log-mult log-inverse log-divide*

12.2 Kullback−Leibler divergence

The Kullback−Leibler divergence is also known as relative entropy or Kullback−Leibler distance.

definition
entropy-density b M N = *log b* ∘ *enn2real* ∘ *RN-deriv M N*

definition
KL-divergence b M N = *integral*^{L} *N* (*entropy-density b M N*)

lemma *measurable-entropy-density*[*measurable*]: *entropy-density b M N* ∈ *borel-measurable M*
unfolding *entropy-density-def* **by** *auto*

lemma (**in** *sigma-finite-measure*) *KL-density*:
fixes *f* :: *'a* ⇒ *real*
assumes *1* < *b*
assumes *f*[*measurable*]: *f* ∈ *borel-measurable M* **and** *nn*: *AE x in M*. *0* ≤ *f x*
shows *KL-divergence b M* (*density M f*) = (∫ *x*. *f x* ∗ *log b* (*f x*) ∂*M*)
unfolding *KL-divergence-def*
proof (*subst integral-real-density*)
show [*measurable*]: *entropy-density b M* (*density M* (λ*x*. *ennreal* (*f x*))) ∈ *borel-measurable M*
using *f*
by (*auto simp*: *comp-def entropy-density-def*)

```
  have density M (RN-deriv M (density M f)) = density M f
    using f nn by (intro density-RN-deriv-density) auto
  then have eq: AE x in M. RN-deriv M (density M f) x = f x
    using f nn by (intro density-unique) auto
  have AE x in M. f x * entropy-density b M (density M (λx. ennreal (f x))) x =
            f x * log b (f x)
    using eq nn by (auto simp: entropy-density-def)
  then show (∫ x. f x * entropy-density b M (density M (λx. ennreal (f x))) x
∂M) = (∫ x. f x * log b (f x) ∂M)
    by (intro integral-cong-AE) measurable
qed fact+

lemma (in sigma-finite-measure) KL-density-density:
  fixes f g :: 'a ⇒ real
  assumes 1 < b
  assumes f: f ∈ borel-measurable M AE x in M. 0 ≤ f x
  assumes g: g ∈ borel-measurable M AE x in M. 0 ≤ g x
  assumes ac: AE x in M. f x = 0 ⟶ g x = 0
  shows KL-divergence b (density M f) (density M g) = (∫ x. g x * log b (g x / f
x) ∂M)
proof −
  interpret Mf: sigma-finite-measure density M f
    using f by (subst sigma-finite-iff-density-finite) auto
  have KL-divergence b (density M f) (density M g) =
    KL-divergence b (density M f) (density (density M f) (λx. g x / f x))
    using f g ac by (subst density-density-divide) simp-all
  also have ... = (∫ x. (g x / f x) * log b (g x / f x) ∂density M f)
    using f g ‹1 < b› by (intro Mf.KL-density) (auto simp: AE-density)
  also have ... = (∫ x. g x * log b (g x / f x) ∂M)
    using ac f g ‹1 < b› by (subst integral-density) (auto intro!: integral-cong-AE)
  finally show ?thesis .
qed

lemma (in information-space) KL-gt-0:
  fixes D :: 'a ⇒ real
  assumes prob-space (density M D)
  assumes D: D ∈ borel-measurable M AE x in M. 0 ≤ D x
  assumes int: integrable M (λx. D x * log b (D x))
  assumes A: density M D ≠ M
  shows 0 < KL-divergence b M (density M D)
proof −
  interpret N: prob-space density M D by fact

  obtain A where A ∈ sets M emeasure (density M D) A ≠ emeasure M A
    using measure-eqI[of density M D M] ‹density M D ≠ M› by auto

  let ?D-set = {x∈space M. D x ≠ 0}
  have [simp, intro]: ?D-set ∈ sets M
    using D by auto
```

```
  have D-neg: (∫⁺ x. ennreal (− D x) ∂M) = 0
    using D by (subst nn-integral-0-iff-AE) (auto simp: ennreal-neg)

  have (∫⁺ x. ennreal (D x) ∂M) = emeasure (density M D) (space M)
    using D by (simp add: emeasure-density cong: nn-integral-cong)
  then have D-pos: (∫⁺ x. ennreal (D x) ∂M) = 1
    using N.emeasure-space-1 by simp

  have integrable M D
   using D D-pos D-neg unfolding real-integrable-def real-lebesgue-integral-def by
simp-all
  then have integral^L M D = 1
    using D D-pos D-neg by (simp add: real-lebesgue-integral-def)

  have 0 ≤ 1 − measure M ?D-set
    using prob-le-1 by (auto simp: field-simps)
  also have ... = (∫ x. D x − indicator ?D-set x ∂M)
    using ‹integrable M D› ‹integral^L M D = 1›
    by (simp add: emeasure-eq-measure)
  also have ... < (∫ x. D x ∗ (ln b ∗ log b (D x)) ∂M)
  proof (rule integral-less-AE)
    show integrable M (λx. D x − indicator ?D-set x)
      using ‹integrable M D› by (auto simp: less-top[symmetric])
  next
    from integrable-mult-left(1)[OF int, of ln b]
    show integrable M (λx. D x ∗ (ln b ∗ log b (D x)))
      by (simp add: ac-simps)
  next
    show emeasure M {x∈space M. D x ≠ 1 ∧ D x ≠ 0} ≠ 0
    proof
      assume eq-0: emeasure M {x∈space M. D x ≠ 1 ∧ D x ≠ 0} = 0
      then have disj: AE x in M. D x = 1 ∨ D x = 0
        using D(1) by (auto intro!: AE-I[OF subset-refl] sets.sets-Collect)

      have emeasure M {x∈space M. D x = 1} = (∫⁺ x. indicator {x∈space M.
D x = 1} x ∂M)
        using D(1) by auto
      also have ... = (∫⁺ x. ennreal (D x) ∂M)
      using disj by (auto intro!: nn-integral-cong-AE simp: indicator-def one-ennreal-def)
      finally have AE x in M. D x = 1
        using D D-pos by (intro AE-I-eq-1) auto
       then have (∫⁺x. indicator A x∂M) = (∫⁺x. ennreal (D x) ∗ indicator A
x∂M)
        by (intro nn-integral-cong-AE) (auto simp: one-ennreal-def[symmetric])
      also have ... = density M D A
        using ‹A ∈ sets M› D by (simp add: emeasure-density)
        finally show False using ‹A ∈ sets M› ‹emeasure (density M D) A ≠
emeasure M A› by simp
```

```
  qed
  show {x∈space M. D x ≠ 1 ∧ D x ≠ 0} ∈ sets M
    using D(1) by (auto intro: sets.sets-Collect-conj)

  have False
    if Dt: t ∈ space M D t ≠ 1 D t ≠ 0 0 ≤ D t
      and eq: D t − indicator ?D-set t = D t * (ln b * log b (D t)) for t
  proof −
    have D t − 1 = D t − indicator ?D-set t
      using Dt by simp
    also note eq
    also have D t * (ln b * log b (D t)) = − D t * ln (1 / D t)
      using b-gt-1 ‹D t ≠ 0› ‹0 ≤ D t›
      by (simp add: log-def ln-div less-le)
    finally have ln (1 / D t) = 1 / D t − 1
      using ‹D t ≠ 0› by (auto simp: field-simps)
    from ln-eq-minus-one[OF - this] ‹D t ≠ 0› ‹0 ≤ D t› ‹D t ≠ 1›
    show False by auto
  qed
  with D(2)
  show AE t in M. t ∈ {x∈space M. D x ≠ 1 ∧ D x ≠ 0} ⟶
    D t − indicator ?D-set t ≠ D t * (ln b * log b (D t))
    by fastforce

  show AE t in M. D t − indicator ?D-set t ≤ D t * (ln b * log b (D t))
    using D(2) AE-space
  proof eventually-elim
    fix t assume t ∈ space M 0 ≤ D t
    show D t − indicator ?D-set t ≤ D t * (ln b * log b (D t))
    proof cases
      assume asm: D t ≠ 0
      then have 0 < D t using ‹0 ≤ D t› by auto
      then have 0 < 1 / D t by auto
      have D t − indicator ?D-set t ≤ − D t * (1 / D t − 1)
        using asm ‹t ∈ space M› by (simp add: field-simps)
      also have − D t * (1 / D t − 1) ≤ − D t * ln (1 / D t)
        using ln-le-minus-one ‹0 < 1 / D t› by (intro mult-left-mono-neg) auto
      also have ... = D t * (ln b * log b (D t))
        using ‹0 < D t› b-gt-1
        by (simp-all add: log-def ln-div)
      finally show ?thesis by simp
    qed simp
  qed
qed
also have ... = (∫ x. ln b * (D x * log b (D x)) ∂M)
  by (simp add: ac-simps)
also have ... = ln b * (∫ x. D x * log b (D x) ∂M)
  using int by simp
finally show ?thesis
```

```
  using b-gt-1 D by (subst KL-density) (auto simp: zero-less-mult-iff)
qed

lemma (in sigma-finite-measure) KL-same-eq-0: KL-divergence b M M = 0
proof −
  have AE x in M. 1 = RN-deriv M M x
  proof (rule RN-deriv-unique)
    show density M (λx. 1) = M
      by (simp add: density-1)
  qed auto
  then have AE x in M. log b (enn2real (RN-deriv M M x)) = 0
    by (elim AE-mp) simp
  from integral-cong-AE[OF - - this]
  have integral^L M (entropy-density b M M) = 0
    by (simp add: entropy-density-def comp-def)
  then show KL-divergence b M M = 0
    unfolding KL-divergence-def
    by auto
qed

lemma (in information-space) KL-eq-0-iff-eq:
  fixes D :: 'a ⇒ real
  assumes prob-space (density M D)
  assumes D: D ∈ borel-measurable M AE x in M. 0 ≤ D x
  assumes int: integrable M (λx. D x ∗ log b (D x))
  shows KL-divergence b M (density M D) = 0 ⟷ density M D = M
  using KL-same-eq-0[of b] KL-gt-0[OF assms]
  by (auto simp: less-le)

lemma (in information-space) KL-eq-0-iff-eq-ac:
  fixes D :: 'a ⇒ real
  assumes prob-space N
  assumes ac: absolutely-continuous M N sets N = sets M
  assumes int: integrable N (entropy-density b M N)
  shows KL-divergence b M N = 0 ⟷ N = M
proof −
  interpret N: prob-space N by fact
  have finite-measure N by unfold-locales
  from real-RN-deriv[OF this ac] obtain D
    where D:
      random-variable borel D
      AE x in M. RN-deriv M N x = ennreal (D x)
      AE x in N. 0 < D x
      ⋀x. 0 ≤ D x
    by this auto

  have N = density M (RN-deriv M N)
    using ac by (rule density-RN-deriv[symmetric])
  also have ... = density M D
```

```
    using D by (auto intro!: density-cong)
  finally have N: N = density M D .

  from absolutely-continuous-AE[OF ac(2,1) D(2)] D b-gt-1 ac measurable-entropy-density
  have integrable N (λx. log b (D x))
    by (intro integrable-cong-AE[THEN iffD2, OF - - - int])
       (auto simp: N entropy-density-def)
  with D b-gt-1 have integrable M (λx. D x * log b (D x))
   by (subst integrable-real-density[symmetric]) (auto simp: N[symmetric] comp-def)
  with ‹prob-space N› D show ?thesis
    unfolding N
    by (intro KL-eq-0-iff-eq) auto
qed

lemma (in information-space) KL-nonneg:
  assumes prob-space (density M D)
  assumes D: D ∈ borel-measurable M AE x in M. 0 ≤ D x
  assumes int: integrable M (λx. D x * log b (D x))
  shows 0 ≤ KL-divergence b M (density M D)
 using KL-gt-0[OF assms] by (cases density M D = M) (auto simp: KL-same-eq-0)

lemma (in sigma-finite-measure) KL-density-density-nonneg:
  fixes f g :: 'a ⇒ real
  assumes 1 < b
  assumes f: f ∈ borel-measurable M AE x in M. 0 ≤ f x prob-space (density M
f)
  assumes g: g ∈ borel-measurable M AE x in M. 0 ≤ g x prob-space (density M
g)
  assumes ac: AE x in M. f x = 0 ⟶ g x = 0
  assumes int: integrable M (λx. g x * log b (g x / f x))
  shows 0 ≤ KL-divergence b (density M f) (density M g)
proof −
  interpret Mf: prob-space density M f by fact
  interpret Mf: information-space density M f b by standard fact
  have eq: density (density M f) (λx. g x / f x) = density M g (is ?DD = -)
    using f g ac by (subst density-density-divide) simp-all

  have 0 ≤ KL-divergence b (density M f) (density (density M f) (λx. g x / f x))
  proof (rule Mf.KL-nonneg)
    show prob-space ?DD unfolding eq by fact
    from f g show (λx. g x / f x) ∈ borel-measurable (density M f)
      by auto
    show AE x in density M f. 0 ≤ g x / f x
      using f g by (auto simp: AE-density)
    show integrable (density M f) (λx. g x / f x * log b (g x / f x))
      using ‹1 < b› f g ac
      by (subst integrable-density)
         (auto intro!: integrable-cong-AE[THEN iffD2, OF - - - int] measurable-If)
  qed
```

also have ... = *KL-divergence b* (*density M f*) (*density M g*)
using *f g ac* **by** (*subst density-density-divide*) *simp-all*
finally show *?thesis* **.**
qed

12.3 Finite Entropy

definition (**in** *information-space*) *finite-entropy* :: *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *real*) ⇒ *bool*
where
finite-entropy S X f ⟷
distributed M S X f ∧
integrable S (λ*x*. *f x* ∗ *log b* (*f x*)) ∧
(∀ *x*∈*space S*. *0* ≤ *f x*)

lemma (**in** *information-space*) *finite-entropy-simple-function*:
assumes *X*: *simple-function M X*
shows *finite-entropy* (*count-space* (*X*‘*space M*)) *X* (λ*a*. *measure M* {*x* ∈ *space M*. *X x* = *a*})
unfolding *finite-entropy-def*
proof *safe*
have [*simp*]: *finite* (*X* ‘ *space M*)
using *X* **by** (*auto simp*: *simple-function-def*)
then show *integrable* (*count-space* (*X* ‘ *space M*))
(λ*x*. *prob* {*xa* ∈ *space M*. *X xa* = *x*} ∗ *log b* (*prob* {*xa* ∈ *space M*. *X xa* = *x*}))
by (*rule integrable-count-space*)
have *d*: *distributed M* (*count-space* (*X* ‘ *space M*)) *X* (λ*x*. *ennreal* (*if x* ∈ *X*‘*space M then prob* {*xa* ∈ *space M*. *X xa* = *x*} *else 0*))
by (*rule distributed-simple-function-superset*[*OF X*]) (*auto intro*!: *arg-cong*[**where** *f*=*prob*])
show *distributed M* (*count-space* (*X* ‘ *space M*)) *X* (λ*x*. *ennreal* (*prob* {*xa* ∈ *space M*. *X xa* = *x*}))
by (*rule distributed-cong-density*[*THEN iffD1*, *OF - - - d*]) *auto*
qed (*rule measure-nonneg*)

lemma *ac-fst*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous S* (*distr* (*S* ⨂$_M$ *T*) *S fst*)
proof −
interpret *sigma-finite-measure T* **by** *fact*
{ fix *A* **assume** *A*: *A* ∈ *sets S emeasure S A* = *0*
then have *fst* −‘ *A* ∩ *space* (*S* ⨂$_M$ *T*) = *A* × *space T*
by (*auto simp*: *space-pair-measure dest*!: *sets.sets-into-space*)
with *A* **have** *emeasure* (*S* ⨂$_M$ *T*) (*fst* −‘ *A* ∩ *space* (*S* ⨂$_M$ *T*)) = *0*
by (*simp add*: *emeasure-pair-measure-Times*) **}**
then show *?thesis*
unfolding *absolutely-continuous-def*
by (*metis emeasure-distr measurable-fst null-setsD1 null-setsD2 null-setsI sets-distr subsetI*)

qed

lemma *ac-snd*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous T* (*distr* (*S* $\bigotimes_M$ *T*) *T snd*)
proof −
interpret *sigma-finite-measure T* **by** *fact*
{ fix *A* **assume** *A*: *A* ∈ *sets T emeasure T A* = *0*
then have *snd* −‘ *A* ∩ *space* (*S* $\bigotimes_M$ *T*) = *space S* × *A*
by (*auto simp*: *space-pair-measure dest*!: *sets.sets-into-space*)
with *A* **have** *emeasure* (*S* $\bigotimes_M$ *T*) (*snd* −‘ *A* ∩ *space* (*S* $\bigotimes_M$ *T*)) = *0*
by (*simp add*: *emeasure-pair-measure-Times*) **}**
then show *?thesis*
unfolding *absolutely-continuous-def*
by (*metis emeasure-distr measurable-snd null-setsD1 null-setsD2 null-setsI sets-distr subsetI*)
qed

lemma (**in** *information-space*) *finite-entropy-integrable*:
finite-entropy S X Px ⟹ *integrable S* (λ*x*. *Px x* ∗ *log b* (*Px x*))
unfolding *finite-entropy-def* **by** *auto*

lemma (**in** *information-space*) *finite-entropy-distributed*:
finite-entropy S X Px ⟹ *distributed M S X Px*
unfolding *finite-entropy-def* **by** *auto*

lemma (**in** *information-space*) *finite-entropy-nn*:
finite-entropy S X Px ⟹ *x* ∈ *space S* ⟹ *0* ≤ *Px x*
by (*auto simp*: *finite-entropy-def*)

lemma (**in** *information-space*) *finite-entropy-measurable*:
finite-entropy S X Px ⟹ *Px* ∈ *S* →$_M$ *borel*
using *distributed-real-measurable*[*of S Px M X*]
finite-entropy-nn[*of S X Px*] *finite-entropy-distributed*[*of S X Px*] **by** *auto*

lemma (**in** *information-space*) *subdensity-finite-entropy*:
fixes *g* :: *'b* ⇒ *real* **and** *f* :: *'c* ⇒ *real*
assumes *T*: *T* ∈ *measurable P Q*
assumes *f*: *finite-entropy P X f*
assumes *g*: *finite-entropy Q Y g*
assumes *Y*: *Y* = *T* ∘ *X*
shows *AE x in P*. *g* (*T x*) = *0* ⟶ *f x* = *0*
using *subdensity*[*OF T*, *of M X* λ*x*. *ennreal* (*f x*) *Y* λ*x*. *ennreal* (*g x*)]
finite-entropy-distributed[*OF f*] *finite-entropy-distributed*[*OF g*]
finite-entropy-nn[*OF f*] *finite-entropy-nn*[*OF g*]
assms
by *auto*

lemma (**in** *information-space*) *finite-entropy-integrable-transform*:

finite-entropy S X Px $\Longrightarrow$ *distributed M T Y Py* $\Longrightarrow$ $(\bigwedge x.\ x \in$ *space T* $\Longrightarrow$ *0* $\leq$ *Py x*) $\Longrightarrow$
X = $(\lambda x.\ f\ (Y\ x))$ $\Longrightarrow$ *f* $\in$ *measurable T S* $\Longrightarrow$ *integrable T* $(\lambda x.$ *Py x* $*$ *log b* (*Px* (*f x*)))
using *distributed-transform-integrable*[*of M T Y Py S X Px f* $\lambda x.$ *log b* (*Px x*)]
using *distributed-real-measurable*[*of S Px M X*]
by (*auto simp*: *finite-entropy-def*)

12.4 Mutual Information

definition (**in** *prob-space*)
mutual-information b S T X Y =
KL-divergence b (*distr M S X* $\bigotimes_M$ *distr M T Y*) (*distr M* (*S* $\bigotimes_M$ *T*) $(\lambda x.$ (*X x*, *Y x*)))

lemma (**in** *information-space*) *mutual-information-indep-vars*:
fixes *S T X Y*
defines *P* $\equiv$ *distr M S X* $\bigotimes_M$ *distr M T Y*
defines *Q* $\equiv$ *distr M* (*S* $\bigotimes_M$ *T*) $(\lambda x.$ (*X x*, *Y x*))
shows *indep-var S X T Y* $\longleftrightarrow$
(*random-variable S X* $\wedge$ *random-variable T Y* $\wedge$
absolutely-continuous P Q $\wedge$ *integrable Q* (*entropy-density b P Q*) $\wedge$
mutual-information b S T X Y = *0*)
unfolding *indep-var-distribution-eq*
proof *safe*
assume *rv*[*measurable*]: *random-variable S X random-variable T Y*

interpret *X*: *prob-space distr M S X*
by (*rule prob-space-distr*) *fact*
interpret *Y*: *prob-space distr M T Y*
by (*rule prob-space-distr*) *fact*
interpret *XY*: *pair-prob-space distr M S X distr M T Y* **by** *standard*
interpret *P*: *information-space P b* **unfolding** *P-def* **by** *standard* (*rule b-gt-1*)

interpret *Q*: *prob-space Q* **unfolding** *Q-def*
by (*rule prob-space-distr*) *simp*

{ assume *distr M S X* $\bigotimes_M$ *distr M T Y* = *distr M* (*S* $\bigotimes_M$ *T*) $(\lambda x.$ (*X x*, *Y x*))
then have [*simp*]: *Q* = *P* **unfolding** *Q-def P-def* **by** *simp*

show *ac*: *absolutely-continuous P Q* **by** (*simp add*: *absolutely-continuous-def*)
then have *ed*: *entropy-density b P Q* $\in$ *borel-measurable P*
by *simp*

have *AE x in P*. *1* = *RN-deriv P Q x*
proof (*rule P.RN-deriv-unique*)
show *density P* $(\lambda x.$ *1*) = *Q*
unfolding ‹*Q* = *P*› **by** (*intro measure-eqI*) (*auto simp*: *emeasure-density*)

qed *auto*
then have *ae-0*: *AE x in P. entropy-density b P Q x = 0*
by (*auto simp*: *entropy-density-def*)
then have *integrable P* (*entropy-density b P Q*) ⟷ *integrable Q* (λ*x*. *0*::*real*)
using *ed* **unfolding** ‹*Q = P*› **by** (*intro integrable-cong-AE*) *auto*
then show *integrable Q* (*entropy-density b P Q*) **by** *simp*

from *ae-0* **have** *mutual-information b S T X Y* = (∫ *x*. *0* ∂*P*)
unfolding *mutual-information-def KL-divergence-def P-def*[*symmetric*] *Q-def*[*symmetric*] ‹*Q = P*›
by (*intro integral-cong-AE*) *auto*
then show *mutual-information b S T X Y = 0*
by *simp* **}**

{ assume *ac*: *absolutely-continuous P Q*
assume *int*: *integrable Q* (*entropy-density b P Q*)
assume *I-eq-0*: *mutual-information b S T X Y = 0*

have *eq*: *Q = P*
proof (*rule P.KL-eq-0-iff-eq-ac*[*THEN iffD1*])
show *prob-space Q* **by** *unfold-locales*
show *absolutely-continuous P Q* **by** *fact*
show *integrable Q* (*entropy-density b P Q*) **by** *fact*
show *sets Q = sets P* **by** (*simp add*: *P-def Q-def sets-pair-measure*)
show *KL-divergence b P Q = 0*
using *I-eq-0* **unfolding** *mutual-information-def* **by** (*simp add*: *P-def Q-def*)
qed
then show *distr M S X* ⨂$_M$ *distr M T Y = distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*))
unfolding *P-def Q-def* **.. }**
qed

abbreviation (**in** *information-space*)
mutual-information-Pow (‹*ℐ'*(- ; -')›) **where**
ℐ(*X* ; *Y*) ≡ *mutual-information b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* × *'c* ⇒ *real* **and** *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Fx*: *finite-entropy S X Px* **and** *Fy*: *finite-entropy T Y Py*
assumes *Fxy*: *finite-entropy* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
defines *f* ≡ λ*x*. *Pxy x* ∗ *log b* (*Pxy x* / (*Px* (*fst x*) ∗ *Py* (*snd x*)))
shows *mutual-information-distr'*: *mutual-information b S T X Y = integral*^{L} (*S* ⨂$_M$ *T*) *f* (**is** *?M = ?R*)
and *mutual-information-nonneg'*: *0* ≤ *mutual-information b S T X Y*
proof −
have *Px*: *distributed M S X Px* **and** *Px-nn*: ⋀*x*. *x* ∈ *space S* ⟹ *0* ≤ *Px x*
using *Fx* **by** (*auto simp*: *finite-entropy-def*)

have *Py*: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
using *Fy* **by** (*auto simp*: *finite-entropy-def*)
have *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*: $\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
$\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
using *Fxy* **by** (*auto simp*: *finite-entropy-def space-pair-measure*)

have [*measurable*]: $Px \in S \rightarrow_M borel$
using *Px Px-nn* **by** (*intro distributed-real-measurable*)
have [*measurable*]: $Py \in T \rightarrow_M borel$
using *Py Py-nn* **by** (*intro distributed-real-measurable*)
have *measurable-Pxy*[*measurable*]: $Pxy \in (S \bigotimes_M T) \rightarrow_M borel$
using *Pxy Pxy-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)

have *X*[*measurable*]: *random-variable S X*
using *Px* **by** *auto*
have *Y*[*measurable*]: *random-variable T Y*
using *Py* **by** *auto*
interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**
interpret *X*: *prob-space distr M S X* **using** *X* **by** (*rule prob-space-distr*)
interpret *Y*: *prob-space distr M T Y* **using** *Y* **by** (*rule prob-space-distr*)
interpret *XY*: *pair-prob-space distr M S X distr M T Y* **..**
let *?P* $= S \bigotimes_M T$
let *?D* $=$ *distr M ?P* $(\lambda x.\ (X\ x,\ Y\ x))$

{ fix *A* **assume** $A \in sets\ S$
with *X*[*THEN measurable-space*] *Y*[*THEN measurable-space*]
have *emeasure* (*distr M S X*) *A* $=$ *emeasure ?D* $(A \times space\ T)$
by (*auto simp*: *emeasure-distr intro*!: *arg-cong*[**where** *f*=*emeasure M*]) **}**
note *marginal-eq1* $=$ *this*
{ fix *A* **assume** $A \in sets\ T$
with *X*[*THEN measurable-space*] *Y*[*THEN measurable-space*]
have *emeasure* (*distr M T Y*) *A* $=$ *emeasure ?D* $(space\ S \times A)$
by (*auto simp*: *emeasure-distr intro*!: *arg-cong*[**where** *f*=*emeasure M*]) **}**
note *marginal-eq2* $=$ *this*

have *distr-eq*: *distr M S X* $\bigotimes_M$ *distr M T Y* $=$ *density ?P* $(\lambda x.\ ennreal\ (Px\ (fst\ x) * Py\ (snd\ x)))$
unfolding *Px*(*1*)[*THEN distributed-distr-eq-density*] *Py*(*1*)[*THEN distributed-distr-eq-density*]
proof (*subst pair-measure-density*)
show $(\lambda x.\ ennreal\ (Px\ x)) \in borel\text{-}measurable\ S$ $(\lambda y.\ ennreal\ (Py\ y)) \in borel\text{-}measurable\ T$
using *Px Py* **by** (*auto simp*: *distributed-def*)
show *sigma-finite-measure* (*density T Py*) **unfolding** *Py*(*1*)[*THEN distributed-distr-eq-density*, *symmetric*] **..**
show *density* $(S \bigotimes_M T)$ $(\lambda(x,\ y).\ ennreal\ (Px\ x) * ennreal\ (Py\ y)) =$
density $(S \bigotimes_M T)$ $(\lambda x.\ ennreal\ (Px\ (fst\ x) * Py\ (snd\ x)))$

using *Px-nn Py-nn* **by** (*auto intro*!: *density-cong simp*: *distributed-def ennreal-mult space-pair-measure*)
qed *fact*

have *M*: *?M = KL-divergence b* (*density ?P* (λ*x*. *ennreal* (*Px* (*fst x*) ∗ *Py* (*snd x*)))) (*density ?P* (λ*x*. *ennreal* (*Pxy x*)))
unfolding *mutual-information-def distr-eq Pxy*(*1*)[*THEN distributed-distr-eq-density*]
..

from *Px Py* **have** *f*: (λ*x*. *Px* (*fst x*) ∗ *Py* (*snd x*)) ∈ *borel-measurable ?P*
by (*intro borel-measurable-times*) (*auto intro*: *distributed-real-measurable measurable-fst'' measurable-snd''*)
have *PxPy-nonneg*: *AE x in ?P*. *0* ≤ *Px* (*fst x*) ∗ *Py* (*snd x*)
using *Px-nn Py-nn* **by** (*auto simp*: *space-pair-measure*)

have *A*: (*AE x in ?P*. *Px* (*fst x*) = *0* ⟶ *Pxy x* = *0*)
by (*rule subdensity-real*[*OF measurable-fst Pxy Px*]) (*insert Px-nn Pxy-nn*, *auto simp*: *space-pair-measure*)
moreover
have *B*: (*AE x in ?P*. *Py* (*snd x*) = *0* ⟶ *Pxy x* = *0*)
by (*rule subdensity-real*[*OF measurable-snd Pxy Py*]) (*insert Py-nn Pxy-nn*, *auto simp*: *space-pair-measure*)
ultimately have *ac*: *AE x in ?P*. *Px* (*fst x*) ∗ *Py* (*snd x*) = *0* ⟶ *Pxy x* = *0*
by *auto*

show *?M* = *?R*
unfolding *M f-def* **using** *Pxy-nn Px-nn Py-nn*
by (*intro ST.KL-density-density b-gt-1 f PxPy-nonneg ac*) (*auto simp*: *space-pair-measure*)

have *X*: *X* = *fst* ∘ (λ*x*. (*X x*, *Y x*)) **and** *Y*: *Y* = *snd* ∘ (λ*x*. (*X x*, *Y x*))
by *auto*

have *integrable* (*S* ⨂$_M$ *T*) (λ*x*. *Pxy x* ∗ *log b* (*Pxy x*) − *Pxy x* ∗ *log b* (*Px* (*fst x*)) − *Pxy x* ∗ *log b* (*Py* (*snd x*)))
using *finite-entropy-integrable*[*OF Fxy*]
using *finite-entropy-integrable-transform*[*OF Fx Pxy*, *of fst*]
using *finite-entropy-integrable-transform*[*OF Fy Pxy*, *of snd*]
by (*simp add*: *Pxy-nn*)
moreover have *f* ∈ *borel-measurable* (*S* ⨂$_M$ *T*)
unfolding *f-def* **using** *Px Py Pxy*
by (*auto intro*: *distributed-real-measurable measurable-fst'' measurable-snd''*
intro!: *borel-measurable-times borel-measurable-log borel-measurable-divide*)
ultimately have *int*: *integrable* (*S* ⨂$_M$ *T*) *f*
proof (*rule integrable-cong-AE-imp*)
show *AE x in S* ⨂$_M$
T. *Pxy x* ∗ *log b* (*Pxy x*) − *Pxy x* ∗ *log b* (*Px* (*fst x*)) − *Pxy x* ∗ *log b* (*Py* (*snd x*))
= *f x*
using *A B*

by (*auto simp*: *f-def field-simps space-pair-measure log-mult log-divide*)
qed

show *0* ≤ *?M* **unfolding** *M*
proof (*intro ST.KL-density-density-nonneg*)
show *prob-space* (*density* (*S* ⨂$_M$ *T*) (λ*x*. *ennreal* (*Pxy x*)))
unfolding *distributed-distr-eq-density*[*OF Pxy*, *symmetric*]
using *distributed-measurable*[*OF Pxy*] **by** (*rule prob-space-distr*)
show *prob-space* (*density* (*S* ⨂$_M$ *T*) (λ*x*. *ennreal* (*Px* (*fst x*) ∗ *Py* (*snd x*))))
unfolding *distr-eq*[*symmetric*] **by** *unfold-locales*
show *integrable* (*S* ⨂$_M$ *T*) (λ*x*. *Pxy x* ∗ *log b* (*Pxy x* / (*Px* (*fst x*) ∗ *Py* (*snd x*))))
using *int* **unfolding** *f-def* **.**
qed (*insert ac*, *auto simp*: *b-gt-1 Px-nn Py-nn Pxy-nn space-pair-measure*)
qed

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* × *'c* ⇒ *real* **and** *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* **and** *Px-nn*: ⋀*x*. *x* ∈ *space S* ⟹ *0* ≤ *Px x*
and *Py*: *distributed M T Y Py* **and** *Py-nn*: ⋀*y*. *y* ∈ *space T* ⟹ *0* ≤ *Py y*
and *Pxy*: *distributed M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*: ⋀*x y*. *x* ∈ *space S* ⟹ *y* ∈ *space T* ⟹ *0* ≤ *Pxy* (*x*, *y*)
defines *f* ≡ λ*x*. *Pxy x* ∗ *log b* (*Pxy x* / (*Px* (*fst x*) ∗ *Py* (*snd x*)))
shows *mutual-information-distr*: *mutual-information b S T X Y* = *integral*^{L} (*S* ⨂$_M$ *T*) *f* (**is** *?M* = *?R*)
and *mutual-information-nonneg*: *integrable* (*S* ⨂$_M$ *T*) *f* ⟹ *0* ≤ *mutual-information b S T X Y*
proof −
have *X*[*measurable*]: *random-variable S X*
using *Px* **by** (*auto simp*: *distributed-def*)
have *Y*[*measurable*]: *random-variable T Y*
using *Py* **by** (*auto simp*: *distributed-def*)
have [*measurable*]: *Px* ∈ *S* →$_M$ *borel*
using *Px Px-nn* **by** (*intro distributed-real-measurable*)
have [*measurable*]: *Py* ∈ *T* →$_M$ *borel*
using *Py Py-nn* **by** (*intro distributed-real-measurable*)
have *measurable-Pxy*[*measurable*]: *Pxy* ∈ (*S* ⨂$_M$ *T*) →$_M$ *borel*
using *Pxy Pxy-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)

interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**
interpret *X*: *prob-space distr M S X* **using** *X* **by** (*rule prob-space-distr*)
interpret *Y*: *prob-space distr M T Y* **using** *Y* **by** (*rule prob-space-distr*)
interpret *XY*: *pair-prob-space distr M S X distr M T Y* **..**
let *?P* = *S* ⨂$_M$ *T*
let *?D* = *distr M ?P* (λ*x*. (*X x*, *Y x*))

{ fix A **assume** $A \in$ *sets* S
with X[*THEN measurable-space*] Y[*THEN measurable-space*]
have *emeasure* (*distr* M S X) A = *emeasure* *?D* ($A \times$ *space* T)
by (*auto simp*: *emeasure-distr intro*!: *arg-cong*[**where** f=*emeasure* M]) **}**
note *marginal-eq1* = *this*
{ fix A **assume** $A \in$ *sets* T
with X[*THEN measurable-space*] Y[*THEN measurable-space*]
have *emeasure* (*distr* M T Y) A = *emeasure* *?D* (*space* $S \times A$)
by (*auto simp*: *emeasure-distr intro*!: *arg-cong*[**where** f=*emeasure* M]) **}**
note *marginal-eq2* = *this*

have *distr-eq*: *distr* M S X $\bigotimes_M$ *distr* M T Y = *density* *?P* (λx. *ennreal* (*Px* (*fst* x) $*$ *Py* (*snd* x)))
unfolding *Px*(*1*)[*THEN distributed-distr-eq-density*] *Py*(*1*)[*THEN distributed-distr-eq-density*]
proof (*subst pair-measure-density*)
show (λx. *ennreal* (*Px* x)) $\in$ *borel-measurable* S (λy. *ennreal* (*Py* y)) $\in$ *borel-measurable* T
using *Px Py* **by** (*auto simp*: *distributed-def*)
show *sigma-finite-measure* (*density* T *Py*) **unfolding** *Py*(*1*)[*THEN distributed-distr-eq-density*, *symmetric*] **..**
show *density* (S $\bigotimes_M$ T) ($\lambda(x, y)$. *ennreal* (*Px* x) $*$ *ennreal* (*Py* y)) =
density (S $\bigotimes_M$ T) (λx. *ennreal* (*Px* (*fst* x)) $*$ *Py* (*snd* x)))
using *Px-nn Py-nn* **by** (*auto intro*!: *density-cong simp*: *distributed-def ennreal-mult space-pair-measure*)
qed *fact*

have *M*: *?M* = *KL-divergence* *b* (*density* *?P* (λx. *ennreal* (*Px* (*fst* x)) $*$ *Py* (*snd* x)))) (*density* *?P* (λx. *ennreal* (*Pxy* x)))
unfolding *mutual-information-def distr-eq Pxy*(*1*)[*THEN distributed-distr-eq-density*]
..

from *Px Py* **have** f: (λx. *Px* (*fst* x) $*$ *Py* (*snd* x)) $\in$ *borel-measurable* *?P*
by (*intro borel-measurable-times*) (*auto intro*: *distributed-real-measurable measurable-fst''* *measurable-snd''*)
have *PxPy-nonneg*: *AE x in ?P*. 0 $\leq$ *Px* (*fst* x) $*$ *Py* (*snd* x)
using *Px-nn Py-nn* **by** (*auto simp*: *space-pair-measure*)

have (*AE x in ?P*. *Px* (*fst* x) = 0 $\longrightarrow$ *Pxy* x = 0)
by (*rule subdensity-real*[*OF measurable-fst Pxy Px*]) (*insert Px-nn Pxy-nn*, *auto simp*: *space-pair-measure*)
moreover
have (*AE x in ?P*. *Py* (*snd* x) = 0 $\longrightarrow$ *Pxy* x = 0)
by (*rule subdensity-real*[*OF measurable-snd Pxy Py*]) (*insert Py-nn Pxy-nn*, *auto simp*: *space-pair-measure*)
ultimately have *ac*: *AE x in ?P*. *Px* (*fst* x) $*$ *Py* (*snd* x) = 0 $\longrightarrow$ *Pxy* x = 0
by *auto*

show *?M* = *?R*
unfolding *M f-def*

using *b-gt-1 f PxPy-nonneg ac Pxy-nn*
by (*intro ST.KL-density-density*) (*auto simp*: *space-pair-measure*)

assume *int*: *integrable* ($S \bigotimes_M T$) *f*
show *0* ≤ *?M* **unfolding** *M*
proof (*intro ST.KL-density-density-nonneg*)
show *prob-space* (*density* ($S \bigotimes_M T$) (λ*x*. *ennreal* (*Pxy x*)))
unfolding *distributed-distr-eq-density*[*OF Pxy*, *symmetric*]
using *distributed-measurable*[*OF Pxy*] **by** (*rule prob-space-distr*)
show *prob-space* (*density* ($S \bigotimes_M T$) (λ*x*. *ennreal* (*Px* (*fst x*) ∗ *Py* (*snd x*))))
unfolding *distr-eq*[*symmetric*] **by** *unfold-locales*
show *integrable* ($S \bigotimes_M T$) (λ*x*. *Pxy x* ∗ *log b* (*Pxy x* / (*Px* (*fst x*) ∗ *Py* (*snd x*))))
using *int* **unfolding** *f-def* **.**
qed (*insert ac*, *auto simp*: *b-gt-1 Px-nn Py-nn Pxy-nn space-pair-measure*)
qed

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* × *'c* ⇒ *real* **and** *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*[*measurable*]: *distributed M S X Px* **and** *Px-nn*: ⋀*x*. *x* ∈ *space S* ⟹ *0* ≤ *Px x*
and *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*: ⋀*x*. *x* ∈ *space T* ⟹ *0* ≤ *Py x*
and *Pxy*[*measurable*]: *distributed M* ($S \bigotimes_M T$) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*: ⋀*x*. *x* ∈ *space* ($S \bigotimes_M T$) ⟹ *0* ≤ *Pxy x*
assumes *ae*: *AE x in S*. *AE y in T*. *Pxy* (*x*, *y*) = *Px x* ∗ *Py y*
shows *mutual-information-eq-0*: *mutual-information b S T X Y* = *0*
proof −
interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**
note
distributed-real-measurable[*OF Px-nn Px*, *measurable*]
distributed-real-measurable[*OF Py-nn Py*, *measurable*]
distributed-real-measurable[*OF Pxy-nn Pxy*, *measurable*]

have *AE x in S* $\bigotimes_M$ *T*. *Px* (*fst x*) = *0* ⟶ *Pxy x* = *0*
by (*rule subdensity-real*[*OF measurable-fst Pxy Px*]) (*auto simp*: *Px-nn Pxy-nn space-pair-measure*)
moreover
have *AE x in S* $\bigotimes_M$ *T*. *Py* (*snd x*) = *0* ⟶ *Pxy x* = *0*
by (*rule subdensity-real*[*OF measurable-snd Pxy Py*]) (*auto simp*: *Py-nn Pxy-nn space-pair-measure*)
moreover
have *AE x in S* $\bigotimes_M$ *T*. *Pxy x* = *Px* (*fst x*) ∗ *Py* (*snd x*)
by (*intro ST.AE-pair-measure*) (*auto simp*: *ae intro*!: *measurable-snd''* *measurable-fst''*)
ultimately have *AE x in S* $\bigotimes_M$ *T*. *Pxy x* ∗ *log b* (*Pxy x* / (*Px* (*fst x*) ∗ *Py*

(snd x))) = 0
by *eventually-elim simp*
then have *(∫ x. Pxy x * log b (Pxy x / (Px (fst x) * Py (snd x))) ∂(S ⨂M T))*
= (∫ x. 0 ∂(S ⨂M T))
by *(intro integral-cong-AE) auto*
then show *?thesis*
by *(subst mutual-information-distr[OF assms(1−8)]) (auto simp add: space-pair-measure)*
qed

lemma **(in** *information-space*) *mutual-information-simple-distributed*:
assumes *X*: *simple-distributed M X Px* **and** *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M (λx. (X x, Y x)) Pxy*
shows *ℐ(X ; Y) = (∑(x, y)∈(λx. (X x, Y x))‘space M. Pxy (x, y) * log b (Pxy (x, y) / (Px x * Py y)))*
proof *(subst mutual-information-distr[OF - - simple-distributed[OF X] - simple-distributed[OF Y] - simple-distributed-joint[OF XY]])*
note *fin = simple-distributed-joint-finite[OF XY, simp]*
show *sigma-finite-measure (count-space (X ‘ space M))*
by *(simp add: sigma-finite-measure-count-space-finite)*
show *sigma-finite-measure (count-space (Y ‘ space M))*
by *(simp add: sigma-finite-measure-count-space-finite)*
let *?Pxy = λx. (if x ∈ (λx. (X x, Y x)) ‘ space M then Pxy x else 0)*
let *?f = λx. ?Pxy x * log b (?Pxy x / (Px (fst x) * Py (snd x)))*
have *⋀x. ?f x = (if x ∈ (λx. (X x, Y x)) ‘ space M then Pxy x * log b (Pxy x / (Px (fst x) * Py (snd x))) else 0)*
by *auto*
with *fin* **show** *(∫ x. ?f x ∂(count-space (X ‘ space M) ⨂M count-space (Y ‘ space M))) =*
*(∑(x, y)∈(λx. (X x, Y x)) ‘ space M. Pxy (x, y) * log b (Pxy (x, y) / (Px x * Py y)))*
by *(auto simp add: pair-measure-count-space lebesgue-integral-count-space-finite sum.If-cases split-beta′*
intro!: sum.cong)
qed *(insert X Y XY, auto simp: simple-distributed-def)*

lemma **(in** *information-space*)
fixes *Pxy :: ′b × ′c ⇒ real* **and** *Px :: ′b ⇒ real* **and** *Py :: ′c ⇒ real*
assumes *Px*: *simple-distributed M X Px* **and** *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M (λx. (X x, Y x)) Pxy*
assumes *ae*: *∀x∈space M. Pxy (X x, Y x) = Px (X x) * Py (Y x)*
shows *mutual-information-eq-0-simple*: *ℐ(X ; Y) = 0*
proof *(subst mutual-information-simple-distributed[OF Px Py Pxy])*
have *(∑(x, y)∈(λx. (X x, Y x)) ‘ space M. Pxy (x, y) * log b (Pxy (x, y) / (Px x * Py y))) =*
(∑(x, y)∈(λx. (X x, Y x)) ‘ space M. 0)
by *(intro sum.cong) (auto simp: ae)*
then show *(∑(x, y)∈(λx. (X x, Y x)) ‘ space M.*
*Pxy (x, y) * log b (Pxy (x, y) / (Px x * Py y))) = 0* **by** *simp*
qed

12.5 Entropy

definition (**in** *prob-space*) *entropy* :: *real* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ *real* **where**
entropy b S X = − *KL-divergence b S* (*distr M S X*)

abbreviation (**in** *information-space*)
entropy-Pow (‹ℋ'(-')›) **where**
ℋ(*X*) ≡ *entropy b* (*count-space* (*X*'*space M*)) *X*

lemma (**in** *prob-space*) *distributed-RN-deriv*:
assumes *X*: *distributed M S X Px*
shows *AE x in S*. *RN-deriv S* (*density S Px*) *x* = *Px x*
proof −
have *distributed M S X* (*RN-deriv S* (*density S Px*))
by (*metis RN-derivI assms borel-measurable-RN-deriv distributed-def*)
then show *?thesis*
using *assms distributed-unique* **by** *blast*
qed

lemma (**in** *information-space*)
fixes *X* :: *'a* ⇒ *'b*
assumes *X*[*measurable*]: *distributed M MX X f* **and** *nn*: ⋀*x*. *x* ∈ *space MX* ⟹ 0 ≤ *f x*
shows *entropy-distr*: *entropy b MX X* = − (∫ *x*. *f x* ∗ *log b* (*f x*) ∂*MX*) (**is** *?eq*)
proof −
note *D* = *distributed-measurable*[*OF X*] *distributed-borel-measurable*[*OF X*]
note *ae* = *distributed-RN-deriv*[*OF X*]
note *distributed-real-measurable*[*OF nn X*, *measurable*]

have *ae-eq*: *AE x in distr M MX X*. *log b* (*enn2real* (*RN-deriv MX* (*distr M MX X*) *x*)) = *log b* (*f x*)
unfolding *distributed-distr-eq-density*[*OF X*]
using *D ae* **by** (*auto simp*: *AE-density*)

have *int-eq*: (∫ *x*. *f x* ∗ *log b* (*f x*) ∂*MX*) = (∫ *x*. *log b* (*f x*) ∂*distr M MX X*)
unfolding *distributed-distr-eq-density*[*OF X*]
using *D*
by (*subst integral-density*) (*auto simp*: *nn*)

show *?eq*
unfolding *entropy-def KL-divergence-def entropy-density-def comp-def int-eq neg-equal-iff-equal*
using *ae-eq* **by** (*intro integral-cong-AE*) (*auto simp*: *nn*)
qed

lemma (**in** *information-space*) *entropy-le*:
fixes *Px* :: *'b* ⇒ *real* **and** *MX* :: *'b measure*
assumes *X*[*measurable*]: *distributed M MX X Px* **and** *Px-nn*[*simp*]: ⋀*x*. *x* ∈ *space MX* ⟹ 0 ≤ *Px x*
and *fin*: *emeasure MX* {*x* ∈ *space MX*. *Px x* ≠ 0} ≠ *top*

and *int*: *integrable MX* ($\lambda x.$ $-$ *Px x* $*$ *log b* (*Px x*))
shows *entropy b MX X* $\leq$ *log b* (*measure MX* $\{x \in$ *space MX*. *Px x* $\neq 0\}$)
proof $-$
note *Px* = *distributed-borel-measurable*[*OF X*]
interpret *X*: *prob-space distr M MX X*
using *distributed-measurable*[*OF X*] **by** (*rule prob-space-distr*)

have $-$ *log b* (*measure MX* $\{x \in$ *space MX*. *Px x* $\neq 0\}$) =
$-$ *log b* ($\int$ *x*. *indicator* $\{x \in$ *space MX*. *Px x* $\neq 0\}$ *x* ∂*MX*)
using *Px Px-nn fin* **by** (*auto simp*: *measure-def*)
also have $-$ *log b* ($\int$ *x*. *indicator* $\{x \in$ *space MX*. *Px x* $\neq 0\}$ *x* ∂*MX*) = $-$ *log b* ($\int$ *x*. *1 / Px x* ∂*distr M MX X*)
proof $-$
have *integral*^{L} *MX* (*indicator* $\{x \in$ *space MX*. *Px x* $\neq 0\}$) = *LINT x*|*MX*. *Px x* $*_R$ (*1 / Px x*)
by (*rule Bochner-Integration.integral-cong*) *auto*
also have ... = *LINT x*|*density MX* ($\lambda x.$ *ennreal* (*Px x*)). *1 / Px x*
by (*rule integral-density* [*symmetric*]) (*use Px Px-nn* **in** *auto*)
finally show *?thesis*
unfolding *distributed-distr-eq-density*[*OF X*] **by** *simp*
qed
also have ... $\leq$ ($\int$ *x*. $-$ *log b* (*1 / Px x*) ∂*distr M MX X*)
proof (*rule X.jensens-inequality*[*of* $\lambda x.$ *1 / Px x* $\{0<..\}$ *0 1* $\lambda x.$ $-$ *log b x*])
show *AE x in distr M MX X*. *1 / Px x* $\in \{0<..\}$
unfolding *distributed-distr-eq-density*[*OF X*]
using *Px* **by** (*auto simp*: *AE-density*)
have [*simp*]: $\bigwedge x.$ *x* $\in$ *space MX* $\Longrightarrow$ *ennreal* (*if Px x = 0 then 0 else 1*) = *indicator* $\{x \in$ *space MX*. *Px x* $\neq 0\}$ *x*
by (*auto simp*: *one-ennreal-def*)
have ($\int^+$ *x*. *ennreal* ($-$ (*if Px x = 0 then 0 else 1*)) ∂*MX*) = ($\int^+$ *x*. *0* ∂*MX*)
by (*intro nn-integral-cong*) (*auto simp*: *ennreal-neg*)
then show *integrable* (*distr M MX X*) ($\lambda x.$ *1 / Px x*)
unfolding *distributed-distr-eq-density*[*OF X*] **using** *Px*
by (*auto simp*: *nn-integral-density real-integrable-def fin ennreal-neg ennreal-mult*[*symmetric*]
cong: *nn-integral-cong*)
have *integrable MX* ($\lambda x.$ *Px x* $*$ *log b* (*1 / Px x*)) =
integrable MX ($\lambda x.$ $-$ *Px x* $*$ *log b* (*Px x*))
using *Px*
by (*intro integrable-cong-AE*) (*auto simp*: *log-divide-pos log-recip*)
then show *integrable* (*distr M MX X*) ($\lambda x.$ $-$ *log b* (*1 / Px x*))
unfolding *distributed-distr-eq-density*[*OF X*]
using *Px int*
by (*subst integrable-real-density*) *auto*
qed (*auto simp*: *minus-log-convex*[*OF b-gt-1*])
also have ... = ($\int$ *x*. *log b* (*Px x*) ∂*distr M MX X*)
unfolding *distributed-distr-eq-density*[*OF X*] **using** *Px*
by (*intro integral-cong-AE*) (*auto simp*: *AE-density log-divide-pos*)
also have ... = $-$ *entropy b MX X*

unfolding *distributed-distr-eq-density*[*OF X*] **using** *Px*
by (*subst entropy-distr*[*OF X*]) (*auto simp*: *integral-density*)
finally show *?thesis*
by *simp*
qed

lemma (**in** *information-space*) *entropy-le-space*:
fixes *Px* :: *'b* ⇒ *real* **and** *MX* :: *'b measure*
assumes *X*: *distributed M MX X Px* **and** *Px-nn*[*simp*]: ⋀*x*. *x* ∈ *space MX* ⟹ *0* ≤ *Px x*
and *fin*: *finite-measure MX*
and *int*: *integrable MX* (λ*x*. − *Px x* ∗ *log b* (*Px x*))
shows *entropy b MX X* ≤ *log b* (*measure MX* (*space MX*))
proof −
note *Px* = *distributed-borel-measurable*[*OF X*]
interpret *finite-measure MX* **by** *fact*
have *entropy b MX X* ≤ *log b* (*measure MX* {*x* ∈ *space MX*. *Px x* ≠ *0*})
using *int X* **by** (*intro entropy-le*) *auto*
also have ... ≤ *log b* (*measure MX* (*space MX*))
using *Px distributed-imp-emeasure-nonzero*[*OF X*]
by (*intro Transcendental.log-mono*)
(*auto intro*!: *finite-measure-mono b-gt-1 less-le*[*THEN iffD2*]
simp: *emeasure-eq-measure cong*: *conj-cong*)
finally show *?thesis* **.**
qed

lemma (**in** *information-space*) *entropy-uniform*:
assumes *X*: *distributed M MX X* (λ*x*. *indicator A x* / *measure MX A*) (**is** *distributed - - - ?f*)
shows *entropy b MX X* = *log b* (*measure MX A*)
proof (*subst entropy-distr*[*OF X*])
have [*simp*]: *emeasure MX A* ≠ ∞
using *uniform-distributed-params*[*OF X*] **by** (*auto simp add*: *measure-def*)
have *eq*: (∫ *x*. *indicator A x* / *measure MX A* ∗ *log b* (*indicator A x* / *measure MX A*) ∂*MX*) =
(∫ *x*. (− *log b* (*measure MX A*) / *measure MX A*) ∗ *indicator A x* ∂*MX*)
using *uniform-distributed-params*[*OF X*]
by (*intro Bochner-Integration.integral-cong*) (*auto split*: *split-indicator simp*: *log-divide-pos zero-less-measure-iff*)
show − (∫ *x*. *indicator A x* / *measure MX A* ∗ *log b* (*indicator A x* / *measure MX A*) ∂*MX*) =
log b (*measure MX A*)
unfolding *eq* **using** *uniform-distributed-params*[*OF X*]
by (*subst Bochner-Integration.integral-mult-right*) (*auto simp*: *measure-def less-top*[*symmetric*] *intro*!: *integrable-real-indicator*)
qed *simp*

lemma (**in** *information-space*) *entropy-simple-distributed*:
simple-distributed M X f ⟹ $\mathcal{H}(X)$ = − (∑ *x*∈*X*‘*space M*. *f x* ∗ *log b* (*f x*))

by (*subst entropy-distr*[*OF simple-distributed*])
(*auto simp add*: *lebesgue-integral-count-space-finite*)

lemma (**in** *information-space*) *entropy-le-card-not-0*:
assumes *X*: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (card\ (X\ `\ space\ M \cap \{x.\ f\ x \neq 0\}))$
proof −
let *?X* = *count-space* (*X*‘*space M*)
have $\mathcal{H}(X) \leq log\ b\ (measure\ ?X\ \{x \in space\ ?X.\ f\ x \neq 0\})$
by (*rule entropy-le*[*OF simple-distributed*[*OF X*]])
(*insert X*, *auto simp*: *simple-distributed-finite*[*OF X*] *subset-eq integrable-count-space emeasure-count-space*)
also have *measure ?X* $\{x \in space\ ?X.\ f\ x \neq 0\} = card\ (X\ `\ space\ M \cap \{x.\ f\ x \neq 0\})$
by (*simp-all add*: *simple-distributed-finite*[*OF X*] *subset-eq emeasure-count-space measure-def Int-def*)
finally show *?thesis* **.**
qed

lemma (**in** *information-space*) *entropy-le-card*:
assumes *X*: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (real\ (card\ (X\ `\ space\ M)))$
proof −
let *?X* = *count-space* (*X*‘*space M*)
have $\mathcal{H}(X) \leq log\ b\ (measure\ ?X\ (space\ ?X))$
by (*rule entropy-le-space*[*OF simple-distributed*[*OF X*]])
(*insert X*, *auto simp*: *simple-distributed-finite*[*OF X*] *subset-eq integrable-count-space emeasure-count-space finite-measure-count-space*)
also have *measure ?X* (*space ?X*) = *card* (*X* ‘ *space M*)
by (*simp-all add*: *simple-distributed-finite*[*OF X*] *subset-eq emeasure-count-space measure-def*)
finally show *?thesis* **.**
qed

12.6 Conditional Mutual Information

definition (**in** *prob-space*)
conditional-mutual-information b MX MY MZ X Y Z ≡
mutual-information b MX (*MY* $\bigotimes_M$ *MZ*) *X* (λ*x*. (*Y x*, *Z x*)) −
mutual-information b MX MZ X Z

abbreviation (**in** *information-space*)
conditional-mutual-information-Pow (‹$\mathcal{I}'$(- ; - | -)›) **where**
$\mathcal{I}(X\ ;\ Y\ |\ Z) \equiv$ *conditional-mutual-information b*
(*count-space* (*X* ‘ *space M*)) (*count-space* (*Y* ‘ *space M*)) (*count-space* (*Z* ‘ *space M*)) *X Y Z*

lemma (**in** *information-space*)
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*:

sigma-finite-measure P
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
assumes *Pz*[*measurable*]: *distributed M P Z Pz*
and *Pz-nn*[*simp*]: $\bigwedge z.\ z \in space\ P \Longrightarrow 0 \leq Pz\ z$
assumes *Pyz*[*measurable*]: *distributed M* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
and *Pyz-nn*[*simp*]: $\bigwedge y\ z.\ y \in space\ T \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pyz\ (y,\ z)$
assumes *Pxz*[*measurable*]: *distributed M* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
and *Pxz-nn*[*simp*]: $\bigwedge x\ z.\ x \in space\ S \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pxz\ (x,\ z)$
assumes *Pxyz*[*measurable*]: *distributed M* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
and *Pxyz-nn*[*simp*]: $\bigwedge x\ y\ z.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pxyz\ (x,\ y,\ z)$
assumes *I1*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Px\ x * Pyz\ (y,\ z))))$
assumes *I2*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxz\ (x,\ z)\ /\ (Px\ x * Pz\ z)))$
shows *conditional-mutual-information-generic-eq*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg*: $0 \leq$ *conditional-mutual-information b S T P X Y Z* (**is** *?nonneg*)
proof −
have [*measurable*]: $Px \in S \rightarrow_M borel$
using *Px Px-nn* **by** (*intro distributed-real-measurable*)
have [*measurable*]: $Pz \in P \rightarrow_M borel$
using *Pz Pz-nn* **by** (*intro distributed-real-measurable*)
have *measurable-Pyz*[*measurable*]: $Pyz \in (T \bigotimes_M P) \rightarrow_M borel$
using *Pyz Pyz-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)
have *measurable-Pxz*[*measurable*]: $Pxz \in (S \bigotimes_M P) \rightarrow_M borel$
using *Pxz Pxz-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)
have *measurable-Pxyz*[*measurable*]: $Pxyz \in (S \bigotimes_M T \bigotimes_M P) \rightarrow_M borel$
using *Pxyz Pxyz-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)

interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *P*: *sigma-finite-measure P* **by** *fact*
interpret *TP*: *pair-sigma-finite T P* ..
interpret *SP*: *pair-sigma-finite S P* ..
interpret *ST*: *pair-sigma-finite S T* ..
interpret *SPT*: *pair-sigma-finite* $S \bigotimes_M P\ T$ ..
interpret *STP*: *pair-sigma-finite* $S\ T \bigotimes_M P$ ..
interpret *TPS*: *pair-sigma-finite* $T \bigotimes_M P\ S$ ..
have *TP*: *sigma-finite-measure* $(T \bigotimes_M P)$ ..
have *SP*: *sigma-finite-measure* $(S \bigotimes_M P)$ ..
have *YZ*: *random-variable* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$
using *Pyz* **by** (*simp add*: *distributed-measurable*)

from *Pxz Pxyz* **have** *distr-eq*: *distr M* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x)) =$
distr (*distr M* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x)))$ $(S \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ (x,\ z))$
by (*simp add*: *comp-def distr-distr*)

have *mutual-information b S P X Z* =
$(\int x.\ Pxz\ x * log\ b\ (Pxz\ x\ /\ (Px\ (fst\ x) * Pz\ (snd\ x)))\ \partial(S \bigotimes_M P))$
by (*rule mutual-information-distr*[*OF S P Px Px-nn Pz Pz-nn Pxz Pxz-nn*])
also have ... = $(\int (x,y,z).\ Pxyz\ (x,y,z) * log\ b\ (Pxz\ (x,z)\ /\ (Px\ x * Pz\ z))\ \partial(S \bigotimes_M T \bigotimes_M P))$
using *b-gt-1 Pxz Px Pz*
by (*subst distributed-transform-integral*[*OF Pxyz - Pxz -*, **where** $T{=}\lambda(x,\ y,\ z).\ (x,\ z)$])
(*auto simp*: *split-beta'* *space-pair-measure*)
finally have *mi-eq*:
mutual-information b S P X Z = $(\int (x,y,z).\ Pxyz\ (x,y,z) * log\ b\ (Pxz\ (x,z)\ /\ (Px\ x * Pz\ z))\ \partial(S \bigotimes_M T \bigotimes_M P))$ **.**

have *ae1*: *AE x in* $S \bigotimes_M T \bigotimes_M P$. $Px\ (fst\ x) = 0 \longrightarrow Pxyz\ x = 0$
by (*intro subdensity-real*[*of fst*, *OF - Pxyz Px*]) (*auto simp*: *space-pair-measure*)
moreover have *ae2*: *AE x in* $S \bigotimes_M T \bigotimes_M P$. $Pz\ (snd\ (snd\ x)) = 0 \longrightarrow Pxyz\ x = 0$
by (*intro subdensity-real*[*of* $\lambda x.\ snd\ (snd\ x)$, *OF - Pxyz Pz*]) (*auto simp*: *space-pair-measure*)
moreover have *ae3*: *AE x in* $S \bigotimes_M T \bigotimes_M P$. $Pxz\ (fst\ x,\ snd\ (snd\ x)) = 0 \longrightarrow Pxyz\ x = 0$
by (*intro subdensity-real*[*of* $\lambda x.\ (fst\ x,\ snd\ (snd\ x))$, *OF - Pxyz Pxz*]) (*auto simp*: *space-pair-measure*)
moreover have *ae4*: *AE x in* $S \bigotimes_M T \bigotimes_M P$. $Pyz\ (snd\ x) = 0 \longrightarrow Pxyz\ x = 0$
by (*intro subdensity-real*[*of snd*, *OF - Pxyz Pyz*]) (*auto simp*: *space-pair-measure*)
ultimately have *ae*: *AE x in* $S \bigotimes_M T \bigotimes_M P$.
$Pxyz\ x * log\ b\ (Pxyz\ x\ /\ (Px\ (fst\ x) * Pyz\ (snd\ x))) -$
$Pxyz\ x * log\ b\ (Pxz\ (fst\ x,\ snd\ (snd\ x))\ /\ (Px\ (fst\ x) * Pz\ (snd\ (snd\ x)))) =$
$Pxyz\ x * log\ b\ (Pxyz\ x * Pz\ (snd\ (snd\ x))\ /\ (Pxz\ (fst\ x,\ snd\ (snd\ x)) * Pyz\ (snd\ x)))$
using *AE-space*
proof *eventually-elim*
case (*elim x*)
show *?case*
proof *cases*
assume $Pxyz\ x \neq 0$
with *elim* **have** $0 < Px\ (fst\ x)$ $0 < Pz\ (snd\ (snd\ x))$ $0 < Pxz\ (fst\ x,\ snd\ (snd\ x))$
$0 < Pyz\ (snd\ x)$ $0 < Pxyz\ x$
by (*auto simp*: *space-pair-measure less-le*)
then show *?thesis*
using *b-gt-1* **by** (*simp add*: *log-simps less-imp-le field-simps*)
qed *simp*

qed
with *I1 I2* **show** *?eq*
unfolding *conditional-mutual-information-def*
apply (*subst mi-eq*)
apply (*subst mutual-information-distr*[*OF S TP Px Px-nn Pyz - Pxyz*])
apply (*auto simp*: *space-pair-measure*)
apply (*subst Bochner-Integration.integral-diff*[*symmetric*])
apply (*auto intro*!: *integral-cong-AE simp*: *split-beta' simp del*: *Bochner-Integration.integral-diff*)
done

let *?P* = *density* (*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*) *Pxyz*
interpret *P*: *prob-space ?P*
unfolding *distributed-distr-eq-density*[*OF Pxyz*, *symmetric*]
by (*rule prob-space-distr*) *simp*

let *?Q* = *density* (*T* $\bigotimes_M$ *P*) *Pyz*
interpret *Q*: *prob-space ?Q*
unfolding *distributed-distr-eq-density*[*OF Pyz*, *symmetric*]
by (*rule prob-space-distr*) *simp*

let *?f* = λ(*x*, *y*, *z*). *Pxz* (*x*, *z*) * (*Pyz* (*y*, *z*) / *Pz z*) / *Pxyz* (*x*, *y*, *z*)

from *subdensity-real*[*of snd*, *OF - Pyz Pz - AE-I2 AE-I2*]
have *aeX1*: *AE x in T* $\bigotimes_M$ *P*. *Pz* (*snd x*) = *0* $\longrightarrow$ *Pyz x* = *0*
by (*auto simp*: *comp-def space-pair-measure*)
have *aeX2*: *AE x in T* $\bigotimes_M$ *P*. *0* $\leq$ *Pz* (*snd x*)
using *Pz* **by** (*intro TP.AE-pair-measure*) (*auto simp*: *comp-def*)

have *aeX3*: *AE y in T* $\bigotimes_M$ *P*. ($\int^+$ *x*. *ennreal* (*Pxz* (*x*, *snd y*)) ∂*S*) = *ennreal* (*Pz* (*snd y*))
using *Pz distributed-marginal-eq-joint2*[*OF P S Pz Pxz*]
by (*intro TP.AE-pair-measure*) *auto*

have ($\int^+$ *x*. *?f x* ∂*?P*) $\leq$ ($\int^+$ (*x*, *y*, *z*). *Pxz* (*x*, *z*) * (*Pyz* (*y*, *z*) / *Pz z*) ∂(*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*))
by (*subst nn-integral-density*)
(*auto intro*!: *nn-integral-mono simp*: *space-pair-measure ennreal-mult*[*symmetric*])
also have ... = ($\int^+$(*y*, *z*). ($\int^+$ *x*. *ennreal* (*Pxz* (*x*, *z*)) * *ennreal* (*Pyz* (*y*, *z*) / *Pz z*) ∂*S*) ∂(*T* $\bigotimes_M$ *P*))
by (*subst STP.nn-integral-snd*[*symmetric*])
(*auto simp add*: *split-beta' ennreal-mult*[*symmetric*] *space-pair-measure intro*!: *nn-integral-cong*)
also have ... = ($\int^+$*x*. *ennreal* (*Pyz x*) * *1* ∂*T* $\bigotimes_M$ *P*)
proof −
have *D*: ($\int^+$ *x*. *ennreal* (*Pxz* (*x*, *b*)) * *ennreal* (*Pyz* (*a*, *b*) / *Pz b*) ∂*S*) = *ennreal* (*Pyz* (*a*, *b*))
if *Pz b* = *0* $\longrightarrow$ *Pyz* (*a*, *b*) = *0 a* $\in$ *space T* $\wedge$ *b* $\in$ *space P*
($\int^+$ *x*. *ennreal* (*Pxz* (*x*, *b*)) ∂*S*) = *ennreal* (*Pz b*)
for *a b*

using *that* **by** (*subst nn-integral-multc*) (*auto split*: *prod.split simp*: *ennreal-mult*[*symmetric*])
show *?thesis*
apply (*rule nn-integral-cong-AE*)
using *aeX1 aeX3*
by (*force simp add*: *space-pair-measure D*)
qed
also have ... = *1*
using *Q.emeasure-space-1 distributed-distr-eq-density*[*OF Pyz*]
by (*subst nn-integral-density*[*symmetric*]) *auto*
finally have *le1*: $(\int^+ x.\ ?f\ x\ \partial ?P) \leq 1$ **.**
also have ... $< \infty$ **by** *simp*
finally have *fin*: $(\int^+ x.\ ?f\ x\ \partial ?P) \neq \infty$ **by** *simp*

have $(\int^+ x.\ ennreal\ (Pxyz\ x) *$
$ennreal\ (Pxz\ (fst\ x,\ snd\ (snd\ x)) * Pyz\ (snd\ x)\ /\ (Pz\ (snd\ (snd\ x)) * Pxyz\ x))$
$\partial(S \bigotimes_M T \bigotimes_M P)) \neq 0$
proof
let $?g = \lambda x.\ ennreal\ (Pxyz\ x) * (Pxz\ (fst\ x,\ snd\ (snd\ x)) * Pyz\ (snd\ x)\ /\ (Pz\ (snd\ (snd\ x)) * Pxyz\ x))$
assume $(\int^+ x.\ ?g\ x\ \partial(S \bigotimes_M T \bigotimes_M P)) = 0$
then have *AE x in* $S \bigotimes_M T \bigotimes_M P.\ ?g\ x = 0$
by (*intro nn-integral-0-iff-AE*[*THEN iffD1*]) *auto*
then have *AE x in* $S \bigotimes_M T \bigotimes_M P.\ Pxyz\ x = 0$
using *ae2 ae3 ae4*
by (*auto split*: *if-split-asm simp*: *mult-le-0-iff divide-le-0-iff space-pair-measure*)
then have $(\int^+ x.\ ennreal\ (Pxyz\ x)\ \partial S \bigotimes_M T \bigotimes_M P) = 0$
by (*subst nn-integral-cong-AE*[*of - λx. 0*]) *auto*
with *P.emeasure-space-1* **show** *False*
by (*subst* (*asm*) *emeasure-density*) (*auto cong*: *nn-integral-cong*)
qed
then have *pos*: $(\int^+ x.\ ?f\ x\ \partial ?P) \neq 0$
by (*subst nn-integral-density*) (*simp-all add*: *split-beta'*)

have *neg*: $(\int^+ x.\ -\ ?f\ x\ \partial ?P) = 0$
by (*subst nn-integral-0-iff-AE*) (*auto simp*: *space-pair-measure ennreal-neg*)

have *I3*: *integrable* $(S \bigotimes_M T \bigotimes_M P)\ (\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z))))$
apply (*rule integrable-cong-AE*[*THEN iffD1*, *OF - - - Bochner-Integration.integrable-diff*[*OF I1 I2*]])
using *ae*
apply (*auto simp*: *split-beta'*)
done

have $-\ log\ b\ 1 \leq -\ log\ b\ (integral^L\ ?P\ ?f)$
proof (*intro le-imp-neg-le Transcendental.log-mono*[*OF b-gt-1*])
have *If*: *integrable ?P ?f*

```
    unfolding real-integrable-def
  proof (intro conjI)
    from neg show (∫⁺ x. − ?f x ∂?P) ≠ ∞
      by simp
    from fin show (∫⁺ x. ?f x ∂?P) ≠ ∞
      by simp
  qed simp
  then have (∫⁺ x. ?f x ∂?P) = (∫ x. ?f x ∂?P)
    apply (rule nn-integral-eq-integral)
    apply (auto simp: space-pair-measure ennreal-neg)
    done
  with pos le1
  show 0 < (∫ x. ?f x ∂?P) (∫ x. ?f x ∂?P) ≤ 1
    by (simp-all add: one-ennreal.rep-eq zero-less-iff-neq-zero[symmetric])
qed
also have − log b (integral^L ?P ?f) ≤ (∫ x. − log b (?f x) ∂?P)
proof (rule P.jensens-inequality[where a=0 and b=1 and I={0<..}])
  show AE x in ?P. ?f x ∈ {0<..}
    unfolding AE-density[OF distributed-borel-measurable[OF Pxyz]]
    using ae1 ae2 ae3 ae4
    by (auto simp: space-pair-measure less-le)
  show integrable ?P ?f
    unfolding real-integrable-def
    using fin neg by (auto simp: split-beta')
  have integrable (S ⨂_M T ⨂_M P) (λx. Pxyz x ∗ − log b (?f x))
    apply (rule integrable-cong-AE[THEN iffD1, OF - - - I3])
    by (auto simp: log-mult log-divide field-simps)
  then
  show integrable ?P (λx. − log b (?f x))
    by (subst integrable-real-density) (auto simp: space-pair-measure)
qed (auto simp: b-gt-1 minus-log-convex)
also have ... = conditional-mutual-information b S T P X Y Z
  unfolding ‹?eq›
  apply (subst integral-real-density)
    apply simp
   apply (force simp: space-pair-measure)
  apply simp
  apply (intro integral-cong-AE)
  by (auto simp: field-simps log-divide)
finally show ?nonneg
  by simp
qed

lemma (in information-space)
  fixes Px :: - ⇒ real
  assumes S: sigma-finite-measure S and T: sigma-finite-measure T and P:
sigma-finite-measure P
  assumes Fx: finite-entropy S X Px
  assumes Fz: finite-entropy P Z Pz
```

assumes *Fyz*: *finite-entropy* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Fxz*: *finite-entropy* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Fxyz*: *finite-entropy* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows *conditional-mutual-information-generic-eq'*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ **(is** *?eq***)**
and *conditional-mutual-information-generic-nonneg'*: $0 \leq$ *conditional-mutual-information b S T P X Y Z* **(is** *?nonneg***)**
proof −
note *Px* = *Fx*[*THEN finite-entropy-distributed*, *measurable*]
note *Pz* = *Fz*[*THEN finite-entropy-distributed*, *measurable*]
note *Pyz* = *Fyz*[*THEN finite-entropy-distributed*, *measurable*]
note *Pxz* = *Fxz*[*THEN finite-entropy-distributed*, *measurable*]
note *Pxyz* = *Fxyz*[*THEN finite-entropy-distributed*, *measurable*]

note *Px-nn* = *Fx*[*THEN finite-entropy-nn*]
note *Pz-nn* = *Fz*[*THEN finite-entropy-nn*]
note *Pyz-nn* = *Fyz*[*THEN finite-entropy-nn*]
note *Pxz-nn* = *Fxz*[*THEN finite-entropy-nn*]
note *Pxyz-nn* = *Fxyz*[*THEN finite-entropy-nn*]

note *Px'* = *Fx*[*THEN finite-entropy-measurable*, *measurable*]
note *Pz'* = *Fz*[*THEN finite-entropy-measurable*, *measurable*]
note *Pyz'* = *Fyz*[*THEN finite-entropy-measurable*, *measurable*]
note *Pxz'* = *Fxz*[*THEN finite-entropy-measurable*, *measurable*]
note *Pxyz'* = *Fxyz*[*THEN finite-entropy-measurable*, *measurable*]

interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *P*: *sigma-finite-measure P* **by** *fact*
interpret *TP*: *pair-sigma-finite T P* **..**
interpret *SP*: *pair-sigma-finite S P* **..**
interpret *ST*: *pair-sigma-finite S T* **..**
interpret *SPT*: *pair-sigma-finite* $S \bigotimes_M P\ T$ **..**
interpret *STP*: *pair-sigma-finite* $S\ T \bigotimes_M P$ **..**
interpret *TPS*: *pair-sigma-finite* $T \bigotimes_M P\ S$ **..**
have *TP*: *sigma-finite-measure* $(T \bigotimes_M P)$ **..**
have *SP*: *sigma-finite-measure* $(S \bigotimes_M P)$ **..**

from *Pxz Pxyz* **have** *distr-eq*: *distr M* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x)) =$
distr (*distr M* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x)))$ $(S \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ (x,\ z))$
by (*simp add*: *distr-distr comp-def*)

have *mutual-information b S P X Z* =
$(\int x.\ Pxz\ x * log\ b\ (Pxz\ x\ /\ (Px\ (fst\ x) * Pz\ (snd\ x)))\ \partial(S \bigotimes_M P))$
using *Px Px-nn Pz Pz-nn Pxz Pxz-nn*
by (*rule mutual-information-distr*[*OF S P*]) (*auto simp*: *space-pair-measure*)

also have ... = ($\int$ (*x*,*y*,*z*). *Pxyz* (*x*,*y*,*z*) * *log b* (*Pxz* (*x*,*z*) / (*Px x* * *Pz z*)) ∂(*S* ⨂$_M$ *T* ⨂$_M$ *P*))
using *b-gt-1 Pxz Pxz-nn Pxyz Pxyz-nn*
by (*subst distributed-transform-integral*[*OF Pxyz - Pxz*, **where** *T*=λ(*x*, *y*, *z*). (*x*, *z*)])
(*auto simp*: *split-beta'*)
finally have *mi-eq*:
mutual-information b S P X Z = ($\int$ (*x*,*y*,*z*). *Pxyz* (*x*,*y*,*z*) * *log b* (*Pxz* (*x*,*z*) / (*Px x* * *Pz z*)) ∂(*S* ⨂$_M$ *T* ⨂$_M$ *P*)) **.**

have *ae1*: *AE x in S* ⨂$_M$ *T* ⨂$_M$ *P*. *Px* (*fst x*) = *0* ⟶ *Pxyz x* = *0*
by (*intro subdensity-finite-entropy*[*of fst*, *OF - Fxyz Fx*]) *auto*
moreover have *ae2*: *AE x in S* ⨂$_M$ *T* ⨂$_M$ *P*. *Pz* (*snd* (*snd x*)) = *0* ⟶ *Pxyz x* = *0*
by (*intro subdensity-finite-entropy*[*of* λ*x*. *snd* (*snd x*), *OF - Fxyz Fz*]) *auto*
moreover have *ae3*: *AE x in S* ⨂$_M$ *T* ⨂$_M$ *P*. *Pxz* (*fst x*, *snd* (*snd x*)) = *0* ⟶ *Pxyz x* = *0*
by (*intro subdensity-finite-entropy*[*of* λ*x*. (*fst x*, *snd* (*snd x*)), *OF - Fxyz Fxz*]) *auto*
moreover have *ae4*: *AE x in S* ⨂$_M$ *T* ⨂$_M$ *P*. *Pyz* (*snd x*) = *0* ⟶ *Pxyz x* = *0*
by (*intro subdensity-finite-entropy*[*of snd*, *OF - Fxyz Fyz*]) *auto*
ultimately have *ae*: *AE x in S* ⨂$_M$ *T* ⨂$_M$ *P*.
Pxyz x * *log b* (*Pxyz x* / (*Px* (*fst x*) * *Pyz* (*snd x*))) −
Pxyz x * *log b* (*Pxz* (*fst x*, *snd* (*snd x*)) / (*Px* (*fst x*) * *Pz* (*snd* (*snd x*)))) =
Pxyz x * *log b* (*Pxyz x* * *Pz* (*snd* (*snd x*)) / (*Pxz* (*fst x*, *snd* (*snd x*)) * *Pyz* (*snd x*)))
using *AE-space*
proof *eventually-elim*
case (*elim x*)
show *?case*
proof *cases*
assume *Pxyz x* ≠ *0*
with *elim* **have** *0* < *Px* (*fst x*) *0* < *Pz* (*snd* (*snd x*)) *0* < *Pxz* (*fst x*, *snd* (*snd x*))
0 < *Pyz* (*snd x*) *0* < *Pxyz x*
using *Px-nn Pz-nn Pxz-nn Pyz-nn Pxyz-nn*
by (*auto simp*: *space-pair-measure less-le*)
then show *?thesis*
using *b-gt-1* **by** (*simp add*: *log-simps less-imp-le field-simps*)
qed *simp*
qed

have *integrable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*)
(λ*x*. *Pxyz x* * *log b* (*Pxyz x*) − *Pxyz x* * *log b* (*Px* (*fst x*)) − *Pxyz x* * *log b* (*Pyz* (*snd x*)))
using *finite-entropy-integrable*[*OF Fxyz*]
using *finite-entropy-integrable-transform*[*OF Fx Pxyz Pxyz-nn*, *of fst*]
using *finite-entropy-integrable-transform*[*OF Fyz Pxyz Pxyz-nn*, *of snd*]

by *simp*
moreover have (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Px x* ∗ *Pyz* (*y*, *z*)))) ∈ *borel-measurable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*)
using *Pxyz Px Pyz* **by** *simp*
ultimately have *I1*: *integrable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*) (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Px x* ∗ *Pyz* (*y*, *z*))))
apply (*rule integrable-cong-AE-imp*)
using *ae1 ae4 Px-nn Pyz-nn Pxyz-nn*
by (*auto simp*: *log-divide log-mult field-simps*)
have *integrable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*)
(λ*x*. *Pxyz x* ∗ *log b* (*Pxz* (*fst x*, *snd* (*snd x*))) − *Pxyz x* ∗ *log b* (*Px* (*fst x*)) − *Pxyz x* ∗ *log b* (*Pz* (*snd* (*snd x*))))
using *finite-entropy-integrable-transform*[*OF Fxz Pxyz Pxyz-nn*, *of* λ*x*. (*fst x*, *snd* (*snd x*))]
using *finite-entropy-integrable-transform*[*OF Fx Pxyz Pxyz-nn*, *of fst*]
using *finite-entropy-integrable-transform*[*OF Fz Pxyz Pxyz-nn*, *of snd* ∘ *snd*]
by *simp*
moreover have (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxz* (*x*, *z*) / (*Px x* ∗ *Pz z*))) ∈ *borel-measurable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*)
using *Pxyz Px Pz*
by *auto*
ultimately have *I2*: *integrable* (*S* ⨂$_M$ *T* ⨂$_M$ *P*) (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxz* (*x*, *z*) / (*Px x* ∗ *Pz z*)))
apply (*rule integrable-cong-AE-imp*)
using *ae1 ae2 ae3 ae4 Px-nn Pz-nn Pxz-nn Pyz-nn Pxyz-nn*
apply(*auto simp*: *field-simps log-divide log-mult*)
done
from *ae I1 I2* **show** *?eq*
unfolding *conditional-mutual-information-def mi-eq*
apply (*subst mutual-information-distr*[*OF S TP Px Px-nn Pyz Pyz-nn Pxyz Pxyz-nn*]; *simp add*: *space-pair-measure*)
apply (*subst Bochner-Integration.integral-diff*[*symmetric*])
apply (*auto intro*!: *integral-cong-AE simp*: *split-beta′ simp del*: *Bochner-Integration.integral-diff*)
done

let *?P* = *density* (*S* ⨂$_M$ *T* ⨂$_M$ *P*) *Pxyz*
interpret *P*: *prob-space ?P*
unfolding *distributed-distr-eq-density*[*OF Pxyz*, *symmetric*] **by** (*rule prob-space-distr*) *simp*

let *?Q* = *density* (*T* ⨂$_M$ *P*) *Pyz*
interpret *Q*: *prob-space ?Q*
unfolding *distributed-distr-eq-density*[*OF Pyz*, *symmetric*] **by** (*rule prob-space-distr*) *simp*

let *?f* = λ(*x*, *y*, *z*). *Pxz* (*x*, *z*) ∗ (*Pyz* (*y*, *z*) / *Pz z*) / *Pxyz* (*x*, *y*, *z*)

from *subdensity-finite-entropy*[*of snd*, *OF - Fyz Fz*]
have *aeX1*: *AE x in T* ⨂$_M$ *P*. *Pz* (*snd x*) = *0* ⟶ *Pyz x* = *0* **by** (*auto simp*:

comp-def)
have *aeX2*: *AE x in T* $\bigotimes_M$ *P. 0* $\leq$ *Pz* (*snd x*)
using *Pz* **by** (*intro TP.AE-pair-measure*) (*auto intro*: *Pz-nn*)

have *aeX3*: *AE y in T* $\bigotimes_M$ *P.* ($\int^+$ *x. ennreal* (*Pxz* (*x*, *snd y*)) ∂S) = *ennreal* (*Pz* (*snd y*))
using *Pz distributed-marginal-eq-joint2*[*OF P S Pz Pxz*]
by (*intro TP.AE-pair-measure*) (*auto*)
have ($\int^+$ *x. ?f x* ∂*?P*) $\leq$ ($\int^+$ (*x*, *y*, *z*). *Pxz* (*x*, *z*) * (*Pyz* (*y*, *z*) / *Pz z*) ∂(*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*))
using *Px-nn Pz-nn Pxz-nn Pyz-nn Pxyz-nn*
by (*subst nn-integral-density*)
(*auto intro*!: *nn-integral-mono simp*: *space-pair-measure ennreal-mult*[*symmetric*])
also have ... = ($\int^+$(*y*, *z*). $\int^+$ *x. ennreal* (*Pxz* (*x*, *z*)) * *ennreal* (*Pyz* (*y*, *z*) / *Pz z*) ∂S ∂T $\bigotimes_M$ *P*)
using *Px-nn Pz-nn Pxz-nn Pyz-nn Pxyz-nn*
by (*subst STP.nn-integral-snd*[*symmetric*])
(*auto simp add*: *split-beta′ ennreal-mult*[*symmetric*] *space-pair-measure intro*!: *nn-integral-cong*)
also have ... = ($\int^+$*x. ennreal* (*Pyz x*) * *1* ∂T $\bigotimes_M$ *P*)
proof −
have *∗*: ($\int^+$ *x. ennreal* (*Pxz* (*x*, *b*)) * *ennreal* (*Pyz* (*a*, *b*) / *Pz b*) ∂S) = *ennreal* (*Pyz* (*a*, *b*))
if *Pz b* = *0* ⟶ *Pyz* (*a*, *b*) = *0 0* $\leq$ *Pz b a* $\in$ *space T* $\wedge$ *b* $\in$ *space P*
($\int^+$ *x. ennreal* (*Pxz* (*x*, *b*)) ∂S) = *ennreal* (*Pz b*) **for** *a b*
using *Pyz-nn*[*of* (*a*,*b*)] *that*
by (*subst nn-integral-multc*) (*auto simp*: *space-pair-measure ennreal-mult*[*symmetric*])
show *?thesis*
using *aeX1 aeX2 aeX3 AE-space*
by (*force simp*: *∗ space-pair-measure intro*: *nn-integral-cong-AE*)
qed
also have ... = *1*
using *Q.emeasure-space-1 Pyz-nn distributed-distr-eq-density*[*OF Pyz*]
by (*subst nn-integral-density*[*symmetric*]) *auto*
finally have *le1*: ($\int^+$ *x. ?f x* ∂*?P*) $\leq$ *1* **.**
also have ... < ∞ **by** *simp*
finally have *fin*: ($\int^+$ *x. ?f x* ∂*?P*) $\neq$ ∞ **by** *simp*

have ($\int^+$ *x. ?f x* ∂*?P*) $\neq$ *0*
using *Pxyz-nn*
apply (*subst nn-integral-density*)
apply (*simp-all add*: *split-beta′ ennreal-mult′*[*symmetric*] *cong*: *nn-integral-cong*)
proof
let *?g* = *λx. ennreal* (*if Pxyz x* = *0 then 0 else Pxz* (*fst x*, *snd* (*snd x*)) * *Pyz* (*snd x*) / *Pz* (*snd* (*snd x*)))
assume ($\int^+$ *x. ?g x* ∂(*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*)) = *0*
then have *AE x in S* $\bigotimes_M$ *T* $\bigotimes_M$ *P. ?g x* = *0*
by (*intro nn-integral-0-iff-AE*[*THEN iffD1*]) *auto*
then have *AE x in S* $\bigotimes_M$ *T* $\bigotimes_M$ *P. Pxyz x* = *0*

```
    using ae1 ae2 ae3 ae4
    by (insert Px-nn Pz-nn Pxz-nn Pyz-nn,
      auto split: if-split-asm simp: mult-le-0-iff divide-le-0-iff space-pair-measure)
  then have (∫⁺ x. ennreal (Pxyz x) ∂S ⨂M T ⨂M P) = 0
    by (subst nn-integral-cong-AE[of - λx. 0]) auto
  with P.emeasure-space-1 show False
    by (subst (asm) emeasure-density) (auto cong: nn-integral-cong)
 qed
 then have pos: 0 < (∫⁺ x. ?f x ∂?P)
  by (simp add: zero-less-iff-neq-zero)

 have neg: (∫⁺ x. − ?f x ∂?P) = 0
  using Pz-nn Pxz-nn Pyz-nn Pxyz-nn
  by (intro nn-integral-0-iff-AE[THEN iffD2])
     (auto simp: split-beta' AE-density space-pair-measure intro!: AE-I2 en-
nreal-neg)

 have I3: integrable (S ⨂M T ⨂M P) (λ(x, y, z). Pxyz (x, y, z) ∗ log b (Pxyz
(x, y, z) / (Pxz (x, z) ∗ (Pyz (y,z) / Pz z))))
  apply (rule integrable-cong-AE[THEN iffD1, OF - - - Bochner-Integration.integrable-diff[OF
I1 I2]])
   using ae
   by (auto simp: split-beta')

 have − log b 1 ≤ − log b (integral^L ?P ?f)
 proof (intro le-imp-neg-le Transcendental.log-mono[OF b-gt-1])
  have If: integrable ?P ?f
   using neg fin by (force simp add: real-integrable-def)
  then have (∫⁺ x. ?f x ∂?P) = (∫ x. ?f x ∂?P)
   using Pz-nn Pxz-nn Pyz-nn Pxyz-nn
   by (intro nn-integral-eq-integral)
     (auto simp: AE-density space-pair-measure)
  with pos le1
  show 0 < (∫ x. ?f x ∂?P) (∫ x. ?f x ∂?P) ≤ 1
   by (simp-all add: )
 qed
 also have − log b (integral^L ?P ?f) ≤ (∫ x. − log b (?f x) ∂?P)
 proof (rule P.jensens-inequality[where a=0 and b=1 and I={0<..}])
  show AE x in ?P. ?f x ∈ {0<..}
   unfolding AE-density[OF distributed-borel-measurable[OF Pxyz]]
   using ae1 ae2 ae3 ae4
    by (insert Pxyz-nn Pyz-nn Pz-nn Pxz-nn, auto simp: space-pair-measure
less-le)
  show integrable ?P ?f
   unfolding real-integrable-def
   using fin neg by (auto simp: split-beta')
  have integrable (S ⨂M T ⨂M P) (λx. Pxyz x ∗ − log b (?f x))
   apply (rule integrable-cong-AE[THEN iffD1, OF - - - I3])
   using Pz-nn Pxz-nn Pyz-nn Pxyz-nn ae2 ae3 ae4
```

```
    by (auto simp: log-divide field-simps)
  then show integrable ?P (λx. − log b (?f x))
    using Pxyz-nn by (auto simp: integrable-real-density)
 qed (auto simp: b-gt-1 minus-log-convex)
 also have ... = conditional-mutual-information b S T P X Y Z
  unfolding ‹?eq›
  using Pz-nn Pxz-nn Pyz-nn Pxyz-nn
  apply (subst integral-real-density)
  apply simp
  apply simp
  apply simp
  apply (intro integral-cong-AE)
  using ae1 ae2 ae3 ae4
  by (auto simp: log-divide zero-less-mult-iff field-simps)
 finally show ?nonneg
  by simp
qed

lemma (in information-space) conditional-mutual-information-eq:
 assumes Pz: simple-distributed M Z Pz
 assumes Pyz: simple-distributed M (λx. (Y x, Z x)) Pyz
 assumes Pxz: simple-distributed M (λx. (X x, Z x)) Pxz
 assumes Pxyz: simple-distributed M (λx. (X x, Y x, Z x)) Pxyz
 shows 𝓘(X ; Y | Z) =
  (∑(x, y, z)∈(λx. (X x, Y x, Z x))‘space M. Pxyz (x, y, z) * log b (Pxyz (x, y,
z) / (Pxz (x, z) * (Pyz (y,z) / Pz z))))
proof (subst conditional-mutual-information-generic-eq[OF - - - - -
   simple-distributed[OF Pz] - simple-distributed-joint[OF Pyz] - simple-distributed-joint[OF
Pxz] -
    simple-distributed-joint2[OF Pxyz]])
 note simple-distributed-joint2-finite[OF Pxyz, simp]
 show sigma-finite-measure (count-space (X ‘ space M))
  by (simp add: sigma-finite-measure-count-space-finite)
 show sigma-finite-measure (count-space (Y ‘ space M))
  by (simp add: sigma-finite-measure-count-space-finite)
 show sigma-finite-measure (count-space (Z ‘ space M))
  by (simp add: sigma-finite-measure-count-space-finite)
 have count-space (X ‘ space M) ⨂M count-space (Y ‘ space M) ⨂M count-space
(Z ‘ space M) =
     count-space (X‘space M × Y‘space M × Z‘space M)
   (is ?P = ?C)
  by (simp add: pair-measure-count-space)

 let ?Px = λx. measure M (X −‘ {x} ∩ space M)
 have (λx. (X x, Z x)) ∈ measurable M (count-space (X ‘ space M) ⨂M count-space
(Z ‘ space M))
  using simple-distributed-joint[OF Pxz] by (rule distributed-measurable)
 from measurable-comp[OF this measurable-fst]
 have random-variable (count-space (X ‘ space M)) X
```

by (*simp add*: *comp-def*)
then have *simple-function M X*
unfolding *simple-function-def* **by** (*auto simp*: *measurable-count-space-eq2*)
then have *simple-distributed M X ?Px*
by (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)
then show *distributed M* (*count-space* (*X* ' *space M*)) *X ?Px*
by (*rule simple-distributed*)

let *?f* = (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*, *Z x*)) ' *space M then Pxyz x else 0*)
let *?g* = (λ*x*. *if x* ∈ (λ*x*. (*Y x*, *Z x*)) ' *space M then Pyz x else 0*)
let *?h* = (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Z x*)) ' *space M then Pxz x else 0*)
show
integrable ?P (λ(*x*, *y*, *z*). *?f* (*x*, *y*, *z*) ∗ *log b* (*?f* (*x*, *y*, *z*) / (*?Px x* ∗ *?g* (*y*, *z*))))
integrable ?P (λ(*x*, *y*, *z*). *?f* (*x*, *y*, *z*) ∗ *log b* (*?h* (*x*, *z*) / (*?Px x* ∗ *Pz z*)))
by (*auto intro*!: *integrable-count-space simp*: *pair-measure-count-space*)
let *?i* = λ*x y z*. *?f* (*x*, *y*, *z*) ∗ *log b* (*?f* (*x*, *y*, *z*) / (*?h* (*x*, *z*) ∗ (*?g* (*y*, *z*) / *Pz z*)))
let *?j* = λ*x y z*. *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Pxz* (*x*, *z*) ∗ (*Pyz* (*y*,*z*) / *Pz z*)))
have (λ(*x*, *y*, *z*). *?i x y z*) = (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*, *Z x*)) ' *space M then ?j* (*fst x*) (*fst* (*snd x*)) (*snd* (*snd x*)) *else 0*)
by (*auto intro*!: *ext*)
then show (∫ (*x*, *y*, *z*). *?i x y z* ∂*?P*) = (∑ (*x*, *y*, *z*)∈(λ*x*. (*X x*, *Y x*, *Z x*)) ' *space M*. *?j x y z*)
by (*auto intro*!: *sum.cong simp add*: ‹*?P* = *?C*› *lebesgue-integral-count-space-finite simple-distributed-finite sum.If-cases split-beta'*)
qed (*insert Pz Pyz Pxz Pxyz*, *auto intro*: *measure-nonneg*)

lemma (**in** *information-space*) *conditional-mutual-information-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **and** *Z*: *simple-function M Z*
shows 0 ≤ $\mathcal{I}$(*X* ; *Y* | *Z*)
proof −
have [*simp*]: *count-space* (*X* ' *space M*) ⨂$_M$ *count-space* (*Y* ' *space M*) ⨂$_M$ *count-space* (*Z* ' *space M*) =
count-space (*X*'*space M* × *Y*'*space M* × *Z*'*space M*)
by (*simp add*: *pair-measure-count-space X Y Z simple-functionD*)
note *sf* = *sigma-finite-measure-count-space-finite*[*OF simple-functionD*(*1*)]
note *sd* = *simple-distributedI*[*OF - - refl*]
note *sp* = *simple-function-Pair*
show *?thesis*
apply (*rule conditional-mutual-information-generic-nonneg*[*OF sf*[*OF X*] *sf*[*OF Y*] *sf*[*OF Z*]])
apply (*force intro*: *simple-distributed*[*OF sd*[*OF X*]])
apply *simp*
apply (*force intro*: *simple-distributed*[*OF sd*[*OF Z*]])
apply *simp*
apply (*force intro*: *simple-distributed-joint*[*OF sd*[*OF sp*[*OF Y Z*]]])

apply *simp*
apply (*force intro*: *simple-distributed-joint*[*OF sd*[*OF sp*[*OF X Z*]]])
apply *simp*
apply (*fastforce intro*: *simple-distributed-joint2*[*OF sd*[*OF sp*[*OF X sp*[*OF Y Z*]]]])
apply (*auto intro*!: *integrable-count-space simp*: *X Y Z simple-functionD*)
done
qed

12.7 Conditional Entropy

definition (**in** *prob-space*)
conditional-entropy b S T X Y = − (∫ (*x*, *y*). *log b* (*enn2real* (*RN-deriv* (*S* ⨂$_M$ *T*) (*distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*))) (*x*, *y*)) /
enn2real (*RN-deriv T* (*distr M T Y*) *y*)) ∂*distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)))

abbreviation (**in** *information-space*)
conditional-entropy-Pow (‹ℋ′(- | -′)›) **where**
ℋ(*X* | *Y*) ≡ *conditional-entropy b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*) *conditional-entropy-generic-eq*:
fixes *Pxy* :: - ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*[*simp*]: ⋀*x*. *x* ∈ *space T* ⟹ 0 ≤ *Py x*
assumes *Pxy*[*measurable*]: *distributed M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*[*simp*]: ⋀*x y*. *x* ∈ *space S* ⟹ *y* ∈ *space T* ⟹ 0 ≤ *Pxy* (*x*, *y*)
shows *conditional-entropy b S T X Y* = − (∫ (*x*, *y*). *Pxy* (*x*, *y*) ∗ *log b* (*Pxy* (*x*, *y*) / *Py y*) ∂(*S* ⨂$_M$ *T*))
proof −
interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**

have [*measurable*]: *Py* ∈ *T* →$_M$ *borel*
using *Py Py-nn* **by** (*intro distributed-real-measurable*)
have *measurable-Pxy*[*measurable*]: *Pxy* ∈ (*S* ⨂$_M$ *T*) →$_M$ *borel*
using *Pxy Pxy-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)

have *AE x in density* (*S* ⨂$_M$ *T*) (λ*x*. *ennreal* (*Pxy x*)). *Pxy x* = *enn2real* (*RN-deriv* (*S* ⨂$_M$ *T*) (*distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*))) *x*)
unfolding *AE-density*[*OF distributed-borel-measurable*, *OF Pxy*]
unfolding *distributed-distr-eq-density*[*OF Pxy*]
using *distributed-RN-deriv*[*OF Pxy*]
by *auto*
moreover
have *AE x in density* (*S* ⨂$_M$ *T*) (λ*x*. *ennreal* (*Pxy x*)). *Py* (*snd x*) = *enn2real*

```
(RN-deriv T (distr M T Y) (snd x))
    unfolding AE-density[OF distributed-borel-measurable, OF Pxy]
    unfolding distributed-distr-eq-density[OF Py]
    using distributed-RN-deriv[OF Py]
    by (force intro: ST.AE-pair-measure)
  ultimately
  have conditional-entropy b S T X Y = - (∫ x. Pxy x * log b (Pxy x / Py (snd
x)) ∂(S ⨂M T))
    unfolding conditional-entropy-def neg-equal-iff-equal
    apply (subst integral-real-density[symmetric])
    apply (auto simp: distributed-distr-eq-density[OF Pxy] space-pair-measure
              intro!: integral-cong-AE)
    done
  then show ?thesis by (simp add: split-beta')
qed

lemma (in information-space) conditional-entropy-eq-entropy:
  fixes Px :: 'b ⇒ real and Py :: 'c ⇒ real
  assumes S: sigma-finite-measure S and T: sigma-finite-measure T
  assumes Py[measurable]: distributed M T Y Py
    and Py-nn[simp]: ⋀x. x ∈ space T ⟹ 0 ≤ Py x
  assumes Pxy[measurable]: distributed M (S ⨂M T) (λx. (X x, Y x)) Pxy
    and Pxy-nn[simp]: ⋀x y. x ∈ space S ⟹ y ∈ space T ⟹ 0 ≤ Pxy (x, y)
  assumes I1: integrable (S ⨂M T) (λx. Pxy x * log b (Pxy x))
  assumes I2: integrable (S ⨂M T) (λx. Pxy x * log b (Py (snd x)))
  shows conditional-entropy b S T X Y = entropy b (S ⨂M T) (λx. (X x, Y x))
- entropy b T Y
proof -
  interpret S: sigma-finite-measure S by fact
  interpret T: sigma-finite-measure T by fact
  interpret ST: pair-sigma-finite S T ..

  have [measurable]: Py ∈ T →M borel
    using Py Py-nn by (intro distributed-real-measurable)
  have measurable-Pxy[measurable]: Pxy ∈ (S ⨂M T) →M borel
   using Pxy Pxy-nn by (intro distributed-real-measurable) (auto simp: space-pair-measure)

  have entropy b T Y = - (∫ y. Py y * log b (Py y) ∂T)
    by (rule entropy-distr[OF Py Py-nn])
  also have ... = - (∫ (x,y). Pxy (x,y) * log b (Py y) ∂(S ⨂M T))
    using b-gt-1
    by (subst distributed-transform-integral[OF Pxy - Py, where T=snd])
       (auto intro!: Bochner-Integration.integral-cong simp: space-pair-measure)
  finally have e-eq: entropy b T Y = - (∫ (x,y). Pxy (x,y) * log b (Py y) ∂(S
⨂M T)) .

  have **: ⋀x. x ∈ space (S ⨂M T) ⟹ 0 ≤ Pxy x
    by (auto simp: space-pair-measure)
```

have *ae2*: *AE x in S* $\bigotimes_M$ *T. Py (snd x) = 0* $\longrightarrow$ *Pxy x = 0*
by (*intro subdensity-real*[*of snd, OF - Pxy Py*])
(*auto intro*: *measurable-Pair simp*: *space-pair-measure*)
moreover have *ae4*: *AE x in S* $\bigotimes_M$ *T. 0* $\leq$ *Py (snd x)*
using *Py* **by** (*intro ST.AE-pair-measure*) (*auto simp*: *comp-def intro*!: *measurable-snd''*)
ultimately have *AE x in S* $\bigotimes_M$ *T. 0* $\leq$ *Pxy x* $\wedge$ *0* $\leq$ *Py (snd x)* $\wedge$
(*Pxy x = 0* $\vee$ (*Pxy x* $\neq$ *0* $\longrightarrow$ *0 < Pxy x* $\wedge$ *0 < Py (snd x)*))
by (*auto simp*: *space-pair-measure less-le*)
then have *ae*: *AE x in S* $\bigotimes_M$ *T.*
*Pxy x * log b (Pxy x) − Pxy x * log b (Py (snd x)) = Pxy x * log b (Pxy x / Py (snd x))*
by (*auto simp*: *log-simps field-simps b-gt-1*)
have *conditional-entropy b S T X Y =*
− (∫ *x. Pxy x * log b (Pxy x) − Pxy x * log b (Py (snd x))* ∂(*S* $\bigotimes_M$ *T*))
unfolding *conditional-entropy-generic-eq*[*OF S T Py Py-nn Pxy Pxy-nn, simplified*] *neg-equal-iff-equal*
using *ae* **by** (*force intro*: *integral-cong-AE*)
also have *... = − (∫ x. Pxy x * log b (Pxy x)* ∂(*S* $\bigotimes_M$ *T*)) − − (∫ *x. Pxy x * log b (Py (snd x))* ∂(*S* $\bigotimes_M$ *T*))
by (*simp add*: *Bochner-Integration.integral-diff*[*OF I1 I2*])
finally show *?thesis*
using *conditional-entropy-generic-eq*[*OF S T Py Py-nn Pxy Pxy-nn, simplified*] *entropy-distr*[*OF Pxy ∗∗, simplified*] *e-eq*
by (*simp add*: *split-beta'*)
qed

lemma (**in** *information-space*) *conditional-entropy-eq-entropy-simple*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{H}(X \mid Y)$ = *entropy b (count-space (X‘space M)* $\bigotimes_M$ *count-space (Y‘space M)) (λx. (X x, Y x)) − $\mathcal{H}$(Y)*
proof −
have *count-space (X ‘ space M)* $\bigotimes_M$ *count-space (Y ‘ space M) = count-space (X‘space M × Y‘space M)*
(**is** *?P = ?C*) **using** *X Y* **by** (*simp add*: *simple-functionD pair-measure-count-space*)
show *?thesis*
by (*rule conditional-entropy-eq-entropy sigma-finite-measure-count-space-finite simple-functionD X Y simple-distributed simple-distributedI*[*OF - - refl*] *simple-distributed-joint simple-function-Pair integrable-count-space measure-nonneg*)+
(*auto simp*: ‹*?P = ?C*› *measure-nonneg intro*!: *integrable-count-space simple-functionD X Y*)
qed

lemma (**in** *information-space*) *conditional-entropy-eq*:
assumes *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M (λx. (X x, Y x)) Pxy*
shows $\mathcal{H}(X \mid Y)$ = − (∑ *(x, y)*∈*(λx. (X x, Y x)) ‘ space M. Pxy (x, y) * log b (Pxy (x, y) / Py y))*

proof (*subst conditional-entropy-generic-eq*[*OF* - -
simple-distributed[*OF Y*] - *simple-distributed-joint*[*OF XY*]])
have *finite* ((λ*x*. (*X x*, *Y x*))‘*space M*)
using *XY* **unfolding** *simple-distributed-def* **by** *auto*
from *finite-imageI*[*OF this*, *of fst*]
have [*simp*]: *finite* (*X*‘*space M*)
by (*simp add*: *image-comp comp-def*)
note *Y*[*THEN simple-distributed-finite*, *simp*]
show *sigma-finite-measure* (*count-space* (*X* ‘ *space M*))
by (*simp add*: *sigma-finite-measure-count-space-finite*)
show *sigma-finite-measure* (*count-space* (*Y* ‘ *space M*))
by (*simp add*: *sigma-finite-measure-count-space-finite*)
let *?f* = (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*)) ‘ *space M then Pxy x else 0*)
have *count-space* (*X* ‘ *space M*) ⨂$_M$ *count-space* (*Y* ‘ *space M*) = *count-space*
(*X*‘*space M* × *Y*‘*space M*)
(**is** *?P* = *?C*)
using *Y* **by** (*simp add*: *simple-distributed-finite pair-measure-count-space*)
have *eq*: (λ(*x*, *y*). *?f* (*x*, *y*) ∗ *log b* (*?f* (*x*, *y*) / *Py y*)) =
(λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*)) ‘ *space M then Pxy x* ∗ *log b* (*Pxy x* / *Py* (*snd x*))
else 0)
by *auto*
from *Y* **show** − (∫ (*x*, *y*). *?f* (*x*, *y*) ∗ *log b* (*?f* (*x*, *y*) / *Py y*) ∂*?P*) =
− (∑(*x*, *y*)∈(λ*x*. (*X x*, *Y x*)) ‘ *space M*. *Pxy* (*x*, *y*) ∗ *log b* (*Pxy* (*x*, *y*) / *Py*
y))
by (*auto intro*!: *sum.cong simp add*: ‹*?P* = *?C*› *lebesgue-integral-count-space-finite*
simple-distributed-finite eq sum.If-cases split-beta′)
qed (*use Y XY* **in** *auto*)

lemma (**in** *information-space*) *conditional-mutual-information-eq-conditional-entropy*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{I}(X ; X \mid Y) = \mathcal{H}(X \mid Y)$
proof −
define *Py* **where** *Py x* = (*if x* ∈ *Y*‘*space M then measure M* (*Y* −‘ {*x*} ∩ *space*
M) *else 0*) **for** *x*
define *Pxy* **where** *Pxy x* =
(*if x* ∈ (λ*x*. (*X x*, *Y x*))‘*space M then measure M* ((λ*x*. (*X x*, *Y x*)) −‘ {*x*} ∩
space M) *else 0*)
for *x*
define *Pxxy* **where** *Pxxy x* =
(*if x* ∈ (λ*x*. (*X x*, *X x*, *Y x*))‘*space M then measure M* ((λ*x*. (*X x*, *X x*, *Y x*))
−‘ {*x*} ∩ *space M*)
else 0)
for *x*
let *?M* = *X*‘*space M* × *X*‘*space M* × *Y*‘*space M*

note *XY* = *simple-function-Pair*[*OF X Y*]
note *XXY* = *simple-function-Pair*[*OF X XY*]
have *Py*: *simple-distributed M Y Py*
using *Y* **by** (*rule simple-distributedI*) (*auto simp*: *Py-def measure-nonneg*)

have *Pxy*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
using *XY* **by** (*rule simple-distributedI*) (*auto simp*: *Pxy-def measure-nonneg*)
have *Pxxy*: *simple-distributed M* (λ*x*. (*X x*, *X x*, *Y x*)) *Pxxy*
using *XXY* **by** (*rule simple-distributedI*) (*auto simp*: *Pxxy-def measure-nonneg*)
have *eq*: (λ*x*. (*X x*, *X x*, *Y x*)) ' *space M* = (λ(*x*, *y*). (*x*, *x*, *y*)) ' (λ*x*. (*X x*, *Y x*)) ' *space M*
by *auto*
have *inj*: ⋀*A*. *inj-on* (λ(*x*, *y*). (*x*, *x*, *y*)) *A*
by (*auto simp*: *inj-on-def*)
have *Pxxy-eq*: ⋀*x y*. *Pxxy* (*x*, *x*, *y*) = *Pxy* (*x*, *y*)
by (*auto simp*: *Pxxy-def Pxy-def intro*!: *arg-cong*[**where** *f*=*prob*])
have *AE x in count-space* ((λ*x*. (*X x*, *Y x*))'*space M*). *Py* (*snd x*) = *0* ⟶ *Pxy x* = *0*
using *Py Pxy*
by (*intro subdensity-real*[*of snd*, *OF - Pxy*[*THEN simple-distributed*] *Py*[*THEN simple-distributed*]])
(*auto intro*: *measurable-Pair simp*: *AE-count-space*)
then show *?thesis*
apply (*subst conditional-mutual-information-eq*[*OF Py Pxy Pxy Pxxy*])
apply (*subst conditional-entropy-eq*[*OF Py Pxy*])
apply (*auto intro*!: *sum.cong simp*: *Pxxy-eq sum-negf*[*symmetric*] *eq sum.reindex*[*OF inj*]
log-simps zero-less-mult-iff zero-le-mult-iff field-simps mult-less-0-iff AE-count-space)
done
qed

lemma (**in** *information-space*) *conditional-entropy-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **shows** *0* ≤ *ℋ*(*X* | *Y*)
using *conditional-mutual-information-eq-conditional-entropy*[*OF X Y*] *conditional-mutual-information-nonne*
X X Y]
by *simp*

12.8 Equalities

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy-distr*:
fixes *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real* **and** *Pxy* :: (*'b* × *'c*) ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: ⋀*x*. *x* ∈ *space S* ⟹ *0* ≤ *Px x*
and *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: ⋀*x*. *x* ∈ *space T* ⟹ *0* ≤ *Py x*
and *Pxy*[*measurable*]: *distributed M* (*S* ⨂*M* *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*[*simp*]: ⋀*x y*. *x* ∈ *space S* ⟹ *y* ∈ *space T* ⟹ *0* ≤ *Pxy* (*x*, *y*)
assumes *Ix*: *integrable*(*S* ⨂*M* *T*) (λ*x*. *Pxy x* ∗ *log b* (*Px* (*fst x*)))
assumes *Iy*: *integrable*(*S* ⨂*M* *T*) (λ*x*. *Pxy x* ∗ *log b* (*Py* (*snd x*)))
assumes *Ixy*: *integrable*(*S* ⨂*M* *T*) (λ*x*. *Pxy x* ∗ *log b* (*Pxy x*))
shows *mutual-information b S T X Y* = *entropy b S X* + *entropy b T Y* −

entropy b $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
proof −
have [*measurable*]: $Px \in S \to_M$ *borel*
using *Px Px-nn* **by** (*intro distributed-real-measurable*)
have [*measurable*]: $Py \in T \to_M$ *borel*
using *Py Py-nn* **by** (*intro distributed-real-measurable*)
have *measurable-Pxy*[*measurable*]: $Pxy \in (S \bigotimes_M T) \to_M$ *borel*
using *Pxy Pxy-nn* **by** (*intro distributed-real-measurable*) (*auto simp*: *space-pair-measure*)

have *X*: *entropy b S X* = − $(\int x.\ Pxy\ x * log\ b\ (Px\ (fst\ x))\ \partial(S \bigotimes_M T))$
using *b-gt-1*
apply (*subst entropy-distr*[*OF Px Px-nn*], *simp*)
apply (*subst distributed-transform-integral*[*OF Pxy - Px*, **where** *T*=*fst*])
apply (*auto intro*!: *integral-cong simp*: *space-pair-measure*)
done

have *Y*: *entropy b T Y* = − $(\int x.\ Pxy\ x * log\ b\ (Py\ (snd\ x))\ \partial(S \bigotimes_M T))$
using *b-gt-1*
apply (*subst entropy-distr*[*OF Py Py-nn*], *simp*)
apply (*subst distributed-transform-integral*[*OF Pxy - Py*, **where** *T*=*snd*])
apply (*auto intro*!: *integral-cong simp*: *space-pair-measure*)
done

interpret *S*: *sigma-finite-measure S* **by** *fact*
interpret *T*: *sigma-finite-measure T* **by** *fact*
interpret *ST*: *pair-sigma-finite S T* **..**
have *ST*: *sigma-finite-measure* $(S \bigotimes_M T)$ **..**

have *XY*: *entropy b* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ = − $(\int x.\ Pxy\ x * log\ b\ (Pxy\ x)\ \partial(S \bigotimes_M T))$
by (*subst entropy-distr*[*OF Pxy*]) (*auto intro*!: *integral-cong simp*: *space-pair-measure*)

have *AE x in* $S \bigotimes_M T$. $Px\ (fst\ x) = 0 \longrightarrow Pxy\ x = 0$
by (*intro subdensity-real*[*of fst*, *OF - Pxy Px*]) (*auto intro*: *measurable-Pair simp*: *space-pair-measure*)
moreover have *AE x in* $S \bigotimes_M T$. $Py\ (snd\ x) = 0 \longrightarrow Pxy\ x = 0$
by (*intro subdensity-real*[*of snd*, *OF - Pxy Py*]) (*auto intro*: *measurable-Pair simp*: *space-pair-measure*)
moreover have *AE x in* $S \bigotimes_M T$. $0 \le Px\ (fst\ x)$
using *Px* **by** (*intro ST.AE-pair-measure*) (*auto simp*: *comp-def intro*!: *measurable-fst''*)
moreover have *AE x in* $S \bigotimes_M T$. $0 \le Py\ (snd\ x)$
using *Py* **by** (*intro ST.AE-pair-measure*) (*auto simp*: *comp-def intro*!: *measurable-snd''*)
ultimately have *AE x in* $S \bigotimes_M T$. $Pxy\ x * log\ b\ (Pxy\ x) - Pxy\ x * log\ b\ (Px\ (fst\ x)) - Pxy\ x * log\ b\ (Py\ (snd\ x)) =$
$Pxy\ x * log\ b\ (Pxy\ x\ /\ (Px\ (fst\ x) * Py\ (snd\ x)))$
(**is** *AE x in -. ?f x* = *?g x*)
using *AE-space*

proof *eventually-elim*
case (*elim x*)
show *?case*
proof *cases*
assume *Pxy x ≠ 0*
with *elim* **have** *0 < Px (fst x) 0 < Py (snd x) 0 < Pxy x*
by (*auto simp*: *space-pair-measure less-le*)
then show *?thesis*
using *b-gt-1* **by** (*simp add*: *log-simps less-imp-le field-simps*)
qed *simp*
qed

have *entropy b S X + entropy b T Y − entropy b* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
$= integral^L\ (S \bigotimes_M T)\ ?f$
unfolding *X Y XY*
apply (*subst Bochner-Integration.integral-diff*)
apply (*intro Bochner-Integration.integrable-diff Ixy Ix Iy*)+
apply (*subst Bochner-Integration.integral-diff*)
apply (*intro Ixy Ix Iy*)+
apply (*simp add*: *field-simps*)
done
also have *... = integral*^{L} $(S \bigotimes_M T)$ *?g*
using ‹*AE x in -. ?f x = ?g x*› **by** (*intro integral-cong-AE*) *auto*
also have *... = mutual-information b S T X Y*
by (*rule mutual-information-distr*[*OF S T Px - Py - Pxy - , symmetric*])
(*auto simp*: *space-pair-measure*)
finally show *?thesis* **..**
qed

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy'*:
fixes *Px* :: *'b ⇒ real* **and** *Py* :: *'c ⇒ real* **and** *Pxy* :: *('b × 'c) ⇒ real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*: *distributed M T Y Py* $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
$\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
assumes *Ix*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes *Iy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes *Ixy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information b S T X Y = entropy b S X − conditional-entropy b S T X Y*
using
mutual-information-eq-entropy-conditional-entropy-distr[*OF S T Px Py Pxy Ix Iy Ixy*]
conditional-entropy-eq-entropy[*OF S T Py Pxy Ixy Iy*]
by (*simp add*: *space-pair-measure*)

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy*:
assumes *sf-X*: *simple-function M X* **and** *sf-Y*: *simple-function M Y*

shows $\mathcal{I}(X\ ;\ Y) = \mathcal{H}(X) - \mathcal{H}(X \mid Y)$
proof −
have *X*: *simple-distributed M X* (λ*x*. *measure M* (*X* −‘ {*x*} ∩ *space M*))
using *sf-X* **by** (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)
have *Y*: *simple-distributed M Y* (λ*x*. *measure M* (*Y* −‘ {*x*} ∩ *space M*))
using *sf-Y* **by** (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)
have *sf-XY*: *simple-function M* (λ*x*. (*X x*, *Y x*))
using *sf-X sf-Y* **by** (*rule simple-function-Pair*)
then have *XY*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) (λ*x*. *measure M* ((λ*x*. (*X x*, *Y x*)) −‘ {*x*} ∩ *space M*))
by (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)
from *simple-distributed-joint-finite*[*OF this*, *simp*]
have *eq*: *count-space* (*X* ‘ *space M*) ⨂$_M$ *count-space* (*Y* ‘ *space M*) = *count-space* (*X* ‘ *space M* × *Y* ‘ *space M*)
by (*simp add*: *pair-measure-count-space*)

have $\mathcal{I}(X\ ;\ Y) = \mathcal{H}(X) + \mathcal{H}(Y) -$ *entropy b* (*count-space* (*X*‘*space M*) ⨂$_M$ *count-space* (*Y*‘*space M*)) (λ*x*. (*X x*, *Y x*))
using *sigma-finite-measure-count-space-finite*
sigma-finite-measure-count-space-finite
simple-distributed[*OF X*] *measure-nonneg simple-distributed*[*OF Y*] *measure-nonneg simple-distributed-joint*[*OF XY*]
by (*rule mutual-information-eq-entropy-conditional-entropy-distr*)
(*auto simp*: *eq integrable-count-space measure-nonneg*)
then show *?thesis*
unfolding *conditional-entropy-eq-entropy-simple*[*OF sf-X sf-Y*] **by** *simp*
qed

lemma (**in** *information-space*) *mutual-information-nonneg-simple*:
assumes *sf-X*: *simple-function M X* **and** *sf-Y*: *simple-function M Y*
shows $0 \leq \mathcal{I}(X\ ;\ Y)$
proof −
have *X*: *simple-distributed M X* (λ*x*. *measure M* (*X* −‘ {*x*} ∩ *space M*))
using *sf-X* **by** (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)
have *Y*: *simple-distributed M Y* (λ*x*. *measure M* (*Y* −‘ {*x*} ∩ *space M*))
using *sf-Y* **by** (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)

have *sf-XY*: *simple-function M* (λ*x*. (*X x*, *Y x*))
using *sf-X sf-Y* **by** (*rule simple-function-Pair*)
then have *XY*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) (λ*x*. *measure M* ((λ*x*. (*X x*, *Y x*)) −‘ {*x*} ∩ *space M*))
by (*rule simple-distributedI*) (*auto simp*: *measure-nonneg*)

from *simple-distributed-joint-finite*[*OF this*, *simp*]
have *eq*: *count-space* (*X* ‘ *space M*) ⨂$_M$ *count-space* (*Y* ‘ *space M*) = *count-space* (*X* ‘ *space M* × *Y* ‘ *space M*)
by (*simp add*: *pair-measure-count-space*)

show *?thesis*

by (*rule mutual-information-nonneg*[*OF - - simple-distributed*[*OF X*] *- simple-distributed*[*OF Y*] *- simple-distributed-joint*[*OF XY*]])
(*simp-all add*: *eq integrable-count-space sigma-finite-measure-count-space-finite measure-nonneg*)
qed

lemma (**in** *information-space*) *conditional-entropy-less-eq-entropy*:
assumes *X*: *simple-function M X* **and** *Z*: *simple-function M Z*
shows $\mathcal{H}(X \mid Z) \leq \mathcal{H}(X)$
proof −
have $0 \leq \mathcal{I}(X ; Z)$ **using** *X Z* **by** (*rule mutual-information-nonneg-simple*)
also have $\mathcal{I}(X ; Z) = \mathcal{H}(X) - \mathcal{H}(X \mid Z)$ **using** *mutual-information-eq-entropy-conditional-entropy*[*OF assms*] **.**
finally show *?thesis* **by** *auto*
qed

lemma (**in** *information-space*)
fixes *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real* **and** *Pxy* :: $('b \times 'c) \Rightarrow$ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*: *finite-entropy S X Px* **and** *Py*: *finite-entropy T Y Py*
assumes *Pxy*: *finite-entropy* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *conditional-entropy b S T X Y* $\leq$ *entropy b S X*
proof −
have $0 \leq$ *mutual-information b S T X Y*
by (*rule mutual-information-nonneg'*) *fact+*
also have ... = *entropy b S X* − *conditional-entropy b S T X Y*
proof (*intro mutual-information-eq-entropy-conditional-entropy'*)
show *integrable* $(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
integrable $(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
integrable $(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
using *assms*
by (*auto intro*!: *finite-entropy-integrable finite-entropy-distributed finite-entropy-integrable-transform*[*OF Px*] *finite-entropy-integrable-transform*[*OF Py*]
intro: *finite-entropy-nn*)
qed (*use assms Px Py Pxy finite-entropy-nn finite-entropy-distributed* **in** *auto*)
finally show *?thesis* **by** *auto*
qed

lemma (**in** *information-space*) *entropy-chain-rule*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x)) = \mathcal{H}(X) + \mathcal{H}(Y|X)$
proof −
note *XY* = *simple-distributedI*[*OF simple-function-Pair*[*OF X Y*] *measure-nonneg refl*]
note *YX* = *simple-distributedI*[*OF simple-function-Pair*[*OF Y X*] *measure-nonneg refl*]
note *simple-distributed-joint-finite*[*OF this*, *simp*]
let *?f* = $\lambda x.$ *prob* $((\lambda x.\ (X\ x,\ Y\ x)) -` \{x\} \cap$ *space M*)

let *?g* = λx. *prob* ((λx. (*Y x*, *X x*)) −‘ {*x*} ∩ *space M*)
let *?h* = λx. *if x* ∈ (λx. (*Y x*, *X x*)) ‘ *space M then prob* ((λx. (*Y x*, *X x*)) −‘ {*x*} ∩ *space M*) *else 0*
have $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x)) = -\ (\sum x \in (\lambda x.\ (X\ x,\ Y\ x))$ ‘ *space M*. *?f x* ∗ *log b* (*?f x*))
using *XY* **by** (*rule entropy-simple-distributed*)
also have ... = − ($\sum x \in (\lambda(x,\ y).\ (y,\ x))$ ‘ (λx. (*X x*, *Y x*)) ‘ *space M*. *?g x* ∗ *log b* (*?g x*))
by (*subst* (*2*) *sum.reindex*) (*auto simp*: *inj-on-def intro*!: *sum.cong arg-cong*[**where** *f*=λA. *prob A* ∗ *log b* (*prob A*)])
also have ... = − ($\sum x \in (\lambda x.\ (Y\ x,\ X\ x))$ ‘ *space M*. *?h x* ∗ *log b* (*?h x*))
by (*auto intro*!: *sum.cong*)
also have ... = *entropy b* (*count-space* (*Y* ‘ *space M*) $\bigotimes_M$ *count-space* (*X* ‘ *space M*)) (λx. (*Y x*, *X x*))
by (*subst entropy-distr*[*OF simple-distributed-joint*[*OF YX*]])
(*auto simp*: *pair-measure-count-space sigma-finite-measure-count-space-finite lebesgue-integral-count-space-finite*
cong del: *sum.cong-simp intro*!: *sum.mono-neutral-left measure-nonneg*)
finally have $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x))$ = *entropy b* (*count-space* (*Y* ‘ *space M*) $\bigotimes_M$ *count-space* (*X* ‘ *space M*)) (λx. (*Y x*, *X x*)) **.**
then show *?thesis*
unfolding *conditional-entropy-eq-entropy-simple*[*OF Y X*] **by** *simp*
qed

lemma (**in** *information-space*) *entropy-partition*:
assumes *X*: *simple-function M X*
shows $\mathcal{H}(X) = \mathcal{H}(f \circ X) + \mathcal{H}(X|f \circ X)$
proof −
note *fX* = *simple-function-compose*[*OF X*, *of f*]
have *eq*: (λx. ((*f* ∘ *X*) *x*, *X x*)) ‘ *space M* = (λx. (*f x*, *x*)) ‘ *X* ‘ *space M* **by** *auto*
have *inj*: $\bigwedge A$. *inj-on* (λx. (*f x*, *x*)) *A*
by (*auto simp*: *inj-on-def*)

have $\mathcal{H}(X) = -\ (\sum x \in X$ ‘ *space M*. *prob* (*X* −‘ {*x*} ∩ *space M*) ∗ *log b* (*prob* (*X* −‘ {*x*} ∩ *space M*)))
by (*simp add*: *entropy-simple-distributed*[*OF simple-distributedI*[*OF X measure-nonneg refl*]])
also have ... = − ($\sum x \in (\lambda x.\ ((f \circ X)\ x,\ X\ x))$ ‘ *space M*.
prob ((λx. ((*f* ∘ *X*) *x*, *X x*)) −‘ {*x*} ∩ *space M*) ∗
log b (*prob* ((λx. ((*f* ∘ *X*) *x*, *X x*)) −‘ {*x*} ∩ *space M*)))
unfolding *eq*
apply (*subst sum.reindex*[*OF inj*])
apply (*auto intro*!: *sum.cong arg-cong*[**where** *f*=λA. *prob A* ∗ *log b* (*prob A*)])
done
also have ... = $\mathcal{H}(\lambda x.\ ((f \circ X)\ x,\ X\ x))$
using *entropy-simple-distributed*[*OF simple-distributedI*[*OF simple-function-Pair*[*OF fX X*] *measure-nonneg refl*]]
by *fastforce*
also have ... = $\mathcal{H}(f \circ X) + \mathcal{H}(X|f \circ X)$

using *X entropy-chain-rule* **by** *blast*
finally show *?thesis* **.**
qed

corollary (**in** *information-space*) *entropy-data-processing*:
assumes *simple-function M X* **shows** $\mathcal{H}(f \circ X) \leq \mathcal{H}(X)$
by (*smt* (*verit*) *assms conditional-entropy-nonneg entropy-partition simple-function-compose*)

corollary (**in** *information-space*) *entropy-of-inj*:
assumes *X*: *simple-function M X* **and** *inj*: *inj-on f* (*X'space M*)
shows $\mathcal{H}(f \circ X) = \mathcal{H}(X)$
proof (*rule antisym*)
show $\mathcal{H}(f \circ X) \leq \mathcal{H}(X)$ **using** *entropy-data-processing*[*OF X*] **.**
next
have *sf*: *simple-function M* $(f \circ X)$
using *X* **by** *auto*
have $\mathcal{H}(X) = \mathcal{H}$(*the-inv-into* (*X'space M*) $f \circ (f \circ X)$)
unfolding *o-assoc*
apply (*subst entropy-simple-distributed*[*OF simple-distributedI*[*OF X measure-nonneg refl*]])
apply (*subst entropy-simple-distributed*[*OF simple-distributedI*[*OF simple-function-compose*[*OF X*]], **where** *f*=$\lambda x.$ *prob* $(X -$ ' $\{x\} \cap$ *space M*)])
apply (*auto intro*!: *sum.cong arg-cong*[**where** *f*=*prob*] *image-eqI simp*: *the-inv-into-f-f*[*OF inj*] *comp-def measure-nonneg*)
done
also have ... $\leq \mathcal{H}(f \circ X)$
using *entropy-data-processing*[*OF sf*] **.**
finally show $\mathcal{H}(X) \leq \mathcal{H}(f \circ X)$ **.**
qed

end

13 Properties of Various Distributions

theory *Distributions*
imports *Convolution Information*
begin

lemma (**in** *prob-space*) *distributed-affine*:
fixes *f* :: *real* ⇒ *ennreal*
assumes *f*: *distributed M lborel X f*
assumes *c*: $c \neq 0$
shows *distributed M lborel* $(\lambda x.\ t + c * X\ x)$ $(\lambda x.\ f\ ((x - t)\ /\ c)\ /\ |c|)$
unfolding *distributed-def*
proof *safe*
have [*measurable*]: *f* ∈ *borel-measurable borel*
using *f* **by** (*simp add*: *distributed-def*)
have [*measurable*]: *X* ∈ *borel-measurable M*
using *f* **by** (*simp add*: *distributed-def*)

```
  show (λx. f ((x − t) / c) / |c|) ∈ borel-measurable lborel
    by simp
  show random-variable lborel (λx. t + c ∗ X x)
    by simp

  have eq: ennreal |c| ∗ (f x / ennreal |c|) = f x for x
    using c
    by (cases f x)
       (auto simp: divide-ennreal ennreal-mult[symmetric] ennreal-top-divide en-
nreal-mult-top)

  have density lborel f = distr M lborel X
    using f by (simp add: distributed-def)
  with c show distr M lborel (λx. t + c ∗ X x) = density lborel (λx. f ((x − t) /
c) / ennreal |c|)
    by (subst (2) lborel-real-affine[where c=c and t=t])
       (simp-all add: density-density-eq density-distr distr-distr field-simps eq cong:
distr-cong)
qed

lemma (in prob-space) distributed-affineI:
  fixes f :: real ⇒ ennreal and c :: real
  assumes f: distributed M lborel (λx. (X x − t) / c) (λx. |c| ∗ f (x ∗ c + t))
  assumes c: c ≠ 0
  shows distributed M lborel X f
proof −
  have eq: f x ∗ ennreal |c| / ennreal |c| = f x for x
    using c by (simp add: ennreal-times-divide[symmetric])

  show ?thesis
    using distributed-affine[OF f c, where t=t] c
    by (simp add: field-simps eq)
qed

lemma (in prob-space) distributed-AE2:
  assumes [measurable]: distributed M N X f Measurable.pred N P
  shows (AE x in M. P (X x)) ⟷ (AE x in N. 0 < f x ⟶ P x)
proof −
  have (AE x in M. P (X x)) ⟷ (AE x in distr M N X. P x)
    by (simp add: AE-distr-iff)
  also have ... ⟷ (AE x in density N f. P x)
    unfolding distributed-distr-eq-density[OF assms(1)] ..
  also have ... ⟷ (AE x in N. 0 < f x ⟶ P x)
    by (rule AE-density) simp
  finally show ?thesis .
qed
```

13.1 Erlang

lemma *nn-intergal-power-times-exp-Icc*:
assumes [*arith*]: $0 \leq a$
shows $(\int^+ x.\ ennreal\ (x\hat{}k * exp\ (-x)) * indicator\ \{0\ ..\ a\}\ x\ \partial lborel) =$
$(1 - (\sum n{\leq}k.\ (a\hat{}n * exp\ (-a))\ /\ fact\ n)) * fact\ k$ (**is** *?I* = -)
proof −
let *?f* = $\lambda k\ x.\ x\hat{}k * exp\ (-x)\ /\ fact\ k$
let *?F* = $\lambda k\ x.\ -\ (\sum n{\leq}k.\ (x\hat{}n * exp\ (-x))\ /\ fact\ n)$
have *?I* $*$ (*inverse* (*real-of-nat* (*fact k*))) =
$(\int^+ x.\ ennreal\ (x\hat{}k * exp\ (-x)) * indicator\ \{0\ ..\ a\}\ x * (inverse\ (real\text{-}of\text{-}nat$
(*fact k*))) *∂lborel*)
by (*intro nn-integral-multc*[*symmetric*]) *auto*
also have ... = $(\int^+ x.\ ennreal\ (?f\ k\ x) * indicator\ \{0\ ..\ a\}\ x\ \partial lborel)$
by (*intro nn-integral-cong*)
(*simp add*: *field-simps ennreal-mult'*[*symmetric*] *indicator-mult-ennreal*)
also have ... = *ennreal* (*?F k a* − *?F k 0*)
proof (*rule nn-integral-FTC-Icc*)
fix *x* **assume** $x \in \{0..a\}$
show *DERIV* (*?F k*) *x* :> *?f k x*
proof(*induction k*)
case *0* **show** *?case* **by** (*auto intro*!: *derivative-eq-intros*)
next
case (*Suc k*)
have *DERIV* ($\lambda x.$ *?F k x* − ($x\hat{}Suc\ k * exp\ (-x)$) / *fact* (*Suc k*)) *x* :>
?f k x − ((*real* (*Suc k*) − *x*) $* x\ \hat{}\ k * exp\ (-\ x))\ /\ (fact\ (Suc\ k))$
by (*intro DERIV-diff Suc*)
(*auto intro*!: *derivative-eq-intros simp del*: *fact-Suc power-Suc*
simp add: *field-simps power-Suc*[*symmetric*])
also have ($\lambda x.$ *?F k x* − ($x\hat{}Suc\ k * exp\ (-x)$) / *fact* (*Suc k*)) = *?F* (*Suc k*)
by *simp*
also have *?f k x* − ((*real* (*Suc k*) − *x*) $* x\ \hat{}\ k * exp\ (-\ x))\ /\ (fact\ (Suc\ k))$
= *?f* (*Suc k*) *x*
by (*auto simp*: *field-simps simp del*: *fact-Suc*)
(*simp-all add*: *of-nat-Suc field-simps*)
finally show *?case* **.**
qed
qed *auto*
also have ... = *ennreal* $(1 - (\sum n{\leq}k.\ (a\hat{}n * exp\ (-a))\ /\ fact\ n))$
by (*auto simp*: *power-0-left if-distrib*[**where** $f{=}\lambda x.\ x\ /\ a$ **for** *a*] *sum.If-cases*)
also have ... = *ennreal* $((1 - (\sum n{\leq}k.\ (a\hat{}n * exp\ (-a))\ /\ fact\ n)) * fact\ k) *$
ennreal (*inverse* (*fact k*))
by (*subst ennreal-mult''*[*symmetric*]) (*auto intro*!: *arg-cong*[**where** *f*=*ennreal*])
finally show *?thesis*
by (*auto simp add*: *mult-right-ennreal-cancel le-less*)
qed

lemma *nn-intergal-power-times-exp-Ici*:
shows $(\int^+ x.\ ennreal\ (x\hat{}k * exp\ (-x)) * indicator\ \{0\ ..\}\ x\ \partial lborel) = real\text{-}of\text{-}nat$
(*fact k*)

proof (*rule LIMSEQ-unique*)
let *?X* = $\lambda n.\ \int^+ x.\ ennreal\ (x\hat{}k * exp\ (-x)) * indicator\ \{0\ ..\ real\ n\}\ x\ \partial lborel$
show *?X* $\longrightarrow$ $(\int^+ x.\ ennreal\ (x\hat{}k * exp\ (-x)) * indicator\ \{0\ ..\}\ x\ \partial lborel)$
apply (*intro nn-integral-LIMSEQ*)
apply (*auto simp*: *incseq-def le-fun-def eventually-sequentially*
split: *split-indicator intro*!: *tendsto-eventually*)
apply (*metis nat-ceiling-le-eq*)
done

have $((\lambda x{::}real.\ (1 - (\sum n{\le}k.\ (x\ \hat{}\ n\ /\ exp\ x)\ /\ (fact\ n))) * fact\ k) \longrightarrow$
$(1 - (\sum n{\le}k.\ 0\ /\ (fact\ n))) * fact\ k)$ *at-top*
by (*intro tendsto-intros tendsto-power-div-exp-0*) *simp*
then show *?X* $\longrightarrow$ *real-of-nat* (*fact k*)
by (*subst nn-intergal-power-times-exp-Icc*)
(*auto simp*: *exp-minus field-simps intro*!: *filterlim-compose*[*OF - filterlim-real-sequentially*])
qed

definition *erlang-density* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *real* $\Rightarrow$ *real* **where**
erlang-density k l x = (*if x* < *0 then 0 else* (*l*$\hat{}$(*Suc k*) * *x*$\hat{}$*k* * *exp* (− *l* * *x*)) / *fact k*)

definition *erlang-CDF* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *real* $\Rightarrow$ *real* **where**
erlang-CDF k l x = (*if x* < *0 then 0 else 1* − ($\sum n{\le}k$. ((*l* * *x*)$\hat{}$*n* * *exp* (− *l* * *x*) / *fact n*)))

lemma *erlang-density-nonneg*[*simp*]: *0* ≤ *l* $\Longrightarrow$ *0* ≤ *erlang-density k l x*
by (*simp add*: *erlang-density-def*)

lemma *borel-measurable-erlang-density*[*measurable*]: *erlang-density k l* ∈ *borel-measurable borel*
by (*auto simp add*: *erlang-density-def*[*abs-def*])

lemma *erlang-CDF-transform*: *0* < *l* $\Longrightarrow$ *erlang-CDF k l a* = *erlang-CDF k 1* (*l* * *a*)
by (*auto simp add*: *erlang-CDF-def mult-less-0-iff*)

lemma *erlang-CDF-nonneg*[*simp*]: **assumes** *0* < *l* **shows** *0* ≤ *erlang-CDF k l x*
unfolding *erlang-CDF-def*
proof (*clarsimp simp*: *not-less*)
assume *0* ≤ *x*
have $(\sum n{\le}k.\ (l * x)\ \hat{}\ n * exp\ (-\ (l * x))\ /\ fact\ n) =$
exp (− (*l* * *x*)) * ($\sum n{\le}k$. (*l* * *x*) $\hat{}$ *n* / *fact n*)
unfolding *sum-distrib-left* **by** (*intro sum.cong*) (*auto simp*: *field-simps*)
also have ... = ($\sum n{\le}k$. (*l* * *x*) $\hat{}$ *n* / *fact n*) / *exp* (*l* * *x*)
by (*simp add*: *exp-minus field-simps*)
also have ... ≤ *1*
proof (*subst divide-le-eq-1-pos*)
show ($\sum n{\le}k$. (*l* * *x*) $\hat{}$ *n* / *fact n*) ≤ *exp* (*l* * *x*)
using ‹*0* < *l*› ‹*0* ≤ *x*› *summable-exp-generic*[*of l* * *x*]

by (*auto simp*: *exp-def divide-inverse ac-simps intro*!: *sum-le-suminf*)
qed *simp*
finally show $(\sum n{\le}k.\ (l * x)\ \hat{}\ n * exp\ (-\ (l * x))\ /\ fact\ n) \le 1$ **.**
qed

lemma *nn-integral-erlang-density*:
assumes [*arith*]: $0 < l$
shows $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x) * indicator\ \{..\ a\}\ x\ \partial lborel) = erlang\text{-}CDF\ k\ l\ a$
proof (*cases* $0 \le a$)
case [*arith*]: *True*
have *eq*: $\bigwedge x.\ indicator\ \{0..a\}\ (x\ /\ l) = indicator\ \{0..a{*}l\}\ x$
by (*simp add*: *field-simps split*: *split-indicator*)
have $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x) * indicator\ \{..\ a\}\ x\ \partial lborel) =$
$(\int^+ x.\ (l/fact\ k) * (ennreal\ ((l{*}x)\hat{}k * exp\ (-\ (l{*}x))) * indicator\ \{0\ ..\ a\}\ x)\ \partial lborel)$
by (*intro nn-integral-cong*)
(*auto simp*: *erlang-density-def power-mult-distrib ennreal-mult*[*symmetric*] *split*: *split-indicator*)
also have $\ldots = (l/fact\ k) * (\int^+ x.\ ennreal\ ((l{*}x)\hat{}k * exp\ (-\ (l{*}x))) * indicator\ \{0\ ..\ a\}\ x\ \partial lborel)$
by (*intro nn-integral-cmult*) *auto*
also have $\ldots = ennreal\ (l/fact\ k) * ((1/l) * (\int^+ x.\ ennreal\ (x\hat{}k * exp\ (-\ x)) * indicator\ \{0\ ..\ l * a\}\ x\ \partial lborel))$
by (*subst nn-integral-real-affine*[**where** *c=1 / l* **and** *t=0*]) (*auto simp*: *field-simps eq*)
also have $\ldots = (1\ -\ (\sum n{\le}k.\ ((l * a)\hat{}n * exp\ (-(l * a)))\ /\ fact\ n))$
by (*subst nn-intergal-power-times-exp-Icc*) (*auto simp*: *ennreal-mult'*[*symmetric*])
also have $\ldots = erlang\text{-}CDF\ k\ l\ a$
by (*auto simp*: *erlang-CDF-def*)
finally show *?thesis* **.**
next
case *False*
then have $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x) * indicator\ \{..\ a\}\ x\ \partial lborel) = (\int^+ x.\ 0\ \partial(lborel::real\ measure))$
by (*intro nn-integral-cong*) (*auto simp*: *erlang-density-def*)
with *False* **show** *?thesis*
by (*simp add*: *erlang-CDF-def*)
qed

lemma *emeasure-erlang-density*:
$0 < l \Longrightarrow emeasure\ (density\ lborel\ (erlang\text{-}density\ k\ l))\ \{..\ a\} = erlang\text{-}CDF\ k\ l\ a$
by (*simp add*: *emeasure-density nn-integral-erlang-density*)

lemma *nn-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: $0 < l$
shows $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x * x\ \hat{}\ i)\ \partial lborel) = fact\ (k + i)\ /$

(*fact k * l ^ i*)
proof −
have *eq*: ⋀*x*. *indicator* {*0*..} (*x* / *l*) = *indicator* {*0*..} *x*
by (*simp add*: *field-simps split*: *split-indicator*)
have (∫⁺ *x*. *ennreal* (*erlang-density k l x * x^i*) *∂lborel*) =
(∫⁺*x*. (*l*/(*fact k * l^i*)) * (*ennreal* ((*l**x*)^(*k*+*i*) * *exp* (− (*l**x*))) * *indicator* {*0*
..} *x*) *∂lborel*)
by (*intro nn-integral-cong*)
(*auto simp*: *erlang-density-def power-mult-distrib power-add ennreal-mult'*[*symmetric*]
split: *split-indicator*)
also have ... = (*l*/(*fact k * l^i*)) * (∫⁺*x*. *ennreal* ((*l**x*)^(*k*+*i*) * *exp* (− (*l**x*)))
* *indicator* {*0* ..} *x ∂lborel*)
by (*intro nn-integral-cmult*) *auto*
also have ... = *ennreal* (*l*/(*fact k * l^i*)) * ((*1*/*l*) * (∫⁺*x*. *ennreal* (*x^(k+i)* *
exp (− *x*)) * *indicator* {*0* ..} *x ∂lborel*))
by (*subst nn-integral-real-affine*[**where** *c=1* / *l* **and** *t=0*]) (*auto simp*: *field-simps*
eq)
also have ... = *fact* (*k* + *i*) / (*fact k * l ^ i*)
by (*subst nn-intergal-power-times-exp-Ici*) (*auto simp*: *ennreal-mult'*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *prob-space-erlang-density*:
assumes *l*[*arith*]: *0* < *l*
shows *prob-space* (*density lborel* (*erlang-density k l*)) (**is** *prob-space ?D*)
proof
show *emeasure ?D* (*space ?D*) = *1*
using *nn-integral-erlang-ith-moment*[*OF l*, **where** *k=k* **and** *i=0*] **by** (*simp*
add: *emeasure-density*)
qed

lemma (**in** *prob-space*) *erlang-distributed-le*:
assumes *D*: *distributed M lborel X* (*erlang-density k l*)
assumes [*simp*, *arith*]: *0* < *l 0* ≤ *a*
shows *P*(*x in M*. *X x* ≤ *a*) = *erlang-CDF k l a*
proof −
have *emeasure M* {*x* ∈ *space M*. *X x* ≤ *a* } = *emeasure* (*distr M lborel X*) {..
a}
using *distributed-measurable*[*OF D*]
by (*subst emeasure-distr*) (*auto intro*!: *arg-cong2*[**where** *f=emeasure*])
also have ... = *emeasure* (*density lborel* (*erlang-density k l*)) {.. *a*}
unfolding *distributed-distr-eq-density*[*OF D*] **..**
also have ... = *erlang-CDF k l a*
by (*auto intro*!: *emeasure-erlang-density*)
finally show *?thesis*
by (*auto simp*: *emeasure-eq-measure measure-nonneg*)
qed

lemma (**in** *prob-space*) *erlang-distributed-gt*:

assumes *D*[*simp*]: *distributed M lborel X* (*erlang-density k l*)
assumes [*arith*]: *0 < l 0 ≤ a*
shows $\mathcal{P}$(*x in M. a < X x*) = *1 −* (*erlang-CDF k l a*)
proof −
have *1 −* (*erlang-CDF k l a*) = *1 −* $\mathcal{P}$(*x in M. X x ≤ a*) **by** (*subst erlang-distributed-le*) *auto*
also have ... = *prob* (*space M −* {*x ∈ space M. X x ≤ a* })
using *distributed-measurable*[*OF D*] **by** (*auto simp*: *prob-compl*)
also have ... = $\mathcal{P}$(*x in M. a < X x*) **by** (*auto intro*!: *arg-cong*[**where** *f*=*prob*] *simp*: *not-le*)
finally show *?thesis* **by** *simp*
qed

lemma *erlang-CDF-at0*: *erlang-CDF k l 0 = 0*
by (*induction k*) (*auto simp*: *erlang-CDF-def*)

lemma *erlang-distributedI*:
assumes *X*[*measurable*]: *X ∈ borel-measurable M* **and** [*arith*]: *0 < l*
and *X-distr*: ⋀*a. 0 ≤ a ⟹ emeasure M* {*x∈space M. X x ≤ a*} = *erlang-CDF k l a*
shows *distributed M lborel X* (*erlang-density k l*)
proof (*rule distributedI-borel-atMost*)
fix *a* :: *real*
{ assume *a ≤ 0*
with *X* **have** *emeasure M* {*x∈space M. X x ≤ a*} ≤ *emeasure M* {*x∈space M. X x ≤ 0*}
by (*intro emeasure-mono*) *auto*
also have ... = *0* **by** (*auto intro*!: *erlang-CDF-at0 simp*: *X-distr*[*of 0*])
finally have *emeasure M* {*x∈space M. X x ≤ a*} ≤ *0* **by** *simp*
then have *emeasure M* {*x∈space M. X x ≤ a*} = *0* **by** *simp*
}
note *eq-0* = *this*

show (∫$^+$ *x. erlang-density k l x ∗ indicator* {..*a*} *x* ∂*lborel*) = *ennreal* (*erlang-CDF k l a*)
using *nn-integral-erlang-density*[*of l k a*]
by (*simp add*: *ennreal-indicator ennreal-mult*)

show *emeasure M* {*x∈space M. X x ≤ a*} = *ennreal* (*erlang-CDF k l a*)
using *X-distr*[*of a*] *eq-0* **by** (*auto simp*: *one-ennreal-def erlang-CDF-def*)
qed *simp-all*

lemma (**in** *prob-space*) *erlang-distributed-iff*:
assumes [*arith*]: *0<l*
shows *distributed M lborel X* (*erlang-density k l*) ⟷
(*X ∈ borel-measurable M* ∧ *0 < l* ∧ (∀ *a*≥*0*. $\mathcal{P}$(*x in M. X x ≤ a*) = *erlang-CDF k l a*))
using
distributed-measurable[*of M lborel X erlang-density k l*]

emeasure-erlang-density[*of l*]
erlang-distributed-le[*of X k l*]
by (*auto intro*!: *erlang-distributedI simp*: *one-ennreal-def emeasure-eq-measure*)

lemma (**in** *prob-space*) *erlang-distributed-mult-const*:
assumes *erlX*: *distributed M lborel X* (*erlang-density k l*)
assumes *a-pos*[*arith*]: $0 < \alpha$ $0 < l$
shows *distributed M lborel* ($\lambda x.\ \alpha * X\ x$) (*erlang-density k* ($l$ / α))
proof (*subst erlang-distributed-iff*, *safe*)
have [*measurable*]: *random-variable borel X* **and** [*arith*]: $0 < l$
and [*simp*]: $\bigwedge a.\ 0 \le a \Longrightarrow$ *prob* $\{x \in$ *space M*. *X x* $\le a\}$ = *erlang-CDF k l a*
by(*insert erlX*, *auto simp*: *erlang-distributed-iff*)

show *random-variable borel* ($\lambda x.\ \alpha * X\ x$) $0 < l$ / α $0 < l$ / α
by (*auto simp*:*field-simps*)

fix *a*:: *real* **assume** [*arith*]: $0 \le a$
obtain *b*:: *real* **where** [*simp*, *arith*]: $b = a/\ \alpha$ **by** *blast*

have [*arith*]: $0 \le b$ **by** (*auto simp*: *divide-nonneg-pos*)

have *prob* $\{x \in$ *space M*. $\alpha * X\ x \le a\}$ = *prob* $\{x \in$ *space M*. *X x* $\le b\}$
by (*rule arg-cong*[**where** *f*= *prob*]) (*auto simp*:*field-simps*)

moreover have *prob* $\{x \in$ *space M*. *X x* $\le b\}$ = *erlang-CDF k l b* **by** *auto*
moreover have *erlang-CDF k* (l / α) *a* = *erlang-CDF k l b* **unfolding** *erlang-CDF-def* **by** *auto*
ultimately show *prob* $\{x \in$ *space M*. $\alpha * X\ x \le a\}$ = *erlang-CDF k* (l / α) *a*
by *fastforce*
qed

lemma (**in** *prob-space*) *has-bochner-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: $0 < l$ **and** *D*: *distributed M lborel X* (*erlang-density k l*)
shows *has-bochner-integral M* ($\lambda x.\ X\ x\ \hat{}\ i$) (*fact* ($k + i$) / (*fact k* $*$ $l\ \hat{}\ i$))
proof (*rule has-bochner-integral-nn-integral*)
show *AE x in M*. $0 \le X\ x\ \hat{}\ i$
by (*subst distributed-AE2*[*OF D*]) (*auto simp*: *erlang-density-def*)
show ($\int^+ x.$ *ennreal* (*X x* $\hat{}$ *i*) ∂M) = *ennreal* (*fact* ($k + i$) / (*fact k* $*$ $l\ \hat{}\ i$))
using *nn-integral-erlang-ith-moment*[*of l k i*]
by (*subst distributed-nn-integral*[*symmetric*, *OF D*]) (*auto simp*: *ennreal-mult'*)
qed (*insert distributed-measurable*[*OF D*], *auto*)

lemma (**in** *prob-space*) *erlang-ith-moment-integrable*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*erlang-density k l*) $\Longrightarrow$ *integrable M* ($\lambda x.\ X\ x\ \hat{}\ i$)
by *rule* (*rule has-bochner-integral-erlang-ith-moment*)

lemma (**in** *prob-space*) *erlang-ith-moment*:

0 < l ⟹ distributed M lborel X (erlang-density k l) ⟹
*expectation (λx. X x ^ i) = fact (k + i) / (fact k * l ^ i)*
by (*rule has-bochner-integral-integral-eq*) (*rule has-bochner-integral-erlang-ith-moment*)

lemma (**in** *prob-space*) *erlang-distributed-variance*:
assumes [*arith*]: *0 < l* **and** *distributed M lborel X (erlang-density k l)*
shows *variance X = (k + 1) / l²*
proof (*subst variance-eq*)
show *integrable M X integrable M (λx. (X x)²)*
using *erlang-ith-moment-integrable*[*OF assms*, *of 1*] *erlang-ith-moment-integrable*[*OF assms*, *of 2*]
by *auto*

show *expectation (λx. (X x)²) − (expectation X)² = real (k + 1) / l²*
using *erlang-ith-moment*[*OF assms*, *of 1*] *erlang-ith-moment*[*OF assms*, *of 2*]
by *simp* (*auto simp*: *power2-eq-square field-simps of-nat-Suc*)
qed

13.2 Exponential distribution

abbreviation *exponential-density* :: *real ⇒ real ⇒ real* **where**
exponential-density ≡ erlang-density 0

lemma *exponential-density-def*:
*exponential-density l x = (if x < 0 then 0 else l * exp (− x * l))*
by (*simp add*: *fun-eq-iff erlang-density-def*)

lemma *erlang-CDF-0*: *erlang-CDF 0 l a = (if 0 ≤ a then 1 − exp (− l * a) else 0)*
by (*simp add*: *erlang-CDF-def*)

lemma *prob-space-exponential-density*: *0 < l ⟹ prob-space* (*density lborel* (*exponential-density l*))
by (*rule prob-space-erlang-density*)

lemma (**in** *prob-space*) *exponential-distributedD-le*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0 ≤ a* **and** *l*: *0 < l*
shows *𝒫*(*x in M. X x ≤ a*) = *1 − exp (− a * l)*
using *erlang-distributed-le*[*OF D l a*] *a* **by** (*simp add*: *erlang-CDF-def*)

lemma (**in** *prob-space*) *exponential-distributedD-gt*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0 ≤ a* **and** *l*: *0 < l*
shows *𝒫*(*x in M. a < X x*) = *exp (− a * l)*
using *erlang-distributed-gt*[*OF D l a*] *a* **by** (*simp add*: *erlang-CDF-def*)

lemma (**in** *prob-space*) *exponential-distributed-memoryless*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0 ≤ a* **and** *l*:

0 < l **and** *t*: *0 ≤ t*
shows $\mathcal{P}(x \text{ in } M.\ a + t < X\ x \mid a < X\ x) = \mathcal{P}(x \text{ in } M.\ t < X\ x)$
proof −
have $\mathcal{P}(x \text{ in } M.\ a + t < X\ x \mid a < X\ x) = \mathcal{P}(x \text{ in } M.\ a + t < X\ x) \ / \ \mathcal{P}(x \text{ in } M.\ a < X\ x)$
using ‹*0 ≤ t*› **by** (*auto simp*: *cond-prob-def intro*!: *arg-cong*[**where** *f*=*prob*] *arg-cong2*[**where** *f*=(/)])
also have ... = *exp* (− (*a* + *t*) ∗ *l*) / *exp* (− *a* ∗ *l*)
using *a t* **by** (*simp add*: *exponential-distributedD-gt*[*OF D* - *l*])
also have ... = *exp* (− *t* ∗ *l*)
using *l* **by** (*auto simp*: *field-simps exp-add*[*symmetric*])
finally show *?thesis*
using *t* **by** (*simp add*: *exponential-distributedD-gt*[*OF D* - *l*])
qed

lemma *exponential-distributedI*:
assumes *X*[*measurable*]: *X* ∈ *borel-measurable M* **and** [*arith*]: *0 < l*
and *X-distr*: ⋀*a*. *0 ≤ a* ⟹ *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *1* − *exp* (− *a* ∗ *l*)
shows *distributed M lborel X* (*exponential-density l*)
proof (*rule erlang-distributedI*)
fix *a* :: *real* **assume** *0 ≤ a* **then show** *emeasure M* {*x* ∈ *space M*. *X x* ≤ *a*} = *ennreal* (*erlang-CDF 0 l a*)
using *X-distr*[*of a*] **by** (*simp add*: *erlang-CDF-def ennreal-minus ennreal-1*[*symmetric*] *del*: *ennreal-1*)
qed *fact*+

lemma (**in** *prob-space*) *exponential-distributed-iff*:
assumes *0 < l*
shows *distributed M lborel X* (*exponential-density l*) ⟷
(*X* ∈ *borel-measurable M* ∧ (∀ *a*≥*0*. $\mathcal{P}(x \text{ in } M.\ X\ x \le a) = 1 - exp\ (-\ a * l)$))
using *assms erlang-distributed-iff*[*of l X 0*] **by** (*auto simp*: *erlang-CDF-0*)

lemma (**in** *prob-space*) *exponential-distributed-expectation*:
0 < l ⟹ *distributed M lborel X* (*exponential-density l*) ⟹ *expectation X = 1 / l*
using *erlang-ith-moment*[*of l X 0 1*] **by** *simp*

lemma *exponential-density-nonneg*: *0 < l* ⟹ *0 ≤ exponential-density l x*
by (*auto simp*: *exponential-density-def*)

lemma (**in** *prob-space*) *exponential-distributed-min*:
assumes *0 < l 0 < u*
assumes *expX*: *distributed M lborel X* (*exponential-density l*)
assumes *expY*: *distributed M lborel Y* (*exponential-density u*)
assumes *ind*: *indep-var borel X borel Y*
shows *distributed M lborel* (λ*x*. *min* (*X x*) (*Y x*)) (*exponential-density* (*l* + *u*))
proof (*subst exponential-distributed-iff*, *safe*)

have *randX*: *random-variable borel X*
using *expX* ‹*0 < l*› **by** (*simp add*: *exponential-distributed-iff*)
moreover have *randY*: *random-variable borel Y*
using *expY* ‹*0 < u*› **by** (*simp add*: *exponential-distributed-iff*)
ultimately show *random-variable borel* (λ*x*. *min* (*X x*) (*Y x*)) **by** *auto*

show *0 < l + u*
using ‹*0 < l*› ‹*0 < u*› **by** *auto*

fix *a*::*real* **assume** *a*[*arith*]: *0 ≤ a*
have *gt1*[*simp*]: $\mathcal{P}$(*x in M*. *a < X x*) = *exp* (− *a* ∗ *l*)
by (*rule exponential-distributedD-gt*[*OF expX a*]) *fact*
have *gt2*[*simp*]: $\mathcal{P}$(*x in M*. *a < Y x*) = *exp* (− *a* ∗ *u*)
by (*rule exponential-distributedD-gt*[*OF expY a*]) *fact*

have $\mathcal{P}$(*x in M*. *a* < (*min* (*X x*) (*Y x*))) = $\mathcal{P}$(*x in M*. *a* < (*X x*) ∧ *a* < (*Y x*))
by (*auto intro*!:*arg-cong*[**where** *f*=*prob*])

also have ... = $\mathcal{P}$(*x in M*. *a* < (*X x*)) ∗ $\mathcal{P}$(*x in M*. *a*< (*Y x*))
using *prob-indep-random-variable*[*OF ind*, *of* {*a* <..} {*a* <..}] **by** *simp*
also have ... = *exp* (− *a* ∗ (*l* + *u*)) **by** (*auto simp*:*field-simps mult-exp-exp*)
finally have *indep-prob*: $\mathcal{P}$(*x in M*. *a* < (*min* (*X x*) (*Y x*))) = *exp* (− *a* ∗ (*l* + *u*)) **.**

have {*x* ∈ *space M*. (*min* (*X x*) (*Y x*)) ≤*a* } = (*space M* − {*x* ∈ *space M*. *a*<(*min* (*X x*) (*Y x*)) })
by *auto*
then have *1* − *prob* {*x* ∈ *space M*. *a* < (*min* (*X x*) (*Y x*))} = *prob* {*x* ∈ *space M*. (*min* (*X x*) (*Y x*)) ≤ *a*}
using *randX randY* **by** (*auto simp*: *prob-compl*)
then show *prob* {*x* ∈ *space M*. (*min* (*X x*) (*Y x*)) ≤ *a*} = *1* − *exp* (− *a* ∗ (*l* + *u*))
using *indep-prob* **by** *auto*
qed

lemma (**in** *prob-space*) *exponential-distributed-Min*:
assumes *finI*: *finite I*
assumes *A*: *I* ≠ {}
assumes *l*: ⋀*i*. *i* ∈ *I* ⟹ *0 < l i*
assumes *expX*: ⋀*i*. *i* ∈ *I* ⟹ *distributed M lborel* (*X i*) (*exponential-density* (*l i*))
assumes *ind*: *indep-vars* (λ*i*. *borel*) *X I*
shows *distributed M lborel* (λ*x*. *Min* ((λ*i*. *X i x*) ‘*I*)) (*exponential-density* (∑ *i*∈*I*. *l i*))
using *assms*
proof (*induct rule*: *finite-ne-induct*)
case (*singleton i*) **then show** *?case* **by** *simp*
next
case (*insert i I*)

then have *distributed M lborel* ($\lambda x.$ *min* (*X i x*) (*Min* ((λ*i*. *X i x*) ‘*I*))) (*exponential-density* (*l i* + (∑ *i*∈*I*. *l i*)))
by (*intro exponential-distributed-min indep-vars-Min insert*)
(*auto intro*: *indep-vars-subset sum-pos*)
then show *?case*
using *insert* **by** *simp*
qed

lemma (**in** *prob-space*) *exponential-distributed-variance*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*exponential-density l*) $\Longrightarrow$ *variance X* = *1* / l^2
using *erlang-distributed-variance*[*of l X 0*] **by** *simp*

lemma *nn-integral-zero'*: *AE x in M*. *f x* = *0* $\Longrightarrow$ (∫$^+$*x*. *f x* ∂*M*) = *0*
by (*simp cong*: *nn-integral-cong-AE*)

lemma *convolution-erlang-density*:
fixes k_1 k_2 :: *nat*
assumes [*simp*, *arith*]: *0* < *l*
shows (λ*x*. ∫$^+$*y*. *ennreal* (*erlang-density* k_1 *l* (*x* − *y*)) ∗ *ennreal* (*erlang-density* k_2 *l y*) ∂*lborel*) =
(*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l*)
(**is** *?LHS* = *?RHS*)
proof
fix *x* :: *real*
have *x* ≤ *0* ∨ *0* < *x*
by *arith*
then show *?LHS x* = *?RHS x*
proof
assume *x* ≤ *0* **then show** *?thesis*
apply (*subst nn-integral-zero'*)
apply (*rule AE-I*[**where** *N*={*0*}])
apply (*auto simp add*: *erlang-density-def not-less*)
done
next
note *zero-le-mult-iff*[*simp*] *zero-le-divide-iff*[*simp*]

have *I-eq1*: *integral*^{N} *lborel* (*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l*) = *1*
using *nn-integral-erlang-ith-moment*[*of l Suc* k_1 + *Suc* k_2 − *1 0*] **by** (*simp del*: *fact-Suc*)

have *1*: (∫$^+$ *x*. *ennreal* (*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l x* ∗ *indicator* {*0*<..} *x*) ∂*lborel*) = *1*
apply (*subst I-eq1*[*symmetric*])
unfolding *erlang-density-def*
by (*auto intro*!: *nn-integral-cong split*:*split-indicator*)

have *prob-space* (*density lborel ?LHS*)
by (*intro prob-space-convolution-density*)
(*auto intro*!: *prob-space-erlang-density erlang-density-nonneg*)

```
  then have 2: integral^N lborel ?LHS = 1
    by (auto dest!: prob-space.emeasure-space-1 simp: emeasure-density)

  let ?I = (integral^N lborel (λy. ennreal ((1 − y)^k1 ∗ y^k2 ∗ indicator {0..1}
y)))
  let ?C = (fact (Suc (k1 + k2))) / ((fact k1) ∗ (fact k2))
  let ?s = Suc k1 + Suc k2 − 1
  let ?L = (λx. ∫+y. ennreal (erlang-density k1 l (x− y) ∗ erlang-density k2 l y
∗ indicator {0..x} y) ∂lborel)

  { fix x :: real assume [arith]: 0 < x
    have ∗: ⋀x y n. (x − y ∗ x::real)^n = x^n ∗ (1 − y)^n
      unfolding power-mult-distrib[symmetric] by (simp add: field-simps)

    have ?LHS x = ?L x
      unfolding erlang-density-def
      by (auto intro!: nn-integral-cong simp: ennreal-mult split:split-indicator)
    also have ... = (λx. ennreal ?C ∗ ?I ∗ erlang-density ?s l x) x
      apply (subst nn-integral-real-affine[where c=x and t = 0])
     apply (simp-all add: nn-integral-cmult[symmetric] nn-integral-multc[symmetric]
del: fact-Suc)
      apply (intro nn-integral-cong)
     apply (auto simp add: erlang-density-def mult-less-0-iff exp-minus field-simps
exp-diff power-add ∗
                           ennreal-mult[symmetric]
                 simp del: fact-Suc split: split-indicator)
      done
    finally have (∫+y. ennreal (erlang-density k1 l (x − y) ∗ erlang-density k2
l y) ∂lborel) =
      (λx. ennreal ?C ∗ ?I ∗ erlang-density ?s l x) x
      by (simp add: ennreal-mult) }
  note ∗ = this

  assume [arith]: 0 < x
  have 3: 1 = integral^N lborel (λxa. ?LHS xa ∗ indicator {0<..} xa)
    by (subst 2[symmetric])
       (auto intro!: nn-integral-cong-AE AE-I[where N={0}]
             simp: erlang-density-def  nn-integral-multc[symmetric] indicator-def
split: if-split-asm)
  also have ... = integral^N lborel (λx. (ennreal (?C) ∗ ?I) ∗ ((erlang-density ?s
l x) ∗ indicator {0<..} x))
    by (auto intro!: nn-integral-cong simp: ennreal-mult[symmetric] ∗ split: split-indicator)
  also have ... = ennreal (?C) ∗ ?I
    using 1
    by (auto simp: nn-integral-cmult)
  finally have  ennreal (?C) ∗ ?I = 1 by presburger

  then show ?thesis
    using ∗ by (simp add: ennreal-mult)
```

qed
qed

lemma (**in** *prob-space*) *sum-indep-erlang*:
assumes *indep*: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: *0 < l*
assumes *erlX*: *distributed M lborel X* (*erlang-density* k_1 *l*)
assumes *erlY*: *distributed M lborel Y* (*erlang-density* k_2 *l*)
shows *distributed M lborel* (λ*x*. *X x* + *Y x*) (*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l*)
using *assms*
apply (*subst convolution-erlang-density*[*symmetric*, *OF* ‹*0*<*l*›])
apply (*intro distributed-convolution*)
apply *auto*
done

lemma (**in** *prob-space*) *erlang-distributed-sum*:
assumes *finI* : *finite I*
assumes *A*: *I* ≠ {}
assumes [*simp*, *arith*]: *0 < l*
assumes *expX*: ⋀*i*. *i* ∈ *I* ⟹ *distributed M lborel* (*X i*) (*erlang-density* (*k i*) *l*)
assumes *ind*: *indep-vars* (λ*i*. *borel*) *X I*
shows *distributed M lborel* (λ*x*. ∑ *i*∈*I*. *X i x*) (*erlang-density* ((∑ *i*∈*I*. *Suc* (*k i*)) − *1*) *l*)
using *assms*
proof (*induct rule*: *finite-ne-induct*)
case (*singleton i*) **then show** *?case* **by** *auto*
next
case (*insert i I*)
then have *distributed M lborel* (λ*x*. (*X i x*) + (∑ *i*∈ *I*. *X i x*)) (*erlang-density* (*Suc* (*k i*) + *Suc* ((∑ *i*∈*I*. *Suc* (*k i*)) − *1*) − *1*) *l*)
by(*intro sum-indep-erlang indep-vars-sum*) (*auto intro*!: *indep-vars-subset*)
also have (λ*x*. (*X i x*) + (∑ *i*∈ *I*. *X i x*)) = (λ*x*. ∑ *i*∈*insert i I*. *X i x*)
using *insert* **by** *auto*
also have *Suc*(*k i*) + *Suc* ((∑ *i*∈*I*. *Suc* (*k i*)) − *1*) − *1* = (∑ *i*∈*insert i I*. *Suc* (*k i*)) − *1*
using *insert* **by** (*auto intro*!: *Suc-pred simp*: *ac-simps*)
finally show *?case* **by** *fast*
qed

lemma (**in** *prob-space*) *exponential-distributed-sum*:
assumes *finI*: *finite I*
assumes *A*: *I* ≠ {}
assumes *l*: *0 < l*
assumes *expX*: ⋀*i*. *i* ∈ *I* ⟹ *distributed M lborel* (*X i*) (*exponential-density l*)
assumes *ind*: *indep-vars* (λ*i*. *borel*) *X I*
shows *distributed M lborel* (λ*x*. ∑ *i*∈*I*. *X i x*) (*erlang-density* ((*card I*) − *1*) *l*)
using *erlang-distributed-sum*[*OF assms*] **by** *simp*

lemma (**in** *information-space*) *entropy-exponential*:
assumes *l*[*simp*, *arith*]: *0 < l*
assumes *D*: *distributed M lborel X* (*exponential-density l*)
shows *entropy b lborel X* = *log b* (*exp 1 / l*)
proof −
have [*simp*]: *integrable lborel* (*exponential-density l*)
using *distributed-integrable*[*OF D*, *of* λ-. *1*] **by** *simp*

have [*simp*]: *integral*^{L} *lborel* (*exponential-density l*) = *1*
using *distributed-integral*[*OF D*, *of* λ-. *1*] **by** (*simp add*: *prob-space*)

have [*simp*]: *integrable lborel* (λ*x*. *exponential-density l x* ∗ *x*)
using *erlang-ith-moment-integrable*[*OF l D*, *of 1*] *distributed-integrable*[*OF D*, *of* λ*x*. *x*] **by** *simp*

have [*simp*]: *integral*^{L} *lborel* (λ*x*. *exponential-density l x* ∗ *x*) = *1 / l*
using *erlang-ith-moment*[*OF l D*, *of 1*] *distributed-integral*[*OF D*, *of* λ*x*. *x*] **by** *simp*

have *entropy b lborel X* = − (∫ *x*. *exponential-density l x* ∗ *log b* (*exponential-density l x*) ∂*lborel*)
using *D* **by** (*rule entropy-distr*) *simp*
also have (∫ *x*. *exponential-density l x* ∗ *log b* (*exponential-density l x*) ∂*lborel*) =
(∫ *x*. (*ln l* ∗ *exponential-density l x* − *l* ∗ (*exponential-density l x* ∗ *x*)) / *ln b* ∂*lborel*)
by (*intro Bochner-Integration.integral-cong*) (*auto simp*: *log-def ln-mult exponential-density-def field-simps*)
also have ... = (*ln l* − *1*) / *ln b*
by *simp*
finally show *?thesis*
by (*simp add*: *log-def ln-div*) (*simp add*: *field-split-simps*)
qed

13.3 Uniform distribution

lemma *uniform-distrI*:
assumes *X*: *X* ∈ *measurable M M′*
and *A*: *A* ∈ *sets M′ emeasure M′ A* ≠ ∞ *emeasure M′ A* ≠ *0*
assumes *distr*: ⋀*B*. *B* ∈ *sets M′* ⟹ *emeasure M* (*X* −‘ *B* ∩ *space M*) = *emeasure M′* (*A* ∩ *B*) / *emeasure M′ A*
shows *distr M M′ X* = *uniform-measure M′ A*
unfolding *uniform-measure-def*
proof (*intro measure-eqI*)
let *?f* = λ*x*. *indicator A x* / *emeasure M′ A*
fix *B* **assume** *B*: *B* ∈ *sets* (*distr M M′ X*)
with *X* **have** *emeasure M* (*X* −‘ *B* ∩ *space M*) = *emeasure M′* (*A* ∩ *B*) / *emeasure M′ A*
by (*simp add*: *distr*[*of B*] *measurable-sets*)

also have ... = (*1 / emeasure M' A*) * *emeasure M' (A ∩ B)*
by (*simp add*: *divide-ennreal-def ac-simps*)
also have ... = (∫⁺ *x*. (*1 / emeasure M' A*) * *indicator (A ∩ B) x* ∂*M'*)
using *A B*
by (*intro nn-integral-cmult-indicator*[*symmetric*]) (*auto intro*!:)
also have ... = (∫⁺ *x*. *?f x* * *indicator B x* ∂*M'*)
by (*rule nn-integral-cong*) (*auto split*: *split-indicator*)
finally show *emeasure* (*distr M M' X*) *B* = *emeasure* (*density M' ?f*) *B*
using *A B X* **by** (*auto simp add*: *emeasure-distr emeasure-density*)
qed *simp*

lemma *uniform-distrI-borel*:
fixes *A* :: *real set*
assumes *X*[*measurable*]: *X* ∈ *borel-measurable M* **and** *A*: *emeasure lborel A* = *ennreal r 0* < *r*
and [*measurable*]: *A* ∈ *sets borel*
assumes *distr*: ⋀*a*. *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *emeasure lborel* (*A* ∩ {.. *a*}) / *r*
shows *distributed M lborel X* (λ*x*. *indicator A x* / *measure lborel A*)
proof (*rule distributedI-borel-atMost*)
let *?f* = λ*x*. *1 / r* * *indicator A x*
fix *a*
have *emeasure lborel* (*A* ∩ {..*a*}) ≤ *emeasure lborel A*
using *A* **by** (*intro emeasure-mono*) *auto*
also have ... < ∞
using *A* **by** *simp*
finally have *fin*: *emeasure lborel* (*A* ∩ {..*a*}) ≠ *top*
by *simp*
from *emeasure-eq-ennreal-measure*[*OF this*]
have *fin-eq*: *emeasure lborel* (*A* ∩ {..*a*}) / *r* = *ennreal* (*measure lborel* (*A* ∩ {..*a*}) / *r*)
using *A* **by** (*simp add*: *divide-ennreal measure-nonneg*)
then show *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *ennreal* (*measure lborel* (*A* ∩ {..*a*}) / *r*)
using *distr* **by** *simp*

have (∫⁺ *x*. *ennreal* (*indicator A x* / *measure lborel A* * *indicator* {..*a*} *x*) ∂*lborel*) =
(∫⁺ *x*. *ennreal* (*1 / measure lborel A*) * *indicator* (*A* ∩ {..*a*}) *x* ∂*lborel*)
by (*auto intro*!: *nn-integral-cong split*: *split-indicator*)
also have ... = *ennreal* (*1 / measure lborel A*) * *emeasure lborel* (*A* ∩ {..*a*})
using ‹*A* ∈ *sets borel*›
by (*intro nn-integral-cmult-indicator*) (*auto simp*: *measure-nonneg*)
also have ... = *ennreal* (*measure lborel* (*A* ∩ {..*a*}) / *r*)
unfolding *emeasure-eq-ennreal-measure*[*OF fin*] **using** *A*
by (*simp add*: *measure-def ennreal-mult'*[*symmetric*])
finally show (∫⁺ *x*. *ennreal* (*indicator A x* / *measure lborel A* * *indicator* {..*a*} *x*) ∂*lborel*) =
ennreal (*measure lborel* (*A* ∩ {..*a*}) / *r*) .

qed (*auto simp*: *measure-nonneg*)

lemma (**in** *prob-space*) *uniform-distrI-borel-atLeastAtMost*:
fixes *a b* :: *real*
assumes *X*: $X \in$ *borel-measurable M* **and** $a < b$
assumes *distr*: $\bigwedge t.\ a \leq t \Longrightarrow t \leq b \Longrightarrow \mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
shows *distributed M lborel X* ($\lambda x.$ *indicator* $\{a..b\}$ *x / measure lborel* $\{a..b\}$)
proof (*rule uniform-distrI-borel*)
fix *t*
have $t < a \vee (a \leq t \wedge t \leq b) \vee b < t$
by *auto*
then show *emeasure M* $\{x \in space\ M.\ X\ x \leq t\}$ = *emeasure lborel* $(\{a\ ..\ b\} \cap \{..t\})\ /\ (b - a)$
proof (*elim disjE conjE*)
assume $t < a$
then have *emeasure M* $\{x \in space\ M.\ X\ x \leq t\} \leq$ *emeasure M* $\{x \in space\ M.\ X\ x \leq a\}$
using *X* **by** (*auto intro*!: *emeasure-mono measurable-sets*)
also have $\ldots = 0$
using *distr*[*of a*] ‹$a < b$› **by** (*simp add*: *emeasure-eq-measure*)
finally have *emeasure M* $\{x \in space\ M.\ X\ x \leq t\} = 0$
by (*simp add*: *antisym measure-nonneg*)
with ‹$t < a$› **show** *?thesis* **by** *simp*
next
assume *bnds*: $a \leq t\ t \leq b$
have $\{a..b\} \cap \{..t\} = \{a..t\}$
using *bnds* **by** *auto*
then show *?thesis* **using** ‹$a \leq t$› ‹$a < b$›
using *distr*[*OF bnds*] **by** (*simp add*: *emeasure-eq-measure divide-ennreal*)
next
assume $b < t$
have *1* = *emeasure M* $\{x \in space\ M.\ X\ x \leq b\}$
using *distr*[*of b*] ‹$a < b$› **by** (*simp add*: *one-ennreal-def emeasure-eq-measure*)
also have $\ldots \leq$ *emeasure M* $\{x \in space\ M.\ X\ x \leq t\}$
using *X* ‹$b < t$› **by** (*auto intro*!: *emeasure-mono measurable-sets*)
finally have *emeasure M* $\{x \in space\ M.\ X\ x \leq t\} = 1$
by (*simp add*: *antisym emeasure-eq-measure*)
with ‹$b < t$› ‹$a < b$› **show** *?thesis* **by** (*simp add*: *measure-def divide-ennreal*)
qed
qed (*insert X* ‹$a < b$›, *auto*)

lemma (**in** *prob-space*) *uniform-distributed-measure*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* ($\lambda x.$ *indicator* $\{a\ ..\ b\}$ *x / measure lborel* $\{a\ ..\ b\}$)
assumes *t*: $a \leq t\ t \leq b$
shows $\mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
proof −

have *emeasure M* {$x \in$ *space M*. *X x* $\leq$ *t*} = *emeasure* (*distr M lborel X*) {.. *t*}
using *distributed-measurable*[*OF D*]
by (*subst emeasure-distr*) (*auto intro*!: *arg-cong2*[**where** *f*=*emeasure*])
also have ... = ($\int^+ x$. *ennreal* (*1* / (*b* − *a*)) ∗ *indicator* {*a* .. *t*} *x* ∂*lborel*)
using *distributed-borel-measurable*[*OF D*] ‹*a* ≤ *t*› ‹*t* ≤ *b*›
unfolding *distributed-distr-eq-density*[*OF D*]
by (*subst emeasure-density*)
(*auto intro*!: *nn-integral-cong simp*: *measure-def split*: *split-indicator*)
also have ... = *ennreal* (*1* / (*b* − *a*)) ∗ (*t* − *a*)
using ‹*a* ≤ *t*› ‹*t* ≤ *b*›
by (*subst nn-integral-cmult-indicator*) *auto*
finally show *?thesis*
using *t* **by** (*simp add*: *emeasure-eq-measure ennreal-mult''*[*symmetric*] *measure-nonneg*)
qed

lemma (**in** *prob-space*) *uniform-distributed-bounds*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a* .. *b*} *x* / *measure lborel* {*a* .. *b*})
shows *a* < *b*
proof (*rule ccontr*)
assume ¬ *a* < *b*
then have {*a* .. *b*} = {} ∨ {*a* .. *b*} = {*a* .. *a*} **by** *simp*
with *uniform-distributed-params*[*OF D*] **show** *False*
by (*auto simp*: *measure-def*)
qed

lemma (**in** *prob-space*) *uniform-distributed-iff*:
fixes *a b* :: *real*
shows *distributed M lborel X* (λ*x*. *indicator* {*a*..*b*} *x* / *measure lborel* {*a*..*b*}) ⟷
(*X* ∈ *borel-measurable M* ∧ *a* < *b* ∧ (∀ *t*∈{*a* .. *b*}. $\mathcal{P}$(*x in M*. *X x* ≤ *t*)= (*t* − *a*) / (*b* − *a*)))
using
uniform-distributed-bounds[*of X a b*]
uniform-distributed-measure[*of X a b*]
distributed-measurable[*of M lborel X*]
by (*auto intro*!: *uniform-distrI-borel-atLeastAtMost simp del*: *content-real-if*)

lemma (**in** *prob-space*) *uniform-distributed-expectation*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a* .. *b*} *x* / *measure lborel* {*a* .. *b*})
shows *expectation X* = (*a* + *b*) / *2*
proof (*subst distributed-integral*[*OF D*, *of* λ*x*. *x*, *symmetric*])
have *a* < *b*
using *uniform-distributed-bounds*[*OF D*] .

have ($\int$ *x*. *indicator* {*a* .. *b*} *x* / *measure lborel* {*a* .. *b*} ∗ *x* ∂*lborel*) =

```
    (∫ x. (x / measure lborel {a .. b}) * indicator {a .. b} x ∂lborel)
    by (intro Bochner-Integration.integral-cong) auto
  also have (∫ x. (x / measure lborel {a .. b}) * indicator {a .. b} x ∂lborel) =
(a + b) / 2
  proof (subst integral-FTC-Icc-real)
    fix x
    show DERIV (λx. x² / (2 * measure lborel {a..b})) x :> x / measure lborel
{a..b}
      using uniform-distributed-params[OF D]
      by (auto intro!: derivative-eq-intros simp del: content-real-if)
    show isCont (λx. x / Sigma-Algebra.measure lborel {a..b}) x
      using uniform-distributed-params[OF D]
      by (auto intro!: isCont-divide)
    have *: b² / (2 * measure lborel {a..b}) - a² / (2 * measure lborel {a..b}) =
      (b*b - a * a) / (2 * (b - a))
      using ‹a < b›
      by (auto simp: measure-def power2-eq-square diff-divide-distrib[symmetric])
    show b² / (2 * measure lborel {a..b}) - a² / (2 * measure lborel {a..b}) = (a
+ b) / 2
      using ‹a < b›
      unfolding * square-diff-square-factored by (auto simp: field-simps)
  qed (insert ‹a < b›, simp)
  finally show (∫ x. indicator {a .. b} x / measure lborel {a .. b} * x ∂lborel) =
(a + b) / 2 .
qed (auto simp: measure-nonneg)

lemma (in prob-space) uniform-distributed-variance:
  fixes a b :: real
  assumes D: distributed M lborel X (λx. indicator {a .. b} x / measure lborel {a
.. b})
  shows variance X = (b - a)² / 12
proof (subst distributed-variance)
  have [arith]: a < b using uniform-distributed-bounds[OF D] .
  let ?μ = expectation X let ?D = λx. indicator {a..b} (x + ?μ) / measure lborel
{a..b}
  have (∫ x. x² * (?D x) ∂lborel) = (∫ x. x² * (indicator {a - ?μ .. b - ?μ} x) /
measure lborel {a .. b} ∂lborel)
    by (intro Bochner-Integration.integral-cong) (auto split: split-indicator)
  also have ... = (b - a)² / 12
    by (simp add: integral-power uniform-distributed-expectation[OF D])
       (simp add: eval-nat-numeral field-simps )
  finally show (∫ x. x² * ?D x ∂lborel) = (b - a)² / 12 .
qed (auto intro: D simp del: content-real-if)
```

13.4 Normal distribution

```
definition normal-density :: real ⇒ real ⇒ real ⇒ real where
  normal-density μ σ x = 1 / sqrt (2 * pi * σ²) * exp (-(x - μ)²/ (2 * σ²))
```

abbreviation *std-normal-density* :: *real* ⇒ *real* **where**
std-normal-density ≡ *normal-density 0 1*

lemma *std-normal-density-def*: *std-normal-density x* = (*1 / sqrt* (*2* ∗ *pi*)) ∗ *exp* (− x^2 / *2*)
unfolding *normal-density-def* **by** *simp*

lemma *normal-density-nonneg*[*simp*]: *0* ≤ *normal-density* μ σ *x*
by (*auto simp*: *normal-density-def*)

lemma *normal-density-pos*: *0* < σ ⟹ *0* < *normal-density* μ σ *x*
by (*auto simp*: *normal-density-def*)

lemma *borel-measurable-normal-density*[*measurable*]: *normal-density* μ σ ∈ *borel-measurable borel*
by (*auto simp*: *normal-density-def*[*abs-def*])

lemma *gaussian-moment-0*:
has-bochner-integral lborel (λ*x*. *indicator* {*0*..} *x* ∗$_R$ *exp* (− x^2)) (*sqrt pi / 2*)
proof −
let *?pI* = λ*f*. (∫$^+$*s*. *f* (*s*::*real*) ∗ *indicator* {*0*..} *s* ∂*lborel*)
let *?gauss* = λ*x*. *exp* (− x^2)

let *?I* = *indicator* {*0*<..} :: *real* ⇒ *real*
let *?ff*= λ*x s*. *x* ∗ *exp* (− x^2 ∗ (*1* + s^2)) :: *real*

have ∗: *?pI ?gauss* = (∫$^+$*x*. *?gauss x* ∗ *?I x* ∂*lborel*)
by (*intro nn-integral-cong-AE AE-I*[**where** *N*={*0*}]) (*auto split*: *split-indicator*)

have *?pI ?gauss* ∗ *?pI ?gauss* = (∫$^+$*x*. ∫$^+$*s*. *?gauss x* ∗ *?gauss s* ∗ *?I s* ∗ *?I x* ∂*lborel* ∂*lborel*)
by (*auto simp*: *nn-integral-cmult*[*symmetric*] *nn-integral-multc*[*symmetric*] ∗ *ennreal-mult*[*symmetric*]
intro!: *nn-integral-cong split*: *split-indicator*)
also have ... = (∫$^+$*x*. ∫$^+$*s*. *?ff x s* ∗ *?I s* ∗ *?I x* ∂*lborel* ∂*lborel*)
proof (*rule nn-integral-cong*, *cases*)
fix *x* :: *real* **assume** *x* ≠ *0*
then show (∫$^+$*s*. *?gauss x* ∗ *?gauss s* ∗ *?I s* ∗ *?I x* ∂*lborel*) = (∫$^+$*s*. *?ff x s* ∗ *?I s* ∗ *?I x* ∂*lborel*)
by (*subst nn-integral-real-affine*[**where** *t*=*0* **and** *c*=*x*])
(*auto simp*: *mult-exp-exp nn-integral-cmult*[*symmetric*] *field-simps zero-less-mult-iff ennreal-mult*[*symmetric*]
intro!: *nn-integral-cong split*: *split-indicator*)
qed *simp*
also have ... = ∫$^+$*s*. ∫$^+$*x*. *?ff x s* ∗ *?I s* ∗ *?I x* ∂*lborel* ∂*lborel*
by (*rule lborel-pair.Fubini'*[*symmetric*]) *auto*
also have ... = *?pI* (λ*s*. *?pI* (λ*x*. *?ff x s*))
by (*rule nn-integral-cong-AE*)
(*auto intro*!: *nn-integral-cong-AE AE-I*[**where** *N*={*0*}] *split*: *split-indicator-asm*)

```
  also have ... = ?pI (λs. ennreal (1 / (2 * (1 + s²))))
  proof (intro nn-integral-cong ennreal-mult-right-cong)
    fix s :: real show ?pI (λx. ?ff x s) = ennreal (1 / (2 * (1 + s²)))
    proof (subst nn-integral-FTC-atLeast)
      have ((λa. - (exp (- (a² * (1 + s²))) / (2 + 2 * s²))) ⟶ (- (0 / (2 +
2 * s²)))) at-top
      apply (intro tendsto-intros filterlim-compose[OF exp-at-bot] filterlim-compose[OF
filterlim-uminus-at-bot-at-top])
        apply (subst mult.commute)
        apply (auto intro!: filterlim-tendsto-pos-mult-at-top
                            filterlim-at-top-mult-at-top[OF filterlim-ident filterlim-ident]
                    simp: add-pos-nonneg  power2-eq-square add-nonneg-eq-0-iff)
        done
      then show ((λa. - (exp (- a² - s² * a²) / (2 + 2 * s²))) ⟶ 0) at-top
        by (simp add: field-simps)
    qed (auto intro!: derivative-eq-intros simp: field-simps add-nonneg-eq-0-iff)
  qed
  also have ... = ennreal (pi / 4)
  proof (subst nn-integral-FTC-atLeast)
    show ((λa. arctan a / 2) ⟶ (pi / 2) / 2 ) at-top
      by (intro tendsto-intros) (simp-all add: tendsto-arctan-at-top)
 qed (auto intro!: derivative-eq-intros simp: add-nonneg-eq-0-iff field-simps power2-eq-square)
  finally have ?pI ?gauss^2 = pi / 4
    by (simp add: power2-eq-square)
  then have ?pI ?gauss = sqrt (pi / 4)
    using power-eq-iff-eq-base[of 2 enn2real (?pI ?gauss) sqrt (pi / 4)]
     by (cases ?pI ?gauss) (auto simp: power2-eq-square ennreal-mult[symmetric]
ennreal-top-mult)
  also have ?pI ?gauss = (∫⁺x. indicator {0..} x *ᵣ exp (- x²) ∂lborel)
    by (intro nn-integral-cong) (simp split: split-indicator)
  also have sqrt (pi / 4) = sqrt pi / 2
    by (simp add: real-sqrt-divide)
  finally show ?thesis
    by (rule has-bochner-integral-nn-integral[rotated 3])
       auto
qed

lemma gaussian-moment-1:
  has-bochner-integral lborel (λx::real. indicator {0..} x *ᵣ (exp (- x²) * x)) (1 /
2)
proof -
  have (∫⁺x. indicator {0..} x *ᵣ (exp (- x²) * x) ∂lborel) =
    (∫⁺x. ennreal (x * exp (- x²)) * indicator {0..} x ∂lborel)
    by (intro nn-integral-cong)
       (auto simp: ac-simps split: split-indicator)
  also have ... = ennreal (0 - (- exp (- 0²) / 2))
  proof (rule nn-integral-FTC-atLeast)
    have ((λx::real. - exp (- x²) / 2) ⟶ - 0 / 2) at-top
      by (intro tendsto-divide tendsto-minus filterlim-compose[OF exp-at-bot]
```

```
                filterlim-compose[OF filterlim-uminus-at-bot-at-top]
                filterlim-pow-at-top filterlim-ident)
        auto
    then show ((λa::real. − exp (− a²) / 2) ⟶ 0) at-top
      by simp
  qed (auto intro!: derivative-eq-intros)
  also have ... = ennreal (1 / 2)
    by simp
  finally show ?thesis
    by (rule has-bochner-integral-nn-integral[rotated 3])
       (auto split: split-indicator)
qed

lemma
  fixes k :: nat
  shows gaussian-moment-even-pos:
     has-bochner-integral lborel (λx::real. indicator {0..} x *R (exp (−x²)*x^(2 *
k)))
       ((sqrt pi / 2) * (fact (2 * k) / (2 ^ (2 * k) * fact k)))
       (is ?even)
    and gaussian-moment-odd-pos:
      has-bochner-integral lborel (λx::real. indicator {0..} x *R (exp (−x²)*x^(2 *
k + 1))) (fact k / 2)
      (is ?odd)
proof −
  let ?M = λk x. exp (− x²) * x^k :: real

  { fix k I assume Mk: has-bochner-integral lborel (λx. indicator {0..} x *R ?M k
x) I
    have 2 ≠ (0::real)
      by linarith
    let ?f = λb. ∫ x. indicator {0..} x *R ?M (k + 2) x * indicator {..b} x ∂lborel
    have ((λb. (k + 1) / 2 * (∫ x. indicator {..b} x *R (indicator {0..} x *R ?M
k x) ∂lborel) − ?M (k + 1) b / 2) ⟶
          (k + 1) / 2 * (∫ x. indicator {0..} x *R ?M k x ∂lborel) − 0 / 2) at-top
(is ?tendsto)
   proof (intro tendsto-intros ‹2 ≠ 0› tendsto-integral-at-top sets-lborel Mk[THEN
integrable.intros])
      show (?M (k + 1) ⟶ 0) at-top
      proof cases
        assume even k
        have ((λx. ((x²)^(k div 2 + 1) / exp (x²)) * (1 / x) :: real) ⟶ 0 * 0)
at-top
        by (intro tendsto-intros tendsto-divide-0[OF tendsto-const] filterlim-compose[OF
tendsto-power-div-exp-0]
                  filterlim-at-top-imp-at-infinity filterlim-ident filterlim-pow-at-top
filterlim-ident)
           auto
        also have (λx. ((x²)^(k div 2 + 1) / exp (x²)) * (1 / x) :: real) = ?M (k
```

+ 1)
using ‹*even k*› **by** (*auto simp*: *fun-eq-iff exp-minus field-simps power2-eq-square power-mult elim*: *evenE*)
finally show *?thesis* **by** *simp*
next
assume *odd k*
have ((λ*x*. ((*x*²)^((*k* − *1*) *div 2 + 1*) / *exp* (*x*²)) :: *real*) ⟶ *0*) *at-top*
by (*intro filterlim-compose*[*OF tendsto-power-div-exp-0*] *filterlim-at-top-imp-at-infinity filterlim-ident filterlim-pow-at-top*)
auto
also have (λ*x*. ((*x*²)^((*k* − *1*) *div 2 + 1*) / *exp* (*x*²)) :: *real*) = *?M* (*k + 1*)
using ‹*odd k*› **by** (*auto simp*: *fun-eq-iff exp-minus field-simps power2-eq-square power-mult elim*: *oddE*)
finally show *?thesis* **by** *simp*
qed
qed
also have *?tendsto* ⟷ ((*?f* ⟶ (*k + 1*) / *2* ∗ (∫ *x*. *indicator* {*0*..} *x* ∗$_R$ *?M k x* ∂*lborel*) − *0* / *2*) *at-top*)
proof (*intro filterlim-cong refl eventually-at-top-linorder*[*THEN iffD2*] *exI*[*of - 0*] *allI impI*)
fix *b* :: *real* **assume** *b*: *0* ≤ *b*
have *Suc k* ∗ (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ *?M k x* ∂*lborel*) = (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ (*exp* (− *x*²) ∗ ((*Suc k*) ∗ *x* ^ *k*)) ∂*lborel*)
unfolding *integral-mult-right-zero*[*symmetric*] **by** (*intro Bochner-Integration.integral-cong*) *auto*
also have ... = *exp* (− *b*²) ∗ *b* ^ (*Suc k*) − *exp* (− *0*²) ∗ *0* ^ (*Suc k*) − (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ (− *2* ∗ *x* ∗ *exp* (− *x*²) ∗ *x* ^ (*Suc k*)) ∂*lborel*)
by (*rule integral-by-parts'*)
(*auto intro*!: *derivative-eq-intros b*
simp: *diff-Suc of-nat-Suc field-simps split*: *nat.split*)
also have ... = *exp* (− *b*²) ∗ *b* ^ (*Suc k*) − (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ (− *2* ∗ (*exp* (− *x*²) ∗ *x* ^ (*k + 2*))) ∂*lborel*)
by (*auto simp*: *intro*!: *Bochner-Integration.integral-cong*)
also have ... = *exp* (− *b*²) ∗ *b* ^ (*Suc k*) + *2* ∗ (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ *?M* (*k + 2*) *x* ∂*lborel*)
unfolding *Bochner-Integration.integral-mult-right-zero* [*symmetric*]
by (*simp del*: *real-scaleR-def*)
finally have *Suc k* ∗ (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ *?M k x* ∂*lborel*) = *exp* (− *b*²) ∗ *b* ^ (*Suc k*) + *2* ∗ (∫ *x*. *indicator* {*0*..*b*} *x* ∗$_R$ *?M* (*k + 2*) *x* ∂*lborel*) .
then show (*k + 1*) / *2* ∗ (∫ *x*. *indicator* {..*b*} *x* ∗$_R$ (*indicator* {*0*..} *x* ∗$_R$ *?M k x*)∂*lborel*) − *exp* (− *b*²) ∗ *b* ^ (*k + 1*) / *2* = *?f b*
by (*simp add*: *field-simps atLeastAtMost-def indicator-inter-arith*)
qed
finally have *int-M-at-top*: ((*?f* ⟶ (*k + 1*) / *2* ∗ (∫ *x*. *indicator* {*0*..} *x* ∗$_R$ *?M k x* ∂*lborel*)) *at-top*)
by *simp*

have *has-bochner-integral lborel* (λ*x*. *indicator* {*0*..} *x* ∗$_R$ *?M* (*k + 2*) *x*) ((*k*

*+ 1) / 2 * I)*
proof (*rule has-bochner-integral-monotone-convergence-at-top*)
fix *y* :: *real*
have *∗*: (*λx. indicator {0..} x ∗R ?M (k + 2) x * indicator {..y} x*::*real*) =
(*λx. indicator {0..y} x ∗R ?M (k + 2) x*)
by *rule* (*simp split*: *split-indicator*)
show *integrable lborel* (*λx. indicator {0..} x ∗R* (*?M (k + 2) x*) *∗ indicator {..y} x*::*real*)
unfolding *∗* **by** (*rule borel-integrable-compact*) (*auto intro*!: *continuous-intros*)
show ((*?f ⟶ (k + 1) / 2 * I*) *at-top*)
using *int-M-at-top has-bochner-integral-integral-eq*[*OF Mk*] **by** *simp*
qed (*auto split*: *split-indicator*) **}**
note *step* = *this*

show *?even*
proof (*induct k*)
case (*Suc k*)
note *step*[*OF this*]
also have (*real (2 * k + 1) / 2 * (sqrt pi / 2 * ((fact (2 * k)) / ((2*::*real)^(2*k) * fact k)))) =*
*sqrt pi / 2 * ((fact (2 * Suc k)) / ((2*::*real)^(2 * Suc k) * fact (Suc k)))*
apply (*simp add*: *field-simps del*: *fact-Suc*)
apply (*simp add*: *of-nat-mult field-simps*)
done
finally show *?case*
by *simp*
qed (*insert gaussian-moment-0*, *simp*)

show *?odd*
proof (*induct k*)
case (*Suc k*)
note *step*[*OF this*]
also have (*real (2 * k + 1 + 1) / (2*::*real) * ((fact k) / 2)) = (fact (Suc k)) / 2*
by (*simp add*: *field-simps of-nat-Suc field-split-simps del*: *fact-Suc*) (*simp add*: *field-simps*)
finally show *?case*
by *simp*
qed (*insert gaussian-moment-1*, *simp*)
qed

context
fixes *k* :: *nat* **and** *μ σ* :: *real* **assumes** [*arith*]: *0 < σ*
begin

lemma *normal-moment-even*:
has-bochner-integral lborel (*λx. normal-density μ σ x * (x − μ) ^ (2 * k)*) (*fact (2 * k) / ((2 / σ*^{2}*)^k * fact k)*)

proof −
have *eq*: $\bigwedge x$::*real*. $x^2 \hat{} k = (x \hat{} k)^2$
by (*simp add*: *power-mult*[*symmetric*] *ac-simps*)

have *has-bochner-integral lborel* (λx. *exp* $(-x^2) * x \hat{} (2 * k)$)
(*sqrt pi* ∗ (*fact* (2 ∗ *k*) / (2 ^ (2 ∗ *k*) ∗ *fact k*)))
using *has-bochner-integral-even-function*[*OF gaussian-moment-even-pos*[**where** *k*=*k*]] **by** *simp*
then have *has-bochner-integral lborel* (λx. (*exp* $(-x^2) * x \hat{} (2 * k)$) ∗ ((2∗$\sigma^2$) ^ *k* / *sqrt* (2 ∗ *pi* ∗ σ^2)))
((*sqrt pi* ∗ (*fact* (2 ∗ *k*) / (2 ^ (2 ∗ *k*) ∗ *fact k*))) ∗ ((2∗σ^2) ^ *k* / *sqrt* (2 ∗ *pi* ∗ σ^2)))
by (*rule has-bochner-integral-mult-left*)
also have (λx. (*exp* $(-x^2) * x \hat{} (2 * k)$) ∗ ((2∗$\sigma^2$) ^ *k* / *sqrt* (2 ∗ *pi* ∗ σ^2))) =
(λx. *exp* (− ((*sqrt* 2 ∗ σ) ∗ *x*)^2 / (2∗σ^2)) ∗ ((*sqrt* 2 ∗ σ) ∗ *x*) ^ (2 ∗ *k*) / *sqrt* (2 ∗ *pi* ∗ σ^2))
by (*auto simp*: *fun-eq-iff field-simps real-sqrt-power*[*symmetric*] *real-sqrt-mult*
real-sqrt-divide power-mult eq)
also have ((*sqrt pi* ∗ (*fact* (2 ∗ *k*) / (2 ^ (2 ∗ *k*) ∗ *fact k*))) ∗ ((2∗σ^2) ^ *k* / *sqrt* (2 ∗ *pi* ∗ σ^2))) =
(*inverse* (*sqrt* 2 ∗ σ) ∗ ((*fact* (2 ∗ *k*))) / ((2/σ^2) ^ *k* ∗ (*fact k*)))
by (*auto simp*: *fun-eq-iff power-mult field-simps real-sqrt-power*[*symmetric*] *real-sqrt-mult*
power2-eq-square)
finally show *?thesis*
unfolding *normal-density-def*
by (*subst lborel-has-bochner-integral-real-affine-iff*[**where** *c*=*sqrt* 2 ∗ σ **and** *t*=μ]) *simp-all*
qed

lemma *normal-moment-abs-odd*:
has-bochner-integral lborel (λx. *normal-density* μ σ *x* ∗ |*x* − μ| ^ (2 ∗ *k* + 1)) (2 ^ *k* ∗ σ ^ (2 ∗ *k* + 1) ∗ *fact k* ∗ *sqrt* (2 / *pi*))
proof −
have *has-bochner-integral lborel* (λx::*real*. *indicator* {0..} *x* ∗$_R$ (*exp* $(-x^2) * |x| \hat{} (2 * k + 1)$)) (*fact k* / 2)
by (*rule has-bochner-integral-cong*[*THEN iffD1*, *OF - - - gaussian-moment-odd-pos*[*of k*]]) *auto*
from *has-bochner-integral-even-function*[*OF this*]
have *has-bochner-integral lborel* (λx::*real*. *exp* $(-x^2) * |x| \hat{} (2 * k + 1)$) (*fact k*)
by *simp*
then have *has-bochner-integral lborel* (λx. (*exp* $(-x^2) * |x| \hat{} (2 * k + 1)$) ∗ (2 ^ *k* ∗ σ ^ (2 ∗ *k* + 1) / *sqrt* (*pi* ∗ σ^2)))
(*fact k* ∗ (2 ^ *k* ∗ σ ^ (2 ∗ *k* + 1) / *sqrt* (*pi* ∗ σ^2)))
by (*rule has-bochner-integral-mult-left*)
also have (λx. (*exp* $(-x^2) * |x| \hat{} (2 * k + 1)$) ∗ (2 ^ *k* ∗ σ ^ (2 ∗ *k* + 1) / *sqrt* (*pi* ∗ σ^2))) =
(λx. *exp* (− (((*sqrt* 2 ∗ σ) ∗ *x*)^2 / (2 ∗ σ^2))) ∗ |*sqrt* 2 ∗ σ ∗ *x*| ^ (2 ∗ *k* + 1) / *sqrt* (2 ∗ *pi* ∗ σ^2))

by (*simp add*: *field-simps abs-mult real-sqrt-power*[*symmetric*] *power-mult real-sqrt-mult*)
also have (*fact k* $*$ (2^k $*$ σ^(2 $*$ k + 1) / *sqrt* (*pi* $*$ σ^2))) =
(*inverse* (*sqrt* 2) $*$ *inverse* σ $*$ (2 ^ k $*$ (σ $*$ σ ^ (2 $*$ k)) $*$ (*fact k*) $*$ *sqrt* (2 / *pi*)))
by (*auto simp*: *fun-eq-iff power-mult field-simps real-sqrt-power*[*symmetric*] *real-sqrt-divide*
real-sqrt-mult)
finally show *?thesis*
unfolding *normal-density-def*
by (*subst lborel-has-bochner-integral-real-affine-iff*[**where** *c*=*sqrt* 2 $*$ σ **and** *t*=μ])
simp-all
qed

lemma *normal-moment-odd*:
has-bochner-integral lborel (λ*x*. *normal-density* μ σ *x* $*$ (*x* − μ)^(2 $*$ *k* + 1)) *0*
proof −
have *has-bochner-integral lborel* (λ*x*. *exp* (− x^2) $*$ x^(2 $*$ *k* + 1)::*real*) *0*
using *gaussian-moment-odd-pos* **by** (*rule has-bochner-integral-odd-function*) *simp*
then have *has-bochner-integral lborel* (λ*x*. (*exp* (−x^2)$*$x^(2 $*$ *k* + 1)) $*$ (2^*k*σ^(2$*$*k*)/*sqrt pi*))
(*0* $*$ (2^*k*$*$$\sigma$^(2$*$*k*)/*sqrt pi*))
by (*rule has-bochner-integral-mult-left*)
also have (λ*x*. (*exp* (−x^2)$*$x^(2 $*$ *k* + 1)) $*$ (2^*k*$*$$\sigma$^(2$*$*k*)/*sqrt pi*)) =
(λ*x*. *exp* (− ((*sqrt* 2 $*$ σ $*$ *x*)^2 / (2 $*$ σ^2))) $*$
(*sqrt* 2 $*$ σ $*$ *x* $*$ (*sqrt* 2 $*$ σ $*$ *x*) ^ (2 $*$ *k*)) /
sqrt (2 $*$ *pi* $*$ σ^2))
unfolding *real-sqrt-mult*
by (*simp add*: *field-simps abs-mult real-sqrt-power*[*symmetric*] *power-mult fun-eq-iff*)
finally show *?thesis*
unfolding *normal-density-def*
by (*subst lborel-has-bochner-integral-real-affine-iff*[**where** *c*=*sqrt* 2 $*$ σ **and** *t*=μ]) *simp-all*
qed

lemma *integral-normal-moment-even*:
integral^L *lborel* (λ*x*. *normal-density* μ σ *x* $*$ (*x* − μ)^(2 $*$ *k*)) = *fact* (2 $*$ *k*) / ((2 / σ^2)^*k* $*$ *fact k*)
using *normal-moment-even* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integral-normal-moment-abs-odd*:
integral^L *lborel* (λ*x*. *normal-density* μ σ *x* $*$ |*x* − μ|^(2 $*$ *k* + 1)) = 2 ^ *k* $*$ σ ^ (2 $*$ *k* + 1) $*$ *fact k* $*$ *sqrt* (2 / *pi*)
using *normal-moment-abs-odd* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integral-normal-moment-odd*:
integral^L *lborel* (λ*x*. *normal-density* μ σ *x* $*$ (*x* − μ)^(2 $*$ *k* + 1)) = *0*
using *normal-moment-odd* **by** (*rule has-bochner-integral-integral-eq*)

end

context
fixes σ :: *real*
assumes σ-*pos*[*arith*]: $0 < \sigma$
begin

lemma *normal-moment-nz-1*: *has-bochner-integral lborel* (λx. *normal-density* μ σ $x * x$) μ
proof −
note *normal-moment-even*[*OF* σ-*pos*, *of* μ *0*]
note *normal-moment-odd*[*OF* σ-*pos*, *of* μ *0*] *has-bochner-integral-mult-left*[*of* μ, *OF this*]
note *has-bochner-integral-add*[*OF this*]
then show *?thesis*
by (*simp add*: *power2-eq-square field-simps*)
qed

lemma *integral-normal-moment-nz-1*:
$integral^L$ *lborel* (λx. *normal-density* μ σ $x * x$) = μ
using *normal-moment-nz-1* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integrable-normal-moment-nz-1*: *integrable lborel* (λx. *normal-density* μ σ $x * x$)
using *normal-moment-nz-1* **by** *rule*

lemma *integrable-normal-moment*: *integrable lborel* (λx. *normal-density* μ σ $x *$ $(x - \mu)\hat{}k$)
proof *cases*
assume *even k* **then show** *?thesis*
using *integrable.intros*[*OF normal-moment-even*] **by** (*auto elim*: *evenE*)
next
assume *odd k* **then show** *?thesis*
using *integrable.intros*[*OF normal-moment-odd*] **by** (*auto elim*: *oddE*)
qed

lemma *integrable-normal-moment-abs*: *integrable lborel* (λx. *normal-density* μ σ x $* |x - \mu|\hat{}k$)
proof *cases*
assume *even k* **then show** *?thesis*
using *integrable.intros*[*OF normal-moment-even*] **by** (*auto simp add*: *power-even-abs elim*: *evenE*)
next
assume *odd k* **then show** *?thesis*
using *integrable.intros*[*OF normal-moment-abs-odd*] **by** (*auto elim*: *oddE*)
qed

lemma *integrable-normal-density*[*simp*, *intro*]: *integrable lborel* (*normal-density* μ σ)
using *integrable-normal-moment*[*of* μ *0*] **by** *simp*

lemma *integral-normal-density*[*simp*]: ($\int$ *x. normal-density* μ σ *x* ∂*lborel*) = *1*
using *integral-normal-moment-even*[*of* σ μ *0*] **by** *simp*

lemma *prob-space-normal-density*:
prob-space (*density lborel* (*normal-density* μ σ))
proof qed (*simp add*: *emeasure-density nn-integral-eq-integral normal-density-nonneg*)

end

context
fixes *k* :: *nat*
begin

lemma *std-normal-moment-even*:
has-bochner-integral lborel (λ*x. std-normal-density x* * *x* ^ (*2* * *k*)) (*fact* (*2* * *k*) / (*2*^*k* * *fact k*))
using *normal-moment-even*[*of 1 0 k*] **by** *simp*

lemma *std-normal-moment-abs-odd*:
has-bochner-integral lborel (λ*x. std-normal-density x* * |*x*|^(*2* * *k* + *1*)) (*sqrt* (*2*/*pi*) * *2*^*k* * *fact k*)
using *normal-moment-abs-odd*[*of 1 0 k*] **by** (*simp add*: *ac-simps*)

lemma *std-normal-moment-odd*:
has-bochner-integral lborel (λ*x. std-normal-density x* * *x*^(*2* * *k* + *1*)) *0*
using *normal-moment-odd*[*of 1 0 k*] **by** *simp*

lemma *integral-std-normal-moment-even*:
integral^L *lborel* (λ*x. std-normal-density x* * *x*^(*2**k)) = *fact* (*2* * *k*) / (*2*^*k* * *fact k*)
using *std-normal-moment-even* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integral-std-normal-moment-abs-odd*:
integral^L *lborel* (λ*x. std-normal-density x* * |*x*|^(*2* * *k* + *1*)) = *sqrt* (*2* / *pi*) * *2*^*k* * *fact k*
using *std-normal-moment-abs-odd* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integral-std-normal-moment-odd*:
integral^L *lborel* (λ*x. std-normal-density x* * *x*^(*2* * *k* + *1*)) = *0*
using *std-normal-moment-odd* **by** (*rule has-bochner-integral-integral-eq*)

lemma *integrable-std-normal-moment-abs*: *integrable lborel* (λ*x. std-normal-density x* * |*x*|^*k*)

using *integrable-normal-moment-abs*[*of 1 0 k*] **by** *simp*

lemma *integrable-std-normal-moment*: *integrable lborel* (λ*x*. *std-normal-density x* ∗ *x*^*k*)
using *integrable-normal-moment*[*of 1 0 k*] **by** *simp*

end

lemma (**in** *prob-space*) *normal-density-affine*:
assumes *X*: *distributed M lborel X* (*normal-density* μ σ)
assumes [*simp*, *arith*]: *0* < σ α ≠ *0*
shows *distributed M lborel* (λ*x*. β + α ∗ *X x*) (*normal-density* (β + α ∗ μ) (|α| ∗ σ))
proof −
have *eq*: ⋀*x*. |α| ∗ *normal-density* (β + α ∗ μ) (|α| ∗ σ) (*x* ∗ α + β) = *normal-density* μ σ *x*
by (*simp add*: *normal-density-def real-sqrt-mult field-simps*)
(*simp add*: *power2-eq-square field-simps*)
show *?thesis*
by (*rule distributed-affineI*[*OF* - ‹α ≠ *0*›, **where** *t*=β])
(*simp-all add*: *eq X ennreal-mult'*[*symmetric*])
qed

lemma (**in** *prob-space*) *normal-standard-normal-convert*:
assumes *pos-var*[*simp*, *arith*]: *0* < σ
shows *distributed M lborel X* (*normal-density* μ σ) = *distributed M lborel* (λ*x*. (*X x* − μ) / σ) *std-normal-density*
proof *auto*
assume *distributed M lborel X* (λ*x*. *ennreal* (*normal-density* μ σ *x*))
then have *distributed M lborel* (λ*x*. −μ / σ + (*1*/σ) ∗ *X x*) (λ*x*. *ennreal* (*normal-density* (−μ / σ + (*1*/σ)∗ μ) (|*1*/σ| ∗ σ) *x*))
by(*rule normal-density-affine*) *auto*

then show *distributed M lborel* (λ*x*. (*X x* − μ) / σ) (λ*x*. *ennreal* (*std-normal-density x*))
by (*simp add*: *diff-divide-distrib*[*symmetric*] *field-simps*)
next
assume ∗: *distributed M lborel* (λ*x*. (*X x* − μ) / σ) (λ*x*. *ennreal* (*std-normal-density x*))
have *distributed M lborel* (λ*x*. μ + σ ∗ ((*X x* − μ) / σ)) (λ*x*. *ennreal* (*normal-density* μ |σ| *x*))
using *normal-density-affine*[*OF* ∗, *of* σ μ] **by** *simp*
then show *distributed M lborel X* (λ*x*. *ennreal* (*normal-density* μ σ *x*)) **by** *simp*
qed

lemma *conv-normal-density-zero-mean*:
assumes [*simp*, *arith*]: *0* < σ *0* < τ
shows (λ*x*. ∫⁺*y*. *ennreal* (*normal-density 0* σ (*x* − *y*) ∗ *normal-density 0* τ *y*) ∂*lborel*) =

normal-density 0 (sqrt ($\sigma^2 + \tau^2$)) (**is** *?LHS = ?RHS*)
proof −
define σ' τ' **where** $\sigma' = \sigma^2$ **and** $\tau' = \tau^2$
then have [*simp*, *arith*]: $0 < \sigma'$ $0 < \tau'$
by *simp-all*
let *?σ = sqrt* ((*σ′ * τ′*) / (*σ′ + τ′*))
have *sqrt*: (*sqrt* (*2 * pi * (σ′ + τ′*)) ** sqrt* (*2 * pi * (σ′ * τ′*) / (*σ′ + τ′*))) =
(*sqrt* (*2 * pi * σ′*) ** sqrt* (*2 * pi * τ′*))
by (*subst power-eq-iff-eq-base*[*symmetric*, **where** *n=2*])
(*simp-all add*: *real-sqrt-mult*[*symmetric*] *power2-eq-square*)
have *?LHS* =
(*λx.* $\int^+ y.$ *ennreal*((*normal-density 0* (*sqrt* (*σ′ + τ′*)) *x*) ** normal-density* (*τ′ * x* / (*σ′ + τ′*)) *?σ y*) *∂lborel*)
apply (*intro ext nn-integral-cong*)
apply (*simp add*: *normal-density-def σ′-def*[*symmetric*] *τ′-def*[*symmetric*] *sqrt mult-exp-exp*)
apply (*simp add*: *divide-simps power2-eq-square*)
apply (*simp add*: *algebra-simps*)
done

also have ... =
(*λx.* (*normal-density 0* (*sqrt* ($\sigma^2 + \tau^2$)) *x*) *** $\int^+ y.$ *ennreal*(*normal-density* (τ^2** *x* / ($\sigma^2 + \tau^2$)) *?σ y*) *∂lborel*)
by (*subst nn-integral-cmult*[*symmetric*])
(*auto simp*: *σ′-def τ′-def normal-density-def ennreal-mult′*[*symmetric*])

also have ... = (*λx.* (*normal-density 0* (*sqrt* ($\sigma^2 + \tau^2$)) *x*))
by (*subst nn-integral-eq-integral*) (*auto simp*: *normal-density-nonneg*)

finally show *?thesis* **by** *fast*
qed

lemma *conv-std-normal-density*:
(*λx.* $\int^+ y.$ *ennreal* (*std-normal-density* (*x − y*) ** std-normal-density y*) *∂lborel*)
=
(*normal-density 0* (*sqrt 2*))
by (*subst conv-normal-density-zero-mean*) *simp-all*

lemma (**in** *prob-space*) *add-indep-normal*:
assumes *indep*: *indep-var borel X borel Y*
assumes *pos-var*[*arith*]: $0 < \sigma$ $0 < \tau$
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density μ σ*)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density ν τ*)
shows *distributed M lborel* (*λx. X x + Y x*) (*normal-density* (*μ + ν*) (*sqrt* (σ^2 + τ^2)))
proof −
have *ind*[*simp*]: *indep-var borel* (*λx. − μ + X x*) *borel* (*λx. − ν + Y x*)
proof −
have *indep-var borel* ((*λx.* $-\mu + x$) *o X*) *borel* ((*λx. − ν + x*) *o Y*)

by (*auto intro*!: *indep-var-compose assms*)
then show *?thesis* **by** (*simp add*: *o-def*)
qed

have *distributed M lborel* (λ*x*. −*μ* + 1 ∗ *X x*) (*normal-density* (− *μ* + 1 ∗ *μ*) (|1| ∗ *σ*))
by(*rule normal-density-affine*[*OF normalX pos-var*(1), *of* 1 −*μ*]) *simp*
then have *1*[*simp*]: *distributed M lborel* (λ*x*. − *μ* + *X x*) (*normal-density 0 σ*) **by** *simp*

have *distributed M lborel* (λ*x*. −*ν* + 1 ∗ *Y x*) (*normal-density* (− *ν* + 1 ∗ *ν*) (|1| ∗ *τ*))
by(*rule normal-density-affine*[*OF normalY pos-var*(2), *of* 1 −*ν*]) *simp*
then have *2*[*simp*]: *distributed M lborel* (λ*x*. − *ν* + *Y x*) (*normal-density 0 τ*) **by** *simp*

have ∗: *distributed M lborel* (λ*x*. (− *μ* + *X x*) + (− *ν* + *Y x*)) (λ*x*. *ennreal* (*normal-density 0* (*sqrt* (*σ*² + *τ*²)) *x*))
using *distributed-convolution*[*OF ind* 1 2] *conv-normal-density-zero-mean*[*OF pos-var*]
by (*simp add*: *ennreal-mult'*[*symmetric*] *normal-density-nonneg*)

have *distributed M lborel* (λ*x*. *μ* + *ν* + 1 ∗ (− *μ* + *X x* + (− *ν* + *Y x*)))
(λ*x*. *ennreal* (*normal-density* (*μ* + *ν* + 1 ∗ 0) (|1| ∗ *sqrt* (*σ*² + *τ*²)) *x*))
by (*rule normal-density-affine*[*OF* ∗, *of* 1 *μ* + *ν*]) (*auto simp*: *add-pos-pos*)

then show *?thesis* **by** *auto*
qed

lemma (**in** *prob-space*) *diff-indep-normal*:
assumes *indep*[*simp*]: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: 0 < *σ* 0 < *τ*
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density μ σ*)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density ν τ*)
shows *distributed M lborel* (λ*x*. *X x* − *Y x*) (*normal-density* (*μ* − *ν*) (*sqrt* (*σ*² + *τ*²)))
proof −
have *distributed M lborel* (λ*x*. 0 + − 1 ∗ *Y x*) (λ*x*. *ennreal* (*normal-density* (0 + − 1 ∗ *ν*) (|− 1| ∗ *τ*) *x*))
by(*rule normal-density-affine*, *auto*)
then have [*simp*]:*distributed M lborel* (λ*x*. − *Y x*) (λ*x*. *ennreal* (*normal-density* (− *ν*) *τ x*)) **by** *simp*

have *distributed M lborel* (λ*x*. *X x* + (− *Y x*)) (*normal-density* (*μ* + − *ν*) (*sqrt* (*σ*² + *τ*²)))
apply (*rule add-indep-normal*)
apply (*rule indep-var-compose*[*unfolded comp-def*, *of borel - borel - λx*. *x - λx*. − *x*])
apply *simp-all*

done
then show *?thesis* **by** *simp*
qed

lemma (**in** *prob-space*) *sum-indep-normal*:
assumes *finite I I* ≠ {} *indep-vars* (λ*i*. *borel*) *X I*
assumes ⋀*i*. *i* ∈ *I* ⟹ *0* < σ *i*
assumes *normal*: ⋀*i*. *i* ∈ *I* ⟹ *distributed M lborel* (*X i*) (*normal-density* (μ *i*) (σ *i*))
shows *distributed M lborel* (λ*x*. ∑ *i*∈*I*. *X i x*) (*normal-density* (∑ *i*∈*I*. μ *i*) (*sqrt* (∑ *i*∈*I*. (σ *i*)²)))
using *assms*
proof (*induct I rule*: *finite-ne-induct*)
case (*singleton i*) **then show** *?case* **by** (*simp add* : *abs-of-pos*)
next
case (*insert i I*)
then have *1*: *distributed M lborel* (λ*x*. (*X i x*) + (∑ *i*∈*I*. *X i x*))
(*normal-density* (μ *i* + *sum* μ *I*) (*sqrt* ((σ *i*)² + (*sqrt* (∑ *i*∈*I*. (σ *i*)²))²)))
apply (*intro add-indep-normal indep-vars-sum insert real-sqrt-gt-zero*)
apply (*auto intro*: *indep-vars-subset intro*!: *sum-pos*)
apply *fastforce*
done
have *2*: (λ*x*. (*X i x*)+ (∑ *i*∈*I*. *X i x*)) = (λ*x*. (∑ *j*∈*insert i I*. *X j x*))
μ *i* + *sum* μ *I* = *sum* μ (*insert i I*)
using *insert* **by** *auto*

have *3*: (*sqrt* ((σ *i*)² + (*sqrt* (∑ *i*∈*I*. (σ *i*)²))²)) = (*sqrt* (∑ *i*∈*insert i I*. (σ *i*)²))
using *insert* **by** (*simp add*: *sum-nonneg*)

show *?case* **using** *1 2 3* **by** *simp*
qed

lemma (**in** *prob-space*) *standard-normal-distributed-expectation*:
assumes *D*: *distributed M lborel X std-normal-density*
shows *expectation X* = *0*
using *integral-std-normal-moment-odd*[*of 0*]
distributed-integral[*OF D*, *of* λ*x*. *x*, *symmetric*]
by *auto*

lemma (**in** *prob-space*) *normal-distributed-expectation*:
assumes σ[*arith*]: *0* < σ
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *expectation X* = μ
using *integral-normal-moment-nz-1*[*OF* σ, *of* μ] *distributed-integral*[*OF D*, *of* λ*x*. *x*, *symmetric*]
by (*auto simp*: *field-simps*)

lemma (**in** *prob-space*) *normal-distributed-variance*:
fixes *a b* :: *real*
assumes [*simp*, *arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *variance X* $= \sigma^2$
using *integral-normal-moment-even*[*of* σ μ *1*]
by (*subst distributed-integral*[*OF D*, *symmetric*])
(*simp-all add*: *eval-nat-numeral normal-distributed-expectation*[*OF assms*])

lemma (**in** *prob-space*) *standard-normal-distributed-variance*:
distributed M lborel X std-normal-density $\Longrightarrow$ *variance X* $= 1$
using *normal-distributed-variance*[*of 1 X 0*] **by** *simp*

lemma (**in** *information-space*) *entropy-normal-density*:
assumes [*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *entropy b lborel X* $=$ *log b* $(2 * pi * exp\ 1 * \sigma^2) / 2$
proof −
have *entropy b lborel X* $= -$ $(\int x.$ *normal-density* μ σ $x *$ *log b* (*normal-density* μ σ x) ∂*lborel*)
using *D* **by** (*rule entropy-distr*) *simp*
also have ... $= -$ $(\int x.$ *normal-density* μ σ $x * (-$ *ln* $(2 * pi * \sigma^2) -$ $(x - \mu)^2 / \sigma^2) /$ $(2 *$ *ln b*) ∂*lborel*)
by (*intro arg-cong*[**where** *f*=*uminus*] *Bochner-Integration.integral-cong*)
(*auto simp*: *normal-density-def field-simps ln-mult log-def ln-div ln-sqrt*)
also have ... $= -$ $(\int x. -$ (*normal-density* μ σ $x *$ (*ln* $(2 * pi * \sigma^2)) +$ (*normal-density* μ σ $x *$ $(x - \mu)^2) / \sigma^2) /$ $(2 *$ *ln b*) ∂*lborel*)
by (*intro arg-cong*[**where** *f*=*uminus*] *Bochner-Integration.integral-cong*) (*auto simp*: *field-split-simps field-simps*)
also have ... $=$ $(\int x.$ *normal-density* μ σ $x *$ (*ln* $(2 * pi * \sigma^2)) +$ (*normal-density* μ σ $x *$ $(x - \mu)^2) / \sigma^2$ ∂*lborel*) / $(2 *$ *ln b*)
by (*simp del*: *minus-add-distrib*)
also have ... $=$ (*ln* $(2 * pi * \sigma^2) + 1) /$ $(2 *$ *ln b*)
using *integral-normal-moment-even*[*of* σ μ *1*] **by** (*simp add*: *integrable-normal-moment fact-numeral*)
also have ... $=$ *log b* $(2 * pi * exp\ 1 * \sigma^2) / 2$
by (*simp add*: *log-def field-simps ln-mult*)
finally show *?thesis* **.**
qed

end

14 Characteristic Functions

theory *Characteristic-Functions*
imports *Weak-Convergence Independent-Family Distributions*
begin

lemma *mult-min-right*: $a \geq 0 \Longrightarrow$ $(a ::$ *real*$) *$ *min b c* $=$ *min* $(a * b)$ $(a * c)$

by (*metis min.absorb-iff2 min-def mult-left-mono*)

lemma *sequentially-even-odd*:
assumes *E*: *eventually* (λ*n*. *P* (*2* ∗ *n*)) *sequentially* **and** *O*: *eventually* (λ*n*. *P* (*2* ∗ *n* + *1*)) *sequentially*
shows *eventually P sequentially*
proof −
from *E* **obtain** *n-e* **where** [*intro*]: ⋀*n*. *n* ≥ *n-e* ⟹ *P* (*2* ∗ *n*)
by (*auto simp*: *eventually-sequentially*)
moreover
from *O* **obtain** *n-o* **where** [*intro*]: ⋀*n*. *n* ≥ *n-o* ⟹ *P* (*Suc* (*2* ∗ *n*))
by (*auto simp*: *eventually-sequentially*)
show *?thesis*
unfolding *eventually-sequentially*
proof (*intro exI allI impI*)
fix *n* **assume** *max* (*2* ∗ *n-e*) (*2* ∗ *n-o* + *1*) ≤ *n* **then show** *P n*
by (*cases even n*) (*auto elim*!: *evenE oddE*)
qed
qed

lemma *limseq-even-odd*:
assumes (λ*n*. *f* (*2* ∗ *n*)) ⟶ (*l* :: *'a* :: *topological-space*)
and (λ*n*. *f* (*2* ∗ *n* + *1*)) ⟶ *l*
shows *f* ⟶ *l*
using *assms* **by** (*auto simp*: *filterlim-iff intro*: *sequentially-even-odd*)

14.1 Application of the FTC: integrating $e^i x$

abbreviation *iexp* :: *real* ⇒ *complex* **where**
iexp ≡ (λ*x*. *exp* (i ∗ *complex-of-real x*))

lemma *isCont-iexp* [*simp*]: *isCont iexp x*
by (*intro continuous-intros*)

lemma *has-vector-derivative-iexp*[*derivative-intros*]:
(*iexp has-vector-derivative* i ∗ *iexp x*) (*at x within s*)
by (*auto intro*!: *derivative-eq-intros simp*: *Re-exp Im-exp has-vector-derivative-complex-iff*)

lemma *interval-integral-iexp*:
fixes *a b* :: *real*
shows (*CLBINT x*=*a*..*b*. *iexp x*) = i ∗ *iexp a* − i ∗ *iexp b*
by (*subst interval-integral-FTC-finite* [**where** *F* = λ*x*. −i ∗ *iexp x*])
(*auto intro*!: *derivative-eq-intros continuous-intros*)

14.2 The Characteristic Function of a Real Measure.

definition
char :: *real measure* ⇒ *real* ⇒ *complex*
where *char M t* ≡ *CLINT x*|*M*. *iexp* (*t* ∗ *x*)

lemma (**in** *real-distribution*) *char-zero*: *char M 0 = 1*
unfolding *char-def* **by** (*simp del*: *space-eq-univ add*: *prob-space*)

lemma (**in** *prob-space*) *integrable-iexp*:
assumes *f*: *f* ∈ *borel-measurable M* ⋀*x*. *Im* (*f x*) = *0*
shows *integrable M* (λ*x*. *exp* (i ∗ (*f x*)))
proof (*intro integrable-const-bound* [*of - 1*])
from *f* **have** ⋀*x*. *of-real* (*Re* (*f x*)) = *f x*
by (*simp add*: *complex-eq-iff*)
then show *AE x in M*. *cmod* (*exp* (i ∗ *f x*)) ≤ *1*
using *norm-exp-i-times*[*of Re* (*f x*) **for** *x*] **by** *simp*
qed (*insert f*, *simp*)

lemma (**in** *real-distribution*) *cmod-char-le-1*: *norm* (*char M t*) ≤ *1*
proof −
have *norm* (*char M t*) ≤ (∫ *x*. *norm* (*iexp* (*t* ∗ *x*)) ∂*M*)
unfolding *char-def* **by** (*intro integral-norm-bound*)
also have . . . ≤ *1*
by (*simp del*: *of-real-mult*)
finally show *?thesis* **.**
qed

lemma (**in** *real-distribution*) *isCont-char*: *isCont* (*char M*) *t*
unfolding *continuous-at-sequentially*
proof *safe*
fix *X* **assume** *X*: *X* ⟶ *t*
show (*char M* ∘ *X*) ⟶ *char M t*
unfolding *comp-def char-def*
by (*rule integral-dominated-convergence*[**where** *w*=λ-. *1*]) (*auto intro*!: *tendsto-intros X*)
qed

lemma (**in** *real-distribution*) *char-measurable* [*measurable*]: *char M* ∈ *borel-measurable borel*
by (*auto intro*!: *borel-measurable-continuous-onI continuous-at-imp-continuous-on isCont-char*)

14.3 Independence

lemma (**in** *prob-space*) *char-distr-add*:
fixes *X1 X2* :: *'a* ⇒ *real* **and** *t* :: *real*
assumes *indep-var borel X1 borel X2*
shows *char* (*distr M borel* (λω. *X1* ω + *X2* ω)) *t* =
char (*distr M borel X1*) *t* ∗ *char* (*distr M borel X2*) *t*
proof −
from *assms* **have** [*measurable*]: *random-variable borel X1* **by** (*elim indep-var-rv1*)
from *assms* **have** [*measurable*]: *random-variable borel X2* **by** (*elim indep-var-rv2*)

have *char* (*distr M borel* (λω. *X1* ω + *X2* ω)) *t* = (*CLINT x*|*M*. *iexp* (*t* ∗ (*X1*

x + X2 x)))
by (*simp add*: *char-def integral-distr*)
also have ... = (*CLINT x*|*M*. *iexp* (*t* ∗ (*X1 x*)) ∗ *iexp* (*t* ∗ (*X2 x*)))
by (*simp add*: *field-simps exp-add*)
also have ... = (*CLINT x*|*M*. *iexp* (*t* ∗ (*X1 x*))) ∗ (*CLINT x*|*M*. *iexp* (*t* ∗ (*X2 x*)))
by (*intro indep-var-lebesgue-integral indep-var-compose*[*unfolded comp-def*, *OF assms*])
(*auto intro*!: *integrable-iexp*)
also have ... = *char* (*distr M borel X1*) *t* ∗ *char* (*distr M borel X2*) *t*
by (*simp add*: *char-def integral-distr*)
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *char-distr-sum*:
indep-vars (λ*i*. *borel*) *X A* ⟹
char (*distr M borel* (λω. ∑ *i*∈*A*. *X i* ω)) *t* = (∏ *i*∈*A*. *char* (*distr M borel* (*X i*)) *t*)
proof (*induct A rule*: *infinite-finite-induct*)
case (*insert x F*) **with** *indep-vars-subset*[*of* λ-. *borel X insert x F F*] **show** *?case*
by (*auto simp add*: *char-distr-add indep-vars-sum*)
qed (*simp-all add*: *char-def integral-distr prob-space del*: *distr-const*)

14.4 Approximations to e^{ix}

Proofs from Billingsley, page 343.

lemma *CLBINT-I0c-power-mirror-iexp*:
fixes *x* :: *real* **and** *n* :: *nat*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows (*CLBINT s*=*0*..*x*. *f s n* ∗ *iexp s*) =
x^*Suc n* / *Suc n* + (i / *Suc n*) ∗ (*CLBINT s*=*0*..*x*. *f s* (*Suc n*) ∗ *iexp s*)
proof −
have *1*:
((λ*s*. *complex-of-real*(−((*x* − *s*) ^ (*Suc n*) / (*Suc n*))) ∗ *iexp s*)
has-vector-derivative complex-of-real((*x* − *s*)^*n*) ∗ *iexp s* + (i ∗ *iexp s*) ∗ *complex-of-real*(−((*x* − *s*) ^ (*Suc n*) / (*Suc n*))))
(*at s within A*) **for** *s A*
by (*intro derivative-eq-intros*) *auto*

let *?F* = λ*s*. *complex-of-real*(−((*x* − *s*) ^ (*Suc n*) / (*Suc n*))) ∗ *iexp s*
have *x*^(*Suc n*) / (*Suc n*) = (*CLBINT s*=*0*..*x*. (*f s n* ∗ *iexp s* + (i ∗ *iexp s*) ∗ −(*f s* (*Suc n*) / (*Suc n*)))) (**is** *?LHS* = *?RHS*)
proof −
have *?RHS* = (*CLBINT s*=*0*..*x*. (*f s n* ∗ *iexp s* + (i ∗ *iexp s*) ∗
complex-of-real(−((*x* − *s*) ^ (*Suc n*) / (*Suc n*)))))
by (*cases* 0 ≤ *x*) (*auto intro*!: *simp*: *f-def*[*abs-def*])
also have ... = *?F x* − *?F* 0
unfolding *zero-ereal-def* **using** *1*
by (*intro interval-integral-FTC-finite*)

```
      (auto simp: f-def add-nonneg-eq-0-iff complex-eq-iff
          intro!: continuous-at-imp-continuous-on continuous-intros)
   finally show ?thesis
     by auto
  qed
  show ?thesis
   unfolding ‹?LHS = ?RHS› f-def interval-lebesgue-integral-mult-right [symmetric]
    by (subst interval-lebesgue-integral-add(2) [symmetric])
       (auto intro!: interval-integrable-isCont continuous-intros simp: zero-ereal-def
complex-eq-iff)
qed

lemma iexp-eq1:
  fixes x :: real
  defines f s m ≡ complex-of-real ((x − s) ^ m)
  shows iexp x =
   (∑ k ≤ n. (i * x)^k / (fact k)) + ((i ^ (Suc n)) / (fact n)) * (CLBINT s=0..x.
(f s n) * (iexp s)) (is ?P n)
proof (induction n)
  show ?P 0
    by (auto simp add: field-simps interval-integral-iexp f-def zero-ereal-def)
next
  fix n assume ih: ?P n
  have **: ⋀a b :: real. a = −b ⟷ a + b = 0
    by linarith
  have *: of-nat n * of-nat (fact n) ≠ − (of-nat (fact n)::complex)
    unfolding of-nat-mult[symmetric]
   by (simp add: complex-eq-iff ** of-nat-add[symmetric] del: of-nat-mult of-nat-fact
of-nat-add)
  show ?P (Suc n)
    unfolding sum.atMost-Suc ih f-def CLBINT-I0c-power-mirror-iexp[of - n]
    by (simp add: divide-simps add-eq-0-iff *) (simp add: field-simps)
qed

lemma iexp-eq2:
  fixes x :: real
  defines f s m ≡ complex-of-real ((x − s) ^ m)
   shows iexp x = (∑ k≤Suc n. (i*x)^k/fact k) + i^Suc n/fact n * (CLBINT
s=0..x. f s n*(iexp s − 1))
proof −
  have isCont-f: isCont (λs. f s n) x for n x
    by (auto simp: f-def)
  let ?F = λs. complex-of-real (−((x − s) ^ (Suc n) / real (Suc n)))
  have calc1: (CLBINT s=0..x. f s n * (iexp s − 1)) =
    (CLBINT s=0..x. f s n * iexp s) − (CLBINT s=0..x. f s n)
    unfolding zero-ereal-def
    by (subst interval-lebesgue-integral-diff(2) [symmetric])
       (simp-all add: interval-integrable-isCont isCont-f field-simps)
```

```
  have calc2: (CLBINT s=0..x. f s n) = x^Suc n / Suc n
    unfolding zero-ereal-def
  proof (subst interval-integral-FTC-finite [where a = 0 and b = x and f = λs.
f s n and F = ?F])
    show (?F has-vector-derivative f y n) (at y within {min 0 x..max 0 x}) for y
      unfolding f-def
      by (intro has-vector-derivative-of-real)
       (auto intro!: derivative-eq-intros simp del: power-Suc simp add: add-nonneg-eq-0-iff)
  qed (auto intro: continuous-at-imp-continuous-on isCont-f)

  have calc3: i ^(Suc (Suc n)) / (fact (Suc n)) = (i ^(Suc n) / (fact n)) * (i /
(Suc n))
    by (simp add: field-simps)

  show ?thesis
    unfolding iexp-eq1 [where n = Suc n and x=x] calc1 calc2 calc3 unfolding
f-def
    by (subst CLBINT-I0c-power-mirror-iexp [where n = n]) auto
qed

lemma abs-LBINT-I0c-abs-power-diff:
  |LBINT s=0..x. |(x - s)^n|| = |x ^(Suc n) / (Suc n)|
proof -
 have |LBINT s=0..x. |(x - s)^n|| = |LBINT s=0..x. (x - s)^n|
  proof cases
    assume 0 ≤ x ∨ even n
    then have (LBINT s=0..x. |(x - s)^n|) = LBINT s=0..x. (x - s)^n
        by (auto simp add: zero-ereal-def power-even-abs power-abs min-absorb1
max-absorb2
            intro!: interval-integral-cong)
    then show ?thesis by simp
  next
    assume ¬ (0 ≤ x ∨ even n)
    then have (LBINT s=0..x. |(x - s)^n|) = LBINT s=0..x. -((x - s)^n)
      by (auto simp add: zero-ereal-def power-abs min-absorb1 max-absorb2
              ereal-min[symmetric] ereal-max[symmetric] power-minus-odd[symmetric]
            simp del: ereal-min ereal-max intro!: interval-integral-cong)
    also have ... = - (LBINT s=0..x. (x - s)^n)
      by (subst interval-lebesgue-integral-uminus, rule refl)
    finally show ?thesis by simp
  qed
  also have LBINT s=0..x. (x - s)^n = x^Suc n / Suc n
  proof -
    let ?F = λt. - ((x - t)^(Suc n) / Suc n)
    have LBINT s=0..x. (x - s)^n = ?F x - ?F 0
      unfolding zero-ereal-def
      by (intro interval-integral-FTC-finite continuous-at-imp-continuous-on
            has-real-derivative-iff-has-vector-derivative[THEN iffD1])
       (auto simp del: power-Suc intro!: derivative-eq-intros simp add: add-nonneg-eq-0-iff)
```

also have ... = $x \hat{} (Suc\ n)$ / $(Suc\ n)$ **by** *simp*
finally show *?thesis* **.**
qed
finally show *?thesis* **.**
qed

lemma *iexp-approx1*: *cmod* $(iexp\ x - (\sum k \leq n.\ (\mathrm{i} * x) \hat{} k\ /\ fact\ k)) \leq |x| \hat{} (Suc\ n)\ /\ fact\ (Suc\ n)$
proof −
have $iexp\ x - (\sum k \leq n.\ (\mathrm{i} * x) \hat{} k\ /\ fact\ k) =$
$((\mathrm{i} \hat{} (Suc\ n))\ /\ (fact\ n)) * (CLBINT\ s{=}0..x.\ (x - s) \hat{} n * (iexp\ s))$ (**is** *?t1 = ?t2*)
by (*subst iexp-eq1* [*of* - *n*], *simp add*: *field-simps*)
then have *cmod* (*?t1*) = *cmod* (*?t2*)
by *simp*
also have ... = $(1\ /\ of\text{-}nat\ (fact\ n)) * cmod\ (CLBINT\ s{=}0..x.\ (x - s) \hat{} n * (iexp\ s))$
by (*simp add*: *norm-mult norm-divide norm-power*)
also have ... $\leq (1\ /\ of\text{-}nat\ (fact\ n)) * |LBINT\ s{=}0..x.\ cmod\ ((x - s) \hat{} n * (iexp\ s))|$
by (*intro mult-left-mono interval-integral-norm2*)
(*auto simp*: *zero-ereal-def intro*: *interval-integrable-isCont*)
also have ... $\leq (1\ /\ of\text{-}nat\ (fact\ n)) * |LBINT\ s{=}0..x.\ |(x - s) \hat{} n||$
by (*simp add*: *norm-mult del*: *of-real-diff of-real-power*)
also have ... $\leq (1\ /\ of\text{-}nat\ (fact\ n)) * |x \hat{} (Suc\ n)\ /\ (Suc\ n)|$
by (*simp add*: *abs-LBINT-I0c-abs-power-diff*)
also have $1\ /\ real\text{-}of\text{-}nat\ (fact\ n{::}nat) * |x \hat{} Suc\ n\ /\ real\ (Suc\ n)| =$
$|x| \hat{} Suc\ n\ /\ fact\ (Suc\ n)$
by (*simp add*: *abs-mult power-abs*)
finally show *?thesis* **.**
qed

lemma *iexp-approx2*: *cmod* $(iexp\ x - (\sum k \leq n.\ (\mathrm{i} * x) \hat{} k\ /\ fact\ k)) \leq 2 * |x| \hat{} n\ /\ fact\ n$
proof (*induction n*) — really cases
case (*Suc n*)
have ∗: $\bigwedge a\ b.\ interval\text{-}lebesgue\text{-}integrable\ lborel\ a\ b\ f \Longrightarrow interval\text{-}lebesgue\text{-}integrable\ lborel\ a\ b\ g \Longrightarrow$
$|LBINT\ s{=}a..b.\ f\ s| \leq |LBINT\ s{=}a..b.\ g\ s|$
if *f*: $\bigwedge s.\ 0 \leq f\ s$ **and** *g*: $\bigwedge s.\ f\ s \leq g\ s$ **for** $f\ g :: - \Rightarrow real$
using *order-trans*[*OF f g*] *f g*
unfolding *interval-lebesgue-integral-def interval-lebesgue-integrable-def set-lebesgue-integral-def set-integrable-def*
by (*auto simp*: *integral-nonneg-AE*[*OF AE-I2*] *intro*!: *integral-mono mult-mono*)

have $iexp\ x - (\sum k \leq Suc\ n.\ (\mathrm{i} * x) \hat{} k\ /\ fact\ k) =$
$((\mathrm{i} \hat{} (Suc\ n))\ /\ (fact\ n)) * (CLBINT\ s{=}0..x.\ (x - s) \hat{} n * (iexp\ s - 1))$ (**is** *?t1 = ?t2*)
unfolding *iexp-eq2* [*of x n*] **by** (*simp add*: *field-simps*)

then have *cmod (?t1) = cmod (?t2)*
by *simp*
also have *... = (1 / (fact n)) * cmod (CLBINT s=0..x. (x − s)^n * (iexp s − 1))*
by (*simp add*: *norm-mult norm-divide norm-power*)
also have *... ≤ (1 / (fact n)) * |LBINT s=0..x. cmod ((x − s)^n * (iexp s − 1))|*
by (*intro mult-left-mono interval-integral-norm2*)
(*auto intro*!: *interval-integrable-isCont simp*: *zero-ereal-def*)
also have *... = (1 / (fact n)) * |LBINT s=0..x. abs ((x − s)^n) * cmod((iexp s − 1))|*
by (*simp add*: *norm-mult del*: *of-real-diff of-real-power*)
also have *... ≤ (1 / (fact n)) * |LBINT s=0..x. abs ((x − s)^n) * 2|*
by (*intro mult-left-mono * order-trans* [*OF norm-triangle-ineq4*])
(*auto simp*: *mult-ac zero-ereal-def intro*!: *interval-integrable-isCont*)
also have *... = (2 / (fact n)) * |x ^ (Suc n) / (Suc n)|*
by (*simp add*: *abs-LBINT-I0c-abs-power-diff abs-mult*)
also have *2 / fact n * |x ^ Suc n / real (Suc n)| = 2 * |x| ^ Suc n / (fact (Suc n))*
by (*simp add*: *abs-mult power-abs*)
finally show *?case* **.**
qed (*insert norm-triangle-ineq4*[*of iexp x 1*], *simp*)

lemma (**in** *real-distribution*) *char-approx1*:
assumes *integrable-moments*: ⋀*k*. *k ≤ n ⟹ integrable M (λx. x^k)*
shows *cmod (char M t − (∑ k ≤ n. ((i * t)^k / fact k) * expectation (λx. x^k))) ≤*
(2|t|^n / fact n) * expectation (λx. |x|^n)* (**is** *cmod (char M t − ?t1) ≤ -*)
proof −
have *integ-iexp*: *integrable M (λx. iexp (t * x))*
by (*intro integrable-const-bound*) *auto*

define *c* **where** [*abs-def*]: *c k x = (i * t)^k / fact k * complex-of-real (x^k)* **for** *k x*
have *integ-c*: ⋀*k*. *k ≤ n ⟹ integrable M (λx. c k x)*
unfolding *c-def* **by** (*intro integrable-mult-right integrable-of-real integrable-moments*)

have *k ≤ n ⟹ expectation (c k) = (i*t) ^ k * (expectation (λx. x ^ k)) / fact k* **for** *k*
unfolding *c-def integral-mult-right-zero integral-complex-of-real* **by** *simp*
then have *norm (char M t − ?t1) = norm (char M t − (CLINT x | M. (∑ k ≤ n. c k x)))*
by (*simp add*: *integ-c*)
also have *... = norm ((CLINT x | M. iexp (t * x) − (∑ k ≤ n. c k x)))*
unfolding *char-def* **by** (*subst Bochner-Integration.integral-diff*[*OF integ-iexp*]) (*auto intro*!: *integ-c*)
also have *... ≤ expectation (λx. cmod (iexp (t * x) − (∑ k ≤ n. c k x)))*
by (*intro integral-norm-bound*)
also have *... ≤ expectation (λx. 2 * |t| ^ n / fact n * |x| ^ n)*

proof (*rule integral-mono*)
show *integrable M* (λ*x*. *cmod* (*iexp* (*t* ∗ *x*) − (∑ *k*≤*n*. *c k x*)))
by (*intro integrable-norm Bochner-Integration.integrable-diff integ-iexp Bochner-Integration.integrable-sum integ-c*) *simp*
show *integrable M* (λ*x*. *2* ∗ |*t*| ^ *n* / *fact n* ∗ |*x*| ^ *n*)
unfolding *power-abs*[*symmetric*]
by (*intro integrable-mult-right integrable-abs integrable-moments*) *simp*
show *cmod* (*iexp* (*t* ∗ *x*) − (∑ *k*≤*n*. *c k x*)) ≤ *2* ∗ |*t*| ^ *n* / *fact n* ∗ |*x*| ^ *n* **for** *x*
using *iexp-approx2*[*of t* ∗ *x n*] **by** (*auto simp add*: *abs-mult field-simps c-def*)
qed
finally show *?thesis*
unfolding *integral-mult-right-zero* **.**
qed

lemma (**in** *real-distribution*) *char-approx2*:
assumes *integrable-moments*: ⋀*k*. *k* ≤ *n* ⟹ *integrable M* (λ*x*. *x* ^ *k*)
shows *cmod* (*char M t* − (∑ *k* ≤ *n*. ((i ∗ *t*)^*k* / *fact k*) ∗ *expectation* (λ*x*. *x*^*k*))) ≤
(|*t*|^*n* / *fact* (*Suc n*)) ∗ *expectation* (λ*x*. *min* (*2* ∗ |*x*|^*n* ∗ *Suc n*) (|*t*| ∗ |*x*|^*Suc n*))
(**is** *cmod* (*char M t* − *?t1*) ≤ -)
proof −
have *integ-iexp*: *integrable M* (λ*x*. *iexp* (*t* ∗ *x*))
by (*intro integrable-const-bound*) *auto*

define *c* **where** [*abs-def*]: *c k x* = (i ∗ *t*)^*k* / *fact k* ∗ *complex-of-real* (*x*^*k*) **for** *k x*
have *integ-c*: ⋀*k*. *k* ≤ *n* ⟹ *integrable M* (λ*x*. *c k x*)
unfolding *c-def* **by** (*intro integrable-mult-right integrable-of-real integrable-moments*)

have ∗: *min* (*2* ∗ |*t* ∗ *x*|^*n* / *fact n*) (|*t* ∗ *x*|^*Suc n* / *fact* (*Suc n*)) =
|*t*|^*n* / *fact* (*Suc n*) ∗ *min* (*2* ∗ |*x*|^*n* ∗ *real* (*Suc n*)) (|*t*| ∗ |*x*|^(*Suc n*)) **for** *x*
apply (*subst mult-min-right*)
apply *simp*
apply (*rule arg-cong2*[**where** *f*=*min*])
apply (*simp-all add*: *field-simps abs-mult del*: *fact-Suc*)
apply (*simp-all add*: *field-simps*)
done

have *k* ≤ *n* ⟹ *expectation* (*c k*) = (i∗*t*) ^ *k* ∗ (*expectation* (λ*x*. *x* ^ *k*)) / *fact k* **for** *k*
unfolding *c-def integral-mult-right-zero integral-complex-of-real* **by** *simp*
then have *norm* (*char M t* − *?t1*) = *norm* (*char M t* − (*CLINT x* | *M*. (∑ *k* ≤ *n*. *c k x*)))
by (*simp add*: *integ-c*)
also have ... = *norm* ((*CLINT x* | *M*. *iexp* (*t* ∗ *x*) − (∑ *k* ≤ *n*. *c k x*)))
unfolding *char-def* **by** (*subst Bochner-Integration.integral-diff*[*OF integ-iexp*]) (*auto intro*!: *integ-c*)

also have ... ≤ *expectation* (λx. *cmod* (*iexp* (*t* ∗ *x*) − (∑ *k* ≤ *n*. *c k x*)))
by (*rule integral-norm-bound*)
also have ... ≤ *expectation* (λx. *min* (2 ∗ |*t* ∗ *x*|^*n* / *fact n*) (|*t* ∗ *x*|^(*Suc n*) / *fact* (*Suc n*)))
(**is** - ≤ *expectation ?f*)
proof (*rule integral-mono*)
show *integrable M* (λx. *cmod* (*iexp* (*t* ∗ *x*) − (∑ *k*≤*n*. *c k x*)))
by (*intro integrable-norm Bochner-Integration.integrable-diff integ-iexp Bochner-Integration.integrable-sum integ-c*) *simp*
show *integrable M ?f*
by (*rule Bochner-Integration.integrable-bound*[**where** *f*=λx. 2 ∗ |*t* ∗ *x*| ^ *n* / *fact n*])
(*auto simp*: *integrable-moments power-abs*[*symmetric*] *power-mult-distrib*)
show *cmod* (*iexp* (*t* ∗ *x*) − (∑ *k*≤*n*. *c k x*)) ≤ *?f x* **for** *x*
using *iexp-approx1*[*of t* ∗ *x n*] *iexp-approx2*[*of t* ∗ *x n*]
by (*auto simp add*: *abs-mult field-simps c-def intro*!: *min.boundedI*)
qed
also have ... = (|*t*|^*n* / *fact* (*Suc n*)) ∗ *expectation* (λx. *min* (2 ∗ |*x*|^*n* ∗ *Suc n*) (|*t*| ∗ |*x*|^*Suc n*))
unfolding ∗
proof (*rule Bochner-Integration.integral-mult-right*)
show *integrable M* (λx. *min* (2 ∗ |*x*| ^ *n* ∗ *real* (*Suc n*)) (|*t*| ∗ |*x*| ^ *Suc n*))
by (*rule Bochner-Integration.integrable-bound*[**where** *f*=λx. 2 ∗ |*x*| ^ *n* ∗ *real* (*Suc n*)])
(*auto simp*: *integrable-moments power-abs*[*symmetric*] *power-mult-distrib*)
qed
finally show *?thesis*
unfolding *integral-mult-right-zero* **.**
qed

lemma (**in** *real-distribution*) *char-approx3*:
fixes *t*
assumes
integrable-1: *integrable M* (λx. *x*) **and**
integral-1: *expectation* (λx. *x*) = *0* **and**
integrable-2: *integrable M* (λx. *x*^*2*) **and**
integral-2: *variance* (λx. *x*) = *σ 2*
shows *cmod* (*char M t* − (*1* − *t*^*2* ∗ *σ 2* / *2*)) ≤
(*t*^*2* / *6*) ∗ *expectation* (λx. *min* (*6* ∗ *x*^*2*) (*abs t* ∗ (*abs x*)^*3*))
proof −
note *real-distribution.char-approx2* [*of M 2 t*, *simplified*]
have [*simp*]: *prob UNIV* = *1* **by** (*metis prob-space space-eq-univ*)
from *integral-2* **have** [*simp*]: *expectation* (λx. *x* ∗ *x*) = *σ 2*
by (*simp add*: *integral-1 numeral-eq-Suc*)
have *1*: *k* ≤ *2* ⟹ *integrable M* (λx. *x*^*k*) **for** *k*
using *assms* **by** (*auto simp*: *eval-nat-numeral le-Suc-eq*)
note *char-approx1*
note *2* = *char-approx1* [*of 2 t*, *OF 1*, *simplified*]
have *cmod* (*char M t* − (∑ *k*≤*2*. (i ∗ *t*) ^ *k* ∗ (*expectation* (λx. *x* ^ *k*)) / (*fact*

$k))) \leq$
$t^2 * expectation\ (\lambda x.\ min\ (6 * x^2)\ (|t| * |x| \hat{}\ 3)) / fact\ (3::nat)$
using *char-approx2* [*of 2 t*, *OF 1*] **by** *simp*
also have $(\sum k{\leq}2.\ (\mathrm{i} * t) \hat{}\ k * expectation\ (\lambda x.\ x \hat{}\ k) / (fact\ k)) = 1 - t\hat{}2 * \sigma 2 / 2$
by (*simp add*: *complex-eq-iff numeral-eq-Suc integral-1 Re-divide Im-divide*)
also have *fact 3 = 6* **by** (*simp add*: *eval-nat-numeral*)
also have $t^2 * expectation\ (\lambda x.\ min\ (6 * x^2)\ (|t| * |x| \hat{}\ 3)) / 6 =$
$t^2 / 6 * expectation\ (\lambda x.\ min\ (6 * x^2)\ (|t| * |x| \hat{}\ 3))$ **by** (*simp add*: *field-simps*)
finally show *?thesis* **.**
qed

This is a more familiar textbook formulation in terms of random variables, but we will use the previous version for the CLT.

lemma (**in** *prob-space*) *char-approx3′*:
fixes μ :: *real measure* **and** *X*
assumes *rv-X* [*simp*]: *random-variable borel X*
and [*simp*]: *integrable M X integrable M* $(\lambda x.\ (X\ x)\hat{}2)$ *expectation X = 0*
and *var-X*: *variance X* $= \sigma 2$
and μ*-def*: μ *= distr M borel X*
shows *cmod* (*char* μ *t* $-$ $(1 - t\hat{}2 * \sigma 2 / 2)) \leq$
$(t\hat{}2 / 6) * expectation\ (\lambda x.\ min\ (6 * (X\ x)\hat{}2)\ (|t| * |X\ x|\hat{}3))$
using *var-X* **unfolding** μ*-def*
apply (*subst integral-distr* [*symmetric*, *OF rv-X*], *simp*)
apply (*intro real-distribution.char-approx3*)
apply (*auto simp add*: *integrable-distr-eq integral-distr*)
done

this is the formulation in the book – in terms of a random variable *with* the distribution, rather the distribution itself. I don't know which is more useful, though in principal we can go back and forth between them.

lemma (**in** *prob-space*) *char-approx1′*:
fixes μ :: *real measure* **and** *X*
assumes *integrable-moments* : $\bigwedge k.\ k \leq n \Longrightarrow$ *integrable M* $(\lambda x.\ X\ x \hat{}\ k)$
and *rv-X*[*measurable*]: *random-variable borel X*
and μ*-distr* : *distr M borel X* $= \mu$
shows *cmod* (*char* μ *t* $-$ $(\sum k \leq n.\ ((\mathrm{i} * t)\hat{}k / fact\ k) * expectation\ (\lambda x.\ (X\ x)\hat{}k))) \leq$
$(2 * |t|\hat{}n / fact\ n) * expectation\ (\lambda x.\ |X\ x|\hat{}n)$
unfolding μ*-distr*[*symmetric*]
apply (*subst* (*1 2*) *integral-distr* [*symmetric*, *OF rv-X*], *simp*, *simp*)
apply (*intro real-distribution.char-approx1*[*of distr M borel X n t*] *real-distribution-distr rv-X*)
apply (*auto simp*: *integrable-distr-eq integrable-moments*)
done

14.5 Calculation of the Characteristic Function of the Standard Distribution

abbreviation
std-normal-distribution ≡ *density lborel std-normal-density*

lemma *real-dist-normal-dist*: *real-distribution std-normal-distribution*
using *prob-space-normal-density* **by** (*auto simp*: *real-distribution-def real-distribution-axioms-def*)

lemma *std-normal-distribution-even-moments*:
fixes *k* :: *nat*
shows (*LINT x*|*std-normal-distribution*. *x*^(2 ∗ *k*)) = *fact* (2 ∗ *k*) / (2^*k* ∗ *fact k*)
and *integrable std-normal-distribution* (λ*x*. *x*^(2 ∗ *k*))
using *integral-std-normal-moment-even*[*of k*]
by (*subst integral-density*)
(*auto simp*: *normal-density-nonneg integrable-density*
intro: *integrable.intros std-normal-moment-even*)

lemma *integrable-std-normal-distribution-moment*: *integrable std-normal-distribution* (λ*x*. *x*^*k*)
by (*auto simp*: *normal-density-nonneg integrable-std-normal-moment integrable-density*)

lemma *integral-std-normal-distribution-moment-odd*:
odd k ⟹ *integral*^{L} *std-normal-distribution* (λ*x*. *x*^*k*) = *0*
using *integral-std-normal-moment-odd*[*of* (*k* − *1*) *div 2*]
by (*auto simp*: *integral-density normal-density-nonneg elim*: *oddE*)

lemma *std-normal-distribution-even-moments-abs*:
fixes *k* :: *nat*
shows (*LINT x*|*std-normal-distribution*. |*x*|^(2 ∗ *k*)) = *fact* (2 ∗ *k*) / (2^*k* ∗ *fact k*)
using *integral-std-normal-moment-even*[*of k*]
by (*subst integral-density*) (*auto simp*: *normal-density-nonneg power-even-abs*)

lemma *std-normal-distribution-odd-moments-abs*:
fixes *k* :: *nat*
shows (*LINT x*|*std-normal-distribution*. |*x*|^(2 ∗ *k* + *1*)) = *sqrt* (2 / *pi*) ∗ 2 ^ *k* ∗ *fact k*
using *integral-std-normal-moment-abs-odd*[*of k*]
by (*subst integral-density*) (*auto simp*: *normal-density-nonneg*)

theorem *char-std-normal-distribution*:
char std-normal-distribution = (λ*t*. *complex-of-real* (*exp* (− (*t*^2) / 2)))
proof (*intro ext LIMSEQ-unique*)
fix *t* :: *real*
let *?f'* = λ*k*. (i ∗ *t*)^*k* / *fact k* ∗ (*LINT x* | *std-normal-distribution*. *x*^*k*)
let *?f* = λ*n*. (∑ *k* ≤ *n*. *?f' k*)
show *?f* ⟶ *exp* (−(*t*^2) / 2)

proof (*rule limseq-even-odd*)
have (i ∗ *complex-of-real t*) ^ (*2* ∗ *a*) / (*2* ^ *a* ∗ *fact a*) = (− ((*complex-of-real t*)² / *2*)) ^ *a* / *fact a* **for** *a*
by (*subst power-mult*) (*simp add*: *field-simps uminus-power-if power-mult*)
then have ∗: *?f* (*2* ∗ *n*) = *complex-of-real* (∑ *k* < *Suc n*. (*1* / *fact k*) ∗ (− (*t*^*2*) / *2*)^*k*) **for** *n* :: *nat*
unfolding *of-real-sum*
by (*intro sum.reindex-bij-witness-not-neutral*[*symmetric*, **where**
i=λ*n*. *n div 2* **and** *j*=λ*n*. *2* ∗ *n* **and** *T'*={*i*. *i* ≤ *2* ∗ *n* ∧ *odd i*} **and** *S'*={}])
(*auto simp*: *integral-std-normal-distribution-moment-odd std-normal-distribution-even-moments*)
show (λ*n*. *?f* (*2* ∗ *n*)) ⟶ *exp* (−(*t*^*2*) / *2*)
unfolding ∗ **using** *exp-converges*[**where** *'a*=*real*]
by (*intro tendsto-of-real LIMSEQ-Suc*) (*auto simp*: *inverse-eq-divide sums-def* [*symmetric*])
have ∗∗: *?f* (*2* ∗ *n* + *1*) = *?f* (*2* ∗ *n*) **for** *n*
proof −
have *?f* (*2* ∗ *n* + *1*) = *?f* (*2* ∗ *n*) + *?f'* (*2* ∗ *n* + *1*)
by *simp*
also have *?f'* (*2* ∗ *n* + *1*) = *0*
by (*subst integral-std-normal-distribution-moment-odd*) *simp-all*
finally show *?f* (*2* ∗ *n* + *1*) = *?f* (*2* ∗ *n*)
by *simp*
qed
show (λ*n*. *?f* (*2* ∗ *n* + *1*)) ⟶ *exp* (−(*t*^*2*) / *2*)
unfolding ∗∗ **by** *fact*
qed

have ∗∗: (λ*n*. *x* ^ *n* / *fact n*) ⟶ *0* **for** *x* :: *real*
using *summable-LIMSEQ-zero* [*OF summable-exp*] **by** (*auto simp add*: *inverse-eq-divide*)

let *?F* = λ*n*. *2* ∗ |*t*| ^ *n* / *fact n* ∗ (*LINT x*|*std-normal-distribution*. |*x*| ^ *n*)

show *?f* ⟶ *char std-normal-distribution t*
proof (*rule metric-tendsto-imp-tendsto*[*OF limseq-even-odd*])
show (λ*n*. *?F* (*2* ∗ *n*)) ⟶ *0*
proof (*rule Lim-transform-eventually*)
show ∀ *F n in sequentially*. *2* ∗ ((*t*^*2* / *2*)^*n* / *fact n*) = *?F* (*2* ∗ *n*)
unfolding *std-normal-distribution-even-moments-abs* **by** (*simp add*: *power-mult power-divide*)
qed (*intro tendsto-mult-right-zero* ∗∗)

have ∗: *?F* (*2* ∗ *n* + *1*) = (*2* ∗ |*t*| ∗ *sqrt* (*2* / *pi*)) ∗ ((*2* ∗ *t*^*2*)^*n* ∗ *fact n* / *fact* (*2* ∗ *n* + *1*)) **for** *n*
unfolding *std-normal-distribution-odd-moments-abs*
by (*simp add*: *field-simps power-mult*[*symmetric*] *power-even-abs*)
have *norm* ((*2* ∗ *t*²) ^ *n* ∗ *fact n* / *fact* (*2* ∗ *n* + *1*)) ≤ (*2* ∗ *t*²) ^ *n* / *fact n* **for** *n*

using *mult-mono*[*OF - square-fact-le-2-fact*, *of 1 1 + 2 * real n n*]
by (*auto simp add*: *divide-simps intro*!: *mult-left-mono*)
then show (λ*n*. *?F* (*2 * n + 1*)) ⟶ *0*
unfolding * **by** (*intro tendsto-mult-right-zero Lim-null-comparison* [*OF - ** [*of 2 * t*^{2}]]) *auto*

show ∀$_F$ *n in sequentially*. *dist* (*?f n*) (*char std-normal-distribution t*) ≤ *dist* (*?F n*) *0*
using *real-distribution.char-approx1*[*OF real-dist-normal-dist integrable-std-normal-distribution-moment*]
by (*auto simp*: *dist-norm integral-nonneg-AE norm-minus-commute*)
qed
qed

end

15 Helly's selection theorem

The set of bounded, monotone, right continuous functions is sequentially compact

theory *Helly-Selection*
imports *HOL−Library.Diagonal-Subsequence Weak-Convergence*
begin

lemma *minus-one-less*: *x − 1 < (x::real)*
by *simp*

theorem *Helly-selection*:
fixes *f* :: *nat* ⇒ *real* ⇒ *real*
assumes *rcont*: ⋀*n x*. *continuous* (*at-right x*) (*f n*)
assumes *mono*: ⋀*n*. *mono* (*f n*)
assumes *bdd*: ⋀*n x*. |*f n x*| ≤ *M*
shows ∃ *s*. *strict-mono* (*s::nat* ⇒ *nat*) ∧ (∃ *F*. (∀ *x*. *continuous* (*at-right x*) *F*) ∧ *mono F* ∧ (∀ *x*. |*F x*| ≤ *M*) ∧
(∀ *x*. *continuous* (*at x*) *F* ⟶ (λ*n*. *f* (*s n*) *x*) ⟶ *F x*))
proof −
obtain *m* :: *real* ⇒ *nat* **where** *bij-betw m* ℚ *UNIV*
using *countable-rat Rats-infinite* **by** (*erule countableE-infinite*)
then obtain *r* :: *nat* ⇒ *real* **where** *bij*: *bij-betw r UNIV* ℚ
using *bij-betw-inv* **by** *blast*

have *dense-r*: ⋀*x y*. *x < y* ⟹ ∃ *n*. *x < r n* ∧ *r n < y*
by (*metis Rats-dense-in-real bij f-the-inv-into-f bij-betw-def*)

let *?P* = λ*n*. λ*s*. *convergent* (λ*k*. *f* (*s k*) (*r n*))
interpret *nat*: *subseqs ?P*
proof (*unfold convergent-def*, *unfold subseqs-def*, *auto*)
fix *n* :: *nat* **and** *s* :: *nat* ⇒ *nat* **assume** *s*: *strict-mono s*
have *bounded* {−*M*..*M*}

```
    using bounded-closed-interval by auto
  moreover have ⋀k. f (s k) (r n) ∈ {−M..M}
    using bdd by (simp add: abs-le-iff minus-le-iff)
  ultimately have ∃ l s'. strict-mono s' ∧ ((λk. f (s k) (r n)) ∘ s') ⟶ l
    using compact-Icc compact-imp-seq-compact seq-compactE by metis
  thus ∃ s'. strict-mono (s'::nat⇒nat) ∧ (∃ l. (λk. f (s (s' k)) (r n)) ⟶ l)
    by (auto simp: comp-def)
 qed
 define d where d = nat.diagseq
 have subseq: strict-mono d
  unfolding d-def using nat.subseq-diagseq by auto
 have rat-cnv: ?P n d for n
 proof −
  have Pn-seqseq: ?P n (nat.seqseq (Suc n))
    by (rule nat.seqseq-holds)
  have 1: (λk. f ((nat.seqseq (Suc n) ∘ (λk. nat.fold-reduce (Suc n) k
    (Suc n + k))) k) (r n)) = (λk. f (nat.seqseq (Suc n) k) (r n)) ∘
    (λk. nat.fold-reduce (Suc n) k (Suc n + k))
    by auto
  have 2: ?P n (d ∘ ((+) (Suc n)))
    unfolding d-def nat.diagseq-seqseq 1
    by (intro convergent-subseq-convergent Pn-seqseq nat.subseq-diagonal-rest)
  then obtain L where 3: (λna. f (d (na + Suc n)) (r n)) ⟶ L
    by (auto simp: add.commute dest: convergentD)
  then have (λk. f (d k) (r n)) ⟶ L
    by (rule LIMSEQ-offset)
  then show ?thesis
    by (auto simp: convergent-def)
 qed
 let ?f = λn. λk. f (d k) (r n)
 have lim-f: ?f n ⟶ lim (?f n) for n
  using rat-cnv convergent-LIMSEQ-iff by auto
 have lim-bdd: lim (?f n) ∈ {−M..M} for n
 proof −
  have closed {−M..M} using closed-real-atLeastAtMost by auto
   hence (∀ i. ?f n i ∈ {−M..M}) ∧ ?f n ⟶ lim (?f n) ⟶ lim (?f n) ∈
{−M..M}
   unfolding closed-sequential-limits by (drule-tac x = λk. f (d k) (r n) in spec)
blast
  moreover have ∀ i. ?f n i ∈ {−M..M}
    using bdd by (simp add: abs-le-iff minus-le-iff)
  ultimately show lim (?f n) ∈ {−M..M}
    using lim-f by auto
 qed
 then have limset-bdd: ⋀x. {lim (?f n) |n. x < r n} ⊆ {−M..M}
  by auto
 then have bdd-below: bdd-below {lim (?f n) |n. x < r n} for x
  by (metis (mono-tags) bdd-below-Icc bdd-below-mono)
 have r-unbdd: ∃ n. x < r n for x
```

```
    using dense-r[OF less-add-one, of x] by auto
  then have nonempty: {lim (?f n) |n. x < r n} ≠ {} for x
    by auto

  define F where F x = Inf {lim (?f n) |n. x < r n} for x
  have F-eq: ereal (F x) = (INF n∈{n. x < r n}. ereal (lim (?f n))) for x
     unfolding F-def by (subst ereal-Inf'[OF bdd-below nonempty]) (simp add:
setcompr-eq-image image-comp)
  have mono-F: mono F
     using nonempty by (auto intro!: cInf-superset-mono simp: F-def bdd-below
mono-def)
  moreover have ⋀x. continuous (at-right x) F
   unfolding continuous-within order-tendsto-iff eventually-at-right[OF less-add-one]
  proof safe
    show F x < u ⟹ ∃ b>x. ∀ y>x. y < b ⟶ F y < u for x u
      unfolding F-def cInf-less-iff[OF nonempty bdd-below] by auto
  next
    show ∃ b>x. ∀ y>x. y < b ⟶ l < F y if l: l < F x for x l
    using less-le-trans[OF l mono-F[THEN monoD, of x]] by (auto intro: less-add-one)
  qed
  moreover
  { fix x
    have F x ∈ {−M..M}
    unfolding F-def using limset-bdd bdd-below r-unbdd by (intro closed-subset-contains-Inf)
auto
    then have |F x| ≤ M by auto }
  moreover have (λn. f (d n) x) ⟶⟶ F x if cts: continuous (at x) F for x
  proof (rule limsup-le-liminf-real)
    show limsup (λn. f (d n) x) ≤ F x
    proof (rule tendsto-lowerbound)
      show (F ⟶⟶ ereal (F x)) (at-right x)
        using cts unfolding continuous-at-split by (auto simp: continuous-within)
      show ∀ F i in at-right x. limsup (λn. f (d n) x) ≤ F i
        unfolding eventually-at-right[OF less-add-one]
      proof (rule, rule, rule less-add-one, safe)
        fix y assume y: x < y
        with dense-r obtain N where x < r N r N < y by auto
        have ∗: y < r n' ⟹ lim (?f N) ≤ lim (?f n') for n'
            using ‹r N < y› by (intro LIMSEQ-le[OF lim-f lim-f]) (auto intro!:
mono[THEN monoD])
        have limsup (λn. f (d n) x) ≤ limsup (?f N)
              using ‹x < r N› by (auto intro!: Limsup-mono always-eventually
mono[THEN monoD])
        also have ... = lim (λn. ereal (?f N n))
          using rat-cnv[of N] by (force intro!: convergent-limsup-cl simp: conver-
gent-def)
        also have ... ≤ F y
          by (auto intro!: INF-greatest ∗ simp: convergent-real-imp-convergent-ereal
rat-cnv F-eq)
```

finally show *limsup* $(\lambda n.\ f\ (d\ n)\ x) \leq F\ y$ **.**
qed
qed *simp*
show $F\ x \leq$ *liminf* $(\lambda n.\ f\ (d\ n)\ x)$
proof (*rule tendsto-upperbound*)
show $(F \longrightarrow$ *ereal* $(F\ x))$ (*at-left x*)
using *cts* **unfolding** *continuous-at-split* **by** (*auto simp*: *continuous-within*)
show $\forall_F\ i$ *in at-left x*. $F\ i \leq$ *liminf* $(\lambda n.\ f\ (d\ n)\ x)$
unfolding *eventually-at-left*[*OF minus-one-less*]
proof (*rule*, *rule*, *rule minus-one-less*, *safe*)
fix y **assume** y: $y < x$
with *dense-r* **obtain** N **where** $y < r\ N\ r\ N < x$ **by** *auto*
have $F\ y \leq$ *liminf* (*?f N*)
using ‹$y < r\ N$› **by** (*auto simp*: *F-eq convergent-real-imp-convergent-ereal rat-cnv convergent-liminf-cl intro*!: *INF-lower2*)
also have $\ldots \leq$ *liminf* $(\lambda n.\ f\ (d\ n)\ x)$
using ‹$r\ N < x$› **by** (*auto intro*!: *Liminf-mono monoD*[*OF mono*] *always-eventually*)
finally show $F\ y \leq$ *liminf* $(\lambda n.\ f\ (d\ n)\ x)$ **.**
qed
qed *simp*
qed
ultimately show *?thesis* **using** *subseq* **by** *auto*
qed

definition
tight :: (*nat* ⇒ *real measure*) ⇒ *bool*
where
tight $\mu \equiv (\forall n.$ *real-distribution* $(\mu\ n)) \wedge (\forall (\varepsilon$::*real*$)>0.\ \exists a\ b$::*real*. $a < b \wedge (\forall n.$ *measure* $(\mu\ n)\ \{a<..b\} > 1 - \varepsilon))$

theorem *tight-imp-convergent-subsubsequence*:
assumes μ: *tight* μ *strict-mono s*
shows $\exists r\ M.$ *strict-mono* $(r :: nat \Rightarrow nat) \wedge$ *real-distribution M* $\wedge$ *weak-conv-m* $(\mu \circ s \circ r)\ M$
proof −
define f **where** $f\ k =$ *cdf* $(\mu\ (s\ k))$ **for** k
interpret μ: *real-distribution* $\mu\ k$ **for** k
using μ **unfolding** *tight-def* **by** *auto*

have *rcont*: $\bigwedge x.$ *continuous* (*at-right x*) (*f k*)
and *mono*: *mono* (*f k*)
and *top*: (*f k* ⟶ *1*) *at-top*
and *bot*: (*f k* ⟶ *0*) *at-bot* **for** k
unfolding *f-def mono-def*
using μ*.cdf-nondecreasing* μ*.cdf-is-right-cont* μ*.cdf-lim-at-top-prob* μ*.cdf-lim-at-bot*

by *auto*
have *bdd*: $|f\ k\ x| \leq 1$ **for** $k\ x$
by (*auto simp add*: *abs-le-iff minus-le-iff f-def* μ*.cdf-nonneg* μ*.cdf-bounded-prob*)

from *Helly-selection*[*OF rcont mono bdd*, *of* $\lambda x.\ x$] **obtain** $r\ F$
where F: *strict-mono* r $\bigwedge x.$ *continuous* (*at-right* x) F *mono* F $\bigwedge x.\ |F\ x| \leq 1$
and *lim-F*: $\bigwedge x.$ *continuous* (*at* x) $F \Longrightarrow (\lambda n.\ f\ (r\ n)\ x) \longrightarrow F\ x$
by *blast*

have $0 \leq f\ n\ x$ **for** $n\ x$
unfolding *f-def* **by** (*rule* μ*.cdf-nonneg*)
have *F-nonneg*: $0 \leq F\ x$ **for** x
proof −
obtain y **where** $y < x$ *isCont* $F\ y$
using *open-minus-countable*[*OF mono-ctble-discont*[*OF* ‹*mono* F›], *of* {..< x}] **by** *auto*
then have $0 \leq F\ y$
by (*intro LIMSEQ-le-const*[*OF lim-F*]) (*auto simp*: *f-def* μ*.cdf-nonneg*)
also have $\ldots \leq F\ x$
using ‹$y < x$› **by** (*auto intro*!: *monoD*[*OF* ‹*mono* F›])
finally show $0 \leq F\ x$ **.**
qed

have *Fab*: $\exists\, a\ b.\ (\forall\, x{\geq}b.\ F\ x \geq 1 - \varepsilon) \wedge (\forall\, x{\leq}a.\ F\ x \leq \varepsilon)$ **if** ε: $0 < \varepsilon$ **for** ε
proof *auto*
obtain $a'\ b'$ **where** $a'b'$: $a' < b'$ $\bigwedge k.$ *measure* $(\mu\ k)$ {a'<..b'} $> 1 - \varepsilon$
using ε μ **by** (*auto simp*: *tight-def*)
obtain a **where** a: $a < a'$ *isCont* $F\ a$
using *open-minus-countable*[*OF mono-ctble-discont*[*OF* ‹*mono* F›], *of* {..< a'}] **by** *auto*
obtain b **where** b: $b' < b$ *isCont* $F\ b$
using *open-minus-countable*[*OF mono-ctble-discont*[*OF* ‹*mono* F›], *of* {b' <..}] **by** *auto*
have $a < b$
using $a\ b\ a'b'$ **by** *simp*

let $?\mu = \lambda k.$ *measure* $(\mu\ (s\ (r\ k)))$
have *ab*: $?\mu\ k$ {a<..b} $> 1 - \varepsilon$ **for** k
proof −
have $?\mu\ k$ {a'<..b'} $\leq ?\mu\ k$ {a<..b}
using $a\ b$ **by** (*intro* μ*.finite-measure-mono*) *auto*
then show *?thesis*
using $a'b'(2)$ **by** (*metis less-eq-real-def less-trans*)
qed

have $(\lambda k.\ ?\mu\ k$ {..b}) $\longrightarrow F\ b$
using $b(2)$ *lim-F* **unfolding** *f-def cdf-def o-def* **by** *auto*
then have $1 - \varepsilon \leq F\ b$
proof (*rule tendsto-lowerbound*, *intro always-eventually allI*)

fix k
have $1 - \varepsilon < ?\mu\ k\ \{a<..b\}$
using *ab* **by** *auto*
also have $\ldots \leq ?\mu\ k\ \{..b\}$
by (*auto intro*!: *μ.finite-measure-mono*)
finally show $1 - \varepsilon \leq ?\mu\ k\ \{..b\}$
by (*rule less-imp-le*)
qed (*rule sequentially-bot*)
then show $\exists\, b.\ \forall\, x{\geq}b.\ 1 - \varepsilon \leq F\ x$
using F **unfolding** *mono-def* **by** (*metis order.trans*)

have $(\lambda k.\ ?\mu\ k\ \{..a\}) \longrightarrow F\ a$
using *a*(*2*) *lim-F* **unfolding** *f-def cdf-def o-def* **by** *auto*
then have *Fa*: $F\ a \leq \varepsilon$
proof (*rule tendsto-upperbound*, *intro always-eventually allI*)
fix k
have $?\mu\ k\ \{..a\} + ?\mu\ k\ \{a<..b\} \leq 1$
by (*subst μ.finite-measure-Union*[*symmetric*]) *auto*
then show $?\mu\ k\ \{..a\} \leq \varepsilon$
using *ab*[*of k*] **by** *simp*
qed (*rule sequentially-bot*)
then show $\exists\, a.\ \forall\, x{\leq}a.\ F\ x \leq \varepsilon$
using F **unfolding** *mono-def* **by** (*metis order.trans*)
qed

have $(F \longrightarrow 1)$ *at-top*
proof (*rule order-tendstoI*)
show $1 < y \Longrightarrow \forall_F\ x$ *in at-top*. $F\ x < y$ **for** y
using ‹$\bigwedge x.\ |F\ x| \leq 1$› ‹$\bigwedge x.\ 0 \leq F\ x$› **by** (*auto intro*: *le-less-trans always-eventually*)
fix y :: *real* **assume** $y < 1$
then obtain z **where** $y < z\ z < 1$
using *dense*[*of y 1*] **by** *auto*
with *Fab*[*of 1 − z*] **show** $\forall_F\ x$ *in at-top*. $y < F\ x$
by (*auto simp*: *eventually-at-top-linorder intro*: *less-le-trans*)
qed
moreover
have $(F \longrightarrow 0)$ *at-bot*
proof (*rule order-tendstoI*)
show $y < 0 \Longrightarrow \forall_F\ x$ *in at-bot*. $y < F\ x$ **for** y
using ‹$\bigwedge x.\ 0 \leq F\ x$› **by** (*auto intro*: *less-le-trans always-eventually*)
fix y :: *real* **assume** $0 < y$
then obtain z **where** $0 < z\ z < y$
using *dense*[*of 0 y*] **by** *auto*
with *Fab*[*of z*] **show** $\forall_F\ x$ *in at-bot*. $F\ x < y$
by (*auto simp*: *eventually-at-bot-linorder intro*: *le-less-trans*)
qed
ultimately have *M*: *real-distribution* (*interval-measure F*) *cdf* (*interval-measure F*) = F

using *F* **by** (*auto intro*!: *real-distribution-interval-measure cdf-interval-measure simp*: *mono-def*)
with *lim-F LIMSEQ-subseq-LIMSEQ M* **have** *weak-conv-m* ($\mu \circ s \circ r$) (*interval-measure F*)
by (*auto simp*: *weak-conv-def weak-conv-m-def f-def comp-def*)
then show $\exists\, r\, M.$ *strict-mono* (r :: *nat* $\Rightarrow$ *nat*) $\wedge$ (*real-distribution M* $\wedge$ *weak-conv-m* ($\mu \circ s \circ r$) *M*)
using *F M* **by** *auto*
qed

corollary *tight-subseq-weak-converge*:
fixes μ :: *nat* $\Rightarrow$ *real measure* **and** *M* :: *real measure*
assumes $\bigwedge n.$ *real-distribution* (μ *n*) *real-distribution M* **and** *tight*: *tight* μ **and**
subseq: $\bigwedge s\ \nu.$ *strict-mono s* $\Longrightarrow$ *real-distribution* ν $\Longrightarrow$ *weak-conv-m* ($\mu \circ s$) ν $\Longrightarrow$ *weak-conv-m* ($\mu \circ s$) *M*
shows *weak-conv-m* μ *M*
proof (*rule ccontr*)
define *f* **where** *f n* = *cdf* (μ *n*) **for** *n*
define *F* **where** *F* = *cdf M*

assume ¬ *weak-conv-m* μ *M*
then obtain *x* **where** *x*: *isCont F x* ¬ ($\lambda n.$ *f n x*) $\longrightarrow$ *F x*
by (*auto simp*: *weak-conv-m-def weak-conv-def f-def F-def*)
then obtain ε **where** $\varepsilon > 0$ **and** *infinite* $\{n.$ ¬ *dist* (*f n x*) (*F x*) $< \varepsilon\}$
by (*auto simp*: *tendsto-iff not-eventually INFM-iff-infinite cofinite-eq-sequentially*[*symmetric*])
then obtain *s* :: *nat* $\Rightarrow$ *nat* **where** *s*: $\bigwedge n.$ ¬ *dist* (*f* (*s n*) *x*) (*F x*) $< \varepsilon$ **and** *strict-mono s*
using *enumerate-in-set enumerate-mono* **by** (*fastforce simp*: *strict-mono-def*)
then obtain *r* ν **where** *r*: *strict-mono r real-distribution* ν *weak-conv-m* ($\mu \circ s \circ r$) ν
using *tight-imp-convergent-subsubsequence*[*OF tight*] **by** *blast*
then have *weak-conv-m* ($\mu \circ (s \circ r)$) *M*
using ‹*strict-mono s*› *r* **by** (*intro subseq strict-mono-o*) (*auto simp*: *comp-assoc*)
then have ($\lambda n.$ *f* (*s* (*r n*)) *x*) $\longrightarrow$ *F x*
using *x* **by** (*auto simp*: *weak-conv-m-def weak-conv-def F-def f-def*)
then show *False*
using *s* ‹$\varepsilon > 0$› **by** (*auto dest*: *tendstoD*)
qed

end

16 Integral of sinc

theory *Sinc-Integral*
imports *Distributions*
begin

16.1 Various preparatory integrals

Naming convention The theorem name consists of the following parts:

- Kind of integral: *has-bochner-integral / integrable / LBINT*
- Interval: Interval (0 / infinity / open / closed) (infinity / open / closed)
- Name of the occurring constants: power, exp, m (for minus), scale, sin, ...

lemma *has-bochner-integral-I0i-power-exp-m'*:
has-bochner-integral lborel (*λx. x^k * exp* (*−x*) *∗ indicator* {*0 ..*} *x*::*real*) (*fact k*)
using *nn-intergal-power-times-exp-Ici*[*of k*]
by (*intro has-bochner-integral-nn-integral*)
(*auto simp*: *nn-integral-set-ennreal split*: *split-indicator*)

lemma *has-bochner-integral-I0i-power-exp-m*:
has-bochner-integral lborel (*λx. x^k * exp* (*−x*) *∗ indicator* {*0 <..*} *x*::*real*) (*fact k*)
using *AE-lborel-singleton*[*of 0*]
by (*intro has-bochner-integral-cong-AE*[*THEN iffD1*, *OF - - - has-bochner-integral-I0i-power-exp-m'*])
(*auto split*: *split-indicator*)

lemma *integrable-I0i-exp-mscale*: *0 < u*::*real* *⟹ set-integrable lborel* {*0 <..*} (*λx. exp* (*−*(*x ∗ u*)))
using *lborel-integrable-real-affine-iff*[*of u λx. indicator* {*0 <..*} *x ∗R exp* (*− x*) *0*]
has-bochner-integral-I0i-power-exp-m[*of 0*]
by (*simp add*: *indicator-def zero-less-mult-iff mult-ac integrable.intros set-integrable-def*)

lemma *LBINT-I0i-exp-mscale*: *0 < u*::*real* *⟹ LBINT x*=*0..∞. exp* (*−*(*x ∗ u*)) = *1 / u*
using *lborel-integrable-real-affine*[*of u λx. indicator* {*0<..*} *x ∗R exp* (*− x*) *0*]
has-bochner-integral-I0i-power-exp-m[*of 0*]
by (*auto simp*: *indicator-def zero-less-mult-iff interval-lebesgue-integral-0-infty set-lebesgue-integral-def field-simps*
dest!: *has-bochner-integral-integral-eq*)

lemma *LBINT-I0c-exp-mscale-sin*:
LBINT x=*0..t. exp* (*−*(*u ∗ x*)) *∗ sin x* =
(*1 / *(*1 + u^2*)) *∗* (*1 − exp* (*−*(*u ∗ t*)) *∗* (*u ∗ sin t* + *cos t*)) (**is** *- = ?F t*)
unfolding *zero-ereal-def*
proof (*subst interval-integral-FTC-finite*)
show (*?F has-vector-derivative exp* (*−* (*u ∗ x*)) *∗ sin x*) (*at x within* {*min 0 t..max 0 t*}) **for** *x*
by (*auto intro*!: *derivative-eq-intros*
simp: *has-real-derivative-iff-has-vector-derivative*[*symmetric*] *power2-eq-square*)
(*simp-all add*: *field-simps add-nonneg-eq-0-iff*)

qed (*auto intro*: *continuous-at-imp-continuous-on*)

lemma *LBINT-I0i-exp-mscale-sin*:
assumes $0 < x$
shows *LBINT* $u{=}0..\infty.\ |exp\ (-u * x) * sin\ x| = |sin\ x|\ /\ x$
proof (*subst interval-integral-FTC-nonneg*)
let *?F* $= \lambda u.\ 1\ /\ x * (1 - exp\ (-\ u * x)) * |sin\ x|$
show $\bigwedge t.$ (*?F has-real-derivative* $|exp\ (-\ t * x) * sin\ x|$) (*at t*)
using ‹$0 < x$› **by** (*auto intro*!: *derivative-eq-intros simp*: *abs-mult*)
show ((*?F* ∘ *real-of-ereal*) ⟶ 0) (*at-right 0*)
using ‹$0 < x$› **by** (*auto simp*: *zero-ereal-def ereal-tendsto-simps intro*!: *tendsto-eq-intros*)
have ∗: $((\lambda t.\ exp\ (-\ t * x)) \longrightarrow 0)$ *at-top*
using ‹$0 < x$›
by (*auto intro*!: *exp-at-bot*[*THEN filterlim-compose*] *filterlim-tendsto-pos-mult-at-top filterlim-ident*
simp: *filterlim-uminus-at-bot mult.commute*[*of - x*])
show ((*?F* ∘ *real-of-ereal*) ⟶ $|sin\ x|\ /\ x$) (*at-left* ∞)
using ‹$0 < x$› **unfolding** *ereal-tendsto-simps*
by (*intro filterlim-compose*[*OF - ∗*]) (*auto intro*!: *tendsto-eq-intros filterlim-ident*)
qed *auto*

lemma
shows *integrable-inverse-1-plus-square*:
set-integrable lborel (*einterval* $(-\infty)$ ∞) $(\lambda x.\ inverse\ (1 + x\hat{}2))$
and *LBINT-inverse-1-plus-square*:
LBINT $x{=}-\infty..\infty.\ inverse\ (1 + x\hat{}2) = pi$
proof −
have *1*: $-\ (pi\ /\ 2) < x \Longrightarrow x * 2 < pi \Longrightarrow$ (*tan has-real-derivative* $1 + (tan\ x)^2$) (*at x*) **for** *x*
using *cos-gt-zero-pi*[*of x*] **by** (*subst tan-sec*) (*auto intro*!: *DERIV-tan simp*: *power-inverse*)
have *2*: $-\ (pi\ /\ 2) < x \Longrightarrow x * 2 < pi \Longrightarrow$ *isCont* $(\lambda x.\ 1 + (tan\ x)^2)\ x$ **for** *x*
using *cos-gt-zero-pi*[*of x*] **by** *auto*
have *3*: $[\![-\ (pi\ /\ 2) < x;\ x * 2 < pi]\!] \Longrightarrow$ *isCont* $(\lambda x.\ inverse\ (1 + x^2))$ (*tan x*) **for** *x*
by (*rule continuous-intros* | *simp add*: *add-nonneg-eq-0-iff*)+
show *LBINT* $x{=}-\infty..\infty.\ inverse\ (1 + x\hat{}2) = pi$
by (*subst interval-integral-substitution-nonneg*[*of* $-pi/2$ $pi/2$ *tan* $\lambda x.\ 1 + (tan\ x)\hat{}2$])
(*auto intro*: *derivative-eq-intros 1 2 3 filterlim-tan-at-right*
simp add: *ereal-tendsto-simps filterlim-tan-at-left add-nonneg-eq-0-iff set-integrable-def*)
show *set-integrable lborel* (*einterval* $(-\infty)$ ∞) $(\lambda x.\ inverse\ (1 + x\hat{}2))$
by (*subst interval-integral-substitution-nonneg*[*of* $-pi/2$ $pi/2$ *tan* $\lambda x.\ 1 + (tan\ x)\hat{}2$])
(*auto intro*: *derivative-eq-intros 1 2 3 filterlim-tan-at-right*
simp add: *ereal-tendsto-simps filterlim-tan-at-left add-nonneg-eq-0-iff set-integrable-def*)

qed

lemma
shows *integrable-I0i-1-div-plus-square*:
interval-lebesgue-integrable lborel 0 ∞ (λx. 1 / (1 + x^2))
and *LBINT-I0i-1-div-plus-square*:
LBINT x=0..∞. 1 / (1 + x^2) = pi / 2
proof −
have *1*: *0 < x ⟹ x * 2 < pi ⟹ (tan has-real-derivative 1 + (tan x)^2) (at x)*
for *x*
using *cos-gt-zero-pi*[*of x*] **by** (*subst tan-sec*) (*auto intro*!: *DERIV-tan simp*: *power-inverse*)
have *2*: *0 < x ⟹ x * 2 < pi ⟹ isCont (λx. 1 + (tan x)^2) x* **for** *x*
using *cos-gt-zero-pi*[*of x*] **by** *auto*
show *LBINT x=0..∞. 1 / (1 + x^2) = pi / 2*
by (*subst interval-integral-substitution-nonneg*[*of 0 pi/2 tan λx. 1 + (tan x)^2*])
(*auto intro*: *derivative-eq-intros 1 2 tendsto-eq-intros*
simp add: *ereal-tendsto-simps filterlim-tan-at-left zero-ereal-def add-nonneg-eq-0-iff set-integrable-def*)
show *interval-lebesgue-integrable lborel 0 ∞ (λx. 1 / (1 + x^2))*
unfolding *interval-lebesgue-integrable-def*
by (*subst interval-integral-substitution-nonneg*[*of 0 pi/2 tan λx. 1 + (tan x)^2*])
(*auto intro*: *derivative-eq-intros 1 2 tendsto-eq-intros*
simp add: *ereal-tendsto-simps filterlim-tan-at-left zero-ereal-def add-nonneg-eq-0-iff set-integrable-def*)
qed

17 The sinc function, and the sine integral (Si)

abbreviation *sinc* :: *real ⇒ real* **where**
sinc ≡ (λx. if x = 0 then 1 else sin x / x)

lemma *sinc-at-0*: *((λx. sin x / x::real) ⟶ 1) (at 0)*
using *DERIV-sin* [*of 0*] **by** (*auto simp add*: *has-field-derivative-def field-has-derivative-at*)

lemma *isCont-sinc*: *isCont sinc x*
proof *cases*
assume *x = 0* **then show** *?thesis*
using *LIM-equal* [**where** *g = λx. sin x / x* **and** *a=0* **and** *f=sinc* **and** *l=1*]
by (*auto simp*: *isCont-def sinc-at-0*)
next
assume *x ≠ 0* **show** *?thesis*
by (*rule continuous-transform-within* [**where** *δ = abs x* **and** *f = λx. sin x / x*])
(*auto simp add*: *dist-real-def ‹x ≠ 0›*)
qed

lemma *continuous-on-sinc*[*continuous-intros*]:
continuous-on S f ⟹ continuous-on S (λx. sinc (f x))

using *continuous-on-compose*[*of S f sinc*, *OF - continuous-at-imp-continuous-on*]
by (*auto simp*: *isCont-sinc*)

lemma *borel-measurable-sinc*[*measurable*]: *sinc* ∈ *borel-measurable borel*
by (*intro borel-measurable-continuous-onI continuous-at-imp-continuous-on ballI isCont-sinc*)

lemma *sinc-AE*: *AE x in lborel. sin x / x = sinc x*
by (*rule AE-I* [**where** *N* = {*0*}], *auto*)

definition *Si* :: *real* ⇒ *real* **where** *Si t* ≡ *LBINT x=0..t. sin x / x*

lemma *sinc-neg* [*simp*]: *sinc* (− *x*) = *sinc x*
by *auto*

lemma *Si-alt-def* : *Si t = LBINT x=0..t. sinc x*
proof *cases*
assume *0* ≤ *t* **then show** *?thesis*
using *AE-lborel-singleton*[*of 0*]
by (*auto simp*: *Si-def intro*!: *interval-lebesgue-integral-cong-AE*)
next
assume ¬ *0* ≤ *t* **then show** *?thesis*
unfolding *Si-def* **using** *AE-lborel-singleton*[*of 0*]
by (*subst* (*1 2*) *interval-integral-endpoints-reverse*)
(*auto simp*: *Si-def intro*!: *interval-lebesgue-integral-cong-AE*)
qed

lemma *Si-neg*:
assumes *T* ≥ *0* **shows** *Si* (− *T*) = − *Si T*
proof −
have *LBINT x=ereal 0..T.* −*1* ∗*R sinc* (− *x*) = *LBINT y= ereal* (− *0*)..*ereal* (− *T*). *sinc y*
by (*rule interval-integral-substitution-finite* [*OF assms*])
(*auto intro*: *derivative-intros continuous-at-imp-continuous-on isCont-sinc*)
also have (*LBINT x=ereal 0..T.* −*1* ∗*R sinc* (− *x*)) = −(*LBINT x=ereal 0..T. sinc x*)
by (*subst sinc-neg*) (*simp-all add*: *interval-lebesgue-integral-uminus*)
finally have ∗: −(*LBINT x=ereal 0..T. sinc x*) = *LBINT y= ereal 0..ereal* (− *T*). *sinc y*
by *simp*
show *?thesis*
using *assms* **unfolding** *Si-alt-def*
by (*subst zero-ereal-def*)+ (*auto simp add*: ∗ [*symmetric*])
qed

lemma *integrable-sinc′*:
interval-lebesgue-integrable lborel (*ereal 0*) (*ereal T*) (λ*t. sin* (*t* ∗ *ϑ*) / *t*)
proof −

have $*$: *interval-lebesgue-integrable lborel* (*ereal 0*) (*ereal T*) ($\lambda t.\ \vartheta * sinc\ (t * \vartheta)$)
by (*intro interval-lebesgue-integrable-mult-right interval-integrable-isCont continuous-within-compose3* [*OF isCont-sinc*])
auto
have $0 \notin$ *einterval* (*min* (*ereal 0*) (*ereal T*)) (*max* (*ereal 0*) (*ereal T*))
by (*smt* (*verit*, *best*) *einterval-iff max-def min-def-raw order-less-le*)
then show *?thesis*
by (*intro interval-lebesgue-integrable-cong-AE*[*THEN iffD1*, *OF - - - *]) *auto*
qed

lemma *DERIV-Si*: (*Si has-real-derivative sinc x*) (*at x*)
proof −
have (*at x within* {*min 0* (*x − 1*)..*max 0* (*x + 1*)}) = *at x*
by (*intro at-within-interior*) *auto*
moreover have *min 0* (*x − 1*) ≤ *x x* ≤ *max 0* (*x + 1*)
by *auto*
ultimately show *?thesis*
using *interval-integral-FTC2*[*of min 0* (*x − 1*) *0 max 0* (*x + 1*) *sinc x*]
by (*auto simp*: *continuous-at-imp-continuous-on isCont-sinc Si-alt-def*[*abs-def*] *zero-ereal-def*
has-real-derivative-iff-has-vector-derivative
split del: *if-split*)
qed

lemma *isCont-Si*: *isCont Si x*
using *DERIV-Si* **by** (*rule DERIV-isCont*)

lemma *borel-measurable-Si*[*measurable*]: *Si* ∈ *borel-measurable borel*
by (*auto intro*: *isCont-Si continuous-at-imp-continuous-on borel-measurable-continuous-onI*)

lemma *Si-at-top-LBINT*:
((λ*t*. (*LBINT x=0..∞*. *exp* (−(*x * t*)) * (*x * sin t + cos t*) / (*1 + x^2*))) ⟶ *0*) *at-top*
proof −
let *?F* = λ*x t*. *exp* (− (*x * t*)) * (*x * sin t + cos t*) / (*1 + x*^{2}) :: *real*
have *int*: *set-integrable lborel* {*0<..*} (λ*x*. *exp* (− *x*) * (*x + 1*) :: *real*)
unfolding *distrib-left*
using *has-bochner-integral-I0i-power-exp-m*[*of 0*] *has-bochner-integral-I0i-power-exp-m*[*of 1*]
by (*intro set-integral-add*) (*auto dest*!: *integrable.intros simp*: *ac-simps set-integrable-def*)

have ((λ*t*::*real*. *LBINT x*:{*0<..*}. *?F x t*) ⟶ (*LBINT x*::*real*:{*0<..*}. *0*)) *at-top*
unfolding *set-lebesgue-integral-def*
proof (*rule integral-dominated-convergence-at-top*[*OF - - int* [*unfolded set-integrable-def*]],
simp-all del: *abs-divide split*: *split-indicator*)
have $*$: *0 < x* ⟹ |*x * sin t + cos t*| / (*1 + x*^{2}) ≤ (*x * 1 + 1*) / *1* **for** *x t* ::

```
real
      by (intro frac-le abs-triangle-ineq[THEN order-trans] add-mono)
        (auto simp add: abs-mult simp del: mult-1-right intro!: mult-mono)
    then have 1 ≤ t ⟹ 0 < x ⟹ |?F x t| ≤ exp (− x) * (x + 1) for x t :: real
        by (auto simp add: abs-mult times-divide-eq-right[symmetric] simp del:
times-divide-eq-right
              intro!: mult-mono)
    then show ∀F i in at-top. AE x in lborel. 0 < x ⟶ |?F x i| ≤ exp (− x) *
(x + 1)
      by (auto intro: eventually-mono eventually-ge-at-top[of 1::real])
    show AE x in lborel. 0 < x ⟶ (?F x ⟶ 0) at-top
    proof (intro always-eventually impI allI)
      fix x :: real assume 0 < x
      show ((λt. exp (− (x * t)) * (x * sin t + cos t) / (1 + x²)) ⟶ 0) at-top
      proof (rule Lim-null-comparison)
        show ∀F t in at-top. norm (?F x t) ≤ exp (− (x * t)) * (x + 1)
          using eventually-ge-at-top[of 1::real] * ‹0 < x›
          by (auto simp add: abs-mult times-divide-eq-right[symmetric] simp del:
times-divide-eq-right
                intro!: mult-mono elim: eventually-mono)
        show ((λt. exp (− (x * t)) * (x + 1)) ⟶ 0) at-top
          by (auto simp: filterlim-uminus-at-top [symmetric]
                  intro!: filterlim-tendsto-pos-mult-at-top[OF tendsto-const] ‹0<x›
filterlim-ident
                      tendsto-mult-left-zero filterlim-compose[OF exp-at-bot])
      qed
    qed
  qed
  then show ((λt. (LBINT x=0..∞. exp (−(x * t)) * (x * sin t + cos t) / (1 +
x^2))) ⟶ 0) at-top
    by (simp add: interval-lebesgue-integral-0-infty set-lebesgue-integral-def)
qed

lemma Si-at-top-integrable:
  assumes t ≥ 0
  shows interval-lebesgue-integrable lborel 0 ∞ (λx. exp (− (x * t)) * (x * sin t +
cos t) / (1 + x²))
  using ‹0 ≤ t› unfolding le-less
proof
  assume 0 = t then show ?thesis
    using integrable-I0i-1-div-plus-square by simp
next
  assume [arith]: 0 < t
  have *: 0 ≤ a ⟹ 0 < x ⟹ a / (1 + x²) ≤ a for a x :: real
    using zero-le-power2[of x, arith] using divide-left-mono[of 1 1+x² a] by auto
  have set-integrable lborel {0<..} (λx. (exp (− x) * x) * (sin t/t) + exp (− x) *
cos t)
    using has-bochner-integral-I0i-power-exp-m[of 0] has-bochner-integral-I0i-power-exp-m[of
1]
```

by (*intro set-integral-add set-integrable-mult-left*)
(*auto dest*!: *integrable.intros simp*: *ac-simps set-integrable-def*)
from *lborel-integrable-real-affine*[*OF this* [*unfolded set-integrable-def*], *of t 0*]
show *?thesis*
unfolding *interval-lebesgue-integral-0-infty set-integrable-def*
by (*rule Bochner-Integration.integrable-bound*) (*auto simp*: *field-simps* ∗ *split*: *split-indicator*)
qed

lemma *Si-at-top*: (*Si* ⟶ *pi* / *2*) *at-top*
proof −
have †: ∀$_F$ *t in at-top*. *pi* / *2* − (*LBINT u*=*0*..∞. *exp* (−(*u* ∗ *t*)) ∗ (*u* ∗ *sin t* + *cos t*) / (*1* + *u*^*2*)) = *Si t*
using *eventually-ge-at-top*[*of 0*]
proof *eventually-elim*
fix *t* :: *real* **assume** *t* ≥ *0*
have *Si t* = *LBINT x*=*0*..*t*. *sin x* ∗ (*LBINT u*=*0*..∞. *exp* (−(*u* ∗ *x*)))
unfolding *Si-def* **using** ‹*0* ≤ *t*›
by (*intro interval-integral-discrete-difference*[**where** *X*={*0*}]) (*auto simp*: *LBINT-I0i-exp-mscale*)
also have ... = (*LBINT x*. (*LBINT u*=*ereal 0*..∞. *indicator* {*0* <..< *t*} *x* ∗$_R$ *sin x* ∗ *exp* (−(*u* ∗ *x*))))
using ‹*0*≤*t*› **by** (*simp add*: *zero-ereal-def interval-lebesgue-integral-le-eq mult-ac set-lebesgue-integral-def*)
also have ... = (*LBINT x*. (*LBINT u*. *indicator* ({*0*<..} × {*0* <..< *t*}) (*u*, *x*) ∗$_R$ (*sin x* ∗ *exp* (−(*u* ∗ *x*)))))
apply (*subst interval-integral-Ioi*)
unfolding *set-lebesgue-integral-def* **by**(*simp-all add*: *indicator-times ac-simps*)
also have ... = (*LBINT u*. (*LBINT x*. *indicator* ({*0*<..} × {*0* <..< *t*}) (*u*, *x*) ∗$_R$ (*sin x* ∗ *exp* (−(*u* ∗ *x*)))))
proof (*intro lborel-pair.Fubini-integral*[*symmetric*] *lborel-pair.Fubini-integrable*)
show (λ(*x*, *y*). *indicator* ({*0*<..} × {*0*<..<*t*}) (*y*, *x*) ∗$_R$ (*sin x* ∗ *exp* (− (*y* ∗ *x*))))
∈ *borel-measurable* (*lborel* ⨂$_M$ *lborel*) (**is** *?f* ∈ *borel-measurable* -)
by *measurable*

have *AE x in lborel*. *indicator* {*0*..*t*} *x* ∗$_R$ *abs* (*sinc x*) = (*LBINT y*. *norm* (*?f* (*x*, *y*)))
using *AE-lborel-singleton*[*of 0*] *AE-lborel-singleton*[*of t*]
proof *eventually-elim*
fix *x* :: *real* **assume** *x*: *x* ≠ *0 x* ≠ *t*
have (*LBINT y*. |*indicator* ({*0*<..} × {*0*<..<*t*}) (*y*, *x*) ∗$_R$ (*sin x* ∗ *exp* (−(*y* ∗ *x*)))|) =
(*LBINT y*. |*sin x*| ∗ *exp* (− (*y* ∗ *x*)) ∗ *indicator* {*0*<..} *y* ∗ *indicator* {*0*<..<*t*} *x*)
by (*intro Bochner-Integration.integral-cong*) (*auto split*: *split-indicator simp*: *abs-mult*)
also have ... = |*sin x*| ∗ *indicator* {*0*<..<*t*} *x* ∗ (*LBINT y*=*0*..∞. *exp* (−

```
(y * x)))
        by (cases x > 0)
         (auto simp: field-simps interval-lebesgue-integral-0-infty set-lebesgue-integral-def
split: split-indicator)
       also have ... = |sin x| * indicator {0<..<t} x * (1 / x)
        by (cases x > 0) (auto simp add: LBINT-I0i-exp-mscale)
       also have ... = indicator {0..t} x *R |sinc x|
        using x by (simp add: field-simps split: split-indicator)
       finally show indicator {0..t} x *R abs (sinc x) = (LBINT y. norm (?f (x,
y)))
        by simp
     qed
     moreover have integrable lborel (λx. indicat-real {0..t} x *R |sinc x|)
     by (auto intro!: borel-integrable-compact continuous-intros simp del: real-scaleR-def)
     ultimately show integrable lborel (λx. LBINT y. norm (?f (x, y)))
      by (rule integrable-cong-AE-imp[rotated 2]) simp

     have 0 < x ⟹ set-integrable lborel {0<..} (λy. sin x * exp (− (y * x))) for
x :: real
        by (intro set-integrable-mult-right integrable-I0i-exp-mscale)
     then show AE x in lborel. integrable lborel (λy. ?f (x, y))
     by (intro AE-I2) (auto simp: indicator-times set-integrable-def split: split-indicator)
   qed
   also have ... = (LBINT u=0..∞. (LBINT x=0..t. exp (−(u * x)) * sin x))
     using ‹0≤t›
    by (auto simp: interval-lebesgue-integral-def set-lebesgue-integral-def zero-ereal-def
ac-simps
             split: split-indicator intro!: Bochner-Integration.integral-cong)
   also have ... = (LBINT u=0..∞. 1 / (1 + u²) − 1 / (1 + u²) * (exp (− (u
* t)) * (u * sin t + cos t)))
    by (auto simp: divide-simps LBINT-I0c-exp-mscale-sin intro!: interval-integral-cong)
   also have ... = pi / 2 − (LBINT u=0..∞. exp (− (u * t)) * (u * sin t + cos
t) / (1 + u^2))
    using Si-at-top-integrable[OF ‹0≤t›] by (simp add: integrable-I0i-1-div-plus-square
LBINT-I0i-1-div-plus-square)
   finally show pi / 2 − (LBINT u=0..∞. exp (−(u * t)) * (u * sin t + cos t)
/ (1 + u^2)) = Si t ..
 qed
 show ?thesis
   by (rule Lim-transform-eventually [OF - †])
     (auto intro!: tendsto-eq-intros Si-at-top-LBINT)
qed
```

17.1 The final theorems: boundedness and scalability

```
lemma bounded-Si: ∃ B. ∀ T. |Si T| ≤ B
proof −
 have *: 0 ≤ y ⟹ dist x y < z ⟹ abs x ≤ y + z for x y z :: real
   by (simp add: dist-real-def)
```

have *eventually* (λ*T*. *dist* (*Si T*) (*pi* / *2*) < *1*) *at-top*
using *Si-at-top* **by** (*elim tendstoD*) *simp*
then have *eventually* (λ*T*. *0* ≤ *T* ∧ |*Si T*| ≤ *pi* / *2* + *1*) *at-top*
using *eventually-ge-at-top*[*of 0*] **by** *eventually-elim* (*insert* ∗[*of pi/2 Si - 1*], *auto*)
then have ∃ *T*. *0* ≤ *T* ∧ (∀ *t* ≥ *T*. |*Si t*| ≤ *pi* / *2* + *1*)
by (*auto simp add*: *eventually-at-top-linorder*)
then obtain *T* **where** *T*: *0* ≤ *T* ⋀*t*. *t* ≥ *T* ⟹ |*Si t*| ≤ *pi* / *2* + *1*
by *auto*
moreover have *t* ≤ − *T* ⟹ |*Si t*| ≤ *pi* / *2* + *1* **for** *t*
using *T*(*1*) *T*(*2*)[*of* −*t*] *Si-neg*[*of* − *t*] **by** *simp*
moreover have ∃ *M*. ∀ *t*. −*T* ≤ *t* ∧ *t* ≤ *T* ⟶ |*Si t*| ≤ *M*
by (*rule isCont-bounded*) (*auto intro*!: *isCont-Si continuous-intros* ‹*0* ≤ *T*›)
then obtain *M* **where** *M*: ⋀*t*. −*T* ≤ *t* ∧ *t* ≤ *T* ⟹ |*Si t*| ≤ *M*
by *auto*
ultimately show *?thesis*
by (*intro exI*[*of - max M* (*pi/2 + 1*)]) (*meson le-max-iff-disj linorder-not-le order-le-less*)
qed

lemma *LBINT-I0c-sin-scale-divide*:
assumes *T* ≥ *0*
shows *LBINT t=0..T. sin* (*t* ∗ ϑ) / *t* = *sgn* ϑ ∗ *Si* (*T* ∗ |ϑ|)
proof −
{ assume *0* < ϑ
have (*LBINT t=ereal 0..T. sin* (*t* ∗ ϑ) / *t*) = (*LBINT t=ereal 0..T.* ϑ ∗*R sinc* (*t* ∗ ϑ))
by (*rule interval-integral-discrete-difference*[*of* {*0*}]) *auto*
also have ... = (*LBINT t=ereal* (*0* ∗ ϑ)..*T* ∗ ϑ. *sinc t*)
apply (*rule interval-integral-substitution-finite* [*OF assms*])
apply (*subst mult.commute*, *rule DERIV-subset*)
by (*auto intro*!: *derivative-intros continuous-at-imp-continuous-on isCont-sinc*)
also have ... = (*LBINT t=ereal* (*0* ∗ ϑ)..*T* ∗ ϑ. *sin t* / *t*)
by (*rule interval-integral-discrete-difference*[*of* {*0*}]) *auto*
finally have (*LBINT t=ereal 0..T. sin* (*t* ∗ ϑ) / *t*) = (*LBINT t=ereal 0..T* ∗ ϑ. *sin t* / *t*)
by *simp*
hence (*LBINT x. indicator* {*0*<..<*T*} *x* ∗ *sin* (*x* ∗ ϑ) / *x*) = (*LBINT x. indicator* {*0*<..<*T* ∗ ϑ} *x* ∗ *sin x* / *x*)
using *assms* ‹*0* < ϑ› **unfolding** *interval-lebesgue-integral-def einterval-eq zero-ereal-def*
by (*auto simp*: *ac-simps set-lebesgue-integral-def*)
} note *aux1* = *this*
{ assume *0* > ϑ
have [*simp*]: *integrable lborel* (λ*x*. *sin* (*x* ∗ ϑ) ∗ *indicator* {*0*<..<*T*} *x* / *x*)
using *integrable-sinc'* [*of T* ϑ] *assms*
by (*simp add*: *interval-lebesgue-integrable-def set-integrable-def ac-simps*)
have (*LBINT t=ereal 0..T. sin* (*t* ∗ −ϑ) / *t*) = (*LBINT t=ereal 0..T.* −ϑ ∗*R*

sinc $(t * -\vartheta))$
by (*rule interval-integral-discrete-difference*[*of* $\{0\}$]) *auto*
also have ... = (*LBINT t=ereal* $(0 * -\vartheta)..T * -\vartheta$. *sinc t*)
apply (*rule interval-integral-substitution-finite* [*OF assms*])
apply (*subst mult.commute*, *rule DERIV-subset*)
by (*auto intro*!: *derivative-intros continuous-at-imp-continuous-on isCont-sinc*)
also have ... = (*LBINT t=ereal* $(0 * -\vartheta)..T * -\vartheta$. *sin t* / *t*)
by (*rule interval-integral-discrete-difference*[*of* $\{0\}$]) *auto*
finally have (*LBINT t=ereal 0..T*. *sin* $(t * -\vartheta)$ / *t*) = (*LBINT t=ereal 0..T* $* -\vartheta$. *sin t* / *t*)
by *simp*
hence (*LBINT x*. *indicator* $\{0<..<T\}$ *x* $*$ *sin* $(x * \vartheta)$ / *x*) =
$-$ (*LBINT x*. *indicator* $\{0<..<-(T * \vartheta)\}$ *x* $*$ *sin x* / *x*)
using *assms* ‹*0* $>$ ϑ› **unfolding** *interval-lebesgue-integral-def einterval-eq zero-ereal-def*
by (*auto simp add*: *field-simps mult-le-0-iff set-lebesgue-integral-def split*: *if-split-asm*)
} note *aux2* = *this*
show *?thesis*
using *assms* **unfolding** *Si-def interval-lebesgue-integral-def set-lebesgue-integral-def sgn-real-def einterval-eq zero-ereal-def*
by (*auto simp*: *aux1 aux2*)
qed

end

18 The Levy inversion theorem, and the Levy continuity theorem.

theory *Levy*
imports *Characteristic-Functions Helly-Selection Sinc-Integral*
begin

18.1 The Levy inversion theorem

lemma *Levy-Inversion-aux1*:
fixes *a b* :: *real*
assumes $a \leq b$
shows $((\lambda t.\ (iexp\ (-(t * a)) - iexp\ (-(t * b)))\ /\ (\mathrm{i} * t)) \longrightarrow b - a)\ (at\ 0)$
(**is** (*?F* $\longrightarrow$ -) (*at* -))
proof −
have *1*: *cmod* (*?F t* − (*b* − *a*)) ≤ *a*^*2* / *2* $*$ *abs t* + *b*^*2* / *2* $*$ *abs t* **if** $t \neq 0$ **for** *t*
proof −
have *cmod* (*?F t* − (*b* − *a*)) = *cmod* (
(*iexp* $(-(t * a))$ − (*1* + i $*$ $-(t * a)$)) / (i $*$ *t*) −
(*iexp* $(-(t * b))$ − (*1* + i $*$ $-(t * b)$)) / (i $*$ *t*))
(**is** - = *cmod* (*?one* / (i $*$ *t*) − *?two* / (i $*$ *t*)))

```
    using ‹t ≠ 0› by (intro arg-cong[where f=norm]) (simp add: field-simps)
  also have ... ≤ cmod (?one / (i * t)) + cmod (?two / (i * t))
    by (rule norm-triangle-ineq4)
  also have cmod (?one / (i * t)) = cmod ?one / abs t
    by (simp add: norm-divide norm-mult)
  also have cmod (?two / (i * t)) = cmod ?two / abs t
    by (simp add: norm-divide norm-mult)
  also have cmod ?one / abs t + cmod ?two / abs t ≤
      ((- (a * t))^2 / 2) / abs t + ((- (b * t))^2 / 2) / abs t
    using iexp-approx1 [of -(t * -) 1]
    by (intro add-mono divide-right-mono abs-ge-zero) (auto simp: field-simps
eval-nat-numeral)
  also have ... = a^2 / 2 * abs t + b^2 / 2 * abs t
   using ‹t ≠ 0› apply (case-tac t ≥ 0, simp add: field-simps power2-eq-square)
     using ‹t ≠ 0› by (subst (1 2) abs-of-neg, auto simp add: field-simps
power2-eq-square)
  finally show cmod (?F t - (b - a)) ≤ a^2 / 2 * abs t + b^2 / 2 * abs t .
 qed
 show ?thesis
  apply (rule LIM-zero-cancel)
  apply (rule tendsto-norm-zero-cancel)
  apply (rule real-LIM-sandwich-zero [OF - - 1])
  apply (auto intro!: tendsto-eq-intros)
  done
qed

lemma Levy-Inversion-aux2:
 fixes a b t :: real
 assumes a ≤ b and t ≠ 0
 shows cmod ((iexp (t * b) - iexp (t * a)) / (i * t)) ≤ b - a (is ?F ≤ -)
proof -
 have ?F = cmod (iexp (t * a) * (iexp (t * (b - a)) - 1) / (i * t))
   using ‹t ≠ 0› by (intro arg-cong[where f=norm]) (simp add: field-simps
exp-diff exp-minus)
 also have ... = cmod (iexp (t * (b - a)) - 1) / abs t
   unfolding norm-divide norm-mult norm-exp-i-times using ‹t ≠ 0›
   by (simp add: complex-eq-iff norm-mult)
 also have ... ≤ abs (t * (b - a)) / abs t
   using iexp-approx1 [of t * (b - a) 0]
   by (intro divide-right-mono) (auto simp add: field-simps eval-nat-numeral)
 also have ... = b - a
   using assms by (auto simp add: abs-mult)
 finally show ?thesis .
qed


theorem (in real-distribution) Levy-Inversion:
 fixes a b :: real
 assumes a ≤ b
```

defines $\mu \equiv$ *measure M* **and** $\varphi \equiv$ *char M*
assumes $\mu\ \{a\} = 0$ **and** $\mu\ \{b\} = 0$
shows $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ (iexp\ (-(t * a)) - iexp\ (-(t * b)))\ /\ (\mathrm{i} * t) * \varphi\ t))$
$\longrightarrow \mu\ \{a{<}..b\}$
(**is** $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ ?F\ t * \varphi\ t)) \longrightarrow of\text{-}real\ (\mu\ \{a{<}..b\}))$
proof −
interpret *P*: *pair-sigma-finite lborel M* **..**
from *bounded-Si* **obtain** *B* **where** *Bprop*: $\bigwedge T.\ abs\ (Si\ T) \le B$ **by** *auto*
from *Bprop* [*of 0*] **have** [*simp*]: $B \ge 0$ **by** *auto*
let *?f* $= \lambda t\ x :: real.\ (iexp\ (t * (x - a)) - iexp(t * (x - b)))\ /\ (\mathrm{i} * t)$
{ fix *T* :: *real*
assume $T \ge 0$
let *?f′* $= \lambda(t, x).$ *indicator* $\{-T{<}..{<}T\}\ t *_R$ *?f t x*
{ fix *x*
have *int*: *interval-lebesgue-integrable lborel* (*ereal 0*) (*ereal T*) ($\lambda t.\ 2 * (sin\ (t * (x{-}w))\ /\ t)$) **for** *w*
using *integrable-sinc′ interval-lebesgue-integrable-mult-right* **by** *blast*
have *1*: *complex-interval-lebesgue-integrable lborel u v* ($\lambda t.$ *?f t x*) **for** *u v* :: *real*
using *Levy-Inversion-aux2*[*of x − b x − a*]
apply (*simp add*: *interval-lebesgue-integrable-def set-integrable-def del*: *times-divide-eq-left*)
apply (*intro integrableI-bounded-set-indicator*[**where** $B{=}b - a$] *conjI impI*)
apply (*auto intro*!: *AE-I* [*of - - {0}*] *simp*: *assms*)
done
have (*CLBINT t. ?f′* $(t, x)) = (CLBINT\ t{=}{-}T..T.$ *?f t x*)
using ‹$T \ge 0$› **by** (*simp add*: *interval-lebesgue-integral-def set-lebesgue-integral-def*)
also have $\ldots = (CLBINT\ t{=}{-}T..(0 :: real).$ *?f t x*) + (*CLBINT t*=(*0* :: *real*)..*T. ?f t x*)
(**is** *- = - + ?t*)
using *1* **by** (*intro interval-integral-sum*[*symmetric*]) (*simp add*: *min-absorb1 max-absorb2* ‹$T \ge 0$›)
also have (*CLBINT t*=$-T$..(*0* :: *real*). *?f t x*) = (*CLBINT t*=(*0*::*real*)..*T. ?f* $(-t)\ x$)
by (*subst interval-integral-reflect*) *auto*
also have $\ldots$ + *?t* = (*CLBINT t*=(*0*::*real*)..*T. ?f* $(-t)\ x$ + *?f t x*)
using *1*
by (*intro interval-lebesgue-integral-add*(*2*) [*symmetric*] *interval-integrable-mirror*[*THEN iffD2*]) *simp-all*
also have $\ldots = (CLBINT\ t{=}(0{::}real)..T.\ ((iexp(t * (x - a)) - iexp\ (-(t * (x - a)))) -$
$(iexp(t * (x - b)) - iexp\ (-(t * (x - b)))))\ /\ (\mathrm{i} * t))$
using ‹$T \ge 0$› **by** (*intro interval-integral-cong*) (*auto simp add*: *field-split-simps*)
also have $\ldots = (CLBINT\ t{=}(0{::}real)..T.$ *complex-of-real*(
$2 * (sin\ (t * (x - a))\ /\ t) - 2 * (sin\ (t * (x - b))\ /\ t))$)
using ‹$T \ge 0$›
by (*intro interval-integral-cong*) (*simp add*: *divide-simps Im-divide Re-divide*

Im-exp Re-exp complex-eq-iff)
also have ... = *complex-of-real* (*LBINT t*=(*0*::*real*)..*T*.
2 ∗ (*sin* (*t* ∗ (*x* − *a*)) / *t*) − *2* ∗ (*sin* (*t* ∗ (*x* − *b*)) / *t*))
by (*rule interval-lebesgue-integral-of-real*)
also have ... = *complex-of-real* ((*LBINT t*=*ereal 0*..*ereal T*. *2* ∗ (*sin* (*t* ∗ (*x* − *a*)) / *t*)) −
(*LBINT t*=*ereal 0*..*ereal T*. *2* ∗ (*sin* (*t* ∗ (*x* − *b*)) / *t*)))
unfolding *interval-lebesgue-integral-diff*
using *int* **by** *auto*
also have ... = *complex-of-real* (*2* ∗ (*sgn* (*x* − *a*) ∗ *Si* (*T* ∗ *abs* (*x* − *a*)) −
sgn (*x* − *b*) ∗ *Si* (*T* ∗ *abs* (*x* − *b*))))
unfolding *interval-lebesgue-integral-mult-right*
by (*simp add*: *zero-ereal-def*[*symmetric*] *LBINT-I0c-sin-scale-divide*[*OF* ‹*T* ≥ *0*›])
finally have (*CLBINT t*. *?f′* (*t*, *x*)) =
2 ∗ (*sgn* (*x* − *a*) ∗ *Si* (*T* ∗ *abs* (*x* − *a*)) − *sgn* (*x* − *b*) ∗ *Si* (*T* ∗ *abs* (*x* − *b*))) **.**
} note *main-eq* = *this*
have (*CLBINT t*=−*T*..*T*. *?F t* ∗ *φ t*) =
(*CLBINT t*. (*CLINT x* | *M*. *?F t* ∗ *iexp* (*t* ∗ *x*) ∗ *indicator* {−*T*<..<*T*} *t*))
using ‹*T* ≥ *0*› **unfolding** *φ-def char-def interval-lebesgue-integral-def set-lebesgue-integral-def*
by (*auto split*: *split-indicator intro*!: *Bochner-Integration.integral-cong*)
also have ... = (*CLBINT t*. (*CLINT x* | *M*. *?f′* (*t*, *x*)))
by (*auto intro*!: *Bochner-Integration.integral-cong simp*: *field-simps exp-diff exp-minus split*: *split-indicator*)
also have ... = (*CLINT x* | *M*. (*CLBINT t*. *?f′* (*t*, *x*)))
proof (*intro P.Fubini-integral* [*symmetric*] *integrableI-bounded-set* [**where** *B*=*b* − *a*])
show *emeasure* (*lborel* ⨂*M* *M*) ({− *T*<..<*T*} × *space M*) < ∞
using ‹*T* ≥ *0*›
by (*subst emeasure-pair-measure-Times*)
(*auto simp*: *ennreal-mult-less-top not-less top-unique*)
show *AE x*∈{− *T*<..<*T*} × *space M in lborel* ⨂*M* *M*. *cmod* (*case x of* (*t*, *x*) ⇒ *?f′* (*t*, *x*)) ≤ *b* − *a*
using *Levy-Inversion-aux2*[*of x* − *b x* − *a* **for** *x*] ‹*a* ≤ *b*›
by (*intro AE-I* [*of* - - {*0*} × *UNIV*]) (*force simp*: *emeasure-pair-measure-Times*)+
qed (*auto split*: *split-indicator split-indicator-asm*)
also have ... = (*CLINT x* | *M*. (*complex-of-real* (*2* ∗ (*sgn* (*x* − *a*) ∗
Si (*T* ∗ *abs* (*x* − *a*)) − *sgn* (*x* − *b*) ∗ *Si* (*T* ∗ *abs* (*x* − *b*))))))
using *main-eq* **by** (*intro Bochner-Integration.integral-cong*, *auto*)
also have ... = *complex-of-real* (*LINT x* | *M*. (*2* ∗ (*sgn* (*x* − *a*) ∗
Si (*T* ∗ *abs* (*x* − *a*)) − *sgn* (*x* − *b*) ∗ *Si* (*T* ∗ *abs* (*x* − *b*)))))
by (*rule integral-complex-of-real*)
finally have (*CLBINT t*=−*T*..*T*. *?F t* ∗ *φ t*) =
complex-of-real (*LINT x* | *M*. (*2* ∗ (*sgn* (*x* − *a*) ∗
Si (*T* ∗ *abs* (*x* − *a*)) − *sgn* (*x* − *b*) ∗ *Si* (*T* ∗ *abs* (*x* − *b*))))) **.**
} note *main-eq2* = *this*

```
  have (λT :: nat. LINT x | M. (2 * (sgn (x − a) *
        Si (T * abs (x − a)) − sgn (x − b) * Si (T * abs (x − b))))) ⟶
       (LINT x | M. 2 * pi * indicator {a<..b} x)
  proof (rule integral-dominated-convergence [where w=λx. 4 * B])
    show integrable M (λx. 4 * B)
      by (rule integrable-const-bound [of - 4 * B]) auto
  next
    let ?S = λn::nat. λx. sgn (x − a) * Si (n * |x − a|) − sgn (x − b) * Si (n *
|x − b|)
    { fix n x
      have norm (?S n x) ≤ norm (sgn (x − a) * Si (n * |x − a|)) + norm (sgn
(x − b) * Si (n * |x − b|))
        by (rule norm-triangle-ineq4)
      also have ... ≤ B + B
        using Bprop by (intro add-mono) (auto simp: abs-mult abs-sgn-eq)
      finally have norm (2 * ?S n x) ≤ 4 * B
        by simp }
    then show ⋀n. AE x in M. norm (2 * ?S n x) ≤ 4 * B
      by auto
    have AE x in M. x ≠ a AE x in M. x ≠ b
      using prob-eq-0[of {a}] prob-eq-0[of {b}] ‹μ {a} = 0› ‹μ {b} = 0› by (auto
simp: μ-def)
    then show AE x in M. (λn. 2 * ?S n x) ⟶ 2 * pi * indicator {a<..b} x
    proof eventually-elim
      fix x assume x: x ≠ a x ≠ b
      then have (λn. 2 * (sgn (x − a) * Si (|x − a| * n) − sgn (x − b) * Si (|x
− b| * n)))
          ⟶ 2 * (sgn (x − a) * (pi / 2) − sgn (x − b) * (pi / 2))
        by (intro tendsto-intros filterlim-compose[OF Si-at-top]
          filterlim-tendsto-pos-mult-at-top[OF tendsto-const] filterlim-real-sequentially)
            auto
      also have (λn. 2 * (sgn (x − a) * Si (|x − a| * n) − sgn (x − b) * Si (|x −
b| * n))) = (λn. 2 * ?S n x)
        by (auto simp: ac-simps)
      also have 2 * (sgn (x − a) * (pi / 2) − sgn (x − b) * (pi / 2)) = 2 * pi *
indicator {a<..b} x
        using x ‹a ≤ b› by (auto split: split-indicator)
      finally show (λn. 2 * ?S n x) ⟶ 2 * pi * indicator {a<..b} x .
    qed
  qed simp-all
  also have (LINT x | M. 2 * pi * indicator {a<..b} x) = 2 * pi * μ {a<..b}
    by (simp add: μ-def)
  finally have (λT. LINT x | M. (2 * (sgn (x − a) *
        Si (T * abs (x − a)) − sgn (x − b) * Si (T * abs (x − b))))) ⟶
       2 * pi * μ {a<..b} .
  with main-eq2 show ?thesis
    by (auto intro!: tendsto-eq-intros)
qed
```

theorem *Levy-uniqueness*:
fixes *M1 M2* :: *real measure*
assumes *real-distribution M1 real-distribution M2* **and**
char M1 = *char M2*
shows *M1* = *M2*
proof −
interpret *M1*: *real-distribution M1* **by** (*rule assms*)
interpret *M2*: *real-distribution M2* **by** (*rule assms*)
have *countable* ({*x*. *measure M1* {*x*} ≠ *0*} ∪ {*x*. *measure M2* {*x*} ≠ *0*})
by (*intro countable-Un M2.countable-support M1.countable-support*)
then have *count*: *countable* {*x*. *measure M1* {*x*} ≠ *0* ∨ *measure M2* {*x*} ≠ *0*}
by (*simp add*: *Un-def*)

have *cdf M1* = *cdf M2*
proof (*rule ext*)
fix *x*
let *?D* = λ*x*. |*cdf M1 x* − *cdf M2 x*|

{ **fix** ε :: *real*
assume ε > *0*
have (*?D* ⟶ *0*) *at-bot*
using *M1.cdf-lim-at-bot M2.cdf-lim-at-bot* **by** (*intro tendsto-eq-intros*) *auto*
then have *eventually* (λ*y*. *?D y* < ε / *2* ∧ *y* ≤ *x*) *at-bot*
using ‹ε > *0*› **by** (*simp only*: *tendsto-iff dist-real-def diff-0-right eventually-conj eventually-le-at-bot abs-idempotent*)
then obtain *M* **where** ⋀*y*. *y* ≤ *M* ⟹ *?D y* < ε / *2 M* ≤ *x*
unfolding *eventually-at-bot-linorder* **by** *auto*
with *open-minus-countable*[*OF count*, *of* {..< *M*}] **obtain** *a* **where**
measure M1 {*a*} = *0 measure M2* {*a*} = *0 a* < *M a* ≤ *x* **and** *1*: *?D a* < ε / *2*
by *auto*

have (*?D* ⟶ *?D x*) (*at-right x*)
using *M1.cdf-is-right-cont* [*of x*] *M2.cdf-is-right-cont* [*of x*]
by (*intro tendsto-intros*) (*auto simp add*: *continuous-within*)
then have *eventually* (λ*y*. |*?D y* − *?D x*| < ε / *2*) (*at-right x*)
using ‹ε > *0*› **by** (*simp only*: *tendsto-iff dist-real-def eventually-conj eventually-at-right-less*)
then obtain *N* **where** *N* > *x* ⋀*y*. *x* < *y* ⟹ *y* < *N* ⟹ |*?D y* − *?D x*| < ε / *2*
by (*auto simp add*: *eventually-at-right*[*OF less-add-one*])
with *open-minus-countable*[*OF count*, *of* {*x* <..< *N*}] **obtain** *b* **where** *x* < *b b* < *N*
measure M1 {*b*} = *0 measure M2* {*b*} = *0* **and** *2*: |*?D b* − *?D x*| < ε / *2*
by (*auto simp*: *abs-minus-commute*)
from ‹*a* ≤ *x*› ‹*x* < *b*› **have** *a* < *b a* ≤ *b* **by** *auto*

from ‹*char M1* = *char M2*›
M1.Levy-Inversion [*OF* ‹*a* ≤ *b*› ‹*measure M1* {*a*} = *0*› ‹*measure M1* {*b*}

= 0›]
M2.Levy-Inversion [*OF ‹a ≤ b› ‹measure M2 {a} = 0› ‹measure M2 {b}*
= 0›]
have *complex-of-real* (*measure M1 {a<..b}*) = *complex-of-real* (*measure M2*
{a<..b})
by (*intro LIMSEQ-unique*) *auto*
then have *?D a = ?D b*
unfolding *of-real-eq-iff M1.cdf-diff-eq* [*OF ‹a < b›*, *symmetric*] *M2.cdf-diff-eq*
[*OF ‹a < b›*, *symmetric*] **by** *simp*
then have *?D x* = |(*?D b − ?D x*) − *?D a*|
by *simp*
also have ... ≤ |*?D b − ?D x*| + |*?D a*|
by (*rule abs-triangle-ineq4*)
also have ... ≤ *ε / 2 + ε / 2*
using *1 2* **by** (*intro add-mono*) *auto*
finally have *?D x* ≤ *ε* **by** *simp* **}**
then show *cdf M1 x = cdf M2 x*
by (*metis abs-le-zero-iff dense-ge eq-iff-diff-eq-0*)
qed
thus *?thesis*
by (*rule cdf-unique* [*OF ‹real-distribution M1› ‹real-distribution M2›*])
qed

18.2 The Levy continuity theorem

theorem *levy-continuity1*:
fixes *M* :: *nat* ⇒ *real measure* **and** *M′* :: *real measure*
assumes ⋀*n*. *real-distribution* (*M n*) *real-distribution M′ weak-conv-m M M′*
shows (*λn*. *char* (*M n*) *t*) ⟶ *char M′ t*
unfolding *char-def* **using** *assms* **by** (*rule weak-conv-imp-integral-bdd-continuous-conv*)
auto

theorem *levy-continuity*:
fixes *M* :: *nat* ⇒ *real measure* **and** *M′* :: *real measure*
assumes *real-distr-M* : ⋀*n*. *real-distribution* (*M n*)
and *real-distr-M′*: *real-distribution M′*
and *char-conv*: ⋀*t*. (*λn*. *char* (*M n*) *t*) ⟶ *char M′ t*
shows *weak-conv-m M M′*
proof −
interpret *Mn*: *real-distribution M n* **for** *n* **by** *fact*
interpret *M′*: *real-distribution M′* **by** *fact*

have ∗: ⋀*u x*. *u > 0* ⟹ *x ≠ 0* ⟹ (*CLBINT t*:{−*u*..*u*}. *1 − iexp* (*t ∗ x*)) =
2 ∗ (*u − sin* (*u ∗ x*) */ x*)
proof −
fix *u* :: *real* **and** *x* :: *real*
assume *u > 0* **and** *x ≠ 0*
hence (*CLBINT t*:{−*u*..*u*}. *1 − iexp* (*t ∗ x*)) = (*CLBINT t*=−*u*..*u*. *1 − iexp*
(*t ∗ x*))

by (*subst interval-integral-Icc*, *auto*)
also have ... = (*CLBINT t=−u..ereal 0*. *1* − *iexp* (*t* ∗ *x*)) + (*CLBINT t=ereal 0..u*. *1* − *iexp* (*t* ∗ *x*))
using ‹*u* > *0*›
by (*subst interval-integral-sum*; *force simp add*: *interval-integrable-isCont*)
also have ... = (*CLBINT t=ereal 0..u*. *1* − *iexp* (*t* ∗ −*x*)) + (*CLBINT t=ereal 0..u*. *1* − *iexp* (*t* ∗ *x*))
by (*subst interval-integral-reflect*, *auto*)
also have ... = *CLBINT xa=ereal 0..ereal u*. *1* − *iexp* (*xa* ∗ − *x*) + (*1* − *iexp* (*xa* ∗ *x*))
by (*subst interval-lebesgue-integral-add* (*2*) [*symmetric*]) (*auto simp*: *interval-integrable-isCont*)
also have ... = (*LBINT t=ereal 0..u*. *2* − *2* ∗ *cos* (*t* ∗ *x*))
unfolding *exp-Euler cos-of-real* **by** (*simp flip*: *interval-lebesgue-integral-of-real*)
also have ... = *2* ∗ *u* − *2* ∗ *sin* (*u* ∗ *x*) / *x*
by (*subst interval-lebesgue-integral-diff*)
(*auto intro*!: *interval-integrable-isCont*
simp: *interval-lebesgue-integral-of-real integral-cos* [*OF* ‹*x* ≠ *0*›] *mult.commute*[*of* - *x*])
finally show (*CLBINT t*:{−*u*..*u*}. *1* − *iexp* (*t* ∗ *x*)) = *2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*)
by (*simp add*: *field-simps*)
qed
have *main-bound*: ⋀*u n*. *u* > *0* ⟹ *Re* (*CLBINT t*:{−*u*..*u*}. *1* − *char* (*M n*) *t*) ≥
u ∗ *measure* (*M n*) {*x*. *abs x* ≥ *2* / *u*}
proof −
fix *u* :: *real* **and** *n*
assume *u* > *0*
interpret *P*: *pair-sigma-finite M n lborel* **..**

have *Mn1* [*simp*]: *measure* (*M n*) *UNIV* = *1* **by** (*metis Mn.prob-space Mn.space-eq-univ*)

have *Mn2* [*simp*]: ⋀*x*. *complex-integrable* (*M n*) (λ*t*. *exp* (i ∗ *complex-of-real* (*x* ∗ *t*)))
by (*rule Mn.integrable-const-bound* [**where** *B* = *1*], *auto*)
have *Mn3*: *set-integrable* (*M n* ⨂*M lborel*) (*UNIV* × {− *u*..*u*}) (λ*a*. *1* − *exp* (i ∗ *complex-of-real* (*snd a* ∗ *fst a*)))
using ‹*0* < *u*›
unfolding *set-integrable-def*
by (*intro integrableI-bounded-set-indicator* [**where** *B*=*2*])
(*auto simp*: *lborel.emeasure-pair-measure-Times ennreal-mult-less-top not-less top-unique*
split: *split-indicator*
intro!: *order-trans* [*OF norm-triangle-ineq4*])
have (*CLBINT t*:{−*u*..*u*}. *1* − *char* (*M n*) *t*) =
(*CLBINT t*:{−*u*..*u*}. (*CLINT x* | *M n*. *1* − *iexp* (*t* ∗ *x*)))
unfolding *char-def* **by** (*rule set-lebesgue-integral-cong*, *auto simp del*: *of-real-mult*)
also have ... = (*CLBINT t*. (*CLINT x* | *M n*. *indicator* {−*u*..*u*} *t* ∗$_R$ (*1* −

iexp (*t* ∗ *x*))))
unfolding *set-lebesgue-integral-def*
by (*rule Bochner-Integration.integral-cong*) (*auto split*: *split-indicator*)
also have ... = (*CLINT x* | *M n*. (*CLBINT t*:{−*u*..*u*}. *1* − *iexp* (*t* ∗ *x*)))
using *Mn3* **by** (*subst P.Fubini-integral*) (*auto simp*: *indicator-times split-beta′ set-integrable-def set-lebesgue-integral-def*)
also have ... = (*CLINT x* | *M n*. (*if x* = *0 then 0 else 2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*)))
using ‹*u* > *0*› **by** (*intro Bochner-Integration.integral-cong*, *auto simp add*: ∗ *simp del*: *of-real-mult*)
also have ... = (*LINT x* | *M n*. (*if x* = *0 then 0 else 2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*)))
by (*rule integral-complex-of-real*)
finally have *Re* (*CLBINT t*:{−*u*..*u*}. *1* − *char* (*M n*) *t*) =
(*LINT x* | *M n*. (*if x* = *0 then 0 else 2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*))) **by** *simp*
also have ... ≥ (*LINT x* : {*x*. *abs x* ≥ *2* / *u*} | *M n*. *u*)
proof −
have *complex-integrable* (*M n*) (λ*x*. *CLBINT t*:{−*u*..*u*}. *1* − *iexp* (*snd* (*x*, *t*) ∗ *fst* (*x*, *t*)))
using *Mn3* **unfolding** *set-integrable-def set-lebesgue-integral-def*
by (*intro P.integrable-fst*) (*simp add*: *indicator-times split-beta′*)
hence *complex-integrable* (*M n*) (λ*x*. *if x* = *0 then 0 else 2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*))
using ‹*u* > *0*›
unfolding *set-integrable-def*
by (*subst Bochner-Integration.integrable-cong*) (*auto simp add*: ∗ *simp del*: *of-real-mult*)
hence ∗∗: *integrable* (*M n*) (λ*x*. *if x* = *0 then 0 else 2* ∗ (*u* − *sin* (*u* ∗ *x*) / *x*))
unfolding *complex-of-real-integrable-eq* **.**
have *2* ∗ *sin x* ≤ *x* **if** *2* ≤ *x* **for** *x* :: *real*
by (*rule order-trans*[*OF* - ‹*2* ≤ *x*›]) *auto*
moreover have *x* ≤ *2* ∗ *sin x* **if** *x* ≤ − *2* **for** *x* :: *real*
by (*rule order-trans*[*OF* ‹*x* ≤ − *2*›]) *auto*
moreover have *x* < *0* ⟹ *x* ≤ *sin x* **for** *x* :: *real*
using *sin-x-le-x*[*of* −*x*] **by** *simp*
ultimately show *?thesis*
using ‹*u* > *0*› **unfolding** *set-lebesgue-integral-def*
by (*intro integral-mono* [*OF* - ∗∗])
(*auto simp*: *divide-simps sin-x-le-x mult.commute*[*of u*] *mult-neg-pos top-unique less-top*[*symmetric*]
split: *split-indicator*)
qed
also (*xtrans*) **have** (*LINT x* : {*x*. *abs x* ≥ *2* / *u*} | *M n*. *u*) = *u* ∗ *measure* (*M n*) {*x*. *abs x* ≥ *2* / *u*}
unfolding *set-lebesgue-integral-def*
by (*simp add*: *Mn.emeasure-eq-measure*)
finally show *Re* (*CLBINT t*:{−*u*..*u*}. *1* − *char* (*M n*) *t*) ≥ *u* ∗ *measure* (*M n*) {*x*. *abs x* ≥ *2* / *u*} **.**

qed

have *tight-aux*: $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow \exists\, a\ b.\ a < b \wedge (\forall\, n.\ 1 - \varepsilon < \mathit{measure}\ (M\ n)\ \{a<..b\})$
proof −
fix ε :: *real*
assume $\varepsilon > 0$
with *M'.isCont-char* [*of 0*]
obtain *d* **where** *d0*: $d>0$ **and** $\forall\, x'.\ \mathit{dist}\ x'\ 0 < d \longrightarrow \mathit{dist}\ (\mathit{char}\ M'\ x')\ (\mathit{char}\ M'\ 0) < \varepsilon/4$
unfolding *continuous-at-eps-delta* **by** (*metis ‹0 < ε› divide-pos-pos zero-less-numeral*)
then have *d1*: $\bigwedge t.\ \mathit{abs}\ t < d \Longrightarrow \mathit{cmod}\ (\mathit{char}\ M'\ t - 1) < \varepsilon\ /\ 4$
by (*simp add*: *M'.char-zero dist-norm*)
have *1*: $\bigwedge x.\ \mathit{cmod}\ (1 - \mathit{char}\ M'\ x) \leq 2$
by (*rule order-trans* [*OF norm-triangle-ineq4*], *auto simp add*: *M'.cmod-char-le-1*)
then have *2*: $\bigwedge u\ v.\ \mathit{complex\text{-}set\text{-}integrable}\ \mathit{lborel}\ \{u..v\}\ (\lambda x.\ 1 - \mathit{char}\ M'\ x)$
unfolding *set-integrable-def*
by (*intro integrableI-bounded-set-indicator*[**where** *B=2*]) (*auto simp*: *emeasure-lborel-Icc-eq*)
have *3*: $\bigwedge u\ v.\ \mathit{integrable}\ \mathit{lborel}\ (\lambda x.\ \mathit{indicat\text{-}real}\ \{u..v\}\ x *_R \mathit{cmod}\ (1 - \mathit{char}\ M'\ x))$
by (*intro borel-integrable-compact*[*OF compact-Icc*] *continuous-at-imp-continuous-on continuous-intros ballI M'.isCont-char continuous-intros*)
have $\mathit{cmod}\ (\mathit{CLBINT}\ t{:}\{-d/2..d/2\}.\ 1 - \mathit{char}\ M'\ t) \leq (\mathit{LBINT}\ t{:}\{-d/2..d/2\}.\ \mathit{cmod}\ (1 - \mathit{char}\ M'\ t))$
unfolding *set-lebesgue-integral-def*
using *integral-norm-bound*[*of* - $\lambda x.\ \mathit{indicator}\ \{u..v\}\ x *_R (1 - \mathit{char}\ M'\ x)$ **for** *u v*] **by** *simp*
also have *4*: $\ldots \leq (\mathit{LBINT}\ t{:}\{-d/2..d/2\}.\ \varepsilon\ /\ 4)$
unfolding *set-lebesgue-integral-def*
proof (*rule integral-mono* [*OF 3*])

show $\mathit{indicat\text{-}real}\ \{-\ d\ /\ 2..d\ /\ 2\}\ x *_R \mathit{cmod}\ (1 - \mathit{char}\ M'\ x) \leq \mathit{indicat\text{-}real}\ \{-\ d\ /\ 2..d\ /\ 2\}\ x *_R (\varepsilon\ /\ 4)$
if $x \in \mathit{space}\ \mathit{lborel}$ **for** *x*
proof (*cases* $x \in \{-d/2..d/2\}$)
case *True*
show *?thesis*
using *d0 d1 that True* [*simplified*]
by (*smt* (*verit*, *best*) *field-sum-of-halves minus-diff-eq norm-minus-cancel indicator-pos-le scaleR-left-mono*)
qed *auto*
qed (*simp add*: *emeasure-lborel-Icc-eq*)
also from *d0 4* **have** $\ldots = d * \varepsilon\ /\ 4$
unfolding *set-lebesgue-integral-def* **by** *simp*
finally have *bound*: $\mathit{cmod}\ (\mathit{CLBINT}\ t{:}\{-d/2..d/2\}.\ 1 - \mathit{char}\ M'\ t) \leq d * \varepsilon\ /\ 4$.
have $\mathit{cmod}\ (1 - \mathit{char}\ (M\ n)\ x) \leq 2$ **for** *n x*
by (*rule order-trans* [*OF norm-triangle-ineq4*], *auto simp add*: *Mn.cmod-char-le-1*)

then have (λn. *CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) $\longrightarrow$ (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)
unfolding *set-lebesgue-integral-def*
apply (*intro integral-dominated-convergence*[**where** w=λx. *indicator* {$-d/2..d/2$} $x *_R 2$])
apply (*auto intro*!: *char-conv tendsto-intros*
simp: *emeasure-lborel-Icc-eq*
split: *split-indicator*)
done
hence *eventually* (λn. *cmod* ((*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) − (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)) < $d * \varepsilon / 4$) *sequentially*
using *d0* ‹$\varepsilon > 0$› **apply** (*subst* (*asm*) *tendsto-iff*)
by (*subst* (*asm*) *dist-complex-def*, *drule spec*, *erule mp*, *auto*)
hence $\exists N.\ \forall n \geq N.$ *cmod* ((*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) − (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)) < $d * \varepsilon / 4$ **by** (*simp add*: *eventually-sequentially*)
then obtain N
where $\forall n{\geq}N.$ *cmod* ((*CLBINT* t:{$-\ d\ /\ 2..d\ /\ 2$}. $1 - char\ (M\ n)\ t$) − (*CLBINT* t:{$-\ d\ /\ 2..d\ /\ 2$}. $1 - char\ M'\ t$)) < $d * \varepsilon / 4$ **..**
hence N: $\bigwedge n.\ n \geq N \Longrightarrow$ *cmod* ((*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) −
(*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)) < $d * \varepsilon / 4$ **by** *auto*
{ fix n
assume $n \geq N$
have *cmod* (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) =
cmod ((*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) − (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)
+ (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)) **by** *simp*
also have $\ldots \leq$ *cmod* ((*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) − (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)) + *cmod*(*CLBINT* t:{$-d/2..d/2$}. $1 - char\ M'\ t$)
by (*rule norm-triangle-ineq*)
also have $\ldots < d * \varepsilon / 4 + d * \varepsilon / 4$
by (*rule add-less-le-mono* [*OF N* [*OF* ‹$n \geq N$›] *bound*])
also have $\ldots = d * \varepsilon / 2$ **by** *auto*
finally have *cmod* (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$) < $d * \varepsilon / 2$ **.**
hence $d * \varepsilon / 2 > Re$ (*CLBINT* t:{$-d/2..d/2$}. $1 - char\ (M\ n)\ t$)
by (*rule order-le-less-trans* [*OF complex-Re-le-cmod*])
hence $d * \varepsilon / 2 > Re$ (*CLBINT* t:{$-(d/2)..d/2$}. $1 - char\ (M\ n)\ t$) (**is** - > *?lhs*) **by** *simp*
also have *?lhs* $\geq (d\ /\ 2) *$ *measure* ($M\ n$) {x. *abs* $x \geq 2\ /\ (d\ /\ 2)$}
using *d0* **by** (*intro main-bound*, *simp*)
finally (*xtrans*) **have** $d * \varepsilon / 2 > (d\ /\ 2) *$ *measure* ($M\ n$) {x. *abs* $x \geq 2\ /\ (d\ /\ 2)$} **.**
with *d0* ‹$\varepsilon > 0$› **have** $\varepsilon >$ *measure* ($M\ n$) {x. *abs* $x \geq 2\ /\ (d\ /\ 2)$} **by** (*simp add*: *field-simps*)
hence $\varepsilon > 1 -$ *measure* ($M\ n$) (*UNIV* − {x. *abs* $x \geq 2\ /\ (d\ /\ 2)$})
apply (*subst Mn.borel-UNIV* [*symmetric*])

```
      by (subst Mn.prob-compl, auto)
    also have UNIV − {x. abs x ≥ 2 / (d / 2)} = {x. −(4 / d) < x ∧ x < (4
/ d)}
      using d0 by (simp add: set-eq-iff divide-simps abs-if) (smt (verit, best)
mult-less-0-iff)
    finally have measure (M n) {x. −(4 / d) < x ∧ x < (4 / d)} > 1 − ε
      by auto
  } note 6 = this
  { fix n :: nat
    have ∗: (UN (k :: nat). {− real k<..real k}) = UNIV
     by (auto, metis leI le-less-trans less-imp-le minus-less-iff reals-Archimedean2)
    have (λk. measure (M n) {− real k<..real k}) ⟶
        measure (M n) (UN (k :: nat). {− real k<..real k})
      by (rule Mn.finite-Lim-measure-incseq, auto simp add: incseq-def)
    hence (λk. measure (M n) {− real k<..real k}) ⟶ 1
      using Mn.prob-space unfolding ∗ Mn.borel-UNIV by simp
   hence eventually (λk. measure (M n) {− real k<..real k} > 1 − ε) sequentially
      using ‹ε > 0› order-tendstoD by fastforce
  } note 7 = this
  { fix n :: nat
    have eventually (λk. ∀ m < n. measure (M m) {− real k<..real k} > 1 − ε)
sequentially
      (is ?P n)
    proof (induct n)
      case (Suc n) with 7[of n] show ?case
        by eventually-elim (auto simp add: less-Suc-eq)
    qed simp
  } note 8 = this
  from 8 [of N] have ∃ K :: nat. ∀ k ≥ K. ∀ m<N. 1 − ε <
      Sigma-Algebra.measure (M m) {− real k<..real k}
    by (auto simp add: eventually-sequentially)
   hence ∃ K :: nat. ∀ m<N. 1 − ε < Sigma-Algebra.measure (M m) {− real
K<..real K} by auto
  then obtain K :: nat where
    ∀ m<N. 1 − ε < Sigma-Algebra.measure (M m) {− real K<..real K} ..
   hence K: ⋀m. m < N ⟹ 1 − ε < Sigma-Algebra.measure (M m) {− real
K<..real K}
    by auto
  let ?K′ = max K (4 / d)
   have 1 − ε < measure (M n) {− max (real K) (4 / d)<..max (real K) (4 /
d)} for n
  proof (cases n < N)
    case True
    then show ?thesis
      by (force intro: order-less-le-trans [OF K Mn.finite-measure-mono])
  next
    case False
    then show ?thesis
      by (force intro: order-less-le-trans [OF 6 Mn.finite-measure-mono])
```

qed
then have $-?K' < ?K' \wedge (\forall\, n.\ 1 - \varepsilon < measure\ (M\ n)\ \{-?K'<..?K'\})$
using *d0* **by** (*simp add*: *less-max-iff-disj minus-less-iff*)
thus $\exists\, a\ b.\ a < b \wedge (\forall\, n.\ 1 - \varepsilon < measure\ (M\ n)\ \{a<..b\})$ **by** (*intro exI*)
qed
have *tight*: *tight M*
by (*auto simp*: *tight-def intro*: *assms tight-aux*)
show *?thesis*
proof (*rule tight-subseq-weak-converge* [*OF real-distr-M real-distr-M' tight*])
fix *s* ν
assume *s*: *strict-mono* (*s* :: *nat* $\Rightarrow$ *nat*)
assume *nu*: *weak-conv-m* (*M* $\circ$ *s*) ν
assume $*$: *real-distribution* ν
have *2*: $\bigwedge n$. *real-distribution* ((*M* $\circ$ *s*) *n*) **unfolding** *comp-def* **by** (*rule assms*)
have *3*: $\bigwedge t$. (λn. *char* ((*M* $\circ$ *s*) *n*) *t*) $\longrightarrow$ *char* ν *t* **by** (*intro levy-continuity1* [*OF 2* $*$ *nu*])
have *4*: $\bigwedge t$. (λn. *char* ((*M* $\circ$ *s*) *n*) *t*) = ((λn. *char* (*M n*) *t*) $\circ$ *s*) **by** (*rule ext*, *simp*)
have *5*: $\bigwedge t$. (λn. *char* ((*M* $\circ$ *s*) *n*) *t*) $\longrightarrow$ *char M' t*
by (*subst 4*, *rule LIMSEQ-subseq-LIMSEQ* [*OF - s*], *rule assms*)
hence *char* ν = *char M'* **by** (*intro ext*, *intro LIMSEQ-unique* [*OF 3 5*])
hence ν = *M'* **by** (*rule Levy-uniqueness* [*OF* $*$ ‹*real-distribution M'*›])
thus *weak-conv-m* (*M* $\circ$ *s*) *M'*
by (*elim subst*) (*rule nu*)
qed
qed

end

19 The Central Limit Theorem

theory *Central-Limit-Theorem*
imports *Levy*
begin

theorem (**in** *prob-space*) *central-limit-theorem-zero-mean*:
fixes *X* :: *nat* $\Rightarrow$ *'a* $\Rightarrow$ *real*
and μ :: *real measure*
and σ :: *real*
and *S* :: *nat* $\Rightarrow$ *'a* $\Rightarrow$ *real*
assumes *X-indep*: *indep-vars* (λi. *borel*) *X UNIV*
and *X-mean-0*: $\bigwedge n$. *expectation* (*X n*) = *0*
and σ*-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n$. *integrable M* (λx. (*X n x*)^2)
and *X-variance*: $\bigwedge n$. *variance* (*X n*) = σ^2
and *X-distrib*: $\bigwedge n$. *distr M borel* (*X n*) = μ
defines *S n* $\equiv$ λx. $\sum i<n$. *X i x*
shows *weak-conv-m* (λn. *distr M borel* (λx. *S n x* / *sqrt* (*n* $*$ σ^2))) *std-normal-distribution*
proof −

let *?S′* = $\lambda n\ x.\ S\ n\ x\ /\ sqrt\ (real\ n * \sigma^2)$
define φ **where** $\varphi\ n$ = *char* (*distr M borel* (*?S′ n*)) **for** *n*
define ψ **where** $\psi\ n\ t$ = *char* μ $(t\ /\ sqrt\ (\sigma^2 * n))$ **for** *n t*

have *X-rv* [*simp*, *measurable*]: *random-variable borel* (*X n*) **for** *n*
using *X-indep* **unfolding** *indep-vars-def2* **by** *simp*
have *X-integrable* [*simp*, *intro*]: *integrable M* (*X n*) **for** *n*
by (*rule square-integrable-imp-integrable*[*OF - X-square-integrable*]) *simp-all*
interpret μ: *real-distribution* μ
by (*subst X-distrib* [*symmetric*, *of 0*], *rule real-distribution-distr*, *simp*)

have μ*-integrable* [*simp*]: *integrable* μ $(\lambda x.\ x)$
and μ*-mean-integrable* [*simp*]: μ*.expectation* $(\lambda x.\ x) = 0$
and μ*-square-integrable* [*simp*]: *integrable* μ $(\lambda x.\ x\hat{}2)$
and μ*-variance* [*simp*]: μ*.expectation* $(\lambda x.\ x\hat{}2) = \sigma^2$
using *assms* **by** (*simp-all add*: *X-distrib* [*symmetric*, *of 0*] *integrable-distr-eq integral-distr*)

have *main*: $\forall_F\ n$ *in sequentially.*
cmod $(\varphi\ n\ t - (1 + (-(t\hat{}2)\ /\ 2)\ /\ n)\hat{}n) \leq$
$t^2\ /\ (6 * \sigma^2) * (LINT\ x|\mu.\ min\ (6 * x^2)\ (|t\ /\ sqrt\ (\sigma^2 * n)| * |x|\ \hat{}\ 3))$ **for** *t*
proof (*rule eventually-sequentiallyI*)
fix *n* :: *nat*
assume $n \geq nat\ (ceiling\ (t\hat{}2\ /\ 4))$
hence *n*: $n \geq t\hat{}2\ /\ 4$ **by** (*subst nat-ceiling-le-eq* [*symmetric*])
let *?t* = $t\ /\ sqrt\ (\sigma^2 * n)$

define ψ' **where** $\psi'\ n\ i$ = *char* $(distr\ M\ borel\ (\lambda x.\ X\ i\ x\ /\ sqrt\ (\sigma^2 * n)))$ **for** *n i*
have $*$: $\bigwedge n\ i\ t.\ \psi'\ n\ i\ t = \psi\ n\ t$
unfolding ψ*-def* ψ'*-def char-def*
by (*subst X-distrib* [*symmetric*]) (*auto simp*: *integral-distr*)

have $\varphi\ n\ t$ = *char* $(distr\ M\ borel\ (\lambda x.\ \sum i{<}n.\ X\ i\ x\ /\ sqrt\ (\sigma^2 * real\ n)))\ t$
by (*auto simp*: φ*-def S-def sum-divide-distrib ac-simps*)
also have ... = $(\prod\ i < n.\ \psi'\ n\ i\ t)$
unfolding ψ'*-def*
apply (*rule char-distr-sum*)
apply (*rule indep-vars-compose2*[**where** *X*=*X*])
apply (*rule indep-vars-subset*)
apply (*rule X-indep*)
apply *auto*
done
also have ... = $(\psi\ n\ t)\hat{}n$
by (*auto simp add*: $*$ *prod-constant*)
finally have φ*-eq*: $\varphi\ n\ t = (\psi\ n\ t)\hat{}n$ **.**

have *norm* $(\psi\ n\ t - (1 - ?t\hat{}2 * \sigma^2\ /\ 2)) \leq ?t^2\ /\ 6 * (LINT\ x|\mu.\ min\ (6 *$

```
x^2) (|?t| * |x| ^ 3))
      unfolding ψ-def by (rule μ.char-approx3, auto)
    also have ?t^2 * σ^2 = t^2 / n
      using σ-pos by (simp add: power-divide)
    also have t^2 / n / 2 = (t^2 / 2) / n
      by simp
    finally have **: norm (ψ n t - (1 + (-(t^2) / 2) / n)) ≤
      ?t^2 / 6 * (LINT x|μ. min (6 * x^2) (|?t| * |x| ^ 3)) by simp

    have norm (φ n t - (complex-of-real (1 + (-(t^2) / 2) / n))^n) ≤
        n * norm (ψ n t - (complex-of-real (1 + (-(t^2) / 2) / n)))
      using n
      by (auto intro!: norm-power-diff μ.cmod-char-le-1 abs-leI
              simp del: of-real-diff simp: of-real-diff[symmetric] divide-le-eq φ-eq
ψ-def)
    also have ... ≤ n * (?t^2 / 6 * (LINT x|μ. min (6 * x^2) (|?t| * |x| ^ 3)))
      by (rule mult-left-mono [OF **], simp)
    also have ... = (t^2 / (6 * σ^2) * (LINT x|μ. min (6 * x^2) (|?t| * |x| ^ 3)))
      using σ-pos by (simp add: field-simps min-absorb2)
    finally show norm (φ n t - (1 + (-(t^2) / 2) / n)^n) ≤
       (t^2 / (6 * σ^2) * (LINT x|μ. min (6 * x^2) (|?t| * |x| ^ 3)))
      by simp
  qed

  show ?thesis
  proof (rule levy-continuity)
    fix t
    let ?t = λn. t / sqrt (σ^2 * n)
    have ⋀x. (λn. min (6 * x^2) (|t| * |x| ^ 3 / |sqrt (σ^2 * real n)|)) ⟶ 0
      using σ-pos
      by (auto simp: real-sqrt-mult min-absorb2
               intro!: tendsto-min[THEN tendsto-eq-rhs] sqrt-at-top[THEN filter-
lim-compose]
                    filterlim-tendsto-pos-mult-at-top filterlim-at-top-imp-at-infinity
                    tendsto-divide-0 filterlim-real-sequentially)
    then have (λn. LINT x|μ. min (6 * x^2) (|?t n| * |x| ^ 3)) ⟶ (LINT x|μ.
0)
      by (intro integral-dominated-convergence [where w = λx. 6 * x^2]) auto
    then have *: (λn. t^2 / (6 * σ^2) * (LINT x|μ. min (6 * x^2) (|?t n| * |x| ^ 3)))
⟶ 0
      by (simp only: integral-zero tendsto-mult-right-zero)

    have (λn. complex-of-real ((1 + (-(t^2) / 2) / n)^n)) ⟶ complex-of-real
(exp (-(t^2) / 2))
      by (rule isCont-tendsto-compose [OF - tendsto-exp-limit-sequentially]) auto
    then have (λn. φ n t) ⟶ complex-of-real (exp (-(t^2) / 2))
      by (rule Lim-transform) (rule Lim-null-comparison [OF main *])
  then show (λn. char (distr M borel (?S' n)) t) ⟶ char std-normal-distribution
t
```

by (*simp add*: φ*-def char-std-normal-distribution*)
qed (*auto intro*!: *real-dist-normal-dist simp*: *S-def*)
qed

theorem (**in** *prob-space*) *central-limit-theorem*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: *indep-vars* $(\lambda i.\ borel)\ X\ UNIV$
and *X-mean*: $\bigwedge n.\ expectation\ (X\ n) = m$
and σ*-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $X'\ i\ x \equiv X\ i\ x - m$
shows *weak-conv-m* $(\lambda n.\ distr\ M\ borel\ (\lambda x.\ (\sum i{<}n.\ X'\ i\ x)\ /\ sqrt\ (n{*}\sigma^2)))$
std-normal-distribution
proof (*intro central-limit-theorem-zero-mean*)
show *indep-vars* $(\lambda i.\ borel)\ X'\ UNIV$
unfolding *X'-def*[*abs-def*] **using** *X-indep* **by** (*rule indep-vars-compose2*) *auto*
have *X-rv* [*simp*, *measurable*]: *random-variable borel* $(X\ n)$ **for** n
using *X-indep* **unfolding** *indep-vars-def2* **by** *simp*
have *X-integrable* [*simp*, *intro*]: *integrable M* $(X\ n)$ **for** n
by (*rule square-integrable-imp-integrable*[*OF - X-square-integrable*]) *simp-all*
show *expectation* $(X'\ n) = 0$ **for** n
using *X-mean* **by** (*auto simp*: *X'-def*[*abs-def*] *prob-space*)
show $\sigma > 0$ *integrable M* $(\lambda x.\ (X'\ n\ x)^2)$ *variance* $(X'\ n) = \sigma^2$ **for** n
using ‹$0 < \sigma$› *X-integrable X-mean X-square-integrable X-variance* **unfolding** *X'-def*
by (*auto simp*: *prob-space power2-diff*)
show *distr M borel* $(X'\ n) = distr\ \mu\ borel\ (\lambda x.\ x - m)$ **for** n
unfolding *X-distrib*[*of n*, *symmetric*] **using** *X-integrable*
by (*subst distr-distr*) (*auto simp*: *X'-def*[*abs-def*] *comp-def*)
qed

end

theory *Discrete-Topology*
imports *HOL−Analysis.Analysis*
begin

Copy of discrete types with discrete topology. This space is polish.

typedef *'a discrete* = *UNIV*::*'a set*
morphisms *of-discrete discrete*
..

instantiation *discrete* :: (*type*) *metric-space*

begin

definition *dist-discrete* :: *'a discrete ⇒ 'a discrete ⇒ real*
where *dist-discrete n m = (if n = m then 0 else 1)*

definition *uniformity-discrete* :: *('a discrete × 'a discrete) filter* **where**
(uniformity::('a discrete × 'a discrete) filter) = (INF e∈{0 <..}. principal {(x, y). dist x y < e})

definition *open-discrete* :: *'a discrete set ⇒ bool* **where**
(open::'a discrete set ⇒ bool) U ⟷ (∀x∈U. eventually (λ(x', y). x' = x ⟶ y ∈ U) uniformity)

instance proof qed (*auto simp*: *uniformity-discrete-def open-discrete-def dist-discrete-def intro*: *exI*[**where** *x=1*])
end

lemma *open-discrete*: *open* (*S* :: *'a discrete set*)
unfolding *open-dist dist-discrete-def* **by** (*auto intro*!: *exI*[*of - 1 / 2*])

instance *discrete* :: (*type*) *complete-space*
proof
fix *X*::*nat*⇒*'a discrete*
assume *Cauchy X*
then obtain *n* **where** *∀m≥n. X n = X m*
by (*force simp*: *dist-discrete-def Cauchy-def split*: *if-split-asm dest*:*spec*[**where** *x=1*])
thus *convergent X*
by (*intro convergentI*[**where** *L=X n*] *tendstoI eventually-sequentiallyI*[*of n*])
(*simp add*: *dist-discrete-def*)
qed

instance *discrete* :: (*countable*) *countable*
proof
have *inj* (*λc::'a discrete. to-nat* (*of-discrete c*))
by (*simp add*: *inj-on-def of-discrete-inject*)
thus *∃f::'a discrete ⇒ nat. inj f* **by** *blast*
qed

instance *discrete* :: (*countable*) *second-countable-topology*
proof
let *?B = range* (*λn::'a discrete.* {*n*})
have *⋀S. generate-topology ?B* (*⋃x∈S.* {*x*})
by (*intro generate-topology-Union*) (*auto intro*: *generate-topology.intros*)
then have *open = generate-topology ?B*
by (*auto intro*!: *ext simp*: *open-discrete*)
moreover have *countable ?B* **by** *simp*
ultimately show *∃B::'a discrete set set. countable B ∧ open = generate-topology B* **by** *blast*

qed

instance *discrete* :: (*countable*) *polish-space* **..**

end

20 Probability mass function

theory *Probability-Mass-Function*
imports
Giry-Monad
HOL−Library.Multiset
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** ‹*abs'-summable'-on*› *46*)

lemma *AE-emeasure-singleton*:
assumes *x*: *emeasure M* {*x*} ≠ *0* **and** *ae*: *AE x in M*. *P x* **shows** *P x*
proof −
from *x* **have** *x-M*: {*x*} ∈ *sets M*
by (*auto intro*: *emeasure-notin-sets*)
from *ae* **obtain** *N* **where** *N*: {*x*∈*space M*. ¬ *P x*} ⊆ *N emeasure M N* = *0 N* ∈ *sets M*
by (*auto elim*: *AE-E*)
{ assume ¬ *P x*
with *x-M*[*THEN sets.sets-into-space*] *N* **have** *emeasure M* {*x*} ≤ *emeasure M N*
by (*intro emeasure-mono*) *auto*
with *x N* **have** *False*
by (*auto simp*:) **}**
then show *P x* **by** *auto*
qed

lemma *AE-measure-singleton*: *measure M* {*x*} ≠ *0* ⟹ *AE x in M*. *P x* ⟹ *P x*
by (*metis AE-emeasure-singleton measure-def emeasure-empty measure-empty*)

lemma (**in** *finite-measure*) *AE-support-countable*:
assumes [*simp*]: *sets M* = *UNIV*
shows (*AE x in M*. *measure M* {*x*} ≠ *0*) ⟷ (∃ *S*. *countable S* ∧ (*AE x in M*. *x* ∈ *S*))
proof
assume ∃ *S*. *countable S* ∧ (*AE x in M*. *x* ∈ *S*)
then obtain *S* **where** *S*[*intro*]: *countable S* **and** *ae*: *AE x in M*. *x* ∈ *S*
by *auto*
then have *emeasure M* (⋃ *x*∈{*x*∈*S*. *emeasure M* {*x*} ≠ *0*}. {*x*}) =
(∫⁺ *x*. *emeasure M* {*x*} ∗ *indicator* {*x*∈*S*. *emeasure M* {*x*} ≠ *0*} *x* ∂*count-space UNIV*)
by (*subst emeasure-UN-countable*)

(*auto simp*: *disjoint-family-on-def nn-integral-restrict-space*[*symmetric*] *restrict-count-space*)
also have ... = ($\int^+ x.$ *emeasure M* {*x*} ∗ *indicator S x* ∂*count-space UNIV*)
by (*auto intro*!: *nn-integral-cong split*: *split-indicator*)
also have ... = *emeasure M* ($\bigcup x{\in}S.$ {*x*})
by (*subst emeasure-UN-countable*)
(*auto simp*: *disjoint-family-on-def nn-integral-restrict-space*[*symmetric*] *restrict-count-space*)
also have ... = *emeasure M* (*space M*)
using *ae* **by** (*intro emeasure-eq-AE*) *auto*
finally have *emeasure M* {*x* ∈ *space M*. *x*∈*S* ∧ *emeasure M* {*x*} ≠ *0*} = *emeasure M* (*space M*)
by (*simp add*: *emeasure-single-in-space cong*: *rev-conj-cong*)
with *finite-measure-compl*[*of* {*x* ∈ *space M*. *x*∈*S* ∧ *emeasure M* {*x*} ≠ *0*}]
have *AE x in M*. *x* ∈ *S* ∧ *emeasure M* {*x*} ≠ *0*
by (*intro AE-I*[*OF order-refl*]) (*auto simp*: *emeasure-eq-measure measure-nonneg set-diff-eq cong*: *conj-cong*)
then show *AE x in M*. *measure M* {*x*} ≠ *0*
by (*auto simp*: *emeasure-eq-measure*)
qed (*auto intro*!: *exI*[*of* - {*x*. *measure M* {*x*} ≠ *0*}] *countable-support*)

20.1 PMF as measure

typedef *'a pmf* = {*M* :: *'a measure*. *prob-space M* ∧ *sets M* = *UNIV* ∧ (*AE x in M*. *measure M* {*x*} ≠ *0*)}
morphisms *measure-pmf Abs-pmf*
by (*intro exI*[*of* - *uniform-measure* (*count-space UNIV*) {*undefined*}])
(*auto intro*!: *prob-space-uniform-measure AE-uniform-measureI*)

declare [[*coercion measure-pmf*]]

lemma *prob-space-measure-pmf*: *prob-space* (*measure-pmf p*)
using *pmf.measure-pmf*[*of p*] **by** *auto*

interpretation *measure-pmf*: *prob-space measure-pmf M* **for** *M*
by (*rule prob-space-measure-pmf*)

interpretation *measure-pmf*: *subprob-space measure-pmf M* **for** *M*
by (*rule prob-space-imp-subprob-space*) *unfold-locales*

lemma *subprob-space-measure-pmf*: *subprob-space* (*measure-pmf x*)
by *unfold-locales*

locale *pmf-as-measure*
begin

setup-lifting *type-definition-pmf*

end

context
begin

interpretation *pmf-as-measure* **.**

lemma *sets-measure-pmf*[*simp*]: *sets* (*measure-pmf p*) = *UNIV*
by *transfer blast*

lemma *sets-measure-pmf-count-space*[*measurable-cong*]:
sets (*measure-pmf M*) = *sets* (*count-space UNIV*)
by *simp*

lemma *space-measure-pmf*[*simp*]: *space* (*measure-pmf p*) = *UNIV*
using *sets-eq-imp-space-eq*[*of measure-pmf p count-space UNIV*] **by** *simp*

lemma *measure-pmf-UNIV* [*simp*]: *measure* (*measure-pmf p*) *UNIV* = *1*
using *measure-pmf.prob-space*[*of p*] **by** *simp*

lemma *measure-pmf-in-subprob-algebra*[*measurable* (*raw*)]: *measure-pmf x* ∈ *space* (*subprob-algebra* (*count-space UNIV*))
by (*simp add*: *space-subprob-algebra subprob-space-measure-pmf*)

lemma *measurable-pmf-measure1*[*simp*]: *measurable* (*M* :: *'a pmf*) *N* = *UNIV* → *space N*
by (*auto simp*: *measurable-def*)

lemma *measurable-pmf-measure2*[*simp*]: *measurable N* (*M* :: *'a pmf*) = *measurable N* (*count-space UNIV*)
by (*intro measurable-cong-sets*) *simp-all*

lemma *measurable-pair-restrict-pmf2*:
assumes *countable A*
assumes [*measurable*]: ⋀*y*. *y* ∈ *A* ⟹ (λ*x*. *f* (*x*, *y*)) ∈ *measurable M L*
shows *f* ∈ *measurable* (*M* ⨂$_M$ *restrict-space* (*measure-pmf N*) *A*) *L* (**is** *f* ∈ *measurable ?M* -)
proof −
have [*measurable-cong*]: *sets* (*restrict-space* (*count-space UNIV*) *A*) = *sets* (*count-space A*)
by (*simp add*: *restrict-count-space*)

show *?thesis*
by (*intro measurable-compose-countable'*[**where** *f*=λ*a b*. *f* (*fst b*, *a*) **and** *g*=*snd* **and** *I*=*A*,
unfolded prod.collapse] *assms*)
measurable
qed

lemma *measurable-pair-restrict-pmf1*:

assumes *countable A*
assumes [*measurable*]: $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x,\ y)) \in$ *measurable N L*
shows $f \in$ *measurable* (*restrict-space* (*measure-pmf M*) $A \bigotimes_M N$) *L*
proof −
have [*measurable-cong*]: *sets* (*restrict-space* (*count-space UNIV*) *A*) = *sets* (*count-space A*)
by (*simp add*: *restrict-count-space*)

show *?thesis*
by (*intro measurable-compose-countable'*[**where** $f{=}\lambda a\ b.\ f\ (a,\ snd\ b)$ **and** $g{=}fst$ **and** $I{=}A$,
unfolded prod.collapse] *assms*)
measurable
qed

lift-definition *pmf* :: *'a pmf* ⇒ *'a* ⇒ *real* **is** $\lambda M\ x.$ *measure M* $\{x\}$ **.**

lift-definition *set-pmf* :: *'a pmf* ⇒ *'a set* **is** $\lambda M.\ \{x.$ *measure M* $\{x\} \neq 0\}$ **.**
declare [[*coercion set-pmf*]]

lemma *AE-measure-pmf*: *AE x in* (*M*::*'a pmf*). $x \in M$
by *transfer simp*

lemma *emeasure-pmf-single-eq-zero-iff*:
fixes *M* :: *'a pmf*
shows *emeasure M* $\{y\} = 0 \longleftrightarrow y \notin M$
unfolding *set-pmf.rep-eq* **by** (*simp add*: *measure-pmf.emeasure-eq-measure*)

lemma *AE-measure-pmf-iff*: (*AE x in measure-pmf M. P x*) ⟷ $(\forall y{\in}M.\ P\ y)$
using *AE-measure-singleton*[*of M*] *AE-measure-pmf*[*of M*]
by (*auto simp*: *set-pmf.rep-eq*)

lemma *AE-pmfI*: $(\bigwedge y.\ y \in$ *set-pmf M* $\Longrightarrow P\ y) \Longrightarrow$ *almost-everywhere* (*measure-pmf M*) *P*
by(*simp add*: *AE-measure-pmf-iff*)

lemma *countable-set-pmf* [*simp*]: *countable* (*set-pmf p*)
by *transfer* (*metis prob-space.finite-measure finite-measure.countable-support*)

lemma *pmf-positive*: $x \in$ *set-pmf p* $\Longrightarrow 0 <$ *pmf p x*
by *transfer* (*simp add*: *less-le*)

lemma *pmf-nonneg*[*simp*]: $0 \leq$ *pmf p x*
by *transfer simp*

lemma *pmf-not-neg* [*simp*]: ¬*pmf p x* < *0*
by (*simp add*: *not-less pmf-nonneg*)

lemma *pmf-pos* [*simp*]: *pmf p x* ≠ *0* ⟹ *pmf p x* > *0*

using *pmf-nonneg*[*of p x*] **by** *linarith*

lemma *pmf-le-1*: *pmf p x* ≤ *1*
by (*simp add*: *pmf.rep-eq*)

lemma *set-pmf-not-empty*: *set-pmf M* ≠ {}
using *AE-measure-pmf*[*of M*] **by** (*intro notI*) *simp*

lemma *set-pmf-iff*: *x* ∈ *set-pmf M* ⟷ *pmf M x* ≠ *0*
by *transfer simp*

lemma *pmf-positive-iff*: *0* < *pmf p x* ⟷ *x* ∈ *set-pmf p*
unfolding *less-le* **by** (*simp add*: *set-pmf-iff*)

lemma *set-pmf-eq*: *set-pmf M* = {*x*. *pmf M x* ≠ *0*}
by (*auto simp*: *set-pmf-iff*)

lemma *set-pmf-eq'*: *set-pmf p* = {*x*. *pmf p x* > *0*}
proof *safe*
fix *x* **assume** *x* ∈ *set-pmf p*
hence *pmf p x* ≠ *0* **by** (*auto simp*: *set-pmf-eq*)
with *pmf-nonneg*[*of p x*] **show** *pmf p x* > *0* **by** *simp*
qed (*auto simp*: *set-pmf-eq*)

lemma *emeasure-pmf-single*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*x*} = *pmf M x*
by *transfer* (*simp add*: *finite-measure.emeasure-eq-measure*[*OF prob-space.finite-measure*])

lemma *measure-pmf-single*: *measure* (*measure-pmf M*) {*x*} = *pmf M x*
using *emeasure-pmf-single*[*of M x*] **by**(*simp add*: *measure-pmf.emeasure-eq-measure pmf-nonneg measure-nonneg*)

lemma *emeasure-measure-pmf-finite*: *finite S* ⟹ *emeasure* (*measure-pmf M*) *S* = (∑ *s*∈*S*. *pmf M s*)
by (*subst emeasure-eq-sum-singleton*) (*auto simp*: *emeasure-pmf-single pmf-nonneg*)

lemma *measure-measure-pmf-finite*: *finite S* ⟹ *measure* (*measure-pmf M*) *S* = *sum* (*pmf M*) *S*
using *emeasure-measure-pmf-finite*[*of S M*]
by (*simp add*: *measure-pmf.emeasure-eq-measure measure-nonneg sum-nonneg pmf-nonneg*)

lemma *sum-pmf-eq-1*:
assumes *finite A set-pmf p* ⊆ *A*
shows (∑ *x*∈*A*. *pmf p x*) = *1*
proof −
have (∑ *x*∈*A*. *pmf p x*) = *measure-pmf.prob p A*
by (*simp add*: *measure-measure-pmf-finite assms*)

also from *assms* **have** ... = *1*
by (*subst measure-pmf.prob-eq-1*) (*auto simp*: *AE-measure-pmf-iff*)
finally show *?thesis* **.**
qed

lemma *nn-integral-measure-pmf-support*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite A* **and** *nn*: ⋀*x*. *x* ∈ *A* ⟹ *0* ≤ *f x* ⋀*x*. *x* ∈ *set-pmf M* ⟹ *x* ∉ *A* ⟹ *f x* = *0*
shows (∫⁺*x*. *f x* ∂*measure-pmf M*) = (∑ *x*∈*A*. *f x* ∗ *pmf M x*)
proof −
have (∫⁺*x*. *f x* ∂*M*) = (∫⁺*x*. *f x* ∗ *indicator A x* ∂*M*)
using *nn* **by** (*intro nn-integral-cong-AE*) (*auto simp*: *AE-measure-pmf-iff split*: *split-indicator*)
also have ... = (∑ *x*∈*A*. *f x* ∗ *emeasure M* {*x*})
using *assms* **by** (*intro nn-integral-indicator-finite*) *auto*
finally show *?thesis*
by (*simp add*: *emeasure-measure-pmf-finite*)
qed

lemma *nn-integral-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite* (*set-pmf M*) **and** *nn*: ⋀*x*. *x* ∈ *set-pmf M* ⟹ *0* ≤ *f x*
shows (∫⁺*x*. *f x* ∂*measure-pmf M*) = (∑ *x*∈*set-pmf M*. *f x* ∗ *pmf M x*)
using *assms* **by** (*intro nn-integral-measure-pmf-support*) *auto*

lemma *integrable-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *finite* (*set-pmf M*) ⟹ *integrable M f*
by (*auto intro*!: *integrableI-bounded simp*: *nn-integral-measure-pmf-finite ennreal-mult-less-top*)

lemma *integral-measure-pmf-real*:
assumes [*simp*]: *finite A* **and** ⋀*a*. *a* ∈ *set-pmf M* ⟹ *f a* ≠ *0* ⟹ *a* ∈ *A*
shows (∫ *x*. *f x* ∂*measure-pmf M*) = (∑ *a*∈*A*. *f a* ∗ *pmf M a*)
proof −
have (∫ *x*. *f x* ∂*measure-pmf M*) = (∫ *x*. *f x* ∗ *indicator A x* ∂*measure-pmf M*)
using *assms*(*2*) **by** (*intro integral-cong-AE*) (*auto split*: *split-indicator simp*: *AE-measure-pmf-iff*)
also have ... = (∑ *a*∈*A*. *f a* ∗ *pmf M a*)
by (*subst integral-indicator-finite-real*)
(*auto simp*: *measure-def emeasure-measure-pmf-finite pmf-nonneg*)
finally show *?thesis* **.**
qed

lemma *integrable-pmf*: *integrable* (*count-space X*) (*pmf M*)
proof −
have (∫⁺ *x*. *pmf M x* ∂*count-space X*) = (∫⁺ *x*. *pmf M x* ∂*count-space* (*M* ∩ *X*))
by (*auto simp add*: *nn-integral-count-space-indicator set-pmf-iff intro*!: *nn-integral-cong*

split: *split-indicator*)
then have *integrable* (*count-space X*) (*pmf M*) = *integrable* (*count-space* (*M* ∩ *X*)) (*pmf M*)
by (*simp add*: *integrable-iff-bounded pmf-nonneg*)
then show *?thesis*
by (*simp add*: *pmf.rep-eq measure-pmf.integrable-measure disjoint-family-on-def*)
qed

lemma *integral-pmf*: (∫ *x*. *pmf M x* ∂*count-space X*) = *measure M X*
proof −
have (∫ *x*. *pmf M x* ∂*count-space X*) = (∫⁺*x*. *pmf M x* ∂*count-space X*)
by (*simp add*: *pmf-nonneg integrable-pmf nn-integral-eq-integral*)
also have ... = (∫⁺*x*. *emeasure M* {*x*} ∂*count-space* (*X* ∩ *M*))
by (*auto intro*!: *nn-integral-cong-AE split*: *split-indicator*
simp: *pmf.rep-eq measure-pmf.emeasure-eq-measure nn-integral-count-space-indicator AE-count-space set-pmf-iff*)
also have ... = *emeasure M* (*X* ∩ *M*)
by (*rule emeasure-countable-singleton*[*symmetric*]) (*auto intro*: *countable-set-pmf*)
also have ... = *emeasure M X*
by (*auto intro*!: *emeasure-eq-AE simp*: *AE-measure-pmf-iff*)
finally show *?thesis*
by (*simp add*: *measure-pmf.emeasure-eq-measure measure-nonneg integral-nonneg pmf-nonneg*)
qed

lemma *integral-pmf-restrict*:
(*f*::′*a* ⇒ ′*b*::{*banach*, *second-countable-topology*}) ∈ *borel-measurable* (*count-space UNIV*) ⟹
(∫ *x*. *f x* ∂*measure-pmf M*) = (∫ *x*. *f x* ∂*restrict-space M M*)
by (*auto intro*!: *integral-cong-AE simp add*: *integral-restrict-space AE-measure-pmf-iff*)

lemma *emeasure-pmf*: *emeasure* (*M*::′*a pmf*) *M* = *1*
proof −
have *emeasure* (*M*::′*a pmf*) *M* = *emeasure* (*M*::′*a pmf*) (*space M*)
by (*intro emeasure-eq-AE*) (*simp-all add*: *AE-measure-pmf*)
then show *?thesis*
using *measure-pmf.emeasure-space-1* **by** *simp*
qed

lemma *emeasure-pmf-UNIV* [*simp*]: *emeasure* (*measure-pmf M*) *UNIV* = *1*
using *measure-pmf.emeasure-space-1*[*of M*] **by** *simp*

lemma *in-null-sets-measure-pmfI*:
A ∩ *set-pmf p* = {} ⟹ *A* ∈ *null-sets* (*measure-pmf p*)
using *emeasure-eq-0-AE*[**where** *?P*=λ*x*. *x* ∈ *A* **and** *M*=*measure-pmf p*]
by(*auto simp add*: *null-sets-def AE-measure-pmf-iff*)

lemma *measure-subprob*: *measure-pmf M* ∈ *space* (*subprob-algebra* (*count-space UNIV*))

by (*simp add*: *space-subprob-algebra subprob-space-measure-pmf*)

20.2 Monad Interpretation

lemma *measurable-measure-pmf*[*measurable*]:
(λ*x*. *measure-pmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
by (*auto simp*: *space-subprob-algebra intro*!: *prob-space-imp-subprob-space*) *unfold-locales*

lemma *bind-measure-pmf-cong*:
assumes ⋀*x*. *A x* ∈ *space* (*subprob-algebra N*) ⋀*x*. *B x* ∈ *space* (*subprob-algebra N*)
assumes ⋀*i*. *i* ∈ *set-pmf x* ⟹ *A i* = *B i*
shows *bind* (*measure-pmf x*) *A* = *bind* (*measure-pmf x*) *B*
proof (*rule measure-eqI*)
show *sets* (*measure-pmf x* ⤜ *A*) = *sets* (*measure-pmf x* ⤜ *B*)
using *assms* **by** (*subst* (*1 2*) *sets-bind*) (*auto simp*: *space-subprob-algebra*)
next
fix *X* **assume** *X* ∈ *sets* (*measure-pmf x* ⤜ *A*)
then have *X*: *X* ∈ *sets N*
using *assms* **by** (*subst* (*asm*) *sets-bind*) (*auto simp*: *space-subprob-algebra*)
show *emeasure* (*measure-pmf x* ⤜ *A*) *X* = *emeasure* (*measure-pmf x* ⤜ *B*) *X*
using *assms*
by (*subst* (*1 2*) *emeasure-bind*[**where** *N*=*N*, *OF* - - *X*])
(*auto intro*!: *nn-integral-cong-AE simp*: *AE-measure-pmf-iff*)
qed

lift-definition *bind-pmf* :: *'a pmf* ⇒ (*'a* ⇒ *'b pmf*) ⇒ *'b pmf* **is** *bind*
proof (*clarify*, *intro conjI*)
fix *f* :: *'a measure* **and** *g* :: *'a* ⇒ *'b measure*
assume *prob-space f*
then interpret *f*: *prob-space f* **.**
assume *sets f* = *UNIV* **and** *ae-f*: *AE x in f*. *measure f* {*x*} ≠ *0*
then have *s-f*[*simp*]: *sets f* = *sets* (*count-space UNIV*)
by *simp*
assume *g*: ⋀*x*. *prob-space* (*g x*) ∧ *sets* (*g x*) = *UNIV* ∧ (*AE y in g x*. *measure* (*g x*) {*y*} ≠ *0*)
then have *g*: ⋀*x*. *prob-space* (*g x*) **and** *s-g*[*simp*]: ⋀*x*. *sets* (*g x*) = *sets* (*count-space UNIV*)
and *ae-g*: ⋀*x*. *AE y in g x*. *measure* (*g x*) {*y*} ≠ *0*
by *auto*

have [*measurable*]: *g* ∈ *measurable f* (*subprob-algebra* (*count-space UNIV*))
by (*auto simp*: *measurable-def space-subprob-algebra prob-space-imp-subprob-space g*)

show *prob-space* (*f* ⤜ *g*)
using *g* **by** (*intro f.prob-space-bind*[**where** *S*=*count-space UNIV*]) *auto*

then interpret *fg*: *prob-space f* ⤜ *g* **.**
show [*simp*]: *sets* (*f* ⤜ *g*) = *UNIV*
using *sets-eq-imp-space-eq*[*OF s-f*]
by (*subst sets-bind*[**where** *N*=*count-space UNIV*]) *auto*
show *AE x in f* ⤜ *g*. *measure* (*f* ⤜ *g*) {*x*} ≠ *0*
apply (*simp add*: *fg.prob-eq-0 AE-bind*[**where** *B*=*count-space UNIV*])
using *ae-f*
apply *eventually-elim*
using *ae-g*
apply *eventually-elim*
apply (*auto dest*: *AE-measure-singleton*)
done
qed

adhoc-overloading *Monad-Syntax.bind* ⇌ *bind-pmf*

lemma *ennreal-pmf-bind*: *pmf* (*bind-pmf N f*) *i* = (∫⁺*x*. *pmf* (*f x*) *i* ∂*measure-pmf N*)
unfolding *pmf.rep-eq bind-pmf.rep-eq*
by (*auto simp*: *measure-pmf.measure-bind*[**where** *N*=*count-space UNIV*] *measure-subprob measure-nonneg*
intro!: *nn-integral-eq-integral*[*symmetric*] *measure-pmf.integrable-const-bound*[**where** *B*=*1*])

lemma *pmf-bind*: *pmf* (*bind-pmf N f*) *i* = (∫ *x*. *pmf* (*f x*) *i* ∂*measure-pmf N*)
using *ennreal-pmf-bind*[*of N f i*]
by (*subst* (*asm*) *nn-integral-eq-integral*)
(*auto simp*: *pmf-nonneg pmf-le-1 pmf-nonneg integral-nonneg*
intro!: *nn-integral-eq-integral*[*symmetric*] *measure-pmf.integrable-const-bound*[**where** *B*=*1*])

lemma *bind-pmf-const*[*simp*]: *bind-pmf M* (λ*x*. *c*) = *c*
by *transfer* (*simp add*: *bind-const' prob-space-imp-subprob-space*)

lemma *set-bind-pmf*[*simp*]: *set-pmf* (*bind-pmf M N*) = (⋃ *M*∈*set-pmf M*. *set-pmf* (*N M*))
proof −
have *set-pmf* (*bind-pmf M N*) = {*x*. *ennreal* (*pmf* (*bind-pmf M N*) *x*) ≠ *0*}
by (*simp add*: *set-pmf-eq pmf-nonneg*)
also have ... = (⋃ *M*∈*set-pmf M*. *set-pmf* (*N M*))
unfolding *ennreal-pmf-bind*
by (*subst nn-integral-0-iff-AE*) (*auto simp*: *AE-measure-pmf-iff pmf-nonneg set-pmf-eq*)
finally show *?thesis* **.**
qed

lemma *bind-pmf-cong* [*fundef-cong*]:
assumes *p* = *q*
shows (⋀*x*. *x* ∈ *set-pmf q* ⟹ *f x* = *g x*) ⟹ *bind-pmf p f* = *bind-pmf q g*

unfolding ‹$p = q$›[*symmetric*] *measure-pmf-inject*[*symmetric*] *bind-pmf.rep-eq*
by (*auto simp*: *AE-measure-pmf-iff Pi-iff space-subprob-algebra subprob-space-measure-pmf sets-bind*[**where** *N*=*count-space UNIV*] *emeasure-bind*[**where** *N*=*count-space UNIV*]
intro!: *nn-integral-cong-AE measure-eqI*)

lemma *bind-pmf-cong-simp*:
$p = q \Longrightarrow (\bigwedge x.\ x \in$ *set-pmf* q =*simp*=> $f\ x = g\ x) \Longrightarrow$ *bind-pmf* $p\ f =$ *bind-pmf* $q\ g$
by (*simp add*: *simp-implies-def cong*: *bind-pmf-cong*)

lemma *measure-pmf-bind*: *measure-pmf* (*bind-pmf* $M\ f$) = (*measure-pmf* $M \gg=$ ($\lambda x.$ *measure-pmf* ($f\ x$)))
by *transfer simp*

lemma *nn-integral-bind-pmf*[*simp*]: $(\int^+ x.\ f\ x\ \partial$*bind-pmf* $M\ N) = (\int^+ x.\ \int^+ y.\ f\ y\ \partial N\ x\ \partial M)$
using *measurable-measure-pmf*[*of N*]
unfolding *measure-pmf-bind*
apply (*intro nn-integral-bind*[**where** *B*=*count-space UNIV*])
apply *auto*
done

lemma *emeasure-bind-pmf*[*simp*]: *emeasure* (*bind-pmf* $M\ N$) $X = (\int^+ x.$ *emeasure* $(N\ x)\ X\ \partial M)$
using *measurable-measure-pmf*[*of N*]
unfolding *measure-pmf-bind*
by (*subst emeasure-bind*[**where** *N*=*count-space UNIV*]) *auto*

lift-definition *return-pmf* :: *'a* ⇒ *'a pmf* **is** *return* (*count-space UNIV*)
by (*auto intro*!: *prob-space-return simp*: *AE-return measure-return*)

lemma *bind-return-pmf*: *bind-pmf* (*return-pmf* x) $f = f\ x$
by *transfer*
(*auto intro*!: *prob-space-imp-subprob-space bind-return*[**where** *N*=*count-space UNIV*]
simp: *space-subprob-algebra*)

lemma *set-return-pmf*[*simp*]: *set-pmf* (*return-pmf* x) = $\{x\}$
by *transfer* (*auto simp add*: *measure-return split*: *split-indicator*)

lemma *bind-return-pmf'*: *bind-pmf N return-pmf* = *N*
proof (*transfer*, *clarify*)
fix *N* :: *'a measure* **assume** *sets N* = *UNIV* **then show** *N* ≫= *return* (*count-space UNIV*) = *N*
by (*subst return-sets-cong*[**where** *N*=*N*]) (*simp-all add*: *bind-return'*)
qed

lemma *bind-assoc-pmf*: *bind-pmf* (*bind-pmf A B*) *C* = *bind-pmf A* ($\lambda x.$ *bind-pmf*

(*B x*) *C*)
by *transfer*
(*auto intro*!: *bind-assoc*[**where** *N*=*count-space UNIV* **and** *R*=*count-space UNIV*]
simp: *measurable-def space-subprob-algebra prob-space-imp-subprob-space*)

definition *map-pmf f M* = *bind-pmf M* (λ*x*. *return-pmf* (*f x*))

lemma *map-bind-pmf*: *map-pmf f* (*bind-pmf M g*) = *bind-pmf M* (λ*x*. *map-pmf f* (*g x*))
by (*simp add*: *map-pmf-def bind-assoc-pmf*)

lemma *bind-map-pmf*: *bind-pmf* (*map-pmf f M*) *g* = *bind-pmf M* (λ*x*. *g* (*f x*))
by (*simp add*: *map-pmf-def bind-assoc-pmf bind-return-pmf*)

lemma *map-pmf-transfer*[*transfer-rule*]:
rel-fun (=) (*rel-fun cr-pmf cr-pmf*) (λ*f M*. *distr M* (*count-space UNIV*) *f*) *map-pmf*
proof −
have *rel-fun* (=) (*rel-fun pmf-as-measure.cr-pmf pmf-as-measure.cr-pmf*)
(λ*f M*. *M* ⤜ (*return* (*count-space UNIV*) *o f*)) *map-pmf*
unfolding *map-pmf-def*[*abs-def*] *comp-def* **by** *transfer-prover*
then show *?thesis*
by (*force simp*: *rel-fun-def cr-pmf-def bind-return-distr*)
qed

lemma *map-pmf-rep-eq*:
measure-pmf (*map-pmf f M*) = *distr* (*measure-pmf M*) (*count-space UNIV*) *f*
unfolding *map-pmf-def bind-pmf.rep-eq comp-def return-pmf.rep-eq*
using *bind-return-distr*[*of M f count-space UNIV*] **by** (*simp add*: *comp-def*)

lemma *map-pmf-id*[*simp*]: *map-pmf id* = *id*
by (*rule*, *transfer*) (*auto simp*: *emeasure-distr measurable-def intro*!: *measure-eqI*)

lemma *map-pmf-ident*[*simp*]: *map-pmf* (λ*x*. *x*) = (λ*x*. *x*)
using *map-pmf-id* **unfolding** *id-def* **.**

lemma *map-pmf-compose*: *map-pmf* (*f* ∘ *g*) = *map-pmf f* ∘ *map-pmf g*
by (*rule*, *transfer*) (*simp add*: *distr-distr*[*symmetric*, **where** *N*=*count-space UNIV*] *measurable-def*)

lemma *map-pmf-comp*: *map-pmf f* (*map-pmf g M*) = *map-pmf* (λ*x*. *f* (*g x*)) *M*
using *map-pmf-compose*[*of f g*] **by** (*simp add*: *comp-def*)

lemma *map-pmf-cong*: *p* = *q* ⟹ (⋀*x*. *x* ∈ *set-pmf q* ⟹ *f x* = *g x*) ⟹ *map-pmf f p* = *map-pmf g q*
unfolding *map-pmf-def* **by** (*rule bind-pmf-cong*) *auto*

lemma *pmf-set-map*: *set-pmf* ∘ *map-pmf f* = (‘) *f* ∘ *set-pmf*

by (*auto simp add*: *comp-def fun-eq-iff map-pmf-def*)

lemma *set-map-pmf*[*simp*]: *set-pmf* (*map-pmf f M*) = *f*‘*set-pmf M*
using *pmf-set-map*[*of f*] **by** (*auto simp*: *comp-def fun-eq-iff*)

lemma *emeasure-map-pmf*[*simp*]: *emeasure* (*map-pmf f M*) *X* = *emeasure M* (*f* −‘ *X*)
unfolding *map-pmf-rep-eq* **by** (*subst emeasure-distr*) *auto*

lemma *measure-map-pmf*[*simp*]: *measure* (*map-pmf f M*) *X* = *measure M* (*f* −‘ *X*)
using *emeasure-map-pmf*[*of f M X*] **by**(*simp add*: *measure-pmf.emeasure-eq-measure measure-nonneg*)

lemma *nn-integral-map-pmf*[*simp*]: (∫⁺*x*. *f x* ∂*map-pmf g M*) = (∫⁺*x*. *f* (*g x*) ∂*M*)
unfolding *map-pmf-rep-eq* **by** (*intro nn-integral-distr*) *auto*

lemma *ennreal-pmf-map*: *pmf* (*map-pmf f p*) *x* = (∫⁺ *y*. *indicator* (*f* −‘ {*x*}) *y* ∂*measure-pmf p*)
proof (*transfer fixing*: *f x*)
fix *p* :: *'b measure*
presume *prob-space p*
then interpret *prob-space p* **.**
presume *sets p* = *UNIV*
then show *ennreal* (*measure* (*distr p* (*count-space UNIV*) *f*) {*x*}) = *integral*^{N} *p* (*indicator* (*f* −‘ {*x*}))
by(*simp add*: *measure-distr measurable-def emeasure-eq-measure*)
qed *simp-all*

lemma *pmf-map*: *pmf* (*map-pmf f p*) *x* = *measure p* (*f* −‘ {*x*})
proof (*transfer fixing*: *f x*)
fix *p* :: *'b measure*
presume *prob-space p*
then interpret *prob-space p* **.**
presume *sets p* = *UNIV*
then show *measure* (*distr p* (*count-space UNIV*) *f*) {*x*} = *measure p* (*f* −‘ {*x*})
by(*simp add*: *measure-distr measurable-def emeasure-eq-measure*)
qed *simp-all*

lemma *nn-integral-pmf*: (∫⁺ *x*. *pmf p x* ∂*count-space A*) = *emeasure* (*measure-pmf p*) *A*
proof −
have (∫⁺ *x*. *pmf p x* ∂*count-space A*) = (∫⁺ *x*. *pmf p x* ∂*count-space* (*A* ∩ *set-pmf p*))
by(*auto simp add*: *nn-integral-count-space-indicator indicator-def set-pmf-iff intro*: *nn-integral-cong*)
also have ... = *emeasure* (*measure-pmf p*) (⋃*x*∈*A* ∩ *set-pmf p*. {*x*})
by(*subst emeasure-UN-countable*)(*auto simp add*: *emeasure-pmf-single disjoint-family-on-def*)

also have ... = *emeasure* (*measure-pmf p*) (($\bigcup x{\in}A \cap$ *set-pmf p*. {*x*}) ∪ {*x*. *x* ∈ *A* ∧ *x* ∉ *set-pmf p*})
by(*rule emeasure-Un-null-set*[*symmetric*])(*auto intro*: *in-null-sets-measure-pmfI*)
also have ... = *emeasure* (*measure-pmf p*) *A*
by(*auto intro*: *arg-cong2*[**where** *f*=*emeasure*])
finally show *?thesis* **.**
qed

lemma *integral-map-pmf*[*simp*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integral*^{L} (*map-pmf g p*) *f* = *integral*^{L} *p* (λ*x*. *f* (*g x*))
by (*simp add*: *integral-distr map-pmf-rep-eq*)

lemma *integrable-map-pmf-eq* [*simp*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integrable* (*map-pmf f p*) *g* ⟷ *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*))
by (*subst map-pmf-rep-eq*, *subst integrable-distr-eq*) *auto*

lemma *integrable-map-pmf* [*intro*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*)) ⟹ *integrable* (*map-pmf f p*) *g*
by (*subst integrable-map-pmf-eq*)

lemma *pmf-abs-summable* [*intro*]: *pmf p abs-summable-on A*
by (*rule abs-summable-on-subset*[*OF - subset-UNIV*])
(*auto simp*: *abs-summable-on-def integrable-iff-bounded nn-integral-pmf*)

lemma *measure-pmf-conv-infsetsum*: *measure* (*measure-pmf p*) *A* = *infsetsum* (*pmf p*) *A*
unfolding *infsetsum-def* **by** (*simp add*: *integral-eq-nn-integral nn-integral-pmf measure-def*)

lemma *infsetsum-pmf-eq-1*:
assumes *set-pmf p* ⊆ *A*
shows *infsetsum* (*pmf p*) *A* = *1*
proof −
have *infsetsum* (*pmf p*) *A* = *lebesgue-integral* (*count-space UNIV*) (*pmf p*)
using *assms* **unfolding** *infsetsum-altdef set-lebesgue-integral-def*
by (*intro Bochner-Integration.integral-cong*) (*auto simp*: *indicator-def set-pmf-eq*)
also have ... = *1*
by (*subst integral-eq-nn-integral*) (*auto simp*: *nn-integral-pmf*)
finally show *?thesis* **.**
qed

lemma *map-return-pmf* [*simp*]: *map-pmf f* (*return-pmf x*) = *return-pmf* (*f x*)
by *transfer* (*simp add*: *distr-return*)

lemma *map-pmf-const*[*simp*]: *map-pmf* (λ-. *c*) *M* = *return-pmf c*
by *transfer* (*auto simp*: *prob-space.distr-const*)

lemma *pmf-return* [*simp*]: *pmf* (*return-pmf x*) *y* = *indicator* {*y*} *x*
by *transfer* (*simp add*: *measure-return*)

lemma *nn-integral-return-pmf*[*simp*]: *0* ≤ *f x* ⟹ (∫⁺*x*. *f x* ∂*return-pmf x*) = *f x*
unfolding *return-pmf.rep-eq* **by** (*intro nn-integral-return*) *auto*

lemma *emeasure-return-pmf*[*simp*]: *emeasure* (*return-pmf x*) *X* = *indicator X x*
unfolding *return-pmf.rep-eq* **by** (*intro emeasure-return*) *auto*

lemma *measure-return-pmf* [*simp*]: *measure-pmf.prob* (*return-pmf x*) *A* = *indicator A x*
proof −
have *ennreal* (*measure-pmf.prob* (*return-pmf x*) *A*) =
emeasure (*measure-pmf* (*return-pmf x*)) *A*
by (*simp add*: *measure-pmf.emeasure-eq-measure*)
also have ... = *ennreal* (*indicator A x*) **by** (*simp add*: *ennreal-indicator*)
finally show *?thesis* **by** *simp*
qed

lemma *return-pmf-inj*[*simp*]: *return-pmf x* = *return-pmf y* ⟷ *x* = *y*
by (*metis insertI1 set-return-pmf singletonD*)

lemma *map-pmf-eq-return-pmf-iff*:
map-pmf f p = *return-pmf x* ⟷ (∀ *y* ∈ *set-pmf p*. *f y* = *x*)
proof
assume *map-pmf f p* = *return-pmf x*
then have *set-pmf* (*map-pmf f p*) = *set-pmf* (*return-pmf x*) **by** *simp*
then show ∀ *y* ∈ *set-pmf p*. *f y* = *x* **by** *auto*
next
assume ∀ *y* ∈ *set-pmf p*. *f y* = *x*
then show *map-pmf f p* = *return-pmf x*
unfolding *map-pmf-const*[*symmetric*, *of* - *p*] **by** (*intro map-pmf-cong*) *auto*
qed

definition *pair-pmf A B* = *bind-pmf A* (λ*x*. *bind-pmf B* (λ*y*. *return-pmf* (*x*, *y*)))

lemma *pmf-pair*: *pmf* (*pair-pmf M N*) (*a*, *b*) = *pmf M a* ∗ *pmf N b*
unfolding *pair-pmf-def pmf-bind pmf-return*
apply (*subst integral-measure-pmf-real*[**where** *A*={*b*}])
apply (*auto simp*: *indicator-eq-0-iff*)
apply (*subst integral-measure-pmf-real*[**where** *A*={*a*}])
apply (*auto simp*: *indicator-eq-0-iff sum-nonneg-eq-0-iff pmf-nonneg*)
done

lemma *set-pair-pmf*[*simp*]: *set-pmf* (*pair-pmf A B*) = *set-pmf A* × *set-pmf B*
unfolding *pair-pmf-def set-bind-pmf set-return-pmf* **by** *auto*

lemma *measure-pmf-in-subprob-space*[*measurable* (*raw*)]:

measure-pmf M ∈ *space* (*subprob-algebra* (*count-space UNIV*))
by (*simp add*: *space-subprob-algebra*) *intro-locales*

lemma *nn-integral-pair-pmf'*: (∫⁺*x*. *f x* ∂*pair-pmf A B*) = (∫⁺*a*. ∫⁺*b*. *f* (*a*, *b*) ∂*B* ∂*A*)
proof −
have (∫⁺*x*. *f x* ∂*pair-pmf A B*) = (∫⁺*x*. *f x* ∗ *indicator* (*A* × *B*) *x* ∂*pair-pmf A B*)
by (*auto simp*: *AE-measure-pmf-iff intro*!: *nn-integral-cong-AE*)
also have ... = (∫⁺*a*. ∫⁺*b*. *f* (*a*, *b*) ∗ *indicator* (*A* × *B*) (*a*, *b*) ∂*B* ∂*A*)
by (*simp add*: *pair-pmf-def*)
also have ... = (∫⁺*a*. ∫⁺*b*. *f* (*a*, *b*) ∂*B* ∂*A*)
by (*auto intro*!: *nn-integral-cong-AE simp*: *AE-measure-pmf-iff*)
finally show *?thesis* **.**
qed

lemma *bind-pair-pmf*:
assumes *M*[*measurable*]: *M* ∈ *measurable* (*count-space UNIV* ⨂*M* *count-space UNIV*) (*subprob-algebra N*)
shows *measure-pmf* (*pair-pmf A B*) ⤜ *M* = (*measure-pmf A* ⤜ (λ*x*. *measure-pmf B* ⤜ (λ*y*. *M* (*x*, *y*))))
(**is** *?L* = *?R*)
proof (*rule measure-eqI*)
have *M'*[*measurable*]: *M* ∈ *measurable* (*pair-pmf A B*) (*subprob-algebra N*)
using *M*[*THEN measurable-space*] **by** (*simp-all add*: *space-pair-measure*)

note *measurable-bind*[**where** *N*=*count-space UNIV*, *measurable*]
note *measure-pmf-in-subprob-space*[*simp*]

have *sets-eq-N*: *sets ?L* = *N*
by (*subst sets-bind*[*OF sets-kernel*[*OF M'*]]) *auto*
show *sets ?L* = *sets ?R*
using *measurable-space*[*OF M*]
by (*simp add*: *sets-eq-N space-pair-measure space-subprob-algebra*)
fix *X* **assume** *X* ∈ *sets ?L*
then have *X*[*measurable*]: *X* ∈ *sets N*
unfolding *sets-eq-N* **.**
then show *emeasure ?L X* = *emeasure ?R X*
apply (*simp add*: *emeasure-bind*[*OF* - *M' X*])
apply (*simp add*: *nn-integral-bind*[**where** *B*=*count-space UNIV*] *pair-pmf-def measure-pmf-bind*[*of A*]
nn-integral-measure-pmf-finite)
apply (*subst emeasure-bind*[*OF* - - *X*])
apply *measurable*
apply (*subst emeasure-bind*[*OF* - - *X*])
apply *measurable*
done
qed

lemma *map-fst-pair-pmf*: *map-pmf fst* (*pair-pmf A B*) = *A*
by (*simp add*: *pair-pmf-def map-pmf-def bind-assoc-pmf bind-return-pmf bind-return-pmf'*)

lemma *map-snd-pair-pmf*: *map-pmf snd* (*pair-pmf A B*) = *B*
by (*simp add*: *pair-pmf-def map-pmf-def bind-assoc-pmf bind-return-pmf bind-return-pmf'*)

lemma *nn-integral-pmf'*:
inj-on f A ⟹ (∫⁺*x*. *pmf p* (*f x*) ∂*count-space A*) = *emeasure p* (*f* ‘ *A*)
by (*subst nn-integral-bij-count-space*[**where** *g*=*f* **and** *B*=*f*‘*A*])
(*auto simp*: *bij-betw-def nn-integral-pmf*)

lemma *pmf-le-0-iff*[*simp*]: *pmf M p* ≤ *0* ⟷ *pmf M p* = *0*
using *pmf-nonneg*[*of M p*] **by** *arith*

lemma *min-pmf-0*[*simp*]: *min* (*pmf M p*) *0* = *0 min 0* (*pmf M p*) = *0*
using *pmf-nonneg*[*of M p*] **by** *arith+*

lemma *pmf-eq-0-set-pmf*: *pmf M p* = *0* ⟷ *p* ∉ *set-pmf M*
unfolding *set-pmf-iff* **by** *simp*

lemma *pmf-map-inj*: *inj-on f* (*set-pmf M*) ⟹ *x* ∈ *set-pmf M* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
by (*auto simp*: *pmf.rep-eq map-pmf-rep-eq measure-distr AE-measure-pmf-iff inj-onD*
intro!: *measure-pmf.finite-measure-eq-AE*)

lemma *pair-return-pmf* [*simp*]: *pair-pmf* (*return-pmf x*) (*return-pmf y*) = *return-pmf* (*x*, *y*)
by (*auto simp*: *pair-pmf-def bind-return-pmf*)

lemma *pmf-map-inj'*: *inj f* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
apply(*cases x* ∈ *set-pmf M*)
apply(*simp add*: *pmf-map-inj*[*OF inj-on-subset*])
apply(*simp add*: *pmf-eq-0-set-pmf*[*symmetric*])
apply(*auto simp add*: *pmf-eq-0-set-pmf dest*: *injD*)
done

lemma *expectation-pair-pmf-fst* [*simp*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*fst x*)) = *measure-pmf.expectation p f*
proof −
have *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*fst x*)) =
measure-pmf.expectation (*map-pmf fst* (*pair-pmf p q*)) *f* **by** *simp*
also have *map-pmf fst* (*pair-pmf p q*) = *p*
by (*simp add*: *map-fst-pair-pmf*)
finally show *?thesis* **.**
qed

lemma *expectation-pair-pmf-snd* [*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
shows *measure-pmf.expectation* (*pair-pmf p q*) ($\lambda x.$ *f* (*snd x*)) = *measure-pmf.expectation q f*
proof −
have *measure-pmf.expectation* (*pair-pmf p q*) ($\lambda x.$ *f* (*snd x*)) =
measure-pmf.expectation (*map-pmf snd* (*pair-pmf p q*)) *f* **by** *simp*
also have *map-pmf snd* (*pair-pmf p q*) = *q*
by (*simp add*: *map-snd-pair-pmf*)
finally show *?thesis* **.**
qed

lemma *pmf-map-outside*: $x \notin f$ ‘ *set-pmf M* $\Longrightarrow$ *pmf* (*map-pmf f M*) *x* = *0*
unfolding *pmf-eq-0-set-pmf* **by** *simp*

lemma *measurable-set-pmf*[*measurable*]: *Measurable.pred* (*count-space UNIV*) ($\lambda x.$ *x* ∈ *set-pmf M*)
by *simp*

20.3 PMFs as function

context
fixes $f :: 'a \Rightarrow real$
assumes *nonneg*: $\bigwedge x.\ 0 \le f\ x$
assumes *prob*: $(\int^+ x.\ f\ x\ \partial count\text{-}space\ UNIV) = 1$
begin

lift-definition *embed-pmf* :: *'a pmf* **is** *density* (*count-space UNIV*) (*ennreal* ∘ *f*)
proof (*intro conjI*)
have ∗[*simp*]: $\bigwedge x\ y.$ *ennreal* (*f y*) ∗ *indicator* {*x*} *y* = *ennreal* (*f x*) ∗ *indicator* {*x*} *y*
by (*simp split*: *split-indicator*)
show *AE x in density* (*count-space UNIV*) (*ennreal* ∘ *f*).
measure (*density* (*count-space UNIV*) (*ennreal* ∘ *f*)) {*x*} ≠ *0*
by (*simp add*: *AE-density nonneg measure-def emeasure-density max-def*)
show *prob-space* (*density* (*count-space UNIV*) (*ennreal* ∘ *f*))
by *standard* (*simp add*: *emeasure-density prob*)
qed *simp*

lemma *pmf-embed-pmf*: *pmf embed-pmf x* = *f x*
proof *transfer*
have ∗[*simp*]: $\bigwedge x\ y.$ *ennreal* (*f y*) ∗ *indicator* {*x*} *y* = *ennreal* (*f x*) ∗ *indicator* {*x*} *y*
by (*simp split*: *split-indicator*)
fix *x* **show** *measure* (*density* (*count-space UNIV*) (*ennreal* ∘ *f*)) {*x*} = *f x*
by *transfer* (*simp add*: *measure-def emeasure-density nonneg max-def*)
qed

lemma *set-embed-pmf*: *set-pmf embed-pmf* = {*x*. *f x* ≠ *0*}

by(*auto simp add*: *set-pmf-eq pmf-embed-pmf*)

end

lemma *embed-pmf-transfer*:
rel-fun (*eq-onp* (λ*f*. (∀ *x*. *0* ≤ *f x*) ∧ (∫⁺*x*. *ennreal* (*f x*) ∂*count-space UNIV*) = *1*)) *pmf-as-measure.cr-pmf* (λ*f*. *density* (*count-space UNIV*) (*ennreal* ∘ *f*)) *embed-pmf*
by (*auto simp*: *rel-fun-def eq-onp-def embed-pmf.transfer*)

lemma *measure-pmf-eq-density*: *measure-pmf p* = *density* (*count-space UNIV*) (*pmf p*)
proof (*transfer*, *elim conjE*)
fix *M* :: *'a measure* **assume** [*simp*]: *sets M* = *UNIV* **and** *ae*: *AE x in M*. *measure M* {*x*} ≠ *0*
assume *prob-space M* **then interpret** *prob-space M* **.**
show *M* = *density* (*count-space UNIV*) (λ*x*. *ennreal* (*measure M* {*x*}))
proof (*rule measure-eqI*)
fix *A* :: *'a set*
have (∫⁺ *x*. *ennreal* (*measure M* {*x*}) ∗ *indicator A x* ∂*count-space UNIV*) =
(∫⁺ *x*. *emeasure M* {*x*} ∗ *indicator* (*A* ∩ {*x*. *measure M* {*x*} ≠ *0*}) *x* ∂*count-space UNIV*)
by (*auto intro*!: *nn-integral-cong simp*: *emeasure-eq-measure split*: *split-indicator*)
also have ... = (∫⁺ *x*. *emeasure M* {*x*} ∂*count-space* (*A* ∩ {*x*. *measure M* {*x*} ≠ *0*}))
by (*subst nn-integral-restrict-space*[*symmetric*]) (*auto simp*: *restrict-count-space*)
also have ... = *emeasure M* (⋃*x*∈(*A* ∩ {*x*. *measure M* {*x*} ≠ *0*}). {*x*})
by (*intro emeasure-UN-countable*[*symmetric*] *countable-Int2 countable-support*)
(*auto simp*: *disjoint-family-on-def*)
also have ... = *emeasure M A*
using *ae* **by** (*intro emeasure-eq-AE*) *auto*
finally show *emeasure M A* = *emeasure* (*density* (*count-space UNIV*) (λ*x*. *ennreal* (*measure M* {*x*}))) *A*
using *emeasure-space-1* **by** (*simp add*: *emeasure-density*)
qed *simp*
qed

lemma *td-pmf-embed-pmf*:
type-definition pmf embed-pmf {*f*::*'a* ⇒ *real*. (∀ *x*. *0* ≤ *f x*) ∧ (∫⁺*x*. *ennreal* (*f x*) ∂*count-space UNIV*) = *1*}
unfolding *type-definition-def*
proof *safe*
fix *p* :: *'a pmf*
have (∫⁺ *x*. *1* ∂*measure-pmf p*) = *1*
using *measure-pmf.emeasure-space-1*[*of p*] **by** *simp*
then show ∗: (∫⁺ *x*. *ennreal* (*pmf p x*) ∂*count-space UNIV*) = *1*
by (*simp add*: *measure-pmf-eq-density nn-integral-density pmf-nonneg del*: *nn-integral-const*)

show *embed-pmf* (*pmf p*) = *p*

by (*intro measure-pmf-inject*[*THEN iffD1*])
(*simp add*: ∗ *embed-pmf.rep-eq pmf-nonneg measure-pmf-eq-density*[*of p*] *comp-def*)
next
fix *f* :: *'a* ⇒ *real* **assume** ∀ *x*. *0* ≤ *f x* (∫⁺*x*. *f x* ∂*count-space UNIV*) = *1*
then show *pmf* (*embed-pmf f*) = *f*
by (*auto intro*!: *pmf-embed-pmf*)
qed (*rule pmf-nonneg*)

end

lemma *nn-integral-measure-pmf*: (∫⁺ *x*. *f x* ∂*measure-pmf p*) = ∫⁺ *x*. *ennreal* (*pmf p x*) ∗ *f x* ∂*count-space UNIV*
by(*simp add*: *measure-pmf-eq-density nn-integral-density pmf-nonneg*)

lemma *integral-measure-pmf*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
assumes *A*: *finite A*
shows (⋀*a*. *a* ∈ *set-pmf M* ⟹ *f a* ≠ *0* ⟹ *a* ∈ *A*) ⟹ (*LINT x*|*M*. *f x*) = (∑ *a*∈*A*. *pmf M a* ∗*R f a*)
unfolding *measure-pmf-eq-density*
apply (*simp add*: *integral-density*)
apply (*subst lebesgue-integral-count-space-finite-support*)
apply (*auto intro*!: *finite-subset*[*OF* - ‹*finite A*›] *sum.mono-neutral-left simp*: *pmf-eq-0-set-pmf*)
done

lemma *expectation-return-pmf* [*simp*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*return-pmf x*) *f* = *f x*
by (*subst integral-measure-pmf*[*of* {*x*}]) *simp-all*

lemma *pmf-expectation-bind*:
fixes *p* :: *'a pmf* **and** *f* :: *'a* ⇒ *'b pmf*
and *h* :: *'b* ⇒ *'c*::{*banach*, *second-countable-topology*}
assumes *finite A* ⋀*x*. *x* ∈ *A* ⟹ *finite* (*set-pmf* (*f x*)) *set-pmf p* ⊆ *A*
shows *measure-pmf.expectation* (*p* ⤜ *f*) *h* =
(∑ *a*∈*A*. *pmf p a* ∗*R measure-pmf.expectation* (*f a*) *h*)
proof −
have *measure-pmf.expectation* (*p* ⤜ *f*) *h* = (∑ *a*∈(⋃ *x*∈*A*. *set-pmf* (*f x*)). *pmf* (*p* ⤜ *f*) *a* ∗*R h a*)
using *assms* **by** (*intro integral-measure-pmf*) *auto*
also have ... = (∑ *x*∈(⋃ *x*∈*A*. *set-pmf* (*f x*)). (∑ *a*∈*A*. (*pmf p a* ∗ *pmf* (*f a*) *x*) ∗*R h x*))
proof (*intro sum.cong refl*, *goal-cases*)
case (*1 x*)
thus *?case*
by (*subst pmf-bind*, *subst integral-measure-pmf*[*of A*])
(*insert assms*, *auto simp*: *scaleR-sum-left*)

```
  qed
  also have ... = (∑j∈A. pmf p j *R (∑ i∈(⋃x∈A. set-pmf (f x)). pmf (f j) i
*R h i))
    by (subst sum.swap) (simp add: scaleR-sum-right)
  also have ... = (∑j∈A. pmf p j *R measure-pmf.expectation (f j) h)
  proof (intro sum.cong refl, goal-cases)
    case (1 x)
    thus ?case
      by (subst integral-measure-pmf[of (⋃x∈A. set-pmf (f x))])
        (insert assms, auto simp: scaleR-sum-left)
  qed
  finally show ?thesis .
qed

lemma continuous-on-LINT-pmf: — This is dominated convergence!?
  fixes f :: 'i ⇒ 'a::topological-space ⇒ 'b::{banach, second-countable-topology}
  assumes f: ⋀i. i ∈ set-pmf M ⟹ continuous-on A (f i)
    and bnd: ⋀a i. a ∈ A ⟹ i ∈ set-pmf M ⟹ norm (f i a) ≤ B
  shows continuous-on A (λa. LINT i|M. f i a)
proof cases
  assume finite M with f show ?thesis
    using integral-measure-pmf[OF ‹finite M›]
    by (subst integral-measure-pmf[OF ‹finite M›])
      (auto intro!: continuous-on-sum continuous-on-scaleR continuous-on-const)
next
  assume infinite M
  let ?f = λi x. pmf (map-pmf (to-nat-on M) M) i *R f (from-nat-into M i) x

  show ?thesis
  proof (rule uniform-limit-theorem)
    show ∀F n in sequentially. continuous-on A (λa. ∑ i<n. ?f i a)
     by (intro always-eventually allI continuous-on-sum continuous-on-scaleR con-
tinuous-on-const f
            from-nat-into set-pmf-not-empty)
   show uniform-limit A (λn a. ∑ i<n. ?f i a) (λa. LINT i|M. f i a) sequentially
   proof (subst uniform-limit-cong[where g=λn a. ∑ i<n. ?f i a])
     fix a assume a ∈ A
    have 1: (LINT i|M. f i a) = (LINT i|map-pmf (to-nat-on M) M. f (from-nat-into
M i) a)
       by (auto intro!: integral-cong-AE AE-pmfI)
     have 2: ... = (LINT i|count-space UNIV. pmf (map-pmf (to-nat-on M) M)
i *R f (from-nat-into M i) a)
       by (simp add: measure-pmf-eq-density integral-density)
     have (λn. ?f n a) sums (LINT i|M. f i a)
       unfolding 1 2
     proof (intro sums-integral-count-space-nat)
       have A: integrable M (λi. f i a)
        using ‹a∈A› by (auto intro!: measure-pmf.integrable-const-bound AE-pmfI
bnd)
```

```
      have integrable (map-pmf (to-nat-on M) M) (λi. f (from-nat-into M i) a)
        by (auto simp add: map-pmf-rep-eq integrable-distr-eq intro!: AE-pmfI
integrable-cong-AE-imp[OF A])
      then show integrable (count-space UNIV) (λn. ?f n a)
        by (simp add: measure-pmf-eq-density integrable-density)
    qed
    then show (LINT i|M. f i a) = (∑ n. ?f n a)
      by (simp add: sums-unique)
  next
    show uniform-limit A (λn a. ∑ i<n. ?f i a) (λa. (∑ n. ?f n a)) sequentially
    proof (rule Weierstrass-m-test)
      fix n a assume a∈A
      then show norm (?f n a) ≤ pmf (map-pmf (to-nat-on M) M) n * B
      using bnd by (auto intro!: mult-mono simp: from-nat-into set-pmf-not-empty)
    next
      have integrable (map-pmf (to-nat-on M) M) (λn. B)
        by auto
      then show summable (λn. pmf (map-pmf (to-nat-on (set-pmf M)) M) n *
B)
        by (fastforce simp add: measure-pmf-eq-density integrable-density inte-
grable-count-space-nat-iff summable-mult2)
    qed
  qed simp
 qed simp
qed

lemma continuous-on-LBINT:
 fixes f :: real ⇒ real
 assumes f: ⋀b. a ≤ b ⟹ set-integrable lborel {a..b} f
 shows continuous-on UNIV (λb. LBINT x:{a..b}. f x)
proof (subst set-borel-integral-eq-integral)
 { fix b :: real assume a ≤ b
   from f[OF this] have continuous-on {a..b} (λb. integral {a..b} f)
     by (intro indefinite-integral-continuous-1 set-borel-integral-eq-integral) }
 note ∗ = this

 have continuous-on (⋃ b∈{a..}. {a <..< b}) (λb. integral {a..b} f)
 proof (intro continuous-on-open-UN)
   show b ∈ {a..} ⟹ continuous-on {a<..<b} (λb. integral {a..b} f) for b
     using ∗[of b] by (rule continuous-on-subset) auto
 qed simp
 also have (⋃ b∈{a..}. {a <..< b}) = {a <..}
   by (auto simp: lt-ex gt-ex less-imp-le) (simp add: Bex-def less-imp-le gt-ex cong:
rev-conj-cong)
 finally have continuous-on {a+1 ..} (λb. integral {a..b} f)
   by (rule continuous-on-subset) auto
 moreover have continuous-on {a..a+1} (λb. integral {a..b} f)
   by (rule ∗) simp
 moreover
```

have $x \leq a \Longrightarrow \{a..x\} = (\mathit{if}\ a = x\ \mathit{then}\ \{a\}\ \mathit{else}\ \{\})$ **for** x
by *auto*
then have *continuous-on* $\{..a\}$ $(\lambda b.\ \mathit{integral}\ \{a..b\}\ f)$
by (*subst continuous-on-cong*[*OF refl*, **where** $g{=}\lambda x.\ 0$]) (*auto intro*!: *continuous-on-const*)
ultimately have *continuous-on* $(\{..a\} \cup \{a..a{+}1\} \cup \{a{+}1\ ..\})$ $(\lambda b.\ \mathit{integral}\ \{a..b\}\ f)$
by (*intro continuous-on-closed-Un*) *auto*
also have $\{..a\} \cup \{a..a{+}1\} \cup \{a{+}1\ ..\} = \mathit{UNIV}$
by *auto*
finally show *continuous-on UNIV* $(\lambda b.\ \mathit{integral}\ \{a..b\}\ f)$
by *auto*
next
show *set-integrable lborel* $\{a..b\}$ f **for** b
using f **by** (*cases* $a \leq b$) *auto*
qed

locale *pmf-as-function*
begin

setup-lifting *td-pmf-embed-pmf*

lemma *set-pmf-transfer*[*transfer-rule*]:
assumes *bi-total A*
shows *rel-fun* (*pcr-pmf A*) (*rel-set A*) $(\lambda f.\ \{x.\ f\,x \neq 0\})$ *set-pmf*
using ‹*bi-total A*›
by (*auto simp*: *pcr-pmf-def cr-pmf-def rel-fun-def rel-set-def bi-total-def Bex-def set-pmf-iff*)
metis+

end

context
begin

interpretation *pmf-as-function* **.**

lemma *pmf-eqI*: $(\bigwedge i.\ \mathit{pmf}\ M\ i = \mathit{pmf}\ N\ i) \Longrightarrow M = N$
by *transfer auto*

lemma *pmf-eq-iff*: $M = N \longleftrightarrow (\forall\, i.\ \mathit{pmf}\ M\ i = \mathit{pmf}\ N\ i)$
by (*auto intro*: *pmf-eqI*)

lemma *pmf-neq-exists-less*:
assumes $M \neq N$
shows $\exists\, x.\ \mathit{pmf}\ M\ x < \mathit{pmf}\ N\ x$
proof (*rule ccontr*)
assume $\neg(\exists\, x.\ \mathit{pmf}\ M\ x < \mathit{pmf}\ N\ x)$
hence *ge*: $\mathit{pmf}\ M\ x \geq \mathit{pmf}\ N\ x$ **for** x **by** (*auto simp*: *not-less*)

from *assms* **obtain** *x* **where** *pmf M x* ≠ *pmf N x* **by** (*auto simp*: *pmf-eq-iff*)
with *ge*[*of x*] **have** *gt*: *pmf M x* > *pmf N x* **by** *simp*
have *1* = *measure* (*measure-pmf M*) *UNIV* **by** *simp*
also have ... = *measure* (*measure-pmf N*) {*x*} + *measure* (*measure-pmf N*) (*UNIV* − {*x*})
by (*subst measure-pmf.finite-measure-Union* [*symmetric*]) *simp-all*
also from *gt* **have** *measure* (*measure-pmf N*) {*x*} < *measure* (*measure-pmf M*) {*x*}
by (*simp add*: *measure-pmf-single*)
also have *measure* (*measure-pmf N*) (*UNIV* − {*x*}) ≤ *measure* (*measure-pmf M*) (*UNIV* − {*x*})
by (*subst* (*1 2*) *integral-pmf* [*symmetric*])
(*intro integral-mono integrable-pmf*, *simp-all add*: *ge*)
also have *measure* (*measure-pmf M*) {*x*} + ... = *1*
by (*subst measure-pmf.finite-measure-Union* [*symmetric*]) *simp-all*
finally show *False* **by** *simp-all*
qed

lemma *bind-commute-pmf*: *bind-pmf A* (λ*x*. *bind-pmf B* (*C x*)) = *bind-pmf B* (λ*y*. *bind-pmf A* (λ*x*. *C x y*))
unfolding *pmf-eq-iff pmf-bind*
proof
fix *i*
interpret *B*: *prob-space restrict-space B B*
by (*intro prob-space-restrict-space measure-pmf.emeasure-eq-1-AE*)
(*auto simp*: *AE-measure-pmf-iff*)
interpret *A*: *prob-space restrict-space A A*
by (*intro prob-space-restrict-space measure-pmf.emeasure-eq-1-AE*)
(*auto simp*: *AE-measure-pmf-iff*)

interpret *AB*: *pair-prob-space restrict-space A A restrict-space B B*
by *unfold-locales*

have (∫ *x*. ∫ *y*. *pmf* (*C x y*) *i* ∂*B* ∂*A*) = (∫ *x*. (∫ *y*. *pmf* (*C x y*) *i* ∂*restrict-space B B*) ∂*A*)
by (*rule Bochner-Integration.integral-cong*) (*auto intro*!: *integral-pmf-restrict*)
also have ... = (∫ *x*. (∫ *y*. *pmf* (*C x y*) *i* ∂*restrict-space B B*) ∂*restrict-space A A*)
by (*intro integral-pmf-restrict B.borel-measurable-lebesgue-integral measurable-pair-restrict-pmf2*
countable-set-pmf borel-measurable-count-space)
also have ... = (∫ *y*. ∫ *x*. *pmf* (*C x y*) *i* ∂*restrict-space A A* ∂*restrict-space B B*)
by (*rule AB.Fubini-integral*[*symmetric*])
(*auto intro*!: *AB.integrable-const-bound*[**where** *B=1*] *measurable-pair-restrict-pmf2*
simp: *pmf-nonneg pmf-le-1 measurable-restrict-space1*)
also have ... = (∫ *y*. ∫ *x*. *pmf* (*C x y*) *i* ∂*restrict-space A A* ∂*B*)
by (*intro integral-pmf-restrict*[*symmetric*] *A.borel-measurable-lebesgue-integral measurable-pair-restrict-pmf2*
countable-set-pmf borel-measurable-count-space)

also have ... = ($\int$ *y*. $\int$ *x*. *pmf* (*C x y*) *i* ∂*A* ∂*B*)
by (*rule Bochner-Integration.integral-cong*) (*auto intro*!: *integral-pmf-restrict*[*symmetric*])
finally show ($\int$ *x*. $\int$ *y*. *pmf* (*C x y*) *i* ∂*B* ∂*A*) = ($\int$ *y*. $\int$ *x*. *pmf* (*C x y*) *i* ∂*A* ∂*B*) **.**
qed

lemma *pair-map-pmf1*: *pair-pmf* (*map-pmf f A*) *B* = *map-pmf* (*apfst f*) (*pair-pmf A B*)
proof (*safe intro*!: *pmf-eqI*)
fix *a* :: *'a* **and** *b* :: *'b*
have [*simp*]: ⋀*c d*. *indicator* (*apfst f* −‘ {(*a*, *b*)}) (*c*, *d*) = *indicator* (*f* −‘ {*a*}) *c* ∗ (*indicator* {*b*} *d*::*ennreal*)
by (*auto split*: *split-indicator*)

have *ennreal* (*pmf* (*pair-pmf* (*map-pmf f A*) *B*) (*a*, *b*)) =
ennreal (*pmf* (*map-pmf* (*apfst f*) (*pair-pmf A B*)) (*a*, *b*))
unfolding *pmf-pair ennreal-pmf-map*
by (*simp add*: *nn-integral-pair-pmf' max-def emeasure-pmf-single nn-integral-multc pmf-nonneg*
emeasure-map-pmf[*symmetric*] *ennreal-mult del*: *emeasure-map-pmf*)
then show *pmf* (*pair-pmf* (*map-pmf f A*) *B*) (*a*, *b*) = *pmf* (*map-pmf* (*apfst f*) (*pair-pmf A B*)) (*a*, *b*)
by (*simp add*: *pmf-nonneg*)
qed

lemma *pair-map-pmf2*: *pair-pmf A* (*map-pmf f B*) = *map-pmf* (*apsnd f*) (*pair-pmf A B*)
proof (*safe intro*!: *pmf-eqI*)
fix *a* :: *'a* **and** *b* :: *'b*
have [*simp*]: ⋀*c d*. *indicator* (*apsnd f* −‘ {(*a*, *b*)}) (*c*, *d*) = *indicator* {*a*} *c* ∗ (*indicator* (*f* −‘ {*b*}) *d*::*ennreal*)
by (*auto split*: *split-indicator*)

have *ennreal* (*pmf* (*pair-pmf A* (*map-pmf f B*)) (*a*, *b*)) =
ennreal (*pmf* (*map-pmf* (*apsnd f*) (*pair-pmf A B*)) (*a*, *b*))
unfolding *pmf-pair ennreal-pmf-map*
by (*simp add*: *nn-integral-pair-pmf' max-def emeasure-pmf-single nn-integral-cmult nn-integral-multc pmf-nonneg*
emeasure-map-pmf[*symmetric*] *ennreal-mult del*: *emeasure-map-pmf*)
then show *pmf* (*pair-pmf A* (*map-pmf f B*)) (*a*, *b*) = *pmf* (*map-pmf* (*apsnd f*) (*pair-pmf A B*)) (*a*, *b*)
by (*simp add*: *pmf-nonneg*)
qed

lemma *map-pair*: *map-pmf* (λ(*a*, *b*). (*f a*, *g b*)) (*pair-pmf A B*) = *pair-pmf* (*map-pmf f A*) (*map-pmf g B*)
by (*simp add*: *pair-map-pmf2 pair-map-pmf1 map-pmf-comp split-beta'*)

end

lemma *pair-return-pmf1*: *pair-pmf* (*return-pmf x*) *y* = *map-pmf* (*Pair x*) *y*
by(*simp add*: *pair-pmf-def bind-return-pmf map-pmf-def*)

lemma *pair-return-pmf2*: *pair-pmf x* (*return-pmf y*) = *map-pmf* (λ*x*. (*x*, *y*)) *x*
by(*simp add*: *pair-pmf-def bind-return-pmf map-pmf-def*)

lemma *pair-pair-pmf*: *pair-pmf* (*pair-pmf u v*) *w* = *map-pmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-pmf u* (*pair-pmf v w*))
by(*simp add*: *pair-pmf-def bind-return-pmf map-pmf-def bind-assoc-pmf*)

lemma *pair-commute-pmf*: *pair-pmf x y* = *map-pmf* (λ(*x*, *y*). (*y*, *x*)) (*pair-pmf y x*)
unfolding *pair-pmf-def* **by**(*subst bind-commute-pmf*)(*simp add*: *map-pmf-def bind-assoc-pmf bind-return-pmf*)

lemma *set-pmf-subset-singleton*: *set-pmf p* ⊆ {*x*} ⟷ *p* = *return-pmf x*
proof(*intro iffI pmf-eqI*)
  fix *i*
  assume *x*: *set-pmf p* ⊆ {*x*}
  hence ∗: *set-pmf p* = {*x*} **using** *set-pmf-not-empty*[*of p*] **by** *auto*
 have *ennreal* (*pmf p x*) = ∫⁺ *i*. *indicator* {*x*} *i* ∂*p* **by**(*simp add*: *emeasure-pmf-single*)
 also have ... = ∫⁺ *i*. *1* ∂*p* **by**(*rule nn-integral-cong-AE*)(*simp add*: *AE-measure-pmf-iff* ∗)
  also have ... = *1* **by** *simp*
  finally show *pmf p i* = *pmf* (*return-pmf x*) *i* **using** *x*
    by(*auto split*: *split-indicator simp add*: *pmf-eq-0-set-pmf*)
qed *auto*

lemma *bind-eq-return-pmf*:
  bind-pmf p f = *return-pmf x* ⟷ (∀ *y*∈*set-pmf p*. *f y* = *return-pmf x*)
  (**is** *?lhs* ⟷ *?rhs*)
proof(*intro iffI strip*)
  fix *y*
  assume *y*: *y* ∈ *set-pmf p*
  assume *?lhs*
  hence *set-pmf* (*bind-pmf p f*) = {*x*} **by** *simp*
  hence (⋃ *y*∈*set-pmf p*. *set-pmf* (*f y*)) = {*x*} **by** *simp*
  hence *set-pmf* (*f y*) ⊆ {*x*} **using** *y* **by** *auto*
  thus *f y* = *return-pmf x* **by**(*simp add*: *set-pmf-subset-singleton*)
next
  assume ∗: *?rhs*
  show *?lhs*
  proof(*rule pmf-eqI*)
    fix *i*
    have *ennreal* (*pmf* (*bind-pmf p f*) *i*) = ∫⁺ *y*. *ennreal* (*pmf* (*f y*) *i*) ∂*p*
      by (*simp add*: *ennreal-pmf-bind*)
    also have ... = ∫⁺ *y*. *ennreal* (*pmf* (*return-pmf x*) *i*) ∂*p*
      by(*rule nn-integral-cong-AE*)(*simp add*: *AE-measure-pmf-iff* ∗)

```
    also have ... = ennreal (pmf (return-pmf x) i)
      by simp
    finally show pmf (bind-pmf p f) i = pmf (return-pmf x) i
      by (simp add: pmf-nonneg)
  qed
qed

lemma pmf-False-conv-True: pmf p False = 1 − pmf p True
proof −
  have pmf p False + pmf p True = measure p {False} + measure p {True}
    by(simp add: measure-pmf-single)
  also have ... = measure p ({False} ∪ {True})
    by(subst measure-pmf.finite-measure-Union) simp-all
  also have {False} ∪ {True} = space p by auto
  finally show ?thesis by simp
qed

lemma pmf-True-conv-False: pmf p True = 1 − pmf p False
by(simp add: pmf-False-conv-True)
```

20.4 Conditional Probabilities

```
lemma measure-pmf-zero-iff: measure (measure-pmf p) s = 0 ⟷ set-pmf p ∩ s
= {}
  by (subst measure-pmf.prob-eq-0) (auto simp: AE-measure-pmf-iff)

context
  fixes p :: 'a pmf and s :: 'a set
  assumes not-empty: set-pmf p ∩ s ≠ {}
begin

interpretation pmf-as-measure .

lemma emeasure-measure-pmf-not-zero: emeasure (measure-pmf p) s ≠ 0
proof
  assume emeasure (measure-pmf p) s = 0
  then have AE x in measure-pmf p. x ∉ s
    by (rule AE-I[rotated]) auto
  with not-empty show False
    by (auto simp: AE-measure-pmf-iff)
qed

lemma measure-measure-pmf-not-zero: measure (measure-pmf p) s ≠ 0
 using emeasure-measure-pmf-not-zero by (simp add: measure-pmf.emeasure-eq-measure
measure-nonneg)

lift-definition cond-pmf :: 'a pmf is
  uniform-measure (measure-pmf p) s
proof (intro conjI)
```

show *prob-space* (*uniform-measure* (*measure-pmf p*) *s*)
by (*intro prob-space-uniform-measure*) (*auto simp*: *emeasure-measure-pmf-not-zero*)
show *AE x in uniform-measure* (*measure-pmf p*) *s*. *measure* (*uniform-measure* (*measure-pmf p*) *s*) {*x*} ≠ *0*
by (*simp add*: *emeasure-measure-pmf-not-zero measure-measure-pmf-not-zero AE-uniform-measure*
AE-measure-pmf-iff set-pmf.rep-eq less-top[*symmetric*])
qed *simp*

lemma *pmf-cond*: *pmf cond-pmf x* = (*if x* ∈ *s then pmf p x* / *measure p s else 0*)
by *transfer* (*simp add*: *emeasure-measure-pmf-not-zero pmf.rep-eq*)

lemma *set-cond-pmf*[*simp*]: *set-pmf cond-pmf* = *set-pmf p* ∩ *s*
by (*auto simp add*: *set-pmf-iff pmf-cond measure-measure-pmf-not-zero split*: *if-split-asm*)

end

lemma *measure-pmf-posI*: *x* ∈ *set-pmf p* ⟹ *x* ∈ *A* ⟹ *measure-pmf.prob p A* > *0*
using *measure-measure-pmf-not-zero*[*of p A*] **by** (*subst zero-less-measure-iff*) *blast*

lemma *cond-map-pmf*:
assumes *set-pmf p* ∩ *f* −‘ *s* ≠ {}
shows *cond-pmf* (*map-pmf f p*) *s* = *map-pmf f* (*cond-pmf p* (*f* −‘ *s*))
proof −
have ∗: *set-pmf* (*map-pmf f p*) ∩ *s* ≠ {}
using *assms* **by** *auto*
{ fix *x*
have *ennreal* (*pmf* (*map-pmf f* (*cond-pmf p* (*f* −‘ *s*))) *x*) =
emeasure p (*f* −‘ *s* ∩ *f* −‘ {*x*}) / *emeasure p* (*f* −‘ *s*)
unfolding *ennreal-pmf-map cond-pmf.rep-eq*[*OF assms*] **by** (*simp add*: *nn-integral-uniform-measure*)
also have *f* −‘ *s* ∩ *f* −‘ {*x*} = (*if x* ∈ *s then f* −‘ {*x*} *else* {})
by *auto*
also have *emeasure p* (*if x* ∈ *s then f* −‘ {*x*} *else* {}) / *emeasure p* (*f* −‘ *s*) =
ennreal (*pmf* (*cond-pmf* (*map-pmf f p*) *s*) *x*)
using *measure-measure-pmf-not-zero*[*OF* ∗]
by (*simp add*: *pmf-cond*[*OF* ∗] *ennreal-pmf-map measure-pmf.emeasure-eq-measure*
divide-ennreal pmf-nonneg measure-nonneg zero-less-measure-iff pmf-map)
finally have *ennreal* (*pmf* (*cond-pmf* (*map-pmf f p*) *s*) *x*) = *ennreal* (*pmf* (*map-pmf f* (*cond-pmf p* (*f* −‘ *s*))) *x*)
by *simp* **}**
then show *?thesis*
by (*intro pmf-eqI*) (*simp add*: *pmf-nonneg*)
qed

lemma *bind-cond-pmf-cancel*:
assumes [*simp*]: ⋀*x*. *x* ∈ *set-pmf p* ⟹ *set-pmf q* ∩ {*y*. *R x y*} ≠ {}

assumes [*simp*]: $\bigwedge y.\ y \in$ *set-pmf q* $\Longrightarrow$ *set-pmf p* $\cap \{x.\ R\ x\ y\} \neq \{\}$
assumes [*simp*]: $\bigwedge x\ y.\ x \in$ *set-pmf p* $\Longrightarrow\ y \in$ *set-pmf q* $\Longrightarrow\ R\ x\ y \Longrightarrow$ *measure q* $\{y.\ R\ x\ y\} =$ *measure p* $\{x.\ R\ x\ y\}$
shows *bind-pmf p* $(\lambda x.$ *cond-pmf q* $\{y.\ R\ x\ y\}) = q$
proof (*rule pmf-eqI*)
fix *i*
have *ennreal* (*pmf* (*bind-pmf p* $(\lambda x.$ *cond-pmf q* $\{y.\ R\ x\ y\}))\ i) =$
$(\int^+ x.$ *ennreal* (*pmf q i* / *measure p* $\{x.\ R\ x\ i\}) *$ *ennreal* (*indicator* $\{x.\ R\ x\ i\}\ x)\ \partial p)$
by (*auto simp add*: *ennreal-pmf-bind AE-measure-pmf-iff pmf-cond pmf-eq-0-set-pmf pmf-nonneg measure-nonneg*
intro!: *nn-integral-cong-AE*)
also have $\ldots =$ (*pmf q i* $*$ *measure p* $\{x.\ R\ x\ i\})$ / *measure p* $\{x.\ R\ x\ i\}$
by (*simp add*: *pmf-nonneg measure-nonneg zero-ennreal-def*[*symmetric*] *ennreal-indicator*
nn-integral-cmult measure-pmf.emeasure-eq-measure ennreal-mult[*symmetric*])
also have $\ldots =$ *pmf q i*
by (*cases pmf q i* $= 0$)
(*simp-all add*: *pmf-eq-0-set-pmf measure-measure-pmf-not-zero pmf-nonneg*)
finally show *pmf* (*bind-pmf p* $(\lambda x.$ *cond-pmf q* $\{y.\ R\ x\ y\}))\ i =$ *pmf q i*
by (*simp add*: *pmf-nonneg*)
qed

20.5 Relator

inductive *rel-pmf* :: $('a \Rightarrow 'b \Rightarrow$ *bool*$) \Rightarrow$ *'a pmf* $\Rightarrow$ *'b pmf* $\Rightarrow$ *bool*
for *R p q*
where
$[\![\bigwedge x\ y.\ (x, y) \in$ *set-pmf pq* $\Longrightarrow$ *R x y*;
map-pmf fst pq $=$ *p*; *map-pmf snd pq* $=$ *q* $]\!]$
$\Longrightarrow$ *rel-pmf R p q*

lemma *rel-pmfI*:
assumes *R*: *rel-set R* (*set-pmf p*) (*set-pmf q*)
assumes *eq*: $\bigwedge x\ y.\ x \in$ *set-pmf p* $\Longrightarrow\ y \in$ *set-pmf q* $\Longrightarrow\ R\ x\ y \Longrightarrow$
measure p $\{x.\ R\ x\ y\} =$ *measure q* $\{y.\ R\ x\ y\}$
shows *rel-pmf R p q*
proof
let *?pq* $=$ *bind-pmf p* $(\lambda x.$ *bind-pmf* (*cond-pmf q* $\{y.\ R\ x\ y\})$ $(\lambda y.$ *return-pmf* $(x, y)))$
have $\bigwedge x.\ x \in$ *set-pmf p* $\Longrightarrow$ *set-pmf q* $\cap \{y.\ R\ x\ y\} \neq \{\}$
using *R* **by** (*auto simp*: *rel-set-def*)
then show $\bigwedge x\ y.\ (x, y) \in$ *set-pmf ?pq* $\Longrightarrow$ *R x y*
by *auto*
show *map-pmf fst ?pq* $=$ *p*
by (*simp add*: *map-bind-pmf bind-return-pmf'*)

show *map-pmf snd ?pq* $=$ *q*
using *R eq*

apply (*simp add*: *bind-cond-pmf-cancel map-bind-pmf bind-return-pmf'*)
apply (*rule bind-cond-pmf-cancel*)
apply (*auto simp*: *rel-set-def*)
done
qed

lemma *rel-pmf-imp-rel-set*: *rel-pmf R p q* ⟹ *rel-set R* (*set-pmf p*) (*set-pmf q*)
by (*force simp add*: *rel-pmf.simps rel-set-def*)

lemma *rel-pmfD-measure*:
assumes *rel-R*: *rel-pmf R p q* **and** *R*: ⋀*a b*. *R a b* ⟹ *R a y* ⟷ *R x b*
assumes *x* ∈ *set-pmf p y* ∈ *set-pmf q*
shows *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*}
proof −
from *rel-R* **obtain** *pq* **where** *pq*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *R x y*
and *eq*: *p* = *map-pmf fst pq q* = *map-pmf snd pq*
by (*auto elim*: *rel-pmf.cases*)
have *measure p* {*x*. *R x y*} = *measure pq* {*x*. *R* (*fst x*) *y*}
by (*simp add*: *eq map-pmf-rep-eq measure-distr*)
also have ... = *measure pq* {*y*. *R x* (*snd y*)}
by (*intro measure-pmf.finite-measure-eq-AE*)
(*auto simp*: *AE-measure-pmf-iff R dest*!: *pq*)
also have ... = *measure q* {*y*. *R x y*}
by (*simp add*: *eq map-pmf-rep-eq measure-distr*)
finally show *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*} **.**
qed

lemma *rel-pmf-measureD*:
assumes *rel-pmf R p q*
shows *measure* (*measure-pmf p*) *A* ≤ *measure* (*measure-pmf q*) {*y*. ∃ *x*∈*A*. *R x y*} (**is** *?lhs* ≤ *?rhs*)
using *assms*
proof *cases*
fix *pq*
assume *R*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *R x y*
and *p*[*symmetric*]: *map-pmf fst pq* = *p*
and *q*[*symmetric*]: *map-pmf snd pq* = *q*
have *?lhs* = *measure* (*measure-pmf pq*) (*fst* −‘ *A*) **by**(*simp add*: *p*)
also have ... ≤ *measure* (*measure-pmf pq*) {*y*. ∃ *x*∈*A*. *R x* (*snd y*)}
by(*rule measure-pmf.finite-measure-mono-AE*)(*auto 4 3 simp add*: *AE-measure-pmf-iff dest*: *R*)
also have ... = *?rhs* **by**(*simp add*: *q*)
finally show *?thesis* **.**
qed

lemma *rel-pmf-iff-measure*:
assumes *symp R transp R*
shows *rel-pmf R p q* ⟷
rel-set R (*set-pmf p*) (*set-pmf q*) ∧

(∀ *x*∈*set-pmf p*. ∀ *y*∈*set-pmf q*. *R x y* ⟶ *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*})
by (*safe intro*!: *rel-pmf-imp-rel-set rel-pmfI*)
(*auto intro*!: *rel-pmfD-measure dest*: *sympD*[*OF* ‹*symp R*›] *transpD*[*OF* ‹*transp R*›])

lemma *quotient-rel-set-disjoint*:
equivp R ⟹ *C* ∈ *UNIV* // {(*x*, *y*). *R x y*} ⟹ *rel-set R A B* ⟹ *A* ∩ *C* = {} ⟷ *B* ∩ *C* = {}
using *in-quotient-imp-closed*[*of UNIV* {(*x*, *y*). *R x y*} *C*]
by (*auto 0 0 simp*: *equivp-equiv rel-set-def set-eq-iff elim*: *equivpE*)
(*blast dest*: *equivp-symp*)+

lemma *quotientD*: *equiv X R* ⟹ *A* ∈ *X* // *R* ⟹ *x* ∈ *A* ⟹ *A* = *R* `` {*x*}
by (*metis Image-singleton-iff equiv-class-eq-iff quotientE*)

lemma *rel-pmf-iff-equivp*:
assumes *equivp R*
shows *rel-pmf R p q* ⟷ (∀ *C*∈*UNIV* // {(*x*, *y*). *R x y*}. *measure p C* = *measure q C*)
(**is** - ⟷ (∀ *C*∈-//*?R*. -))
proof (*subst rel-pmf-iff-measure*, *safe*)
show *symp R transp R*
using *assms* **by** (*auto simp*: *equivp-reflp-symp-transp*)
next
fix *C* **assume** *C*: *C* ∈ *UNIV* // *?R* **and** *R*: *rel-set R* (*set-pmf p*) (*set-pmf q*)
assume *eq*: ∀ *x*∈*set-pmf p*. ∀ *y*∈*set-pmf q*. *R x y* ⟶ *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*}

show *measure p C* = *measure q C*
proof (*cases p* ∩ *C* = {})
case *True*
then have *q* ∩ *C* = {}
using *quotient-rel-set-disjoint*[*OF assms C R*] **by** *simp*
with *True* **show** *?thesis*
unfolding *measure-pmf-zero-iff*[*symmetric*] **by** *simp*
next
case *False*
then have *q* ∩ *C* ≠ {}
using *quotient-rel-set-disjoint*[*OF assms C R*] **by** *simp*
with *False* **obtain** *x y* **where** *in-set*: *x* ∈ *set-pmf p y* ∈ *set-pmf q* **and** *in-C*: *x* ∈ *C y* ∈ *C*
by *auto*
then have *R x y*
using *in-quotient-imp-in-rel*[*of UNIV ?R C x y*] *C assms*
by (*simp add*: *equivp-equiv*)
with *in-set eq* **have** *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*}
by *auto*
moreover have {*y*. *R x y*} = *C*

```
    using assms ‹x ∈ C› C quotientD[of UNIV ?R C x] by (simp add: equivp-equiv)
    moreover have {x. R x y} = C
      using assms ‹y ∈ C› C quotientD[of UNIV ?R C y] sympD[of R]
      by (auto simp add: equivp-equiv elim: equivpE)
    ultimately show ?thesis
      by auto
  qed
next
  assume eq: ∀ C∈UNIV // ?R. measure p C = measure q C
  show rel-set R (set-pmf p) (set-pmf q)
    unfolding rel-set-def
  proof safe
    fix x assume x: x ∈ set-pmf p
    have {y. R x y} ∈ UNIV // ?R
      by (auto simp: quotient-def)
    with eq have ∗: measure q {y. R x y} = measure p {y. R x y}
      by auto
    have measure q {y. R x y} ≠ 0
      using x assms unfolding ∗ by (auto simp: measure-pmf-zero-iff set-eq-iff
dest: equivp-reflp)
    then show ∃ y∈set-pmf q. R x y
      unfolding measure-pmf-zero-iff by auto
  next
    fix y assume y: y ∈ set-pmf q
    have {x. R x y} ∈ UNIV // ?R
      using assms by (auto simp: quotient-def dest: equivp-symp)
    with eq have ∗: measure p {x. R x y} = measure q {x. R x y}
      by auto
    have measure p {x. R x y} ≠ 0
      using y assms unfolding ∗ by (auto simp: measure-pmf-zero-iff set-eq-iff
dest: equivp-reflp)
    then show ∃ x∈set-pmf p. R x y
      unfolding measure-pmf-zero-iff by auto
  qed

  fix x y assume x ∈ set-pmf p y ∈ set-pmf q R x y
  have {y. R x y} ∈ UNIV // ?R {x. R x y} = {y. R x y}
   using assms ‹R x y› by (auto simp: quotient-def dest: equivp-symp equivp-transp)
  with eq show measure p {x. R x y} = measure q {y. R x y}
    by auto
qed

bnf pmf: 'a pmf map: map-pmf sets: set-pmf bd : card-suc natLeq rel: rel-pmf
proof −
  show map-pmf id = id by (rule map-pmf-id)
 show ⋀f g. map-pmf (f ∘ g) = map-pmf f ∘ map-pmf g by (rule map-pmf-compose)
  show ⋀f g::'a ⇒ 'b. ⋀p. (⋀x. x ∈ set-pmf p ⟹ f x = g x) ⟹ map-pmf f p =
map-pmf g p
    by (intro map-pmf-cong refl)
```

```
 show ⋀f::'a ⇒ 'b. set-pmf ∘ map-pmf f = (`) f ∘ set-pmf
   by (rule pmf-set-map)

 show card-order (card-suc natLeq) using natLeq-card-order by (rule card-order-card-suc)
 show BNF-Cardinal-Arithmetic.cinfinite (card-suc natLeq)
   using natLeq-Cinfinite natLeq-card-order Cinfinite-card-suc by blast
 show regularCard (card-suc natLeq) using natLeq-card-order natLeq-Cinfinite
   by (rule regularCard-card-suc)

 show (card-of (set-pmf p), card-suc natLeq) ∈ ordLess for p :: 's pmf
 proof −
   have (card-of (set-pmf p), card-of (UNIV :: nat set)) ∈ ordLeq
     by (rule card-of-ordLeqI[where f=to-nat-on (set-pmf p)])
        (auto intro: countable-set-pmf)
   also have (card-of (UNIV :: nat set), natLeq) ∈ ordLeq
     by (metis Field-natLeq card-of-least natLeq-Well-order)
  finally show ?thesis using card-suc-greater natLeq-card-order ordLeq-ordLess-trans
by blast
 qed

 show ⋀R. rel-pmf R = (λx y. ∃z. set-pmf z ⊆ {(x, y). R x y} ∧
   map-pmf fst z = x ∧ map-pmf snd z = y)
    by (auto simp add: fun-eq-iff rel-pmf.simps)

 show rel-pmf R OO rel-pmf S ≤ rel-pmf (R OO S)
   for R :: 'a ⇒ 'b ⇒ bool and S :: 'b ⇒ 'c ⇒ bool
 proof −
   { fix p q r
     assume pq: rel-pmf R p q
       and qr:rel-pmf S q r
     from pq obtain pq where pq: ⋀x y. (x, y) ∈ set-pmf pq ⟹ R x y
       and p: p = map-pmf fst pq and q: q = map-pmf snd pq by cases auto
     from qr obtain qr where qr: ⋀y z. (y, z) ∈ set-pmf qr ⟹ S y z
       and q': q = map-pmf fst qr and r: r = map-pmf snd qr by cases auto

     define pr where pr =
       bind-pmf pq (λxy. bind-pmf (cond-pmf qr {yz. fst yz = snd xy})
         (λyz. return-pmf (fst xy, snd yz)))
     have pr-welldefined: ⋀y. y ∈ q ⟹ qr ∩ {yz. fst yz = y} ≠ {}
       by (force simp: q')

     have rel-pmf (R OO S) p r
     proof (rule rel-pmf.intros)
       fix x z assume (x, z) ∈ pr
       then have ∃y. (x, y) ∈ pq ∧ (y, z) ∈ qr
         by (auto simp: q pr-welldefined pr-def split-beta)
       with pq qr show (R OO S) x z
         by blast
```

next
have *map-pmf snd pr = map-pmf snd (bind-pmf q (λy. cond-pmf qr {yz. fst yz = y}))*
by (*simp add*: *pr-def q split-beta bind-map-pmf map-pmf-def*[*symmetric*] *map-bind-pmf map-pmf-comp*)
then show *map-pmf snd pr = r*
unfolding *r q' bind-map-pmf* **by** (*subst* (*asm*) *bind-cond-pmf-cancel*) (*auto simp*: *eq-commute*)
qed (*simp add*: *pr-def map-bind-pmf split-beta map-pmf-def*[*symmetric*] *p map-pmf-comp*)
}
then show *?thesis*
by(*auto simp add*: *le-fun-def*)
qed
qed

lemma *map-pmf-idI*: (⋀*x*. *x* ∈ *set-pmf p* ⟹ *f x* = *x*) ⟹ *map-pmf f p* = *p*
by(*simp cong*: *pmf.map-cong*)

lemma *rel-pmf-conj*[*simp*]:
rel-pmf (λ*x y*. *P* ∧ *Q x y*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
rel-pmf (λ*x y*. *Q x y* ∧ *P*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
using *set-pmf-not-empty* **by** (*fastforce simp*: *pmf.in-rel subset-eq*)+

lemma *rel-pmf-top*[*simp*]: *rel-pmf top* = *top*
by (*auto simp*: *pmf.in-rel*[*abs-def*] *fun-eq-iff map-fst-pair-pmf map-snd-pair-pmf intro*: *exI*[*of - pair-pmf x y* **for** *x y*])

lemma *rel-pmf-return-pmf1*: *rel-pmf R* (*return-pmf x*) *M* ⟷ (∀ *a*∈*M*. *R x a*)
proof *safe*
fix *a* **assume** *a* ∈ *M rel-pmf R* (*return-pmf x*) *M*
then obtain *pq* **where** ∗: ⋀*a b*. (*a*, *b*) ∈ *set-pmf pq* ⟹ *R a b*
and *eq*: *return-pmf x* = *map-pmf fst pq M* = *map-pmf snd pq*
by (*force elim*: *rel-pmf.cases*)
moreover have *set-pmf* (*return-pmf x*) = {*x*}
by *simp*
with ‹*a* ∈ *M*› **have** (*x*, *a*) ∈ *pq*
by (*force simp*: *eq*)
with ∗ **show** *R x a*
by *auto*
qed (*auto intro*!: *rel-pmf.intros*[**where** *pq*=*pair-pmf* (*return-pmf x*) *M*] *simp*: *map-fst-pair-pmf map-snd-pair-pmf*)

lemma *rel-pmf-return-pmf2*: *rel-pmf R M* (*return-pmf x*) ⟷ (∀ *a*∈*M*. *R a x*)
by (*subst pmf.rel-flip*[*symmetric*]) (*simp add*: *rel-pmf-return-pmf1*)

lemma *rel-return-pmf*[*simp*]: *rel-pmf R* (*return-pmf x1*) (*return-pmf x2*) = *R x1 x2*
unfolding *rel-pmf-return-pmf2 set-return-pmf* **by** *simp*

lemma *rel-pmf-False*[*simp*]: *rel-pmf* (λ*x y*. *False*) *x y* = *False*
unfolding *pmf.in-rel fun-eq-iff* **using** *set-pmf-not-empty* **by** *fastforce*

lemma *rel-pmf-rel-prod*:
rel-pmf (*rel-prod R S*) (*pair-pmf A A'*) (*pair-pmf B B'*) ⟷ *rel-pmf R A B* ∧ *rel-pmf S A' B'*
proof *safe*
assume *rel-pmf* (*rel-prod R S*) (*pair-pmf A A'*) (*pair-pmf B B'*)
then obtain *pq* **where** *pq*: ⋀*a b c d*. ((*a*, *c*), (*b*, *d*)) ∈ *set-pmf pq* ⟹ *R a b* ∧ *S c d*
and *eq*: *map-pmf fst pq* = *pair-pmf A A' map-pmf snd pq* = *pair-pmf B B'*
by (*force elim*: *rel-pmf.cases*)
show *rel-pmf R A B*
proof (*rule rel-pmf.intros*)
let *?f* = λ(*a*, *b*). (*fst a*, *fst b*)
have [*simp*]: (λ*x*. *fst* (*?f x*)) = *fst o fst* (λ*x*. *snd* (*?f x*)) = *fst o snd*
by *auto*

show *map-pmf fst* (*map-pmf ?f pq*) = *A*
by (*simp add*: *map-pmf-comp pmf.map-comp*[*symmetric*] *eq map-fst-pair-pmf*)
show *map-pmf snd* (*map-pmf ?f pq*) = *B*
by (*simp add*: *map-pmf-comp pmf.map-comp*[*symmetric*] *eq map-fst-pair-pmf*)

fix *a b* **assume** (*a*, *b*) ∈ *set-pmf* (*map-pmf ?f pq*)
then obtain *c d* **where** ((*a*, *c*), (*b*, *d*)) ∈ *set-pmf pq*
by *auto*
from *pq*[*OF this*] **show** *R a b* **..**
qed
show *rel-pmf S A' B'*
proof (*rule rel-pmf.intros*)
let *?f* = λ(*a*, *b*). (*snd a*, *snd b*)
have [*simp*]: (λ*x*. *fst* (*?f x*)) = *snd o fst* (λ*x*. *snd* (*?f x*)) = *snd o snd*
by *auto*

show *map-pmf fst* (*map-pmf ?f pq*) = *A'*
by (*simp add*: *map-pmf-comp pmf.map-comp*[*symmetric*] *eq map-snd-pair-pmf*)
show *map-pmf snd* (*map-pmf ?f pq*) = *B'*
by (*simp add*: *map-pmf-comp pmf.map-comp*[*symmetric*] *eq map-snd-pair-pmf*)

fix *c d* **assume** (*c*, *d*) ∈ *set-pmf* (*map-pmf ?f pq*)
then obtain *a b* **where** ((*a*, *c*), (*b*, *d*)) ∈ *set-pmf pq*
by *auto*
from *pq*[*OF this*] **show** *S c d* **..**
qed
next
assume *rel-pmf R A B rel-pmf S A' B'*
then obtain *Rpq Spq*
where *Rpq*: ⋀*a b*. (*a*, *b*) ∈ *set-pmf Rpq* ⟹ *R a b*

```
      map-pmf fst Rpq = A map-pmf snd Rpq = B
    and Spq: ⋀a b. (a, b) ∈ set-pmf Spq ⟹ S a b
      map-pmf fst Spq = A' map-pmf snd Spq = B'
  by (force elim: rel-pmf.cases)

 let ?f = (λ((a, c), (b, d)). ((a, b), (c, d)))
 let ?pq = map-pmf ?f (pair-pmf Rpq Spq)
 have [simp]: (λx. fst (?f x)) = (λ(a, b). (fst a, fst b)) (λx. snd (?f x)) = (λ(a,
b). (snd a, snd b))
  by auto

 show rel-pmf (rel-prod R S) (pair-pmf A A') (pair-pmf B B')
  by (rule rel-pmf.intros[where pq=?pq])
    (auto simp: map-snd-pair-pmf map-fst-pair-pmf map-pmf-comp Rpq Spq
          map-pair)
qed

lemma rel-pmf-reflI:
 assumes ⋀x. x ∈ set-pmf p ⟹ P x x
 shows rel-pmf P p p
 by (rule rel-pmf.intros[where pq=map-pmf (λx. (x, x)) p])
   (auto simp add: pmf.map-comp o-def assms)

lemma rel-pmf-bij-betw:
 assumes f: bij-betw f (set-pmf p) (set-pmf q)
 and eq: ⋀x. x ∈ set-pmf p ⟹ pmf p x = pmf q (f x)
 shows rel-pmf (λx y. f x = y) p q
proof(rule rel-pmf.intros)
 let ?pq = map-pmf (λx. (x, f x)) p
 show map-pmf fst ?pq = p by(simp add: pmf.map-comp o-def)

 have map-pmf f p = q
 proof(rule pmf-eqI)
  fix i
  show pmf (map-pmf f p) i = pmf q i
  proof(cases i ∈ set-pmf q)
   case True
   with f obtain j where i = f j j ∈ set-pmf p
    by(auto simp add: bij-betw-def image-iff)
   thus ?thesis using f by(simp add: bij-betw-def pmf-map-inj eq)
  next
   case False thus ?thesis
    by(subst pmf-map-outside)(auto simp add: set-pmf-iff eq[symmetric])
  qed
 qed
 then show map-pmf snd ?pq = q by(simp add: pmf.map-comp o-def)
qed auto

context
```

begin

interpretation *pmf-as-measure* .

definition *join-pmf M* = *bind-pmf M* (λ*x*. *x*)

lemma *bind-eq-join-pmf*: *bind-pmf M f* = *join-pmf* (*map-pmf f M*)
  unfolding *join-pmf-def bind-map-pmf* ..

lemma *join-eq-bind-pmf*: *join-pmf M* = *bind-pmf M id*
  by (*simp add*: *join-pmf-def id-def*)

lemma *pmf-join*: *pmf* (*join-pmf N*) *i* = (∫ *M*. *pmf M i* ∂*measure-pmf N*)
  unfolding *join-pmf-def pmf-bind* ..

lemma *ennreal-pmf-join*: *ennreal* (*pmf* (*join-pmf N*) *i*) = (∫⁺ *M*. *pmf M i* ∂*measure-pmf N*)
  unfolding *join-pmf-def ennreal-pmf-bind* ..

lemma *set-pmf-join-pmf*[*simp*]: *set-pmf* (*join-pmf f*) = (⋃ *p*∈*set-pmf f*. *set-pmf p*)
  by (*simp add*: *join-pmf-def*)

lemma *join-return-pmf*: *join-pmf* (*return-pmf M*) = *M*
  by (*simp add*: *integral-return pmf-eq-iff pmf-join return-pmf*.*rep-eq*)

lemma *map-join-pmf*: *map-pmf f* (*join-pmf AA*) = *join-pmf* (*map-pmf* (*map-pmf f*) *AA*)
  by (*simp add*: *join-pmf-def map-pmf-def bind-assoc-pmf bind-return-pmf*)

lemma *join-map-return-pmf*: *join-pmf* (*map-pmf return-pmf A*) = *A*
  by (*simp add*: *join-pmf-def map-pmf-def bind-assoc-pmf bind-return-pmf bind-return-pmf'*)

end

lemma *rel-pmf-joinI*:
  assumes *rel-pmf* (*rel-pmf P*) *p q*
  shows *rel-pmf P* (*join-pmf p*) (*join-pmf q*)
proof −
  from *assms* **obtain** *pq* **where** *p*: *p* = *map-pmf fst pq*
    and *q*: *q* = *map-pmf snd pq*
    and *P*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *rel-pmf P x y*
    by *cases auto*
  from *P* **obtain** *PQ*
    where *PQ*: ⋀*x y a b*. ⟦ (*x*, *y*) ∈ *set-pmf pq*; (*a*, *b*) ∈ *set-pmf* (*PQ x y*) ⟧ ⟹ *P a b*
    and *x*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *map-pmf fst* (*PQ x y*) = *x*
    and *y*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *map-pmf snd* (*PQ x y*) = *y*
    by(*metis rel-pmf*.*simps*)

let *?r* = *bind-pmf pq* (λ(*x*, *y*). *PQ x y*)
have ⋀*a b*. (*a*, *b*) ∈ *set-pmf ?r* ⟹ *P a b* **by** (*auto intro*: *PQ*)
moreover have *map-pmf fst ?r* = *join-pmf p map-pmf snd ?r* = *join-pmf q*
by (*simp-all add*: *p q x y join-pmf-def map-bind-pmf bind-map-pmf split-def cong*: *bind-pmf-cong*)
ultimately show *?thesis* **..**
qed

lemma *rel-pmf-bindI*:
assumes *pq*: *rel-pmf R p q*
and *fg*: ⋀*x y*. *R x y* ⟹ *rel-pmf P* (*f x*) (*g y*)
shows *rel-pmf P* (*bind-pmf p f*) (*bind-pmf q g*)
unfolding *bind-eq-join-pmf*
by (*rule rel-pmf-joinI*)
(*auto simp add*: *pmf.rel-map intro*: *pmf.rel-mono*[*THEN le-funD*, *THEN le-funD*, *THEN le-boolD*, *THEN mp*, *OF - pq*] *fg*)

Proof that *rel-pmf* preserves orders. Antisymmetry proof follows Thm. 1 in N. Saheb-Djahromi, Cpo's of measures for nondeterminism, Theoretical Computer Science 12(1):19–37, 1980, https://doi.org/10.1016/0304-3975(80)90003-1

lemma
assumes ∗: *rel-pmf R p q*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *measure-Ici*: *measure p* {*y*. *R x y*} ≤ *measure q* {*y*. *R x y*} (**is** *?thesis1*)
and *measure-Ioi*: *measure p* {*y*. *R x y* ∧ ¬ *R y x*} ≤ *measure q* {*y*. *R x y* ∧ ¬ *R y x*} (**is** *?thesis2*)
proof −
from ∗ **obtain** *pq*
where *pq*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *R x y*
and *p*: *p* = *map-pmf fst pq*
and *q*: *q* = *map-pmf snd pq*
by *cases auto*
show *?thesis1 ?thesis2* **unfolding** *p q map-pmf-rep-eq* **using** *refl trans*
by(*auto 4 3 simp add*: *measure-distr reflpD AE-measure-pmf-iff intro*!: *measure-pmf.finite-measure-mono-AE dest*!: *pq elim*: *transpE*)
qed

lemma *rel-pmf-inf*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *rel-pmf* (*inf R* R^{-1-1}) *p q*
proof (*subst rel-pmf-iff-equivp*, *safe*)
show *equivp* (*inf R* R^{-1-1})
using *trans refl* **by** (*auto simp*: *equivp-reflp-symp-transp intro*: *sympI transpI reflpI dest*: *transpD reflpD*)

fix C **assume** $C \in \mathit{UNIV} \mathbin{//} \{(x, y).\ \mathit{inf}\ R\ R^{-1-1}\ x\ y\}$
then obtain x **where** C: $C = \{y.\ R\ x\ y \wedge R\ y\ x\}$
by (*auto elim*: *quotientE*)

let $?R = \lambda x\ y.\ R\ x\ y \wedge R\ y\ x$
let $?\mu R = \lambda y.\ \mathit{measure}\ q\ \{x.\ ?R\ x\ y\}$
have $\mathit{measure}\ p\ \{y.\ ?R\ x\ y\} = \mathit{measure}\ p\ (\{y.\ R\ x\ y\} - \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\})$
by(*auto intro*!: *arg-cong*[**where** f=*measure p*])
also have $\ldots = \mathit{measure}\ p\ \{y.\ R\ x\ y\} - \mathit{measure}\ p\ \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\}$
by (*rule measure-pmf.finite-measure-Diff*) *auto*
also have $\mathit{measure}\ p\ \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\} = \mathit{measure}\ q\ \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\}$
using *1 2 refl trans* **by**(*auto intro*!: *Orderings.antisym measure-Ioi*)
also have $\mathit{measure}\ p\ \{y.\ R\ x\ y\} = \mathit{measure}\ q\ \{y.\ R\ x\ y\}$
using *1 2 refl trans* **by**(*auto intro*!: *Orderings.antisym measure-Ici*)
also have $\mathit{measure}\ q\ \{y.\ R\ x\ y\} - \mathit{measure}\ q\ \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\} =$
$\mathit{measure}\ q\ (\{y.\ R\ x\ y\} - \{y.\ R\ x\ y \wedge \neg\ R\ y\ x\})$
by(*rule measure-pmf.finite-measure-Diff*[*symmetric*]) *auto*
also have $\ldots = ?\mu R\ x$
by(*auto intro*!: *arg-cong*[**where** f=*measure q*])
finally show $\mathit{measure}\ p\ C = \mathit{measure}\ q\ C$
by (*simp add*: *C conj-commute*)
qed

lemma *rel-pmf-antisym*:
fixes $p\ q :: 'a\ \mathit{pmf}$
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R* **and** *antisym*: *antisymp R*
shows $p = q$
proof −
from *1 2 refl trans* **have** *rel-pmf* $(\mathit{inf}\ R\ R^{-1-1})\ p\ q$ **by**(*rule rel-pmf-inf*)
also have $\mathit{inf}\ R\ R^{-1-1} = (=)$
using *refl antisym* **by** (*auto intro*!: *ext simp add*: *reflpD dest*: *antisympD*)
finally show *?thesis* **unfolding** *pmf.rel-eq* **.**
qed

lemma *reflp-rel-pmf*: *reflp R* $\Longrightarrow$ *reflp* (*rel-pmf R*)
by (*fact pmf.rel-reflp*)

lemma *antisymp-rel-pmf*:
⟦ *reflp R*; *transp R*; *antisymp R* ⟧
$\Longrightarrow$ *antisymp* (*rel-pmf R*)
by(*rule antisympI*)(*blast intro*: *rel-pmf-antisym*)

lemma *transp-rel-pmf*:
assumes *transp R*
shows *transp* (*rel-pmf R*)
using *assms* **by** (*fact pmf.rel-transp*)

20.6 Distributions

context
begin

interpretation *pmf-as-function* **.**

20.6.1 Bernoulli Distribution

lift-definition *bernoulli-pmf* :: *real* ⇒ *bool pmf* **is**
λp b. ((λp. if b then p else 1 − p) ∘ min 1 ∘ max 0) p
by (*auto simp*: *nn-integral-count-space-finite*[**where** *A*={*False*, *True*}] *UNIV-bool split*: *split-max split-min*)

lemma *pmf-bernoulli-True*[*simp*]: *0 ≤ p ⟹ p ≤ 1 ⟹ pmf* (*bernoulli-pmf p*) *True* = *p*
by *transfer simp*

lemma *pmf-bernoulli-False*[*simp*]: *0 ≤ p ⟹ p ≤ 1 ⟹ pmf* (*bernoulli-pmf p*) *False* = *1 − p*
by *transfer simp*

lemma *set-pmf-bernoulli*[*simp*]: *0 < p ⟹ p < 1 ⟹ set-pmf* (*bernoulli-pmf p*) = *UNIV*
by (*auto simp add*: *set-pmf-iff UNIV-bool*)

lemma *nn-integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0 ≤ p p ≤ 1* ⋀*x. 0 ≤ f x*
shows (∫⁺*x. f x ∂bernoulli-pmf p*) = *f True ∗ p + f False ∗ (1 − p)*
by (*subst nn-integral-measure-pmf-support*[*of UNIV*])
(*auto simp*: *UNIV-bool field-simps*)

lemma *integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0 ≤ p p ≤ 1*
shows (∫ *x. f x ∂bernoulli-pmf p*) = *f True ∗ p + f False ∗ (1 − p)*
by (*subst integral-measure-pmf*[*of UNIV*]) (*auto simp*: *UNIV-bool*)

lemma *pmf-bernoulli-half* [*simp*]: *pmf* (*bernoulli-pmf* (*1 / 2*)) *x* = *1 / 2*
by(*cases x*) *simp-all*

lemma *measure-pmf-bernoulli-half*: *measure-pmf* (*bernoulli-pmf* (*1 / 2*)) = *uniform-count-measure UNIV*
by (*rule measure-eqI*)
(*simp-all add*: *nn-integral-pmf*[*symmetric*] *emeasure-uniform-count-measure ennreal-divide-numeral*[*symmetric*]
nn-integral-count-space-finite sets-uniform-count-measure divide-ennreal-def mult-ac
ennreal-of-nat-eq-real-of-nat)

20.6.2 Geometric Distribution

context
fixes $p :: real$ **assumes** $p[arith]: 0 < p\ p \leq 1$
begin

lift-definition *geometric-pmf* :: *nat pmf* **is** $\lambda n.\ (1 - p) \hat{}\ n * p$
proof
have $(\sum i.\ ennreal\ (p * (1 - p)\ \hat{}\ i)) = ennreal\ (p * (1 / (1 - (1 - p))))$
by (*intro suminf-ennreal-eq sums-mult geometric-sums*) *auto*
then show $(\int^+ x.\ ennreal\ ((1 - p) \hat{}\ x * p)\ \partial count\text{-}space\ UNIV) = 1$
by (*simp add*: *nn-integral-count-space-nat field-simps*)
qed *simp*

lemma *pmf-geometric*[*simp*]: *pmf geometric-pmf n* $= (1 - p) \hat{}\ n * p$
by *transfer rule*

end

lemma *geometric-pmf-1* [*simp*]: *geometric-pmf 1 = return-pmf 0*
by (*intro pmf-eqI*) (*auto simp*: *indicator-def*)

lemma *set-pmf-geometric*: $0 < p \Longrightarrow p < 1 \Longrightarrow$ *set-pmf* (*geometric-pmf p*) = *UNIV*
by (*auto simp*: *set-pmf-iff*)

lemma *geometric-sums-times-n*:
fixes $c::'a::\{banach, real\text{-}normed\text{-}field\}$
assumes *norm c < 1*
shows $(\lambda n.\ c \hat{}\ n * of\text{-}nat\ n)\ sums\ (c / (1 - c)^2)$
proof −
have $(\lambda n.\ c * z \hat{}\ n)\ sums\ (c / (1 - z))$ **if** *norm z < 1* **for** *z*
using *geometric-sums sums-mult that* **by** *fastforce*
moreover have $((\lambda z.\ c / (1 - z))\ has\text{-}field\text{-}derivative\ (c / (1 - c)^2))\ (at\ c)$
using *assms* **by** (*auto intro*!: *derivative-eq-intros simp add*: *semiring-normalization-rules*)
ultimately have $(\lambda n.\ diffs\ (\lambda n.\ c)\ n * c \hat{}\ n)\ sums\ (c / (1 - c)^2)$
using *assms* **by** (*intro termdiffs-sums-strong*)
then have $(\lambda n.\ of\text{-}nat\ (Suc\ n) * c \hat{}\ (Suc\ n))\ sums\ (c / (1 - c)^2)$
unfolding *diffs-def* **by** (*simp add*: *power-eq-if mult.assoc*)
then show *?thesis*
by (*subst* (*asm*) *sums-Suc-iff*) (*auto simp add*: *mult.commute*)
qed

lemma *geometric-sums-times-norm*:
fixes $c::'a::\{banach, real\text{-}normed\text{-}field\}$
assumes *norm c < 1*
shows $(\lambda n.\ norm\ (c \hat{}\ n * of\text{-}nat\ n))\ sums\ (norm\ c / (1 - norm\ c)^2)$
proof −
have $norm\ (c \hat{}\ n * of\text{-}nat\ n) = (norm\ c) \hat{}\ n * of\text{-}nat\ n$ **for** $n::nat$
by (*simp add*: *norm-power norm-mult*)

then show *?thesis*
using *geometric-sums-times-n*[*of norm c*] *assms*
by *force*
qed

lemma *integrable-real-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows *integrable* (*geometric-pmf p*) *real*
proof −
have *summable* ($\lambda x.\ p * ((1 - p) \hat{}\ x * real\ x)$)
using *geometric-sums-times-norm*[*of 1 − p*] *assms*
by (*intro summable-mult*) (*auto simp*: *sums-iff*)
hence *summable* ($\lambda x.\ (1 - p) \hat{}\ x * real\ x$)
by (*rule summable-mult-D*) (*use assms* **in** *auto*)
thus *?thesis*
unfolding *measure-pmf-eq-density* **using** *assms*
by (*subst integrable-density*)
(*auto simp*: *integrable-count-space-nat-iff mult-ac*)
qed

lemma *expectation-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*geometric-pmf p*) *real* = (*1 − p*) / *p*
proof −
have ($\lambda n.\ p * ((1 - p) \hat{}\ n * n)$) *sums* ($p * ((1 - p) / p\hat{}2)$)
using *assms geometric-sums-times-n*[*of 1−p*] **by** (*intro sums-mult*) *auto*
moreover have ($\lambda n.\ p * ((1 - p) \hat{}\ n * n)$) = ($\lambda n.\ (1 - p) \hat{}\ n * p\ * real\ n$)
by *auto*
ultimately have ∗: ($\lambda n.\ (1 - p) \hat{}\ n * p\ * real\ n$) *sums* ((*1 − p*) / *p*)
using *assms sums-subst* **by** (*auto simp add*: *power2-eq-square*)
have *measure-pmf.expectation* (*geometric-pmf p*) *real* =
($\int n.$ *pmf* (*geometric-pmf p*) *n* ∗ *real n* ∂*count-space UNIV*)
unfolding *measure-pmf-eq-density* **by** (*subst integral-density*) *auto*
also have *integrable* (*count-space UNIV*) ($\lambda n.$ *pmf* (*geometric-pmf p*) *n* ∗ *real n*)
using ∗ *assms* **unfolding** *integrable-count-space-nat-iff* **by** (*simp add*: *sums-iff*)
hence ($\int n.$ *pmf* (*geometric-pmf p*) *n* ∗ *real n* ∂*count-space UNIV*) = (*1 − p*) / *p*
using ∗ *assms* **by** (*subst integral-count-space-nat*) (*simp-all add*: *sums-iff*)
finally show *?thesis* **by** *auto*
qed

lemma *geometric-bind-pmf-unfold*:
assumes $p \in \{0<..1\}$
shows *geometric-pmf p* =
do {*b* ← *bernoulli-pmf p*;
if b then return-pmf 0 else map-pmf Suc (*geometric-pmf p*)}
proof −
have ∗: (*Suc* −‘ {*i*}) = (*if i = 0 then* {} *else* {*i − 1*}) **for** *i*

```
    by force
  have pmf (geometric-pmf p) i =
        pmf (bernoulli-pmf p ⤜
            (λb. if b then return-pmf 0 else map-pmf Suc (geometric-pmf p)))
            i for i
  proof −
    have pmf (geometric-pmf p) i =
          (if i = 0 then p else (1 − p) ∗ pmf (geometric-pmf p) (i − 1))
      using assms by (simp add: power-eq-if)
   also have ... = (if i = 0 then p else (1 − p) ∗ pmf (map-pmf Suc (geometric-pmf
p)) i)
      by (simp add: pmf-map indicator-def measure-pmf-single ∗)
    also have ... = measure-pmf.expectation (bernoulli-pmf p)
          (λx. pmf (if x then return-pmf 0 else map-pmf Suc (geometric-pmf p)) i)
      using assms by (auto simp add: pmf-map ∗)
    also have ... = pmf (bernoulli-pmf p ⤜
                  (λb. if b then return-pmf 0 else map-pmf Suc (geometric-pmf p)))
                  i
      by (auto simp add: pmf-bind)
    finally show ?thesis .
  qed
  then show ?thesis
    using pmf-eqI by blast
qed
```

20.6.3 Uniform Multiset Distribution

```
context
  fixes M :: 'a multiset assumes M-not-empty: M ≠ {#}
begin

lift-definition pmf-of-multiset :: 'a pmf is λx. count M x / size M
proof
  show (∫+ x. ennreal (real (count M x) / real (size M)) ∂count-space UNIV) =
1
    using M-not-empty
    by (simp add: zero-less-divide-iff nn-integral-count-space nonempty-has-size
                  sum-divide-distrib[symmetric])
       (auto simp: size-multiset-overloaded-eq intro!: sum.cong)
qed simp

lemma pmf-of-multiset[simp]: pmf pmf-of-multiset x = count M x / size M
  by transfer rule

lemma set-pmf-of-multiset[simp]: set-pmf pmf-of-multiset = set-mset M
  by (auto simp: set-pmf-iff)

end
```

20.6.4 Uniform Distribution

context
fixes $S :: 'a\ set$ **assumes** *S-not-empty*: $S \neq \{\}$ **and** *S-finite*: *finite S*
begin

lift-definition *pmf-of-set* :: $'a\ pmf$ **is** $\lambda x.\ indicator\ S\ x\ /\ card\ S$
proof
show $(\int^+ x.\ ennreal\ (indicator\ S\ x\ /\ real\ (card\ S))\ \partial count\text{-}space\ UNIV) = 1$
using *S-not-empty S-finite*
by (*subst nn-integral-count-space'*[*of S*])
(*auto simp*: *ennreal-of-nat-eq-real-of-nat ennreal-mult*[*symmetric*])
qed *simp*

lemma *pmf-of-set*[*simp*]: *pmf pmf-of-set x* = *indicator S x / card S*
by *transfer rule*

lemma *set-pmf-of-set*[*simp*]: *set-pmf pmf-of-set* = *S*
using *S-finite S-not-empty* **by** (*auto simp*: *set-pmf-iff*)

lemma *emeasure-pmf-of-set-space*[*simp*]: *emeasure pmf-of-set S* = *1*
by (*rule measure-pmf.emeasure-eq-1-AE*) (*auto simp*: *AE-measure-pmf-iff*)

lemma *nn-integral-pmf-of-set*: *nn-integral* (*measure-pmf pmf-of-set*) *f* = *sum f S / card S*
by (*subst nn-integral-measure-pmf-finite*)
(*simp-all add*: *sum-distrib-right*[*symmetric*] *card-gt-0-iff S-not-empty S-finite divide-ennreal-def*
divide-ennreal[*symmetric*] *ennreal-of-nat-eq-real-of-nat*[*symmetric*] *ennreal-times-divide*)

lemma *integral-pmf-of-set*: $integral^L$ (*measure-pmf pmf-of-set*) *f* = *sum f S / card S*
by (*subst integral-measure-pmf*[*of S*]) (*auto simp*: *S-finite sum-divide-distrib*)

lemma *emeasure-pmf-of-set*: *emeasure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
by (*subst nn-integral-indicator*[*symmetric*], *simp*)
(*simp add*: *S-finite S-not-empty card-gt-0-iff indicator-def sum.If-cases divide-ennreal*
ennreal-of-nat-eq-real-of-nat nn-integral-pmf-of-set)

lemma *measure-pmf-of-set*: *measure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
using *emeasure-pmf-of-set*[*of A*]
by (*simp add*: *measure-nonneg measure-pmf.emeasure-eq-measure*)

end

lemma *pmf-expectation-bind-pmf-of-set*:

fixes $A :: 'a\ set$ **and** $f :: 'a \Rightarrow 'b\ pmf$
and $h :: 'b \Rightarrow 'c::\{banach, second\text{-}countable\text{-}topology\}$
assumes $A \neq \{\}$ *finite* A $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*set-pmf* ($f\ x$))
shows *measure-pmf.expectation* (*pmf-of-set* $A \gg\!= f$) $h =$
$(\sum a{\in}A.\ measure\text{-}pmf.expectation\ (f\ a)\ h\ /_R\ real\ (card\ A))$
using *assms* **by** (*subst pmf-expectation-bind*[*of A*]) (*auto simp*: *field-split-simps*)

lemma *map-pmf-of-set*:
assumes *finite* A $A \neq \{\}$
shows *map-pmf f* (*pmf-of-set A*) = *pmf-of-multiset* (*image-mset f* (*mset-set A*))
(**is** *?lhs* = *?rhs*)
proof (*intro pmf-eqI*)
fix x
from *assms* **have** *ennreal* (*pmf ?lhs x*) = *ennreal* (*pmf ?rhs x*)
by (*subst ennreal-pmf-map*)
(*simp-all add*: *emeasure-pmf-of-set mset-set-empty-iff count-image-mset Int-commute*)
thus *pmf ?lhs x* = *pmf ?rhs x* **by** *simp*
qed

lemma *pmf-bind-pmf-of-set*:
assumes $A \neq \{\}$ *finite* A
shows *pmf* (*bind-pmf* (*pmf-of-set A*) f) $x =$
$(\sum xa{\in}A.\ pmf\ (f\ xa)\ x)$ / *real-of-nat* (*card A*) (**is** *?lhs* = *?rhs*)
proof −
from *assms* **have** *card A* > *0* **by** *auto*
with *assms* **have** *ennreal ?lhs* = *ennreal ?rhs*
by (*subst ennreal-pmf-bind*)
(*simp-all add*: *nn-integral-pmf-of-set max-def pmf-nonneg divide-ennreal*[*symmetric*]
sum-nonneg ennreal-of-nat-eq-real-of-nat)
thus *?thesis* **by** (*subst* (*asm*) *ennreal-inj*) (*auto intro*!: *sum-nonneg divide-nonneg-nonneg*)
qed

lemma *pmf-of-set-singleton*: *pmf-of-set* $\{x\}$ = *return-pmf x*
by(*rule pmf-eqI*)(*simp add*: *indicator-def*)

lemma *map-pmf-of-set-inj*:
assumes f: *inj-on f A*
and [*simp*]: $A \neq \{\}$ *finite* A
shows *map-pmf f* (*pmf-of-set A*) = *pmf-of-set* (f ‘ A) (**is** *?lhs* = *?rhs*)
proof(*rule pmf-eqI*)
fix i
show *pmf ?lhs i* = *pmf ?rhs i*
proof(*cases* $i \in f$ ‘ A)
case *True*
then obtain i' **where** $i = f\ i'$ $i' \in A$ **by** *auto*
thus *?thesis* **using** f **by**(*simp add*: *card-image pmf-map-inj*)

```
  next
    case False
    hence pmf ?lhs i = 0 by(simp add: pmf-eq-0-set-pmf set-map-pmf)
    moreover have pmf ?rhs i = 0 using False by simp
    ultimately show ?thesis by simp
  qed
qed

lemma map-pmf-of-set-bij-betw:
  assumes bij-betw f A B A ≠ {} finite A
  shows   map-pmf f (pmf-of-set A) = pmf-of-set B
proof −
  have map-pmf f (pmf-of-set A) = pmf-of-set (f ‘ A)
    by (intro map-pmf-of-set-inj assms bij-betw-imp-inj-on[OF assms(1)])
  also from assms have f ‘ A = B by (simp add: bij-betw-def)
  finally show ?thesis .
qed
```

Choosing an element uniformly at random from the union of a disjoint family of finite non-empty sets with the same size is the same as first choosing a set from the family uniformly at random and then choosing an element from the chosen set uniformly at random.

```
lemma pmf-of-set-UN:
  assumes finite (⋃(f ‘ A)) A ≠ {} ⋀x. x ∈ A ⟹ f x ≠ {}
          ⋀x. x ∈ A ⟹ card (f x) = n disjoint-family-on f A
  shows   pmf-of-set (⋃(f ‘ A)) = do {x ← pmf-of-set A; pmf-of-set (f x)}
          (is ?lhs = ?rhs)
proof (intro pmf-eqI)
  fix x
  from assms have [simp]: finite A
    using infinite-disjoint-family-imp-infinite-UNION[of A f] by blast
  from assms have ereal (pmf (pmf-of-set (⋃(f ‘ A))) x) =
    ereal (indicator (⋃x∈A. f x) x / real (card (⋃x∈A. f x)))
    by (subst pmf-of-set) auto
  also from assms have card (⋃x∈A. f x) = card A * n
    by (subst card-UN-disjoint) (auto simp: disjoint-family-on-def)
  also from assms
    have indicator (⋃x∈A. f x) x / real ... =
           indicator (⋃x∈A. f x) x / (n * real (card A))
      by (simp add: sum-divide-distrib [symmetric] mult-ac)
  also from assms have indicator (⋃x∈A. f x) x = (∑y∈A. indicator (f y) x)
    by (intro indicator-UN-disjoint) simp-all
  also from assms have ereal ((∑y∈A. indicator (f y) x) / (real n * real (card
A))) =
                        ereal (pmf ?rhs x)
    by (subst pmf-bind-pmf-of-set) (simp-all add: sum-divide-distrib)
  finally show pmf ?lhs x = pmf ?rhs x by simp
qed
```

lemma *bernoulli-pmf-half-conv-pmf-of-set*: *bernoulli-pmf (1 / 2) = pmf-of-set UNIV*
by (*rule pmf-eqI*) *simp-all*

20.6.5 Poisson Distribution

context
fixes *rate* :: *real* **assumes** *rate-pos*: *0 < rate*
begin

lift-definition *poisson-pmf* :: *nat pmf* **is** *λk. rate ^ k / fact k * exp (−rate)*
proof
have *summable*: *summable (λx::nat. rate ^ x / fact x)* **using** *summable-exp*
by (*simp add*: *field-simps divide-inverse* [*symmetric*])
have *(∫⁺(x::nat). rate ^ x / fact x * exp (−rate) ∂count-space UNIV) =*
*exp (−rate) * (∫⁺(x::nat). rate ^ x / fact x ∂count-space UNIV)*
by (*simp add*: *field-simps nn-integral-cmult*[*symmetric*] *ennreal-mult'*[*symmetric*])
also from *rate-pos* **have** *(∫⁺(x::nat). rate ^ x / fact x ∂count-space UNIV) = (∑ x. rate ^ x / fact x)*
by (*simp-all add*: *nn-integral-count-space-nat suminf-ennreal summable ennreal-suminf-neq-top*)
also have *... = exp rate* **unfolding** *exp-def*
by (*simp add*: *field-simps divide-inverse* [*symmetric*])
also have *ennreal (exp (−rate)) * ennreal (exp rate) = 1*
by (*simp add*: *mult-exp-exp ennreal-mult*[*symmetric*])
finally show *(∫⁺ x. ennreal (rate ^ x / (fact x) * exp (− rate)) ∂count-space UNIV) = 1* **.**
qed (*simp add*: *rate-pos*[*THEN less-imp-le*])

lemma *pmf-poisson*[*simp*]: *pmf poisson-pmf k = rate ^ k / fact k * exp (−rate)*
by *transfer rule*

lemma *set-pmf-poisson*[*simp*]: *set-pmf poisson-pmf = UNIV*
using *rate-pos* **by** (*auto simp*: *set-pmf-iff*)

end

20.6.6 Binomial Distribution

context
fixes *n* :: *nat* **and** *p* :: *real* **assumes** *p-nonneg*: *0 ≤ p* **and** *p-le-1*: *p ≤ 1*
begin

lift-definition *binomial-pmf* :: *nat pmf* **is** *λk. (n choose k) * p^k * (1 − p)^(n − k)*
proof
have *(∫⁺k. ennreal (real (n choose k) * p ^ k * (1 − p) ^ (n − k)) ∂count-space UNIV) =*
*ennreal (∑ k≤n. real (n choose k) * p ^ k * (1 − p) ^ (n − k))*
using *p-le-1 p-nonneg* **by** (*subst nn-integral-count-space'*) *auto*

also have $(\sum k{\le}n.\ real\ (n\ choose\ k) * p\ \widehat{}\ k * (1 - p)\ \widehat{}\ (n - k)) = (p + (1 - p))\ \widehat{}\ n$
by (*subst binomial-ring*) (*simp add*: *atLeast0AtMost*)
finally show $(\int^+ x.\ ennreal\ (real\ (n\ choose\ x) * p\ \widehat{}\ x * (1 - p)\ \widehat{}\ (n - x))\ \partial count\text{-}space\ UNIV) = 1$
by *simp*
qed (*insert p-nonneg p-le-1*, *simp*)

lemma *pmf-binomial*[*simp*]: *pmf binomial-pmf k* = (*n choose k*) * *p*^*k* * (1 − *p*)^(*n* − *k*)
by *transfer rule*

lemma *set-pmf-binomial-eq*: *set-pmf binomial-pmf* = (*if p* = *0 then* {*0*} *else if p* = *1 then* {*n*} *else* {.. *n*})
using *p-nonneg p-le-1* **unfolding** *set-eq-iff set-pmf-iff pmf-binomial* **by** (*auto simp*: *set-pmf-iff*)

end

end

lemma *set-pmf-binomial-0*[*simp*]: *set-pmf* (*binomial-pmf n 0*) = {*0*}
by (*simp add*: *set-pmf-binomial-eq*)

lemma *set-pmf-binomial-1*[*simp*]: *set-pmf* (*binomial-pmf n 1*) = {*n*}
by (*simp add*: *set-pmf-binomial-eq*)

lemma *set-pmf-binomial*[*simp*]: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*binomial-pmf n p*) = {..*n*}
by (*simp add*: *set-pmf-binomial-eq*)

lemma *finite-set-pmf-binomial-pmf* [*intro*]: *p* ∈ {*0*..*1*} ⟹ *finite* (*set-pmf* (*binomial-pmf n p*))
by (*subst set-pmf-binomial-eq*) *auto*

lemma *expectation-binomial-pmf'*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *p*: *p* ∈ {*0*..*1*}
shows *measure-pmf.expectation* (*binomial-pmf n p*) *f* =
($\sum k{\le}n.$ (*real* (*n choose k*) * *p* ^ *k* * (*1* − *p*) ^ (*n* − *k*)) *$_R$ f k*)
using *p* **by** (*subst integral-measure-pmf*[**where** *A* = {..*n*}])
(*auto simp*: *set-pmf-binomial-eq split*: *if-splits*)

lemma *integrable-binomial-pmf* [*simp*, *intro*]:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *p*: *p* ∈ {*0*..*1*}
shows *integrable* (*binomial-pmf n p*) *f*
by (*rule integrable-measure-pmf-finite*) (*use assms* **in** *auto*)

context includes *lifting-syntax*
begin

lemma *bind-pmf-parametric* [*transfer-rule*]:
(*rel-pmf A* ===> (*A* ===> *rel-pmf B*) ===> *rel-pmf B*) *bind-pmf bind-pmf*
by(*blast intro*: *rel-pmf-bindI dest*: *rel-funD*)

lemma *return-pmf-parametric* [*transfer-rule*]: (*A* ===> *rel-pmf A*) *return-pmf return-pmf*
by(*rule rel-funI*) *simp*

end

primrec *replicate-pmf* :: *nat* ⇒ *'a pmf* ⇒ *'a list pmf* **where**
replicate-pmf 0 - = return-pmf []
| *replicate-pmf* (*Suc n*) *p* = *do* {*x* ← *p*; *xs* ← *replicate-pmf n p*; *return-pmf* (*x*#*xs*)}

lemma *replicate-pmf-1*: *replicate-pmf 1 p* = *map-pmf* (λ*x*. [*x*]) *p*
by (*simp add*: *map-pmf-def bind-return-pmf*)

lemma *set-replicate-pmf*:
set-pmf (*replicate-pmf n p*) = {*xs*∈*lists* (*set-pmf p*). *length xs* = *n*}
by (*induction n*) (*auto simp*: *length-Suc-conv*)

lemma *replicate-pmf-distrib*:
replicate-pmf (*m* + *n*) *p* =
do {*xs* ← *replicate-pmf m p*; *ys* ← *replicate-pmf n p*; *return-pmf* (*xs* @ *ys*)}
by (*induction m*) (*simp-all add*: *bind-return-pmf bind-return-pmf' bind-assoc-pmf*)

lemma *power-diff'*:
assumes *b* ≤ *a*
shows *x* ^ (*a* − *b*) = (*if x* = *0* ∧ *a* = *b then 1 else x* ^ *a* / (*x*::*'a*::*field*) ^ *b*)
proof (*cases x* = *0*)
case *True*
with *assms* **show** *?thesis* **by** (*cases a* − *b*) *simp-all*
qed (*insert assms*, *simp-all add*: *power-diff*)

lemma *binomial-pmf-Suc*:
assumes *p* ∈ {*0..1*}
shows *binomial-pmf* (*Suc n*) *p* =
do {*b* ← *bernoulli-pmf p*;
k ← *binomial-pmf n p*;
return-pmf ((*if b then 1 else 0*) + *k*)} (**is** *- = ?rhs*)
proof (*intro pmf-eqI*)
fix *k*
have *A*: *indicator* {*Suc a*} (*Suc b*) = *indicator* {*a*} *b* **for** *a b*
by (*simp add*: *indicator-def*)

show *pmf* (*binomial-pmf* (*Suc n*) *p*) *k* = *pmf ?rhs k*
by (*cases k*; *cases k* > *n*)
(*insert assms*, *auto simp*: *pmf-bind measure-pmf-single A field-split-simps algebra-simps*
not-less less-eq-Suc-le [*symmetric*] *power-diff'*)
qed

lemma *binomial-pmf-0*: *p* ∈ {*0..1*} ⟹ *binomial-pmf 0 p* = *return-pmf 0*
by (*rule pmf-eqI*) (*simp-all add*: *indicator-def*)

lemma *binomial-pmf-altdef*:
assumes *p* ∈ {*0..1*}
shows *binomial-pmf n p* = *map-pmf* (*length* ∘ *filter id*) (*replicate-pmf n* (*bernoulli-pmf p*))
by (*induction n*)
(*insert assms*, *auto simp*: *binomial-pmf-Suc map-pmf-def bind-return-pmf bind-assoc-pmf*
bind-return-pmf' binomial-pmf-0 intro!: *bind-pmf-cong*)

20.7 Negative Binomial distribution

The negative binomial distribution counts the number of times a weighted coin comes up tails before having come up heads n times. In other words: how many failures do we see before seeing the n-th success?

An alternative view is that the negative binomial distribution is the sum of n i.i.d. geometric variables (this is the definition that we use).

Note that there are sometimes different conventions for this distributions in the literature; for instance, sometimes the number of *attempts* is counted instead of the number of failures. This only shifts the entire distribution by a constant number and is thus not a big difference. I think that the convention we use is the most natural one since the support of the distribution starts at 0, whereas for the other convention it starts at n.

primrec *neg-binomial-pmf* :: *nat* ⇒ *real* ⇒ *nat pmf* **where**
neg-binomial-pmf 0 p = *return-pmf 0*
| *neg-binomial-pmf* (*Suc n*) *p* =
map-pmf (λ(*x*,*y*). (*x* + *y*)) (*pair-pmf* (*geometric-pmf p*) (*neg-binomial-pmf n p*))

lemma *neg-binomial-pmf-Suc-0* [*simp*]: *neg-binomial-pmf* (*Suc 0*) *p* = *geometric-pmf p*
by (*auto simp*: *pair-pmf-def bind-return-pmf map-pmf-def bind-assoc-pmf bind-return-pmf'*)

lemmas *neg-binomial-pmf-Suc* [*simp del*] = *neg-binomial-pmf.simps*(*2*)

lemma *neg-binomial-prob-1* [*simp*]: *neg-binomial-pmf n 1* = *return-pmf 0*
by (*induction n*) (*simp-all add*: *neg-binomial-pmf-Suc*)

We can now show the aforementioned intuition about counting the failures before the n-th success with the following recurrence:

lemma *neg-binomial-pmf-unfold*:
assumes *p*: $p \in \{0<..1\}$
shows *neg-binomial-pmf* (*Suc n*) *p* =
do {*b* ← *bernoulli-pmf p*;
if b then neg-binomial-pmf n p else map-pmf Suc (*neg-binomial-pmf* (*Suc n*) *p*)}
(**is** - = *?rhs*)
unfolding *neg-binomial-pmf-Suc*
by (*subst geometric-bind-pmf-unfold*[*OF p*])
(*auto simp*: *map-pmf-def pair-pmf-def bind-assoc-pmf bind-return-pmf bind-return-pmf'*
intro!: *bind-pmf-cong*)

Next, we show an explicit formula for the probability mass function of the negative binomial distribution:

lemma *pmf-neg-binomial*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf* (*neg-binomial-pmf n p*) *k* = *real* ((*k* + *n* − *1*) *choose k*) ∗ *p* ^ *n* ∗ (*1* − *p*) ^ *k*
proof (*induction n arbitrary*: *k*)
case *0*
thus *?case* **using** *assms* **by** (*auto simp*: *indicator-def*)
next
case (*Suc n*)
show *?case*
proof (*cases n* = *0*)
case *True*
thus *?thesis* **using** *assms* **by** *auto*
next
case *False*
let *?f* = *pmf* (*neg-binomial-pmf n p*)
have *pmf* (*neg-binomial-pmf* (*Suc n*) *p*) *k* =
pmf (*geometric-pmf p* ⤜ (λ*x*. *map-pmf* ((+) *x*) (*neg-binomial-pmf n p*))) *k*
by (*auto simp*: *pair-pmf-def bind-return-pmf map-pmf-def bind-assoc-pmf neg-binomial-pmf-Suc*)
also have ... = *measure-pmf.expectation* (*geometric-pmf p*)
(λ*x*. *measure-pmf.prob* (*neg-binomial-pmf n p*) ((+) *x* −‘ {*k*}))
by (*simp add*: *pmf-bind pmf-map*)
also have (λ*x*. (+) *x* −‘ {*k*}) = (λ*x*. *if x* ≤ *k then* {*k* − *x*} *else* {})
by (*auto simp*: *fun-eq-iff*)
also have (λ*x*. *measure-pmf.prob* (*neg-binomial-pmf n p*) (... *x*)) =
(λ*x*. *if x* ≤ *k then ?f*(*k* − *x*) *else 0*)
by (*auto simp*: *fun-eq-iff measure-pmf-single*)
also have *measure-pmf.expectation* (*geometric-pmf p*) ... =
(∑ *i*≤*k*. *pmf* (*neg-binomial-pmf n p*) (*k* − *i*) ∗ *pmf* (*geometric-pmf p*) *i*)
by (*subst integral-measure-pmf-real*[**where** *A* = {..*k*}]) (*auto split*: *if-splits*)

```
    also have ... = p^(n+1) * (1-p)^k * real (∑ i≤k. (k - i + n - 1) choose
(k - i))
      unfolding sum-distrib-left of-nat-sum
    proof (intro sum.cong refl, goal-cases)
      case (1 i)
      have pmf (neg-binomial-pmf n p) (k - i) * pmf (geometric-pmf p) i =
            real ((k - i + n - 1) choose (k - i)) * p^(n+1) * ((1-p)^(k-i) *
(1-p)^i)
        using assms Suc.IH by (simp add: mult-ac)
      also have (1-p)^(k-i) * (1-p)^i = (1-p)^k
        using 1 by (subst power-add [symmetric]) auto
      finally show ?case by simp
    qed
    also have (∑ i≤k. (k - i + n - 1) choose (k - i)) = (∑ i≤k. (n - 1 + i)
choose i)
      by (intro sum.reindex-bij-witness[of - λi. k - i λi. k - i])
        (use ‹n ≠ 0› in ‹auto simp: algebra-simps›)
    also have ... = (n + k) choose k
      by (subst sum-choose-lower) (use ‹n ≠ 0› in auto)
    finally show ?thesis
      by (simp add: add-ac)
  qed
qed


lemma gbinomial-0-left: 0 gchoose k = (if k = 0 then 1 else 0)
  by (cases k) auto
```

The following alternative formula highlights why it is called 'negative binomial distribution':

```
lemma pmf-neg-binomial':
  assumes p: p ∈ {0<..1}
  shows   pmf (neg-binomial-pmf n p) k = (-1) ^ k * ((-real n) gchoose k) * p
^ n * (1 - p) ^ k
proof (cases n > 0)
  case n: True
  have pmf (neg-binomial-pmf n p) k = real ((k + n - 1) choose k) * p ^ n * (1
- p) ^ k
    by (rule pmf-neg-binomial) fact+
  also have real ((k + n - 1) choose k) = ((real k + real n - 1) gchoose k)
    using n by (subst binomial-gbinomial) (auto simp: of-nat-diff)
  also have ... = (-1) ^ k * ((-real n) gchoose k)
    by (subst gbinomial-negated-upper) auto
  finally show ?thesis by simp
qed (auto simp: indicator-def gbinomial-0-left)
```

The cumulative distribution function of the negative binomial distribution can be expressed in terms of that of the 'normal' binomial distribution.

```
lemma prob-neg-binomial-pmf-atMost:
```

```
  assumes p: p ∈ {0<..1}
  shows measure-pmf.prob (neg-binomial-pmf n p) {..k} =
        measure-pmf.prob (binomial-pmf (n + k) (1 − p)) {..k}
proof (cases n = 0)
  case [simp]: True
  have set-pmf (binomial-pmf (n + k) (1 − p)) ⊆ {..n+k}
    using p by (subst set-pmf-binomial-eq) auto
  hence measure-pmf.prob (binomial-pmf (n + k) (1 − p)) {..k} = 1
    by (subst measure-pmf.prob-eq-1) (auto intro!: AE-pmfI)
  thus ?thesis by simp
next
  case False
  hence n: n > 0 by auto
  have measure-pmf.prob (binomial-pmf (n + k) (1 − p)) {..k} = (∑ i≤k. pmf
(binomial-pmf (n + k) (1 − p)) i)
    by (intro measure-measure-pmf-finite) auto
  also have ... = (∑ i≤k. real ((n + k) choose i) * p ^(n + k − i) * (1 − p) ^
i)
    using p by (simp add: mult-ac)
  also have ... = p ^n * (∑ i≤k. real ((n + k) choose i) * (1 − p) ^i * p ^(k
− i))
    unfolding sum-distrib-left by (intro sum.cong) (auto simp: algebra-simps simp
flip: power-add)
  also have (∑ i≤k. real ((n + k) choose i) * (1 − p) ^i * p ^(k − i)) =
            (∑ i≤k. ((n + i − 1) choose i) * (1 − p) ^i)
    using gbinomial-partial-sum-poly-xpos[of k real n 1 − p p] n
    by (simp add: binomial-gbinomial add-ac of-nat-diff)
  also have p ^n * ... = (∑ i≤k. pmf (neg-binomial-pmf n p) i)
     using p unfolding sum-distrib-left by (simp add: pmf-neg-binomial alge-
bra-simps)
  also have ... = measure-pmf.prob (neg-binomial-pmf n p) {..k}
    by (intro measure-measure-pmf-finite [symmetric]) auto
  finally show ?thesis ..
qed

lemma prob-neg-binomial-pmf-lessThan:
  assumes p: p ∈ {0<..1}
  shows measure-pmf.prob (neg-binomial-pmf n p) {..<k} =
        measure-pmf.prob (binomial-pmf (n + k − 1) (1 − p)) {..<k}
proof (cases k = 0)
  case False
  hence {..<k} = {..k−1}
    by auto
  thus ?thesis
    using prob-neg-binomial-pmf-atMost[OF p, of n k − 1] False by simp
qed auto
```

The expected value of the negative binomial distribution is $n(1-p)/p$:

```
lemma nn-integral-neg-binomial-pmf-real:
```

assumes *p*: $p \in \{0<..1\}$
shows *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *ennreal* (*n* * (*1* − *p*) / *p*)
proof (*induction n*)
case *0*
thus *?case* **by** *auto*
next
case (*Suc n*)
have *nn-integral* (*measure-pmf* (*neg-binomial-pmf* (*Suc n*) *p*)) *of-nat* =
nn-integral (*measure-pmf* (*geometric-pmf p*)) *of-nat* +
nn-integral (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat*
by (*simp add*: *neg-binomial-pmf-Suc case-prod-unfold nn-integral-add nn-integral-pair-pmf'*)
also have *nn-integral* (*measure-pmf* (*geometric-pmf p*)) *of-nat* = *ennreal* ((*1*−*p*) / *p*)
unfolding *ennreal-of-nat-eq-real-of-nat*
using *expectation-geometric-pmf*[*OF p*] *integrable-real-geometric-pmf*[*OF p*]
by (*subst nn-integral-eq-integral*) *auto*
also have *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *n* * (*1* − *p*) / *p* **using** *p*
by (*subst Suc.IH*)
(*auto simp*: *ennreal-of-nat-eq-real-of-nat ennreal-mult simp flip*: *divide-ennreal ennreal-minus*)
also have *ennreal* ((*1* − *p*) / *p*) + *ennreal* (*real n* * (*1* − *p*) / *p*) =
ennreal ((*1*−*p*) / *p* + *real n* * (*1* − *p*) / *p*)
by (*intro ennreal-plus* [*symmetric*] *divide-nonneg-pos mult-nonneg-nonneg*) (*use p* **in** *auto*)
also have (*1*−*p*) / *p* + *real n* * (*1* − *p*) / *p* = *real* (*Suc n*) * (*1* − *p*) / *p*
using *p* **by** (*auto simp*: *field-simps*)
finally show *?case*
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)
qed

lemma *integrable-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *integrable* (*measure-pmf* (*neg-binomial-pmf n p*)) *real*
using *nn-integral-neg-binomial-pmf-real*[*OF p*, *of n*]
by (*subst integrable-iff-bounded*) (*auto simp flip*: *ennreal-of-nat-eq-real-of-nat*)

lemma *expectation-neg-binomial-pmf*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*neg-binomial-pmf n p*) *real* = *n* * (*1* − *p*) / *p*
proof −
have *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *ennreal* (*n* * (*1* − *p*) / *p*)
by (*intro nn-integral-neg-binomial-pmf-real p*)
also have *of-nat* = (λ*x*. *ennreal* (*real x*))
by (*simp add*: *ennreal-of-nat-eq-real-of-nat fun-eq-iff*)
finally show *?thesis*
using *p* **by** (*subst* (*asm*) *nn-integral-eq-integrable*) *auto*

qed

20.8 PMFs from association lists

definition *pmf-of-list* :: ('*a* × *real*) *list* ⇒ '*a pmf* **where**
pmf-of-list xs = *embed-pmf* (λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))

definition *pmf-of-list-wf* **where**
pmf-of-list-wf xs ⟷ (∀ *x*∈*set* (*map snd xs*) . *x* ≥ *0*) ∧ *sum-list* (*map snd xs*) = *1*

lemma *pmf-of-list-wfI*:
(⋀*x*. *x* ∈ *set* (*map snd xs*) ⟹ *x* ≥ *0*) ⟹ *sum-list* (*map snd xs*) = *1* ⟹ *pmf-of-list-wf xs*
unfolding *pmf-of-list-wf-def* **by** *simp*

context
begin

private lemma *pmf-of-list-aux*:
assumes ⋀*x*. *x* ∈ *set* (*map snd xs*) ⟹ *x* ≥ *0*
assumes *sum-list* (*map snd xs*) = *1*
shows (∫⁺ *x*. *ennreal* (*sum-list* (*map snd* [*z*←*xs* . *fst z* = *x*])) ∂*count-space UNIV*) = *1*
proof −
have (∫⁺ *x*. *ennreal* (*sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*))) ∂*count-space UNIV*) =
(∫⁺ *x*. *ennreal* (*sum-list* (*map* (λ(*x'*,*p*). *indicator* {*x'*} *x* ∗ *p*) *xs*)) ∂*count-space UNIV*)
apply (*intro nn-integral-cong ennreal-cong*, *subst sum-list-map-filter'*)
apply (*rule arg-cong*[**where** *f* = *sum-list*])
apply (*auto cong*: *map-cong*)
done
also have ... = (∑(*x'*,*p*)←*xs*. (∫⁺ *x*. *ennreal* (*indicator* {*x'*} *x* ∗ *p*) ∂*count-space UNIV*))
using *assms*(*1*)
proof (*induction xs*)
case (*Cons x xs*)
from *Cons.prems* **have** *snd x* ≥ *0* **by** *simp*
moreover have *b* ≥ *0* **if** (*a*,*b*) ∈ *set xs* **for** *a b*
using *Cons.prems*[*of b*] *that* **by** *force*
ultimately have (∫⁺ *y*. *ennreal* (∑(*x'*, *p*)←*x* # *xs*. *indicator* {*x'*} *y* ∗ *p*) ∂*count-space UNIV*) =
(∫⁺ *y*. *ennreal* (*indicator* {*fst x*} *y* ∗ *snd x*) +
ennreal (∑(*x'*, *p*)←*xs*. *indicator* {*x'*} *y* ∗ *p*) ∂*count-space UNIV*)
by (*intro nn-integral-cong*, *subst ennreal-plus* [*symmetric*])
(*auto simp*: *case-prod-unfold indicator-def intro*!: *sum-list-nonneg*)
also have ... = (∫⁺ *y*. *ennreal* (*indicator* {*fst x*} *y* ∗ *snd x*) ∂*count-space UNIV*) +

```
                (∫⁺ y. ennreal (∑ (x', p)←xs. indicator {x'} y * p) ∂count-space
UNIV)
     by (intro nn-integral-add)
       (force intro!: sum-list-nonneg AE-I2 intro: Cons simp: indicator-def)+
   also have (∫⁺ y. ennreal (∑ (x', p)←xs. indicator {x'} y * p) ∂count-space
UNIV) =
                (∑ (x', p)←xs. (∫⁺ y. ennreal (indicator {x'} y * p) ∂count-space
UNIV))
     using Cons(1) by (intro Cons) simp-all
   finally show ?case by (simp add: case-prod-unfold)
 qed simp
 also have ... = (∑ (x',p)←xs. ennreal p * (∫⁺ x. indicator {x'} x ∂count-space
UNIV))
   using assms(1)
  by (simp cong: map-cong only: case-prod-unfold, subst nn-integral-cmult [symmetric])
      (auto intro!: assms(1) simp: max-def times-ereal.simps [symmetric] mult-ac
ereal-indicator
          simp del: times-ereal.simps)+
 also from assms have ... = sum-list (map snd xs) by (simp add: case-prod-unfold
sum-list-ennreal)
 also have ... = 1 using assms(2) by simp
 finally show ?thesis .
qed

lemma pmf-pmf-of-list:
 assumes pmf-of-list-wf xs
 shows   pmf (pmf-of-list xs) x = sum-list (map snd (filter (λz. fst z = x) xs))
 using assms pmf-of-list-aux[of xs] unfolding pmf-of-list-def pmf-of-list-wf-def
 by (subst pmf-embed-pmf) (auto intro!: sum-list-nonneg)

end

lemma set-pmf-of-list:
 assumes pmf-of-list-wf xs
 shows   set-pmf (pmf-of-list xs) ⊆ set (map fst xs)
proof clarify
 fix x assume A: x ∈ set-pmf (pmf-of-list xs)
 show x ∈ set (map fst xs)
 proof (rule ccontr)
   assume x ∉ set (map fst xs)
   hence [z←xs . fst z = x] = [] by (auto simp: filter-empty-conv)
   with A assms show False by (simp add: pmf-pmf-of-list set-pmf-eq)
 qed
qed

lemma finite-set-pmf-of-list:
 assumes pmf-of-list-wf xs
 shows   finite (set-pmf (pmf-of-list xs))
 using assms by (rule finite-subset[OF set-pmf-of-list]) simp-all
```

lemma *emeasure-Int-set-pmf*:
emeasure (*measure-pmf p*) (*A* ∩ *set-pmf p*) = *emeasure* (*measure-pmf p*) *A*
by (*rule emeasure-eq-AE*) (*auto simp*: *AE-measure-pmf-iff*)

lemma *measure-Int-set-pmf*:
measure (*measure-pmf p*) (*A* ∩ *set-pmf p*) = *measure* (*measure-pmf p*) *A*
using *emeasure-Int-set-pmf*[*of p A*] **by** (*simp add*: *Sigma-Algebra.measure-def*)

lemma *measure-prob-cong-0*:
assumes ⋀*x*. *x* ∈ *A* − *B* ⟹ *pmf p x* = *0*
assumes ⋀*x*. *x* ∈ *B* − *A* ⟹ *pmf p x* = *0*
shows *measure* (*measure-pmf p*) *A* = *measure* (*measure-pmf p*) *B*
proof −
have *measure-pmf.prob p A* = *measure-pmf.prob p* (*A* ∩ *set-pmf p*)
by (*simp add*: *measure-Int-set-pmf*)
also have *A* ∩ *set-pmf p* = *B* ∩ *set-pmf p*
using *assms* **by** (*auto simp*: *set-pmf-eq*)
also have *measure-pmf.prob p* ... = *measure-pmf.prob p B*
by (*simp add*: *measure-Int-set-pmf*)
finally show *?thesis* **.**
qed

lemma *emeasure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *emeasure* (*pmf-of-list xs*) *A* = *ennreal* (*sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*)))
proof −
have *emeasure* (*pmf-of-list xs*) *A* = *nn-integral* (*measure-pmf* (*pmf-of-list xs*)) (*indicator A*)
by *simp*
also from *assms*
have ... = (∑ *x*∈*set-pmf* (*pmf-of-list xs*) ∩ *A*. *ennreal* (*sum-list* (*map snd* [*z*←*xs* . *fst z* = *x*])))
by (*subst nn-integral-measure-pmf-finite*) (*simp-all add*: *finite-set-pmf-of-list pmf-pmf-of-list Int-def*)
also from *assms*
have ... = *ennreal* (∑ *x*∈*set-pmf* (*pmf-of-list xs*) ∩ *A*. *sum-list* (*map snd* [*z*←*xs* . *fst z* = *x*]))
by (*subst sum-ennreal*) (*auto simp*: *pmf-of-list-wf-def intro*!: *sum-list-nonneg*)
also have ... = *ennreal* (∑ *x*∈*set-pmf* (*pmf-of-list xs*) ∩ *A*.
indicator A x ∗ *pmf* (*pmf-of-list xs*) *x*) (**is** - = *ennreal ?S*)
using *assms* **by** (*intro ennreal-cong sum.cong*) (*auto simp*: *pmf-pmf-of-list*)
also have *?S* = (∑ *x*∈*set-pmf* (*pmf-of-list xs*). *indicator A x* ∗ *pmf* (*pmf-of-list xs*) *x*)
using *assms* **by** (*intro sum.mono-neutral-left set-pmf-of-list finite-set-pmf-of-list*) *auto*
also have ... = (∑ *x*∈*set* (*map fst xs*). *indicator A x* ∗ *pmf* (*pmf-of-list xs*) *x*)
using *assms* **by** (*intro sum.mono-neutral-left set-pmf-of-list*) (*auto simp*: *set-pmf-eq*)

also have ... = ($\sum x{\in}set$ (*map fst xs*). *indicator A x* ∗
sum-list (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))
using *assms* **by** (*simp add*: *pmf-pmf-of-list*)
also have ... = ($\sum x{\in}set$ (*map fst xs*). *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x* ∧ *x* ∈ *A*) *xs*)))
by (*intro sum.cong*) (*auto simp*: *indicator-def*)
also have ... = ($\sum x{\in}set$ (*map fst xs*). ($\sum xa$ = *0*..<*length xs*.
if fst (*xs* ! *xa*) = *x* ∧ *x* ∈ *A then snd* (*xs* ! *xa*) *else 0*))
by (*intro sum.cong refl*, *subst sum-list-map-filter′*, *subst sum-list-sum-nth*) *simp*
also have ... = ($\sum xa$ = *0*..<*length xs*. ($\sum x{\in}set$ (*map fst xs*).
if fst (*xs* ! *xa*) = *x* ∧ *x* ∈ *A then snd* (*xs* ! *xa*) *else 0*))
by (*rule sum.swap*)
also have ... = ($\sum xa$ = *0*..<*length xs*. *if fst* (*xs* ! *xa*) ∈ *A then*
($\sum x{\in}set$ (*map fst xs*). *if x* = *fst* (*xs* ! *xa*) *then snd* (*xs* ! *xa*) *else 0*) *else 0*)
by (*auto intro*!: *sum.cong sum.neutral simp del*: *sum.delta*)
also have ... = ($\sum xa$ = *0*..<*length xs*. *if fst* (*xs* ! *xa*) ∈ *A then snd* (*xs* ! *xa*) *else 0*)
by (*intro sum.cong refl*) (*simp-all add*: *sum.delta*)
also have ... = *sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*))
by (*subst sum-list-map-filter′*, *subst sum-list-sum-nth*) *simp-all*
finally show *?thesis* **.**
qed

lemma *measure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *measure* (*pmf-of-list xs*) *A* = *sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*))
using *assms* **unfolding** *pmf-of-list-wf-def Sigma-Algebra.measure-def*
by (*subst emeasure-pmf-of-list* [*OF assms*], *subst enn2real-ennreal*) (*auto intro*!: *sum-list-nonneg*)

lemma *sum-list-nonneg-eq-zero-iff*:
fixes *xs* :: *'a* :: *linordered-ab-group-add list*
shows (⋀*x*. *x* ∈ *set xs* ⟹ *x* ≥ *0*) ⟹ *sum-list xs* = *0* ⟷ *set xs* ⊆ {*0*}
proof (*induction xs*)
case (*Cons x xs*)
from *Cons.prems* **have** *sum-list* (*x*#*xs*) = *0* ⟷ *x* = *0* ∧ *sum-list xs* = *0*
unfolding *sum-list-simps* **by** (*subst add-nonneg-eq-0-iff*) (*auto intro*: *sum-list-nonneg*)
with *Cons.IH Cons.prems* **show** *?case* **by** *simp*
qed *simp-all*

lemma *sum-list-filter-nonzero*:
sum-list (*filter* (λ*x*. *x* ≠ *0*) *xs*) = *sum-list xs*
by (*induction xs*) *simp-all*

lemma *set-pmf-of-list-eq*:

assumes *pmf-of-list-wf xs* $\bigwedge x.\ x \in$ *snd* ' *set xs* $\Longrightarrow x > 0$
shows *set-pmf* (*pmf-of-list xs*) = *fst* ' *set xs*
proof
{
fix *x* **assume** *A*: $x \in$ *fst* ' *set xs* **and** *B*: $x \notin$ *set-pmf* (*pmf-of-list xs*)
then obtain *y* **where** *y*: (*x*, *y*) $\in$ *set xs* **by** *auto*
from *B* **have** *sum-list* (*map snd* [*z*←*xs*. *fst z* = *x*]) = 0
by (*simp add*: *pmf-pmf-of-list*[*OF assms*(1)] *set-pmf-eq*)
moreover from *y* **have** *y* $\in$ *snd* ' {*xa* $\in$ *set xs*. *fst xa* = *x*} **by** *force*
ultimately have *y* = 0 **using** *assms*(1)
by (*subst* (*asm*) *sum-list-nonneg-eq-zero-iff*) (*auto simp*: *pmf-of-list-wf-def*)
with *assms*(2) *y* **have** *False* **by** *force*
}
thus *fst* ' *set xs* $\subseteq$ *set-pmf* (*pmf-of-list xs*) **by** *blast*
qed (*insert set-pmf-of-list*[*OF assms*(1)], *simp-all*)

lemma *pmf-of-list-remove-zeros*:
assumes *pmf-of-list-wf xs*
defines *xs'* ≡ *filter* (λ*z*. *snd z* ≠ 0) *xs*
shows *pmf-of-list-wf xs' pmf-of-list xs'* = *pmf-of-list xs*
proof −
have *map snd* [*z*←*xs* . *snd z* ≠ 0] = *filter* (λ*x*. *x* ≠ 0) (*map snd xs*)
by (*induction xs*) *simp-all*
with *assms*(1) **show** *wf*: *pmf-of-list-wf xs'*
by (*auto simp*: *pmf-of-list-wf-def xs'-def sum-list-filter-nonzero*)
have *sum-list* (*map snd* [*z*←*xs'* . *fst z* = *i*]) = *sum-list* (*map snd* [*z*←*xs* . *fst z* = *i*]) **for** *i*
unfolding *xs'-def* **by** (*induction xs*) *simp-all*
with *assms*(1) *wf* **show** *pmf-of-list xs'* = *pmf-of-list xs*
by (*intro pmf-eqI*) (*simp-all add*: *pmf-pmf-of-list*)
qed

end

21 Code generation for PMFs

theory *PMF-Impl*
imports *Probability-Mass-Function HOL−Library.AList-Mapping*
begin

21.1 General code generation setup

definition *pmf-of-mapping* :: ('*a*, *real*) *mapping* ⇒ '*a pmf* **where**
pmf-of-mapping m = *embed-pmf* (*Mapping.lookup-default* 0 *m*)

lemma *nn-integral-lookup-default*:
fixes *m* :: ('*a*, *real*) *mapping*
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *x*. *x* ≥ 0)
shows *nn-integral* (*count-space UNIV*) (λ*k*. *ennreal* (*Mapping.lookup-default* 0

m k)) =
ennreal (∑ *k*∈*Mapping.keys m*. *Mapping.lookup-default 0 m k*)
proof −
have *nn-integral* (*count-space UNIV*) (λ*k*. *ennreal* (*Mapping.lookup-default 0 m k*)) =
(∑ *x*∈*Mapping.keys m*. *ennreal* (*Mapping.lookup-default 0 m x*)) **using** *assms*
by (*subst nn-integral-count-space*′[*of Mapping.keys m*])
(*auto simp*: *Mapping.lookup-default-def keys-is-none-rep Option.is-none-def*)
also from *assms* **have** ... = *ennreal* (∑ *k*∈*Mapping.keys m*. *Mapping.lookup-default 0 m k*)
by (*intro sum-ennreal*)
(*auto simp*: *Mapping.lookup-default-def All-mapping-def split*: *option.splits*)
finally show *?thesis* **.**
qed

lemma *pmf-of-mapping*:
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *p*. *p* ≥ *0*)
assumes (∑ *x*∈*Mapping.keys m*. *Mapping.lookup-default 0 m x*) = *1*
shows *pmf* (*pmf-of-mapping m*) *x* = *Mapping.lookup-default 0 m x*
unfolding *pmf-of-mapping-def*
proof (*intro pmf-embed-pmf*)
from *assms* **show** (∫ +*x*. *ennreal* (*Mapping.lookup-default 0 m x*) ∂*count-space UNIV*) = *1*
by (*subst nn-integral-lookup-default*) (*simp-all*)
qed (*insert assms*, *simp add*: *All-mapping-def Mapping.lookup-default-def split*: *option.splits*)

lemma *pmf-of-set-pmf-of-mapping*:
assumes *A* ≠ {} *set xs* = *A distinct xs*
shows *pmf-of-set A* = *pmf-of-mapping* (*Mapping.tabulate xs* (λ-. *1* / *real* (*length xs*)))
(**is** *?lhs* = *?rhs*)
by (*rule pmf-eqI*, *subst pmf-of-mapping*)
(*insert assms*, *auto intro*!: *All-mapping-tabulate*
simp: *Mapping.lookup-default-def lookup-tabulate distinct-card*)

lift-definition *mapping-of-pmf* :: ′*a pmf* ⇒ (′*a*, *real*) *mapping* **is**
λ*p x*. *if pmf p x* = *0 then None else Some* (*pmf p x*) **.**

lemma *lookup-default-mapping-of-pmf*:
Mapping.lookup-default 0 (*mapping-of-pmf p*) *x* = *pmf p x*
by (*simp add*: *mapping-of-pmf.abs-eq lookup-default-def Mapping.lookup.abs-eq*)

context
begin

interpretation *pmf-as-function* **.**

lemma *nn-integral-pmf-eq-1*: $(\int^+ x.\ ennreal\ (pmf\ p\ x)\ \partial count\text{-}space\ UNIV) = 1$
by *transfer simp-all*
end

lemma *pmf-of-mapping-mapping-of-pmf* [*code abstype*]:
$pmf\text{-}of\text{-}mapping\ (mapping\text{-}of\text{-}pmf\ p) = p$
unfolding *pmf-of-mapping-def*
by (*rule pmf-eqI*, *subst pmf-embed-pmf*)
(*insert nn-integral-pmf-eq-1*[*of p*],
auto simp: *lookup-default-mapping-of-pmf split*: *option.splits*)

lemma *mapping-of-pmfI*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\ m\ x = Some\ (pmf\ p\ x)$
assumes $Mapping.keys\ m = set\text{-}pmf\ p$
shows $mapping\text{-}of\text{-}pmf\ p = m$
using *assms* **by** *transfer* (*rule ext*, *auto simp*: *set-pmf-eq*)

lemma *mapping-of-pmfI'*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\text{-}default\ 0\ m\ x = pmf\ p\ x$

assumes $Mapping.keys\ m = set\text{-}pmf\ p$
shows $mapping\text{-}of\text{-}pmf\ p = m$
using *assms* **unfolding** *Mapping.lookup-default-def*
by *transfer* (*rule ext*, *force simp*: *set-pmf-eq*)

lemma *return-pmf-code* [*code abstract*]:
$mapping\text{-}of\text{-}pmf\ (return\text{-}pmf\ x) = Mapping.update\ x\ 1\ Mapping.empty$
by (*intro mapping-of-pmfI*) (*auto simp*: *lookup-update'*)

lemma *pmf-of-set-code-aux*:
assumes $A \neq \{\}$ *set xs* $= A$ *distinct xs*
shows $mapping\text{-}of\text{-}pmf\ (pmf\text{-}of\text{-}set\ A) = Mapping.tabulate\ xs\ (\lambda\text{-}.\ 1\ /\ real\ (length\ xs))$
using *assms*
by (*intro mapping-of-pmfI*, *subst pmf-of-set*)
(*auto simp*: *lookup-tabulate distinct-card*)

definition *pmf-of-set-impl* **where**
$pmf\text{-}of\text{-}set\text{-}impl\ A = mapping\text{-}of\text{-}pmf\ (pmf\text{-}of\text{-}set\ A)$

lemma *pmf-of-set-impl-code-alt*:
assumes $A \neq \{\}$ *finite A*
shows $pmf\text{-}of\text{-}set\text{-}impl\ A =$
$(let\ p = 1\ /\ real\ (card\ A)$
$in\ \ Finite\text{-}Set.fold\ (\lambda x.\ Mapping.update\ x\ p)\ Mapping.empty\ A)$
proof −
define *p* **where** $p = 1\ /\ real\ (card\ A)$
let $?m = Finite\text{-}Set.fold\ (\lambda x.\ Mapping.update\ x\ p)\ Mapping.empty\ A$

interpret *comp-fun-idem* $\lambda x.$ *Mapping.update x p*
by *standard* (*transfer*, *force simp*: *fun-eq-iff*)+
have *keys*: *Mapping.keys ?m* = *A*
using *assms*(*2*) **by** (*induction A rule*: *finite-induct*) *simp-all*
have *lookup*: *Mapping.lookup ?m x* = *Some p* **if** $x \in A$ **for** *x*
using *assms*(*2*) *that* **by** (*induction A rule*: *finite-induct*) (*auto simp*: *lookup-update'*)
from *keys lookup assms* **show** *?thesis* **unfolding** *pmf-of-set-impl-def*
by (*intro mapping-of-pmfI*) (*simp-all add*: *Let-def p-def*)
qed

lemma *pmf-of-set-impl-code* [*code*]:
pmf-of-set-impl (*set xs*) =
(*if xs* = [] *then*
Code.abort (*STR ''pmf-of-set of empty set''*) (λ-. *mapping-of-pmf* (*pmf-of-set* (*set xs*)))
else let xs' = *remdups xs*; *p* = *1* / *real* (*length xs'*) *in*
Mapping.tabulate xs' (λ-. *p*))
unfolding *pmf-of-set-impl-def*
using *pmf-of-set-code-aux*[*of set xs remdups xs*] **by** (*simp add*: *Let-def*)

lemma *pmf-of-set-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-set A*) = *pmf-of-set-impl A*
by (*simp add*: *pmf-of-set-impl-def*)

lemma *pmf-of-multiset-pmf-of-mapping*:
assumes $A \neq \{\#\}$ *set xs* = *set-mset A distinct xs*
shows *mapping-of-pmf* (*pmf-of-multiset A*) = *Mapping.tabulate xs* (λ*x*. *count A x* / *real* (*size A*))
using *assms* **by** (*intro mapping-of-pmfI*) (*auto simp*: *lookup-tabulate*)

definition *pmf-of-multiset-impl* **where**
pmf-of-multiset-impl A = *mapping-of-pmf* (*pmf-of-multiset A*)

lemma *pmf-of-multiset-impl-code-alt*:
assumes $A \neq \{\#\}$
shows *pmf-of-multiset-impl A* =
(*let p* = *1* / *real* (*size A*)
in fold-mset (λ*x*. *Mapping.map-default x 0* ((+) *p*)) *Mapping.empty A*)
proof −
define *p* **where** *p* = *1* / *real* (*size A*)
interpret *comp-fun-commute* λ*x*. *Mapping.map-default x 0* ((+) *p*)
unfolding *Mapping.map-default-def* [*abs-def*]
by (*standard*, *intro mapping-eqI ext*)
(*simp-all add*: *o-def lookup-map-entry' lookup-default' lookup-default-def*)
let *?m* = *fold-mset* (λ*x*. *Mapping.map-default x 0* ((+) *p*)) *Mapping.empty A*
have *keys*: *Mapping.keys ?m* = *set-mset A* **by** (*induction A*) *simp-all*
have *lookup*: *Mapping.lookup-default 0 ?m x* = *real* (*count A x*) ∗ *p* **for** *x*
by (*induction A*)

(*simp-all add*: *lookup-map-default'* *lookup-default-def* *lookup-empty* *ring-distribs*)
from *keys* *lookup* *assms* **show** *?thesis* **unfolding** *pmf-of-multiset-impl-def*
by (*intro* *mapping-of-pmfI'*) (*simp-all add*: *Let-def* *p-def*)
qed

lemma *pmf-of-multiset-impl-code* [*code*]:
pmf-of-multiset-impl (*mset xs*) =
(*if xs* = [] *then*
Code.abort (*STR ''pmf-of-multiset of empty multiset''*)
(λ-. *mapping-of-pmf* (*pmf-of-multiset* (*mset xs*)))
else let xs' = *remdups xs*; *p* = *1* / *real* (*length xs*) *in*
Mapping.tabulate xs' (λ*x*. *real* (*count* (*mset xs*) *x*) ∗ *p*))
using *pmf-of-multiset-pmf-of-mapping*[*of mset xs remdups xs*]
by (*simp add*: *pmf-of-multiset-impl-def*)

lemma *pmf-of-multiset-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-multiset A*) = *pmf-of-multiset-impl A*
by (*simp add*: *pmf-of-multiset-impl-def*)

lemma *bernoulli-pmf-code* [*code abstract*]:
mapping-of-pmf (*bernoulli-pmf p*) =
(*if p* ≤ *0 then Mapping.update False 1 Mapping.empty*
else if p ≥ *1 then Mapping.update True 1 Mapping.empty*
else Mapping.update False (*1* − *p*) (*Mapping.update True p Mapping.empty*))
by (*intro mapping-of-pmfI*) (*auto simp*: *bernoulli-pmf.rep-eq* *lookup-update'* *set-pmf-eq*)

lemma *pmf-code* [*code*]: *pmf p x* = *Mapping.lookup-default 0* (*mapping-of-pmf p*) *x*
unfolding *mapping-of-pmf-def* *Mapping.lookup-default-def*
by (*auto split*: *option.splits simp*: *id-def* *Mapping.lookup.abs-eq*)

lemma *set-pmf-code* [*code*]: *set-pmf p* = *Mapping.keys* (*mapping-of-pmf p*)
by *transfer* (*auto simp*: *dom-def* *set-pmf-eq*)

lemma *keys-mapping-of-pmf* [*simp*]: *Mapping.keys* (*mapping-of-pmf p*) = *set-pmf p*
by *transfer* (*auto simp*: *dom-def* *set-pmf-eq*)

definition *fold-combine-plus* **where**
fold-combine-plus = *comm-monoid-set.F* (*Mapping.combine* ((+) :: *real* ⇒ -)) *Mapping.empty*

context
begin

interpretation *fold-combine-plus*: *combine-mapping-abel-semigroup* (+) :: *real* ⇒
-
by *unfold-locales* (*simp-all add*: *add-ac*)

qualified lemma *lookup-default-fold-combine-plus*:
fixes *A* :: *'b set* **and** *f* :: *'b* ⇒ (*'a*, *real*) *mapping*
assumes *finite A*
shows *Mapping.lookup-default 0* (*fold-combine-plus f A*) *x* =
(∑ *y*∈*A*. *Mapping.lookup-default 0* (*f y*) *x*)
unfolding *fold-combine-plus-def* **using** *assms*
by (*induction A rule*: *finite-induct*)
(*simp-all add*: *lookup-default-empty lookup-default-neutral-combine*)

qualified lemma *keys-fold-combine-plus*:
finite A ⟹ *Mapping.keys* (*fold-combine-plus f A*) = (⋃ *x*∈*A*. *Mapping.keys* (*f x*))
by (*simp add*: *fold-combine-plus-def fold-combine-plus.keys-fold-combine*)

qualified lemma *fold-combine-plus-code* [*code*]:
fold-combine-plus g (*set xs*) = *foldr* (λ*x*. *Mapping.combine* (+) (*g x*)) (*remdups xs*) *Mapping.empty*
by (*simp add*: *fold-combine-plus-def fold-combine-plus.fold-combine-code*)

private lemma *lookup-default-0-map-values*:
assumes *f x 0* = *0*
shows *Mapping.lookup-default 0* (*Mapping.map-values f m*) *x* = *f x* (*Mapping.lookup-default 0 m x*)
unfolding *Mapping.lookup-default-def*
using *assms* **by** *transfer* (*auto split*: *option.splits*)

qualified lemma *mapping-of-bind-pmf*:
assumes *finite* (*set-pmf p*)
shows *mapping-of-pmf* (*bind-pmf p f*) =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) (*set-pmf p*)
using *assms*
by (*intro mapping-of-pmfI'*)
(*auto simp*: *keys-fold-combine-plus lookup-default-fold-combine-plus*
pmf-bind integral-measure-pmf lookup-default-0-map-values
lookup-default-mapping-of-pmf mult-ac)

lift-definition *bind-pmf-aux* :: *'a pmf* ⇒ (*'a* ⇒ *'b pmf*) ⇒ *'a set* ⇒ (*'b*, *real*) *mapping* **is**
λ(*p* :: *'a pmf*) (*f* :: *'a* ⇒ *'b pmf*) (*A*::*'a set*) (*x*::*'b*).
if x ∈ (⋃ *y*∈*A*. *set-pmf* (*f y*)) *then*
Some (*measure-pmf.expectation p* (λ*y*. *indicator A y* ∗ *pmf* (*f y*) *x*))
else None .

lemma *keys-bind-pmf-aux* [*simp*]:

Mapping.keys (*bind-pmf-aux p f A*) = (⋃ *x*∈*A*. *set-pmf* (*f x*))
by *transfer* (*auto split*: *if-splits*)

lemma *lookup-default-bind-pmf-aux*:
Mapping.lookup-default 0 (*bind-pmf-aux p f A*) *x* =
(*if x* ∈ (⋃ *y*∈*A*. *set-pmf* (*f y*)) *then*
measure-pmf.expectation p (λ*y*. *indicator A y* ∗ *pmf* (*f y*) *x*) *else 0*)
unfolding *lookup-default-def* **by** *transfer′ simp-all*

lemma *lookup-default-bind-pmf-aux′* [*simp*]:
Mapping.lookup-default 0 (*bind-pmf-aux p f* (*set-pmf p*)) *x* = *pmf* (*bind-pmf p f*) *x*
unfolding *lookup-default-def*
by *transfer* (*auto simp*: *pmf-bind AE-measure-pmf-iff set-pmf-eq*
intro!: *integral-cong-AE integral-eq-zero-AE*)

lemma *bind-pmf-aux-correct*:
mapping-of-pmf (*bind-pmf p f*) = *bind-pmf-aux p f* (*set-pmf p*)
by (*intro mapping-of-pmfI′*) *simp-all*

lemma *bind-pmf-aux-code-aux*:
assumes *finite A*
shows *bind-pmf-aux p f A* =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) *A* (**is** *?lhs* = *?rhs*)
proof (*intro mapping-eqI′*[**where** *d* = *0*])
fix *x* **assume** *x* ∈ *Mapping.keys ?lhs*
then obtain *y* **where** *y*: *y* ∈ *A x* ∈ *set-pmf* (*f y*) **by** *auto*
hence *Mapping.lookup-default 0 ?lhs x* =
measure-pmf.expectation p (λ*y*. *indicator A y* ∗ *pmf* (*f y*) *x*)
by (*auto simp*: *lookup-default-bind-pmf-aux*)
also from *assms* **have** ... = (∑ *y*∈*A*. *pmf p y* ∗ *pmf* (*f y*) *x*)
by (*subst integral-measure-pmf* [*of A*])
(*auto simp*: *set-pmf-eq indicator-def mult-ac split*: *if-splits*)
also from *assms* **have** ... = *Mapping.lookup-default 0 ?rhs x*
by (*simp add*: *lookup-default-fold-combine-plus lookup-default-0-map-values*
lookup-default-mapping-of-pmf)
finally show *Mapping.lookup-default 0 ?lhs x* = *Mapping.lookup-default 0 ?rhs x* .
qed (*insert assms*, *simp-all add*: *keys-fold-combine-plus*)

lemma *bind-pmf-aux-code* [*code*]:
bind-pmf-aux p f (*set xs*) =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) (*set xs*)
by (*rule bind-pmf-aux-code-aux*) *simp-all*

lemmas *bind-pmf-code* [*code abstract*] = *bind-pmf-aux-correct*

end

hide-const (**open**) *fold-combine-plus*

lift-definition *cond-pmf-impl* :: *'a pmf* ⇒ *'a set* ⇒ (*'a*, *real*) *mapping option* **is**
λ*p A*. *if A* ∩ *set-pmf p* = {} *then None else*
Some (λ*x*. *if x* ∈ *A* ∩ *set-pmf p then Some* (*pmf p x* / *measure-pmf.prob p A*) *else None*) **.**

lemma *cond-pmf-impl-code-alt*:
assumes *finite A*
shows *cond-pmf-impl p A* = (
let C = *A* ∩ *set-pmf p*;
prob = (∑ *x*∈*C*. *pmf p x*)
in if prob = *0 then*
None
else
Some (*Mapping.map-values* (λ- *y*. *y* / *prob*)
(*Mapping.filter* (λ*k* -. *k* ∈ *C*) (*mapping-of-pmf p*))))
proof −
define *C* **where** *C* = *A* ∩ *set-pmf p*
define *prob* **where** *prob* = (∑ *x*∈*C*. *pmf p x*)
also note *C-def*
also from *assms* **have** (∑ *x*∈*A* ∩ *set-pmf p*. *pmf p x*) = (∑ *x*∈*A*. *pmf p x*)
by (*intro sum.mono-neutral-left*) (*auto simp*: *set-pmf-eq*)
finally have *prob1*: *prob* = (∑ *x*∈*A*. *pmf p x*) **.**
hence *prob2*: *prob* = *measure-pmf.prob p A*
using *assms* **by** (*subst measure-measure-pmf-finite*) *simp-all*
have *prob3*: *prob* = *0* ⟷ *A* ∩ *set-pmf p* = {}
by (*subst prob1*, *subst sum-nonneg-eq-0-iff*) (*auto simp*: *set-pmf-eq assms*)
from *assms* **have** *prob4*: *prob* = *measure-pmf.prob p C*
unfolding *prob-def* **by** (*intro measure-measure-pmf-finite* [*symmetric*]) (*simp-all add*: *C-def*)

show *?thesis*
proof (*cases prob* = *0*)
case *True*
hence *A* ∩ *set-pmf p* = {} **by** (*subst* (*asm*) *prob3*)
with *True* **show** *?thesis* **by** (*simp add*: *Let-def prob-def C-def cond-pmf-impl.abs-eq*)
next
case *False*
hence *A*: *C* ≠ {} **unfolding** *C-def* **by** (*subst* (*asm*) *prob3*) *auto*
with *prob3* **have** *prob-nz*: *prob* ≠ *0* **by** (*auto simp*: *C-def*)
fix *x*
have *cond-pmf-impl p A* =
Some (*mapping.Mapping* (λ*x*. *if x* ∈ *C then*
Some (*pmf p x* / *measure-pmf.prob p C*) *else None*))
(**is** - = *Some ?m*)

```
    using A prob2 prob4 unfolding C-def by transfer (auto simp: fun-eq-iff)
  also have ?m = Mapping.map-values (λ- y. y / prob)
              (Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))
    using prob-nz prob4 assms unfolding C-def
    by transfer (auto simp: fun-eq-iff set-pmf-eq)
  finally show ?thesis using False by (simp add: Let-def prob-def C-def)
 qed
qed

lemma cond-pmf-impl-code [code]:
 cond-pmf-impl p (set xs) = (
   let C = set xs ∩ set-pmf p;
       prob = (∑ x∈C. pmf p x)
   in  if prob = 0 then
         None
       else
         Some (Mapping.map-values (λ- y. y / prob)
           (Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
 by (rule cond-pmf-impl-code-alt) simp-all

lemma cond-pmf-code [code abstract]:
 mapping-of-pmf (cond-pmf p A) =
   (case cond-pmf-impl p A of
     None ⇒ Code.abort (STR ''cond-pmf with set of probability 0'')
              (λ-. mapping-of-pmf (cond-pmf p A))
   | Some m ⇒ m)
proof (cases cond-pmf-impl p A)
 case (Some m)
 hence A: set-pmf p ∩ A ≠ {} by transfer (auto split: if-splits)
 from Some have B: Mapping.keys m = set-pmf (cond-pmf p A)
   by (subst set-cond-pmf[OF A], transfer) (auto split: if-splits)
 with Some A have mapping-of-pmf (cond-pmf p A) = m
  by (intro mapping-of-pmfI[OF - B], transfer) (auto split: if-splits simp: pmf-cond)
 with Some show ?thesis by simp
qed simp-all


lemma binomial-pmf-code [code abstract]:
 mapping-of-pmf (binomial-pmf n p) = (
   if p < 0 ∨ p > 1 then
     Code.abort (STR ''binomial-pmf with invalid probability'')
       (λ-. mapping-of-pmf (binomial-pmf n p))
   else if p = 0 then Mapping.update 0 1 Mapping.empty
   else if p = 1 then Mapping.update n 1 Mapping.empty
   else Mapping.tabulate [0..<Suc n] (λk. real (n choose k) * p ^ k * (1 - p) ^
(n - k)))
 by (cases p < 0 ∨ p > 1)
   (simp, intro mapping-of-pmfI,
     auto simp: lookup-update' lookup-empty set-pmf-binomial-eq lookup-tabulate
```

split: *if-splits*)

lemma *pred-pmf-code* [*code*]:
pred-pmf P p = (∀ *x*∈*set-pmf p*. *P x*)
by (*auto simp*: *pred-pmf-def*)

lemma *mapping-of-pmf-pmf-of-list*:
assumes ⋀*x*. *x* ∈ *snd* ‘ *set xs* ⟹ *x* > *0 sum-list* (*map snd xs*) = *1*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))
proof −
from *assms* **have** *wf*: *pmf-of-list-wf xs* **by** (*intro pmf-of-list-wfI*) *force*
with *assms* **have** *set-pmf* (*pmf-of-list xs*) = *fst* ‘ *set xs*
by (*intro set-pmf-of-list-eq*) *auto*
with *wf* **show** *?thesis*
by (*intro mapping-of-pmfI*) (*auto simp*: *lookup-tabulate pmf-pmf-of-list*)
qed

lemma *mapping-of-pmf-pmf-of-list′*:
assumes *pmf-of-list-wf xs*
defines *xs′* ≡ *filter* (λ*z*. *snd z* ≠ *0*) *xs*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs′*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs′*))) (**is** - = *?rhs*)
proof −
have *wf*: *pmf-of-list-wf xs′* **unfolding** *xs′-def* **by** (*rule pmf-of-list-remove-zeros*)
fact
have *pos*: ∀ *x*∈*snd*‘*set xs′*. *x* > *0* **using** *assms*(*1*) **unfolding** *xs′-def*
by (*force simp*: *pmf-of-list-wf-def*)
from *assms* **have** *pmf-of-list xs* = *pmf-of-list xs′*
unfolding *xs′-def* **by** (*subst pmf-of-list-remove-zeros*) *simp-all*
also from *wf pos* **have** *mapping-of-pmf* ... = *?rhs*
by (*intro mapping-of-pmf-pmf-of-list*) (*auto simp*: *pmf-of-list-wf-def*)
finally show *?thesis* **.**
qed

lemma *pmf-of-list-wf-code* [*code*]:
pmf-of-list-wf xs ⟷ *list-all* (λ*z*. *snd z* ≥ *0*) *xs* ∧ *sum-list* (*map snd xs*) = *1*
by (*auto simp add*: *pmf-of-list-wf-def list-all-def*)

lemma *pmf-of-list-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-list xs*) = (
if pmf-of-list-wf xs then
let xs′ = *filter* (λ*z*. *snd z* ≠ *0*) *xs*
in Mapping.tabulate (*remdups* (*map fst xs′*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs′*)))

else
Code.abort (*STR ''Invalid list for pmf-of-list''*) (λ-. *mapping-of-pmf* (*pmf-of-list xs*)))
using *mapping-of-pmf-pmf-of-list'*[*of xs*] **by** (*simp add*: *Let-def*)

lemma *mapping-of-pmf-eq-iff* [*simp*]:
mapping-of-pmf p = *mapping-of-pmf q* ⟷ *p* = (*q* :: *'a pmf*)
proof (*transfer*, *intro iffI pmf-eqI*)
fix *p q* :: *'a pmf* **and** *x* :: *'a*
assume (λ*x*. *if pmf p x* = *0 then None else Some* (*pmf p x*)) =
(λ*x*. *if pmf q x* = *0 then None else Some* (*pmf q x*))
hence (*if pmf p x* = *0 then None else Some* (*pmf p x*)) =
(*if pmf q x* = *0 then None else Some* (*pmf q x*)) **for** *x*
by (*simp add*: *fun-eq-iff*)
from *this*[*of x*] **show** *pmf p x* = *pmf q x* **by** (*auto split*: *if-splits*)
qed (*simp-all cong*: *if-cong*)

21.2 Code abbreviations for integrals and probabilities

Integrals and probabilities are defined for general measures, so we cannot give any code equations directly. We can, however, specialise these constants them to PMFs, give code equations for these specialised constants, and tell the code generator to unfold the original constants to the specialised ones whenever possible.

definition *pmf-integral* **where**
pmf-integral p f = *lebesgue-integral* (*measure-pmf p*) (*f* :: - ⇒ *real*)

definition *pmf-set-integral* **where**
pmf-set-integral p f A = *lebesgue-integral* (*measure-pmf p*) (λ*x*. *indicator A x* ∗ *f x* :: *real*)

definition *pmf-prob* **where**
pmf-prob p A = *measure-pmf.prob p A*

lemma *pmf-prob-compl*: *pmf-prob p* (−*A*) = *1* − *pmf-prob p A*
using *measure-pmf.prob-compl*[*of A p*] **by** (*simp add*: *pmf-prob-def Compl-eq-Diff-UNIV*)

lemma *pmf-integral-pmf-set-integral* [*code*]:
pmf-integral p f = *pmf-set-integral p f* (*set-pmf p*)
unfolding *pmf-integral-def pmf-set-integral-def*
by (*intro integral-cong-AE*) (*simp-all add*: *AE-measure-pmf-iff*)

lemma *pmf-prob-pmf-set-integral*:
pmf-prob p A = *pmf-set-integral p* (λ-. *1*) *A*
by (*simp add*: *pmf-prob-def pmf-set-integral-def*)

lemma *pmf-set-integral-code-alt-finite*:
finite A ⟹ *pmf-set-integral p f A* = (∑ *x*∈*A*. *pmf p x* ∗ *f x*)

unfolding *pmf-set-integral-def*
by (*subst integral-measure-pmf*[*of A*]) (*auto simp*: *indicator-def mult-ac split*: *if-splits*)

lemma *pmf-set-integral-code* [*code*]:
pmf-set-integral p f (*set xs*) = (∑ *x*∈*set xs*. *pmf p x* ∗ *f x*)
by (*rule pmf-set-integral-code-alt-finite*) *simp-all*

lemma *pmf-prob-code-alt-finite*:
finite A ⟹ *pmf-prob p A* = (∑ *x*∈*A*. *pmf p x*)
by (*simp add*: *pmf-prob-pmf-set-integral pmf-set-integral-code-alt-finite*)

lemma *pmf-prob-code* [*code*]:
pmf-prob p (*set xs*) = (∑ *x*∈*set xs*. *pmf p x*)
pmf-prob p (*List.coset xs*) = *1* − (∑ *x*∈*set xs*. *pmf p x*)
by (*simp-all add*: *pmf-prob-code-alt-finite pmf-prob-compl*)

lemma *pmf-prob-code-unfold* [*code-abbrev*]: *pmf-prob p* = *measure-pmf.prob p*
by (*intro ext*) (*simp add*: *pmf-prob-def*)

lemma *pmf-integral-code-unfold* [*code-abbrev*]: *pmf-integral p* = *measure-pmf.expectation p*
by (*intro ext*) (*simp add*: *pmf-integral-def*)

definition *pmf-of-alist xs* = *embed-pmf* (λ*x*. *case map-of xs x of Some p* ⇒ *p* | *None* ⇒ *0*)

lemma *pmf-of-mapping-Mapping* [*code-post*]:
pmf-of-mapping (*Mapping xs*) = *pmf-of-alist xs*
unfolding *pmf-of-mapping-def Mapping.lookup-default-def* [*abs-def*] *pmf-of-alist-def*
by *transfer simp-all*

instantiation *pmf* :: (*equal*) *equal*
begin

definition *equal-pmf p q* = (*mapping-of-pmf p* = *mapping-of-pmf* (*q* :: *'a pmf*))

instance by *standard* (*simp add*: *equal-pmf-def*)
end

definition *single* :: *'a* ⇒ *'a multiset* **where**
single s = {#*s*#}

instantiation *pmf* :: (*random*) *random*
begin

context
includes *state-combinator-syntax* **and** *term-syntax*
begin

definition
pmfify :: ('*b*::*typerep multiset* × (*unit* ⇒ *Code-Evaluation.term*)) ⇒
'*b* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
'*b pmf* × (*unit* ⇒ *Code-Evaluation.term*) **where**
[*code-unfold*]: *pmfify A x* =
Code-Evaluation.valtermify pmf-of-multiset {·}
(*Code-Evaluation.valtermify* (+) {·} *A* {·}
(*Code-Evaluation.valtermify single* {·} *x*))

definition
Quickcheck-Random.random i =
Quickcheck-Random.random i ∘→ (λ*A*.
Quickcheck-Random.random i ∘→ (λ*x*. *Pair* (*pmfify A x*)))

instance ..

end

end

instantiation *pmf* :: (*full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive-pmf* :: ('*a pmf* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-pmf f i =
Quickcheck-Exhaustive.full-exhaustive (λ*A*.
Quickcheck-Exhaustive.full-exhaustive (λ*x*. *f* (*pmfify A x*)) *i*) *i*

instance ..

end

end

22 Finite Maps

theory *Fin-Map*
imports *HOL−Analysis.Finite-Product-Measure HOL−Library.Finite-Map*
begin

The *fmap* type can be instantiated to *polish-space*, needed for the proof of projective limit. *extensional* functions are used for the representation in order to stay close to the developments of (finite) products Pi_E and their sigma-algebra Pi_M.

type-notation *fmap* (‹(‹*notation*=‹*infix fmap*›› - ⇒$_F$ /-)› [*22*, *21*] *21*)

unbundle *fmap.lifting*

22.1 Domain and Application

lift-definition *domain*::('*i* ⇒$_F$ '*a*) ⇒ '*i set* **is** *dom* .

lemma *finite-domain*[*simp*, *intro*]: *finite* (*domain P*)
by *transfer simp*

lift-definition *proj* :: ('*i* ⇒$_F$ '*a*) ⇒ '*i* ⇒ '*a*
(‹(‹*indent*=*1 notation*=‹*mixfix proj*›› '(-')$_F$)› [*0*] *1000*) **is**
λ*f x*. *if x* ∈ *dom f then the* (*f x*) *else undefined* .

declare [[*coercion proj*]]

lemma *extensional-proj*[*simp*, *intro*]: (*P*)$_F$ ∈ *extensional* (*domain P*)
by *transfer* (*auto simp*: *extensional-def*)

lemma *proj-undefined*[*simp*, *intro*]: *i* ∉ *domain P* ⟹ *P i* = *undefined*
using *extensional-proj*[*of P*] **unfolding** *extensional-def* **by** *auto*

lemma *finmap-eq-iff*: *P* = *Q* ⟷ (*domain P* = *domain Q* ∧ (∀ *i*∈*domain P*. *P i* = *Q i*))
apply *transfer*
apply (*safe intro*!: *ext*)
subgoal for *P Q x*
by (*cases x* ∈ *dom P*; *cases P x*) (*auto dest*!: *bspec*[**where** *x*=*x*])
done

22.2 Constructor of Finite Maps

lift-definition *finmap-of*::'*i set* ⇒ ('*i* ⇒ '*a*) ⇒ ('*i* ⇒$_F$ '*a*) **is**
λ*I f x*. *if x* ∈ *I* ∧ *finite I then Some* (*f x*) *else None*
by (*simp add*: *dom-def*)

lemma *proj-finmap-of*[*simp*]:
assumes *finite inds*
shows (*finmap-of inds f*)$_F$ = *restrict f inds*
using *assms*
by *transfer force*

lemma *domain-finmap-of*[*simp*]:
assumes *finite inds*

shows *domain* (*finmap-of inds f*) = *inds*
using *assms*
by *transfer* (*auto split*: *if-splits*)

lemma *finmap-of-eq-iff*[*simp*]:
assumes *finite i finite j*
shows *finmap-of i m* = *finmap-of j n* ⟷ *i* = *j* ∧ (∀ *k*∈*i*. *m k*= *n k*)
using *assms* **by** (*auto simp*: *finmap-eq-iff*)

lemma *finmap-of-inj-on-extensional-finite*:
assumes *finite K*
assumes *S* ⊆ *extensional K*
shows *inj-on* (*finmap-of K*) *S*
proof (*rule inj-onI*)
fix *x y*::'*a* ⇒ '*b*
assume *finmap-of K x* = *finmap-of K y*
hence (*finmap-of K x*)$_F$ = (*finmap-of K y*)$_F$ **by** *simp*
moreover
assume *x* ∈ *S y* ∈ *S* **hence** *x* ∈ *extensional K y* ∈ *extensional K* **using** *assms*
by *auto*
ultimately
show *x* = *y* **using** *assms* **by** (*simp add*: *extensional-restrict*)
qed

22.3 Product set of Finite Maps

This is *Pi* for Finite Maps, most of this is copied

definition *Pi'* :: '*i set* ⇒ ('*i* ⇒ '*a set*) ⇒ ('*i* ⇒$_F$ '*a*) *set* **where**
Pi' I A = { *P*. *domain P* = *I* ∧ (∀ *i*. *i* ∈ *I* ⟶ (*P*)$_F$ *i* ∈ *A i*) }

syntax
-*Pi'* :: [*pttrn*, '*a set*, '*b set*] => ('*a* => '*b*) *set*
(‹(‹*indent*=*3 notation*=‹*binder* Π'››Π'' -∈-./ -)› *10*)
syntax-consts
-*Pi'* == *Pi'*
translations
Π' *x*∈*A*. *B* == *CONST Pi' A* (λ*x*. *B*)

22.3.1 Basic Properties of *Pi'*

lemma *Pi'-I*[*intro*!]: *domain f* = *A* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi' A B*
by (*simp add*: *Pi'-def*)

lemma *Pi'-I'*[*simp*]: *domain f* = *A* ⟹ (⋀*x*. *x* ∈ *A* ⟶ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi' A B*
by (*simp add*:*Pi'-def*)

lemma *Pi'-mem*: *f*∈ *Pi' A B* ⟹ *x* ∈ *A* ⟹ *f x* ∈ *B x*

by (*simp add*: *Pi'-def*)

lemma *Pi'-iff*: $f \in Pi'\ I\ X \longleftrightarrow domain\ f = I \wedge (\forall i{\in}I.\ f\ i \in X\ i)$
unfolding *Pi'-def* **by** *auto*

lemma *Pi'E* [*elim*]:
$f \in Pi'\ A\ B \Longrightarrow (f\ x \in B\ x \Longrightarrow domain\ f = A \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
by(*auto simp*: *Pi'-def*)

lemma *in-Pi'-cong*:
$domain\ f = domain\ g \Longrightarrow (\bigwedge w.\ w \in A \Longrightarrow f\ w = g\ w) \Longrightarrow f \in Pi'\ A\ B \longleftrightarrow g \in Pi'\ A\ B$
by (*auto simp*: *Pi'-def*)

lemma *Pi'-eq-empty*[*simp*]:
assumes *finite A* **shows** $(Pi'\ A\ B) = \{\} \longleftrightarrow (\exists x{\in}A.\ B\ x = \{\})$
using *assms*
apply (*simp add*: *Pi'-def*, *auto*)
apply (*drule-tac x = finmap-of A* ($\lambda u.$ *SOME y.* $y \in B\ u$) **in** *spec*, *auto*)
apply (*cut-tac P*= $\%y.\ y \in B\ i$ **in** *some-eq-ex*, *auto*)
done

lemma *Pi'-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi'\ A\ B \subseteq Pi'\ A\ C$
by (*auto simp*: *Pi'-def*)

lemma *Pi-Pi'*: *finite A* $\Longrightarrow (Pi_E\ A\ B) = proj\ `\ Pi'\ A\ B$
apply (*auto simp*: *Pi'-def Pi-def extensional-def*)
apply (*rule-tac x = finmap-of A* (*restrict x A*) **in** *image-eqI*)
apply *auto*
done

22.4 Topological Space of Finite Maps

instantiation *fmap* :: (*type*, *topological-space*) *topological-space*
begin

definition *open-fmap* :: $('a \Rightarrow_F 'b)\ set \Rightarrow bool$ **where**
[*code del*]: *open-fmap* = *generate-topology* $\{Pi'\ a\ b | a\ b.\ \forall i{\in}a.\ open\ (b\ i)\}$

lemma *open-Pi'I*: $(\bigwedge i.\ i \in I \Longrightarrow open\ (A\ i)) \Longrightarrow open\ (Pi'\ I\ A)$
by (*auto intro*: *generate-topology.Basis simp*: *open-fmap-def*)

instance using *topological-space-generate-topology*
by *intro-classes* (*auto simp*: *open-fmap-def class.topological-space-def*)

end

lemma *open-restricted-space*:

```
  shows open {m. P (domain m)}
proof −
  have {m. P (domain m)} = (⋃ i ∈ Collect P. {m. domain m = i}) by auto
  also have open ...
  proof (rule, safe, cases)
    fix i::'a set
    assume finite i
    hence {m. domain m = i} = Pi' i (λ-. UNIV) by (auto simp: Pi'-def)
    also have open ... by (auto intro: open-Pi'I simp: ‹finite i›)
    finally show open {m. domain m = i} .
  next
    fix i::'a set
    assume ¬ finite i hence {m. domain m = i} = {} by auto
    also have open ... by simp
    finally show open {m. domain m = i} .
  qed
  finally show ?thesis .
qed

lemma closed-restricted-space:
  shows closed {m. P (domain m)}
  using open-restricted-space[of λx. ¬ P x]
  unfolding closed-def by (rule back-subst) auto

lemma tendsto-proj: ((λx. x) ⟶ a) F ⟹ ((λx. (x)_F i) ⟶ (a)_F i) F
  unfolding tendsto-def
proof safe
  fix S::'b set
  let ?S = Pi' (domain a) (λx. if x = i then S else UNIV)
  assume open S hence open ?S by (auto intro!: open-Pi'I)
  moreover assume ∀ S. open S ⟶ a ∈ S ⟶ eventually (λx. x ∈ S) F a i ∈ S
  ultimately have eventually (λx. x ∈ ?S) F by auto
  thus eventually (λx. (x)_F i ∈ S) F
    by eventually-elim (insert ‹a i ∈ S›, force simp: Pi'-iff split: if-split-asm)
qed

lemma continuous-proj:
  shows continuous-on s (λx. (x)_F i)
  unfolding continuous-on-def by (safe intro!: tendsto-proj tendsto-ident-at)

instance fmap :: (type, first-countable-topology) first-countable-topology
proof
  fix x::'a⇒_F'b
  have ∀ i. ∃ A. countable A ∧ (∀ a∈A. x i ∈ a) ∧ (∀ a∈A. open a) ∧
    (∀ S. open S ∧ x i ∈ S ⟶ (∃ a∈A. a ⊆ S)) ∧ (∀ a b. a ∈ A ⟶ b ∈ A ⟶ a
∩ b ∈ A) (is ∀ i. ?th i)
  proof
    fix i from first-countable-basis-Int-stableE[of x i]
    obtain A where
```

countable A
$\bigwedge C.\ C \in A \Longrightarrow (x)_F\ i \in C$
$\bigwedge C.\ C \in A \Longrightarrow open\ C$
$\bigwedge S.\ open\ S \Longrightarrow (x)_F\ i \in S \Longrightarrow \exists\, A{\in}A.\ A \subseteq S$
$\bigwedge C\ D.\ C \in A \Longrightarrow D \in A \Longrightarrow C \cap D \in A$
by *auto*
thus *?th i* **by** (*intro exI*[**where** $x{=}A$]) *simp*
qed
then obtain *A*
where *A*: $\forall\, i.$ *countable* $(A\ i) \wedge$ *Ball* $(A\ i)$ $((\in)$ $((x)_F\ i)) \wedge$ *Ball* $(A\ i)$ *open* $\wedge$ $(\forall\, S.$ *open* $S \wedge (x)_F\ i \in S \longrightarrow (\exists\, a{\in}A\ i.\ a \subseteq S)) \wedge (\forall\, a\ b.\ a \in A\ i \longrightarrow b \in A\ i \longrightarrow a \cap b \in A\ i)$
by (*auto simp*: *choice-iff*)
hence *open-sub*: $\bigwedge i\ S.\ i{\in}domain\ x \Longrightarrow open\ (S\ i) \Longrightarrow x\ i{\in}(S\ i) \Longrightarrow (\exists\, a{\in}A\ i.\ a{\subseteq}(S\ i))$ **by** *auto*
have *A-notempty*: $\bigwedge i.\ i \in domain\ x \Longrightarrow A\ i \neq \{\}$ **using** *open-sub*[*of* - λ-. *UNIV*] **by** *auto*
let *?A* $= (\lambda f.\ Pi'\ (domain\ x)\ f)$ ` $(Pi_E\ (domain\ x)\ A)$
show $\exists\, A{::}nat \Rightarrow ('a{\Rightarrow_F}'b)\ set.\ (\forall\, i.\ x \in (A\ i) \wedge open\ (A\ i)) \wedge (\forall\, S.\ open\ S \wedge x \in S \longrightarrow (\exists\, i.\ A\ i \subseteq S))$
proof (*rule first-countableI*[*of ?A*], *safe*)
show *countable ?A* **using** *A* **by** (*simp add*: *countable-PiE*)
next
fix $S{::}('a \Rightarrow_F 'b)\ set$ **assume** *open S x* $\in S$
thus $\exists\, a{\in}?A.\ a \subseteq S$ **unfolding** *open-fmap-def*
proof (*induct rule*: *generate-topology.induct*)
case *UNIV* **thus** *?case* **by** (*auto simp add*: *ex-in-conv PiE-eq-empty-iff A-notempty*)
next
case (*Int a b*)
then obtain *f g* **where**
$f \in Pi_E\ (domain\ x)\ A\ Pi'\ (domain\ x)\ f \subseteq a\ g \in Pi_E\ (domain\ x)\ A\ Pi'\ (domain\ x)\ g \subseteq b$
by *auto*
thus *?case* **using** *A*
by (*auto simp*: *Pi'-iff PiE-iff extensional-def Int-stable-def intro*!: *bexI*[**where** $x{=}\lambda i.\ f\ i \cap g\ i$])
next
case (*UN B*)
then obtain *b* **where** $x \in b\ b \in B$ **by** *auto*
hence $\exists\, a{\in}?A.\ a \subseteq b$ **using** *UN* **by** *simp*
thus *?case* **using** $\langle b \in B \rangle$ **by** (*metis Sup-upper2*)
next
case (*Basis s*)
then obtain *a b* **where** *xs*: $x{\in}\ Pi'\ a\ b\ s = Pi'\ a\ b\ \bigwedge i.\ i{\in}a \Longrightarrow open\ (b\ i)$ **by** *auto*
have $\forall\, i.\ \exists\, a.\ (i \in domain\ x \wedge open\ (b\ i) \wedge (x)_F\ i \in b\ i) \longrightarrow (a{\in}A\ i \wedge a \subseteq b\ i)$
using *open-sub*[*of* - *b*] **by** *auto*

then obtain b'
where $\bigwedge i.\ i \in \mathit{domain}\ x \Longrightarrow \mathit{open}\ (b\ i) \Longrightarrow (x)_F\ i \in b\ i \Longrightarrow (b'\ i \in_A i \wedge b'\ i \subseteq b\ i)$
unfolding *choice-iff* **by** *auto*
with *xs* **have** $\bigwedge i.\ i \in a \Longrightarrow (b'\ i \in_A i \wedge b'\ i \subseteq b\ i)\ Pi'\ a\ b' \subseteq Pi'\ a\ b$
by (*auto simp*: *Pi'-iff intro*!: *Pi'-mono*)
thus *?case* **using** *xs*
by (*intro bexI*[**where** x=*Pi'* *a* b'])
(*auto simp*: *Pi'-iff intro*!: *image-eqI*[**where** x=*restrict* b' (*domain* x)])
qed
qed (*insert A*,*auto simp*: *PiE-iff intro*!: *open-Pi'I*)
qed

22.5 Metric Space of Finite Maps

instantiation *fmap* :: (*type*, *metric-space*) *dist*
begin

definition *dist-fmap* **where**
dist P Q = *Max* (*range* (λ*i*. *dist* ((*P*)$_F$ *i*) ((*Q*)$_F$ *i*))) + (*if domain P* = *domain Q then 0 else 1*)

instance ..
end

instantiation *fmap* :: (*type*, *metric-space*) *uniformity-dist*
begin

definition [*code del*]:
(*uniformity* :: (('*a*, '*b*) *fmap* × ('*a* ⇒$_F$ '*b*)) *filter*) =
(*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

instance
by *standard* (*rule uniformity-fmap-def*)
end

declare *uniformity-Abort*[**where** '*a*=('*a* ⇒$_F$ '*b*::*metric-space*), *code*]

instantiation *fmap* :: (*type*, *metric-space*) *metric-space*
begin

lemma *finite-proj-image'*: $x \notin \mathit{domain}\ P \Longrightarrow \mathit{finite}\ ((P)_F\ `\ S)$
by (*rule finite-subset*[*of* - *proj P* ` (*domain P* ∩ *S* ∪ {*x*})]) *auto*

lemma *finite-proj-image*: *finite* ((*P*)$_F$ ` *S*)
by (*cases* ∃ *x*. *x* ∉ *domain P*) (*auto intro*: *finite-proj-image' finite-subset*[**where** *B*=*domain P*])

lemma *finite-proj-diag*: *finite* ((λ*i*. *d* ((*P*)$_F$ *i*) ((*Q*)$_F$ *i*)) ` *S*)

proof −
have $(\lambda i.\ d\ ((P)_F\ i)\ ((Q)_F\ i))\ `\ S = (\lambda(i,\ j).\ d\ i\ j)\ `\ ((\lambda i.\ ((P)_F\ i,\ (Q)_F\ i))\ `\ S)$ **by** *auto*
moreover have $((\lambda i.\ ((P)_F\ i,\ (Q)_F\ i))\ `\ S) \subseteq (\lambda i.\ (P)_F\ i)\ `\ S \times (\lambda i.\ (Q)_F\ i)\ `\ S$ **by** *auto*
moreover have *finite* ... **using** *finite-proj-image*[*of P S*] *finite-proj-image*[*of Q S*]
by (*intro finite-cartesian-product*) *simp-all*
ultimately show *?thesis* **by** (*simp add*: *finite-subset*)
qed

lemma *dist-le-1-imp-domain-eq*:
shows $dist\ P\ Q < 1 \Longrightarrow domain\ P = domain\ Q$
by (*simp add*: *dist-fmap-def finite-proj-diag split*: *if-split-asm*)

lemma *dist-proj*:
shows $dist\ ((x)_F\ i)\ ((y)_F\ i) \leq dist\ x\ y$
proof −
have $dist\ (x\ i)\ (y\ i) \leq Max\ (range\ (\lambda i.\ dist\ (x\ i)\ (y\ i)))$
by (*simp add*: *Max-ge-iff finite-proj-diag*)
also have ... $\leq dist\ x\ y$ **by** (*simp add*: *dist-fmap-def*)
finally show *?thesis* **.**
qed

lemma *dist-finmap-lessI*:
assumes $domain\ P = domain\ Q$
assumes $0 < e$
assumes $\bigwedge i.\ i \in domain\ P \Longrightarrow dist\ (P\ i)\ (Q\ i) < e$
shows $dist\ P\ Q < e$
proof −
have $dist\ P\ Q = Max\ (range\ (\lambda i.\ dist\ (P\ i)\ (Q\ i)))$
using *assms* **by** (*simp add*: *dist-fmap-def finite-proj-diag*)
also have ... $< e$
proof (*subst Max-less-iff*, *safe*)
fix *i*
show $dist\ ((P)_F\ i)\ ((Q)_F\ i) < e$ **using** *assms*
by (*cases* $i \in domain\ P$) *simp-all*
qed (*simp add*: *finite-proj-diag*)
finally show *?thesis* **.**
qed

instance
proof
fix S::$('a \Rightarrow_F 'b)\ set$
have $*$: $open\ S = (\forall x{\in}S.\ \exists e{>}0.\ \forall y.\ dist\ y\ x < e \longrightarrow y \in S)$ (**is** - = *?od*)
proof
assume *open S*
thus *?od*
unfolding *open-fmap-def*

proof (*induct rule*: *generate-topology.induct*)
case *UNIV* **thus** *?case* **by** (*auto intro*: *zero-less-one*)
next
case (*Int a b*)
show *?case*
proof *safe*
fix *x* **assume** *x*: $x \in a$ $x \in b$
with *Int x* **obtain** *e1 e2* **where**
$e1{>}0$ $\forall\, y.\ dist\ y\ x < e1 \longrightarrow y \in a$ $e2{>}0$ $\forall\, y.\ dist\ y\ x < e2 \longrightarrow y \in b$ **by** *force*
thus $\exists\, e{>}0.\ \forall\, y.\ dist\ y\ x < e \longrightarrow y \in a \cap b$
by (*auto intro*!: *exI*[**where** *x*=*min e1 e2*])
qed
next
case (*UN K*)
show *?case*
proof *safe*
fix *x X* **assume** $x \in X$ **and** *X*: $X \in K$
with *UN* **obtain** *e* **where** $e{>}0$ $\bigwedge y.\ dist\ y\ x < e \longrightarrow y \in X$ **by** *force*
with *X* **show** $\exists\, e{>}0.\ \forall\, y.\ dist\ y\ x < e \longrightarrow y \in \bigcup K$ **by** *auto*
qed
next
case (*Basis s*) **then obtain** *a b* **where** *s*: $s = Pi'\ a\ b$ **and** *b*: $\bigwedge i.\ i{\in}a \Longrightarrow open\ (b\ i)$ **by** *auto*
show *?case*
proof *safe*
fix *x* **assume** $x \in s$
hence [*simp*]: *finite a* **and** *a-dom*: $a = domain\ x$ **using** *s* **by** (*auto simp*: *Pi'-iff*)
obtain *es* **where** *es*: $\forall\, i \in a.\ es\ i > 0 \wedge (\forall\, y.\ dist\ y\ (proj\ x\ i) < es\ i \longrightarrow y \in b\ i)$
using *b* ‹$x \in s$› **by** *atomize-elim* (*intro bchoice*, *auto simp*: *open-dist s*)
hence *in-b*: $\bigwedge i\ y.\ i \in a \Longrightarrow dist\ y\ (proj\ x\ i) < es\ i \Longrightarrow y \in b\ i$ **by** *auto*
show $\exists\, e{>}0.\ \forall\, y.\ dist\ y\ x < e \longrightarrow y \in s$
proof (*cases*, *rule*, *safe*)
assume $a \neq \{\}$
show $0 < min\ 1\ (Min\ (es\ `\ a))$ **using** *es* **by** (*auto simp*: ‹$a \neq \{\}$›)
fix *y* **assume** *d*: $dist\ y\ x < min\ 1\ (Min\ (es\ `\ a))$
show $y \in s$ **unfolding** *s*
proof
show $domain\ y = a$ **using** *d s* ‹$a \neq \{\}$› **by** (*auto simp*: *dist-le-1-imp-domain-eq a-dom*)
fix *i* **assume** *i*: $i \in a$
hence $dist\ ((y)_F\ i)\ ((x)_F\ i) < es\ i$ **using** *d*
by (*auto simp*: *dist-fmap-def* ‹$a \neq \{\}$› *intro*!: *le-less-trans*[*OF dist-proj*])
with *i* **show** $y\ i \in b\ i$ **by** (*rule in-b*)
qed
next
assume $\neg a \neq \{\}$

```
        thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ s
          using s ‹x ∈ s› by (auto simp: Pi'-def dist-le-1-imp-domain-eq intro!:
exI[where x=1])
       qed
     qed
   qed
  next
   assume ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S
   then obtain e where e-pos: ⋀x. x ∈ S ⟹ e x > 0 and
     e-in: ⋀x y . x ∈ S ⟹ dist y x < e x ⟹ y ∈ S
     unfolding bchoice-iff
     by auto
   have S-eq: S = ⋃{Pi' a b| a b. ∃ x∈S. domain x = a ∧ b = (λi. ball (x i) (e
x))}
   proof safe
     fix x assume x ∈ S
     thus x ∈ ⋃{Pi' a b| a b. ∃ x∈S. domain x = a ∧ b = (λi. ball (x i) (e x))}
       using e-pos by (auto intro!: exI[where x=Pi' (domain x) (λi. ball (x i) (e
x))])
   next
     fix x y
     assume y ∈ S
     moreover
     assume x ∈ (Π' i∈domain y. ball (y i) (e y))
     hence dist x y < e y using e-pos ‹y ∈ S›
       by (auto simp: dist-fmap-def Pi'-iff finite-proj-diag dist-commute)
     ultimately show x ∈ S by (rule e-in)
   qed
   also have open ...
     unfolding open-fmap-def
     by (intro generate-topology.UN) (auto intro: generate-topology.Basis)
   finally show open S .
  qed
  show open S = (∀ x∈S. ∀ F (x', y) in uniformity. x' = x ⟶ y ∈ S)
   unfolding ∗ eventually-uniformity-metric
   by (simp del: split-paired-All add: dist-fmap-def dist-commute eq-commute)
next
  fix P Q::'a ⇒F 'b
  have Max-eq-iff: ⋀A m. finite A ⟹ A ≠ {} ⟹ (Max A = m) = (m ∈ A ∧
(∀ a∈A. a ≤ m))
   by (auto intro: Max-in Max-eqI)
  show dist P Q = 0 ⟷ P = Q
   by (auto simp: finmap-eq-iff dist-fmap-def Max-ge-iff finite-proj-diag Max-eq-iff
       add-nonneg-eq-0-iff
     intro!: Max-eqI image-eqI[where x=undefined])
next
  fix P Q R::'a ⇒F 'b
  let ?dists = λP Q i. dist ((P)F i) ((Q)F i)
  let ?dpq = ?dists P Q and ?dpr = ?dists P R and ?dqr = ?dists Q R
```

let *?dom* = $\lambda P\ Q$. (*if domain P = domain Q then 0 else 1*::*real*)
have *dist P Q* = *Max* (*range ?dpq*) + *?dom P Q*
by (*simp add*: *dist-fmap-def*)
also obtain *t* **where** *t* ∈ *range ?dpq t* = *Max* (*range ?dpq*) **by** (*simp add*: *finite-proj-diag*)
then obtain *i* **where** *Max* (*range ?dpq*) = *?dpq i* **by** *auto*
also have *?dpq i* ≤ *?dpr i* + *?dqr i* **by** (*rule dist-triangle2*)
also have *?dpr i* ≤ *Max* (*range ?dpr*) **by** (*simp add*: *finite-proj-diag*)
also have *?dqr i* ≤ *Max* (*range ?dqr*) **by** (*simp add*: *finite-proj-diag*)
also have *?dom P Q* ≤ *?dom P R* + *?dom Q R* **by** *simp*
finally show *dist P Q* ≤ *dist P R* + *dist Q R* **by** (*simp add*: *dist-fmap-def ac-simps*)
qed

end

22.6 Complete Space of Finite Maps

lemma *tendsto-finmap*:
fixes *f*::*nat* ⇒ ('*i* ⇒$_F$ ('*a*::*metric-space*))
assumes *ind-f*: ⋀*n*. *domain* (*f n*) = *domain g*
assumes *proj-g*: ⋀*i*. *i* ∈ *domain g* ⟹ (λ*n*. (*f n*) *i*) ⟶ *g i*
shows *f* ⟶ *g*
unfolding *tendsto-iff*
proof *safe*
fix *e*::*real* **assume** *0* < *e*
let *?dists* = λ*x i*. *dist* ((*f x*)$_F$ *i*) ((*g*)$_F$ *i*)
have *eventually* (λ*x*. ∀ *i*∈*domain g*. *?dists x i* < *e*) *sequentially*
using *finite-domain*[*of g*] *proj-g*
proof *induct*
case (*insert i G*)
with ‹*0* < *e*› **have** *eventually* (λ*x*. *?dists x i* < *e*) *sequentially* **by** (*auto simp add*: *tendsto-iff*)
moreover
from *insert* **have** *eventually* (λ*x*. ∀ *i*∈*G*. *dist* ((*f x*)$_F$ *i*) ((*g*)$_F$ *i*) < *e*) *sequentially* **by** *simp*
ultimately show *?case* **by** *eventually-elim auto*
qed *simp*
thus *eventually* (λ*x*. *dist* (*f x*) *g* < *e*) *sequentially*
by *eventually-elim* (*auto simp add*: *dist-fmap-def finite-proj-diag ind-f* ‹*0* < *e*›)
qed

instance *fmap* :: (*type*, *complete-space*) *complete-space*
proof
fix *P*::*nat* ⇒ '*a* ⇒$_F$ '*b*
assume *Cauchy P*
then obtain *Nd* **where** *Nd*: ⋀*n*. *n* ≥ *Nd* ⟹ *dist* (*P n*) (*P Nd*) < *1*
by (*force simp*: *Cauchy-altdef2*)
define *d* **where** *d* = *domain* (*P Nd*)

with *Nd* **have** *dim*: $\bigwedge n.\ n \geq Nd \Longrightarrow$ *domain* $(P\ n) = d$ **using** *dist-le-1-imp-domain-eq*
by *auto*
have [*simp*]: *finite d* **unfolding** *d-def* **by** *simp*
define *p* **where** $p\ i\ n = P\ n\ i$ **for** $i\ n$
define *q* **where** $q\ i =$ *lim* $(p\ i)$ **for** *i*
define *Q* **where** $Q =$ *finmap-of d q*
have *q*: $\bigwedge i.\ i \in d \Longrightarrow q\ i = Q\ i$ **by** (*auto simp add*: *Q-def Abs-fmap-inverse*)
{
fix *i* **assume** $i \in d$
have *Cauchy* $(p\ i)$ **unfolding** *Cauchy-altdef2 p-def*
proof *safe*
fix *e*::*real* **assume** $0 < e$
with ‹*Cauchy P*› **obtain** *N* **where** *N*: $\bigwedge n.\ n \geq N \Longrightarrow$ *dist* $(P\ n)\ (P\ N) <$ *min e 1*
by (*force simp*: *Cauchy-altdef2 min-def*)
hence $\bigwedge n.\ n \geq N \Longrightarrow$ *domain* $(P\ n) =$ *domain* $(P\ N)$ **using** *dist-le-1-imp-domain-eq*
by *auto*
with *dim* **have** *dim*: $\bigwedge n.\ n \geq N \Longrightarrow$ *domain* $(P\ n) = d$ **by** (*metis nat-le-linear*)
show $\exists N.\ \forall n{\geq}N.$ *dist* $((P\ n)\ i)\ ((P\ N)\ i) < e$
proof (*safe intro*!: *exI*[**where** $x{=}N$])
fix *n* **assume** $N \leq n$ **have** $N \leq N$ **by** *simp*
have *dist* $((P\ n)\ i)\ ((P\ N)\ i) \leq$ *dist* $(P\ n)\ (P\ N)$
using *dim*[*OF* ‹$N \leq n$›] *dim*[*OF* ‹$N \leq N$›] ‹$i \in d$›
by (*auto intro*!: *dist-proj*)
also have $\ldots < e$ **using** *N*[*OF* ‹$N \leq n$›] **by** *simp*
finally show *dist* $((P\ n)\ i)\ ((P\ N)\ i) < e$ **.**
qed
qed
hence *convergent* $(p\ i)$ **by** (*metis Cauchy-convergent-iff*)
hence $p\ i \longrightarrow q\ i$ **unfolding** *q-def convergent-def* **by** (*metis limI*)
} **note** $p =$ *this*
have $P \longrightarrow Q$
proof (*rule metric-LIMSEQ-I*)
fix *e*::*real* **assume** $0 < e$
have $\exists ni.\ \forall i{\in}d.\ \forall n{\geq}ni\ i.$ *dist* $(p\ i\ n)\ (q\ i) < e$
proof (*safe intro*!: *bchoice*)
fix *i* **assume** $i \in d$
from *p*[*OF* ‹$i \in d$›, *THEN metric-LIMSEQ-D*, *OF* ‹$0 < e$›]
show $\exists no.\ \forall n{\geq}no.$ *dist* $(p\ i\ n)\ (q\ i) < e$ **.**
qed
then obtain *ni* **where** *ni*: $\forall i{\in}d.\ \forall n{\geq}ni\ i.$ *dist* $(p\ i\ n)\ (q\ i) < e$ **..**
define *N* **where** $N =$ *max Nd* (*Max* (*ni* ‘ *d*))
show $\exists N.\ \forall n{\geq}N.$ *dist* $(P\ n)\ Q < e$
proof (*safe intro*!: *exI*[**where** $x{=}N$])
fix *n* **assume** $N \leq n$
hence *dom*: *domain* $(P\ n) = d$ *domain* $Q = d$ *domain* $(P\ n) =$ *domain* Q
using *dim* **by** (*simp-all add*: *N-def Q-def dim-def Abs-fmap-inverse*)
show *dist* $(P\ n)\ Q < e$
proof (*rule dist-finmap-lessI*[*OF dom*(*3*) ‹$0 < e$›])

fix *i*
assume *i* ∈ *domain* (*P n*)
hence *ni i* ≤ *Max* (*ni* ‘ *d*) **using** *dom* **by** *simp*
also have ... ≤ *N* **by** (*simp add*: *N-def*)
finally show *dist* ((*P n*)$_F$ *i*) ((*Q*)$_F$ *i*) < *e* **using** *ni* ‹*i* ∈ *domain* (*P n*)› ‹*N* ≤ *n*› *dom*
by (*auto simp*: *p-def q N-def less-imp-le*)
qed
qed
qed
thus *convergent P* **by** (*auto simp*: *convergent-def*)
qed

22.7 Second Countable Space of Finite Maps

instantiation *fmap* :: (*countable*, *second-countable-topology*) *second-countable-topology*
begin

definition *basis-proj*::*'b set set*
where *basis-proj* = (*SOME B*. *countable B* ∧ *topological-basis B*)

lemma *countable-basis-proj*: *countable basis-proj* **and** *basis-proj*: *topological-basis basis-proj*
unfolding *basis-proj-def* **by** (*intro is-basis countable-basis*)+

definition *basis-finmap*::(*'a* ⇒$_F$ *'b*) *set set*
where *basis-finmap* = {*Pi' I S*|*I S*. *finite I* ∧ (∀ *i* ∈ *I*. *S i* ∈ *basis-proj*)}

lemma *in-basis-finmapI*:
assumes *finite I* **assumes** ⋀*i*. *i* ∈ *I* ⟹ *S i* ∈ *basis-proj*
shows *Pi' I S* ∈ *basis-finmap*
using *assms* **unfolding** *basis-finmap-def* **by** *auto*

lemma *basis-finmap-eq*:
assumes *basis-proj* ≠ {}
shows *basis-finmap* = (λ*f*. *Pi'* (*domain f*) (λ*i*. *from-nat-into basis-proj* ((*f*)$_F$ *i*))) ‘
(*UNIV*::(*'a* ⇒$_F$ *nat*) *set*) (**is** - = *?f* ‘ -)
unfolding *basis-finmap-def*
proof *safe*
fix *I*::*'a set* **and** *S*::*'a* ⇒ *'b set*
assume *finite I* ∀ *i*∈*I*. *S i* ∈ *basis-proj*
hence *Pi' I S* = *?f* (*finmap-of I* (λ*x*. *to-nat-on basis-proj* (*S x*)))
by (*force simp*: *Pi'-def countable-basis-proj*)
thus *Pi' I S* ∈ *range ?f* **by** *simp*
next
fix *x* **and** *f*::*'a* ⇒$_F$ *nat*
show ∃ *I S*. (Π' *i*∈*domain f*. *from-nat-into basis-proj* ((*f*)$_F$ *i*)) = *Pi' I S* ∧
finite I ∧ (∀ *i*∈*I*. *S i* ∈ *basis-proj*)

using *assms* **by** (*auto intro*: *from-nat-into*)
qed

lemma *basis-finmap-eq-empty*: *basis-proj* = {} ⟹ *basis-finmap* = {*Pi'* {} *undefined*}
by (*auto simp*: *Pi'-iff basis-finmap-def*)

lemma *countable-basis-finmap*: *countable basis-finmap*
by (*cases basis-proj* = {}) (*auto simp*: *basis-finmap-eq basis-finmap-eq-empty*)

lemma *finmap-topological-basis*:
topological-basis basis-finmap
proof (*subst topological-basis-iff*, *safe*)
fix *B'* **assume** *B'* ∈ *basis-finmap*
thus *open B'*
by (*auto intro*!: *open-Pi'I topological-basis-open*[*OF basis-proj*]
simp: *topological-basis-def basis-finmap-def Let-def*)
next
fix *O'*::('*a* ⇒$_F$ '*b*) *set* **and** *x*
assume *O'*: *open O' x* ∈ *O'*
then obtain *a* **where** *a*:
x ∈ *Pi'* (*domain x*) *a Pi'* (*domain x*) *a* ⊆ *O'* ⋀*i*. *i*∈*domain x* ⟹ *open* (*a i*)
unfolding *open-fmap-def*
proof (*atomize-elim*, *induct rule*: *generate-topology.induct*)
case (*Int a b*)
let *?p*=λ*a f*. *x* ∈ *Pi'* (*domain x*) *f* ∧ *Pi'* (*domain x*) *f* ⊆ *a* ∧ (∀ *i*. *i* ∈ *domain x* ⟶ *open* (*f i*))
from *Int* **obtain** *f g* **where** *?p a f ?p b g* **by** *auto*
thus *?case* **by** (*force intro*!: *exI*[**where** *x*=λ*i*. *f i* ∩ *g i*] *simp*: *Pi'-def*)
next
case (*UN k*)
then obtain *kk a* **where** *x* ∈ *kk kk* ∈ *k x* ∈ *Pi'* (*domain x*) *a Pi'* (*domain x*) *a* ⊆ *kk*
⋀*i*. *i*∈*domain x* ⟹ *open* (*a i*)
by *force*
thus *?case* **by** *blast*
qed (*auto simp*: *Pi'-def*)
have ∃ *B*.
(∀ *i*∈*domain x*. *x i* ∈ *B i* ∧ *B i* ⊆ *a i* ∧ *B i* ∈ *basis-proj*)
proof (*rule bchoice*, *safe*)
fix *i* **assume** *i* ∈ *domain x*
hence *open* (*a i*) *x i* ∈ *a i* **using** *a* **by** *auto*
from *topological-basisE*[*OF basis-proj this*] **obtain** *b'*
where *b'* ∈ *basis-proj* (*x*)$_F$ *i* ∈ *b' b'* ⊆ *a i*
by *blast*
thus ∃ *y*. *x i* ∈ *y* ∧ *y* ⊆ *a i* ∧ *y* ∈ *basis-proj* **by** *auto*
qed
then obtain *B* **where** *B*: ∀ *i*∈*domain x*. (*x*)$_F$ *i* ∈ *B i* ∧ *B i* ⊆ *a i* ∧ *B i* ∈ *basis-proj*

by *auto*
define B' **where** $B' = Pi'\ (domain\ x)\ (\lambda i.\ (B\ i)::'b\ set)$
have $B' \subseteq Pi'\ (domain\ x)\ a$ **using** B **by** (*auto intro*!: *Pi'-mono simp*: *B'-def*)
also note ‹... ⊆ O'›
finally show $\exists B' \in basis\text{-}finmap.\ x \in B' \wedge B' \subseteq O'$ **using** B
by (*auto intro*!: *bexI*[**where** $x=B'$] *Pi'-mono in-basis-finmapI simp*: *B'-def*)
qed

lemma *range-enum-basis-finmap-imp-open*:
assumes $x \in basis\text{-}finmap$
shows *open x*
using *finmap-topological-basis assms* **by** (*auto simp*: *topological-basis-def*)

instance proof qed (*blast intro*: *finmap-topological-basis countable-basis-finmap topological-basis-imp-subbasis*)

end

22.8 Polish Space of Finite Maps

instance *fmap* :: (*countable*, *polish-space*) *polish-space* **proof qed**

22.9 Product Measurable Space of Finite Maps

definition $PiF\ I\ M \equiv$
$sigma\ (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))\ \{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$

abbreviation
$Pi_F\ I\ M \equiv PiF\ I\ M$

syntax
$\text{-}PiF :: pttrn \Rightarrow 'i\ set \Rightarrow 'a\ measure \Rightarrow ('i => 'a)\ measure$
(‹(‹*indent=3 notation=‹binder* Π_F››Π_F -∈-./ -)› *10*)
syntax-consts
$\text{-}PiF == PiF$
translations
$\Pi_F\ x{\in}I.\ M == CONST\ PiF\ I\ (\%x.\ M)$

lemma *PiF-gen-subset*: $\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
⊆
$Pow\ (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
by (*auto simp*: *Pi'-def*) (*blast dest*: *sets.sets-into-space*)

lemma *space-PiF*: $space\ (PiF\ I\ M) = (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
unfolding *PiF-def* **using** *PiF-gen-subset* **by** (*rule space-measure-of*)

lemma *sets-PiF*:
$sets\ (PiF\ I\ M) = sigma\text{-}sets\ (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
$\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$

unfolding *PiF-def* **using** *PiF-gen-subset* **by** (*rule sets-measure-of*)

lemma *sets-PiF-singleton*:
sets (*PiF* {*I*} *M*) = *sigma-sets* (Π′ *j*∈*I*. *space* (*M j*))
{(Π′ *j*∈*I*. *X j*) |*X*. *X* ∈ (Π *j*∈*I*. *sets* (*M j*))}
unfolding *sets-PiF* **by** *simp*

lemma *in-sets-PiFI*:
assumes *X* = (*Pi*′ *J S*) *J* ∈ *I* ⋀*i*. *i*∈*J* ⟹ *S i* ∈ *sets* (*M i*)
shows *X* ∈ *sets* (*PiF I M*)
unfolding *sets-PiF*
using *assms* **by** *blast*

lemma *product-in-sets-PiFI*:
assumes *J* ∈ *I* ⋀*i*. *i*∈*J* ⟹ *S i* ∈ *sets* (*M i*)
shows (*Pi*′ *J S*) ∈ *sets* (*PiF I M*)
unfolding *sets-PiF*
using *assms* **by** *blast*

lemma *singleton-space-subset-in-sets*:
fixes *J*
assumes *J* ∈ *I*
assumes *finite J*
shows *space* (*PiF* {*J*} *M*) ∈ *sets* (*PiF I M*)
using *assms*
by (*intro in-sets-PiFI*[**where** *J*=*J* **and** *S*=λ*i*. *space* (*M i*)])
(*auto simp*: *product-def space-PiF*)

lemma *singleton-subspace-set-in-sets*:
assumes *A*: *A* ∈ *sets* (*PiF* {*J*} *M*)
assumes *finite J*
assumes *J* ∈ *I*
shows *A* ∈ *sets* (*PiF I M*)
using *A*[*unfolded sets-PiF*]
apply (*induct A*)
unfolding *sets-PiF*[*symmetric*] **unfolding** *space-PiF*[*symmetric*]
using *assms*
by (*auto intro*: *in-sets-PiFI intro*!: *singleton-space-subset-in-sets*)

lemma *finite-measurable-singletonI*:
assumes *finite I*
assumes ⋀*J*. *J* ∈ *I* ⟹ *finite J*
assumes *MN*: ⋀*J*. *J* ∈ *I* ⟹ *A* ∈ *measurable* (*PiF* {*J*} *M*) *N*
shows *A* ∈ *measurable* (*PiF I M*) *N*
unfolding *measurable-def*
proof *safe*
fix *y* **assume** *y* ∈ *sets N*
have *A* −‘ *y* ∩ *space* (*PiF I M*) = (⋃ *J*∈*I*. *A* −‘ *y* ∩ *space* (*PiF* {*J*} *M*))
by (*auto simp*: *space-PiF*)

```
  also have ... ∈ sets (PiF I M)
  proof (rule sets.finite-UN)
    show finite I by fact
    fix J assume J ∈ I
    with assms have finite J by simp
    show A -` y ∩ space (PiF {J} M) ∈ sets (PiF I M)
      by (rule singleton-subspace-set-in-sets[OF measurable-sets[OF assms(3)]])
fact+
  qed
  finally show A -` y ∩ space (PiF I M) ∈ sets (PiF I M) .
next
  fix x assume x ∈ space (PiF I M) thus A x ∈ space N
    using MN[of domain x]
    by (auto simp: space-PiF measurable-space Pi'-def)
qed

lemma countable-finite-comprehension:
  fixes f :: 'a::countable set ⇒ -
  assumes ⋀s. P s ⟹ finite s
  assumes ⋀s. P s ⟹ f s ∈ sets M
  shows ⋃{f s|s. P s} ∈ sets M
proof −
  have ⋃{f s|s. P s} = (⋃n::nat. let s = set (from-nat n) in if P s then f s else
{})
  proof safe
    fix x X s assume ∗: x ∈ f s P s
    with assms obtain l where s = set l using finite-list by blast
   with ∗ show x ∈ (⋃n. let s = set (from-nat n) in if P s then f s else {}) using
‹P s›
      by (auto intro!: exI[where x=to-nat l])
  next
    fix x n assume x ∈ (let s = set (from-nat n) in if P s then f s else {})
    thus x ∈ ⋃{f s|s. P s} using assms by (auto simp: Let-def split: if-split-asm)
  qed
  hence ⋃{f s|s. P s} = (⋃n. let s = set (from-nat n) in if P s then f s else {})
by simp
  also have ... ∈ sets M using assms by (auto simp: Let-def)
  finally show ?thesis .
qed

lemma space-subset-in-sets:
  fixes J::'a::countable set set
  assumes J ⊆ I
  assumes ⋀j. j ∈ J ⟹ finite j
  shows space (PiF J M) ∈ sets (PiF I M)
proof −
  have space (PiF J M) = ⋃{space (PiF {j} M)|j. j ∈ J}
    unfolding space-PiF by blast
  also have ... ∈ sets (PiF I M) using assms
```

by (*intro countable-finite-comprehension*) (*auto simp*: *singleton-space-subset-in-sets*)
finally show *?thesis* **.**
qed

lemma *subspace-set-in-sets*:
fixes J::$'a$::*countable set set*
assumes A: $A \in sets\ (PiF\ J\ M)$
assumes $J \subseteq I$
assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
shows $A \in sets\ (PiF\ I\ M)$
using A[*unfolded sets-PiF*]
apply (*induct A*)
unfolding *sets-PiF*[*symmetric*] **unfolding** *space-PiF*[*symmetric*]
using *assms*
by (*auto intro*: *in-sets-PiFI intro*!: *space-subset-in-sets*)

lemma *countable-measurable-PiFI*:
fixes I::$'a$::*countable set set*
assumes *MN*: $\bigwedge J.\ J \in I \Longrightarrow finite\ J \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
shows $A \in measurable\ (PiF\ I\ M)\ N$
unfolding *measurable-def*
proof *safe*
fix y **assume** $y \in sets\ N$
have $A -` y = (\bigcup \{A -` y \cap \{x.\ domain\ x = J\} | J.\ finite\ J\})$ **by** *auto*
{ fix x::$'a \Rightarrow_F\ 'b$
from *finite-list*[*of domain x*] **obtain** *xs* **where** *set xs* = *domain x* **by** *auto*
hence $\exists n.\ domain\ x = set\ (from\text{-}nat\ n)$
by (*intro exI*[**where** *x*=*to-nat xs*]) *auto* **}**
hence $A -` y \cap space\ (PiF\ I\ M) = (\bigcup n.\ A -` y \cap space\ (PiF\ (\{set\ (from\text{-}nat\ n)\} \cap I)\ M))$
by (*auto simp*: *space-PiF Pi'-def*)
also have $\ldots \in sets\ (PiF\ I\ M)$
apply (*intro sets.Int sets.countable-nat-UN subsetI*, *safe*)
apply (*case-tac set* (*from-nat i*) $\in I$)
apply *simp-all*
apply (*rule singleton-subspace-set-in-sets*[*OF measurable-sets*[*OF MN*]])
using *assms* ‹$y \in sets\ N$›
apply (*auto simp*: *space-PiF*)
done
finally show $A -` y \cap space\ (PiF\ I\ M) \in sets\ (PiF\ I\ M)$ **.**
next
fix x **assume** $x \in space\ (PiF\ I\ M)$ **thus** $A\ x \in space\ N$
using *MN*[*of domain x*] **by** (*auto simp*: *space-PiF measurable-space Pi'-def*)
qed

lemma *measurable-PiF*:
assumes f: $\bigwedge x.\ x \in space\ N \Longrightarrow domain\ (f\ x) \in I \wedge (\forall i \in domain\ (f\ x).\ (f\ x)\ i \in space\ (M\ i))$
assumes S: $\bigwedge J\ S.\ J \in I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow S\ i \in sets\ (M\ i)) \Longrightarrow$

$f -` (Pi' J S) \cap space N \in sets N$
shows $f \in measurable N (PiF I M)$
unfolding *PiF-def*
using *PiF-gen-subset*
apply (*rule measurable-measure-of*)
using *f* **apply** *force*
apply (*insert S*, *auto*)
done

lemma *restrict-sets-measurable*:
assumes *A*: $A \in sets (PiF I M)$ **and** $J \subseteq I$
shows $A \cap \{m.\ domain\ m \in J\} \in sets (PiF J M)$
using *A*[*unfolded sets-PiF*]
proof (*induct A*)
case (*Basic a*)
then obtain *K S* **where** *S*: $a = Pi' K S K \in I\ (\forall i{\in}K.\ S\ i \in sets\ (M\ i))$
by *auto*
show *?case*
proof *cases*
assume $K \in J$
hence $a \cap \{m.\ domain\ m \in J\} \in \{Pi'\ K\ X\ |X\ K.\ K \in J \wedge X \in (\Pi\ j{\in}K.\ sets\ (M\ j))\}$ **using** *S*
by (*auto intro*!: *exI*[**where** *x*=*K*] *exI*[**where** *x*=*S*] *simp*: *Pi'-def*)
also have ... $\subseteq$ *sets* (*PiF J M*) **unfolding** *sets-PiF* **by** *auto*
finally show *?thesis* **.**
next
assume $K \notin J$
hence $a \cap \{m.\ domain\ m \in J\} = \{\}$ **using** *S* **by** (*auto simp*: *Pi'-def*)
also have ... $\in$ *sets* (*PiF J M*) **by** *simp*
finally show *?thesis* **.**
qed
next
case (*Union a*)
have $\bigcup (a\ `\ UNIV) \cap \{m.\ domain\ m \in J\} = (\bigcup i.\ (a\ i \cap \{m.\ domain\ m \in J\}))$
by *simp*
also have ... $\in$ *sets* (*PiF J M*) **using** *Union* **by** (*intro sets.countable-nat-UN*) *auto*
finally show *?case* **.**
next
case (*Compl a*)
have (*space* (*PiF I M*) − *a*) $\cap \{m.\ domain\ m \in J\} = (space\ (PiF\ J\ M) - (a \cap \{m.\ domain\ m \in J\}))$
using ‹$J \subseteq I$› **by** (*auto simp*: *space-PiF Pi'-def*)
also have ... $\in$ *sets* (*PiF J M*) **using** *Compl* **by** *auto*
finally show *?case* **by** (*simp add*: *space-PiF*)
qed *simp*

lemma *measurable-finmap-of*:
assumes *f*: $\bigwedge i.\ (\exists x \in space\ N.\ i \in J\ x) \Longrightarrow (\lambda x.\ f\ x\ i) \in measurable\ N\ (M\ i)$

assumes J: $\bigwedge x.\ x \in space\ N \Longrightarrow J\ x \in I\ \bigwedge x.\ x \in space\ N \Longrightarrow finite\ (J\ x)$
assumes JN: $\bigwedge S.\ \{x.\ J\ x = S\} \cap space\ N \in sets\ N$
shows $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) \in measurable\ N\ (PiF\ I\ M)$
proof (*rule measurable-PiF*)
fix x **assume** $x \in space\ N$
with J[*of* x] *measurable-space*[*OF* f]
show *domain* (*finmap-of* $(J\ x)\ (f\ x)) \in I \wedge$
$(\forall i \in domain\ (finmap\text{-}of\ (J\ x)\ (f\ x)).\ (finmap\text{-}of\ (J\ x)\ (f\ x))\ i \in space\ (M\ i))$
by *auto*
next
fix $K\ S$ **assume** $K \in I$ **and** $*$: $\bigwedge i.\ i \in K \Longrightarrow S\ i \in sets\ (M\ i)$
with J **have** *eq*: $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) -` Pi'\ K\ S \cap space\ N =$
$(if\ \exists x \in space\ N.\ K = J\ x \wedge finite\ K\ then\ if\ K = \{\}\ then\ \{x \in space\ N.\ J\ x = K\}$
$else\ (\bigcap i \in K.\ (\lambda x.\ f\ x\ i) -` S\ i \cap \{x \in space\ N.\ J\ x = K\})\ else\ \{\})$
by (*auto simp*: *Pi'-def*)
have r: $\{x \in space\ N.\ J\ x = K\} = space\ N \cap (\{x.\ J\ x = K\} \cap space\ N)$ **by** *auto*
show $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) -` Pi'\ K\ S \cap space\ N \in sets\ N$
unfolding *eq r*
apply (*simp del*: *INT-simps add*:)
apply (*intro conjI impI sets.finite-INT JN sets.Int*[*OF sets.top*])
apply *simp* **apply** *assumption*
apply (*subst Int-assoc*[*symmetric*])
apply (*rule sets.Int*)
apply (*intro measurable-sets*[*OF* f] $*$) **apply** *force* **apply** *assumption*
apply (*intro JN*)
done
qed

lemma *measurable-PiM-finmap-of*:
assumes *finite* J
shows *finmap-of* $J \in measurable\ (Pi_M\ J\ M)\ (PiF\ \{J\}\ M)$
apply (*rule measurable-finmap-of*)
apply (*rule measurable-component-singleton*)
apply *simp*
apply *rule*
apply (*rule* ‹*finite* J›)
apply *simp*
done

lemma *proj-measurable-singleton*:
assumes $A \in sets\ (M\ i)$
shows $(\lambda x.\ (x)_F\ i) -` A \cap space\ (PiF\ \{I\}\ M) \in sets\ (PiF\ \{I\}\ M)$
proof *cases*
assume $i \in I$
hence $(\lambda x.\ (x)_F\ i) -` A \cap space\ (PiF\ \{I\}\ M) =$
$Pi'\ I\ (\lambda x.\ if\ x = i\ then\ A\ else\ space\ (M\ x))$

```
    using sets.sets-into-space[OF ] ‹A ∈ sets (M i)› assms
    by (auto simp: space-PiF Pi'-def)
  thus ?thesis  using assms ‹A ∈ sets (M i)›
    by (intro in-sets-PiFI) auto
next
  assume i ∉ I
  hence (λx. (x)_F i) -` A ∩ space (PiF {I} M) =
    (if undefined ∈ A then space (PiF {I} M) else {}) by (auto simp: space-PiF
Pi'-def)
  thus ?thesis by simp
qed

lemma measurable-proj-singleton:
  assumes i ∈ I
  shows (λx. (x)_F i) ∈ measurable (PiF {I} M) (M i)
   by (unfold measurable-def, intro CollectI conjI ballI proj-measurable-singleton
assms)
     (insert ‹i ∈ I›, auto simp: space-PiF)

lemma measurable-proj-countable:
  fixes I::'a::countable set set
  assumes y ∈ space (M i)
  shows (λx. if i ∈ domain x then (x)_F i else y) ∈ measurable (PiF I M) (M i)
proof (rule countable-measurable-PiFI)
  fix J assume J ∈ I finite J
  show (λx. if i ∈ domain x then x i else y) ∈ measurable (PiF {J} M) (M i)
    unfolding measurable-def
  proof safe
    fix z assume z ∈ sets (M i)
    have (λx. if i ∈ domain x then x i else y) -` z ∩ space (PiF {J} M) =
      (λx. if i ∈ J then (x)_F i else y) -` z ∩ space (PiF {J} M)
      by (auto simp: space-PiF Pi'-def)
    also have ... ∈ sets (PiF {J} M) using ‹z ∈ sets (M i)› ‹finite J›
      by (cases i ∈ J) (auto intro!: measurable-sets[OF measurable-proj-singleton])
    finally show (λx. if i ∈ domain x then x i else y) -` z ∩ space (PiF {J} M) ∈
      sets (PiF {J} M) .
  qed (insert ‹y ∈ space (M i)›, auto simp: space-PiF Pi'-def)
qed

lemma measurable-restrict-proj:
  assumes J ∈ II finite J
  shows finmap-of J ∈ measurable (PiM J M) (PiF II M)
  using assms
  by (intro measurable-finmap-of measurable-component-singleton) auto

lemma measurable-proj-PiM:
  fixes J K ::'a::countable set and I::'a set set
  assumes finite J J ∈ I
  assumes x ∈ space (PiM J M)
```

shows *proj* ∈ *measurable* (*PiF* {*J*} *M*) (*PiM J M*)
proof (*rule measurable-PiM-single*)
show *proj* ∈ *space* (*PiF* {*J*} *M*) → (Π$_E$ *i* ∈ *J*. *space* (*M i*))
using *assms* **by** (*auto simp add*: *space-PiM space-PiF extensional-def sets-PiF Pi'-def*)
next
fix *A i* **assume** *A*: *i* ∈ *J A* ∈ *sets* (*M i*)
show {ω ∈ *space* (*PiF* {*J*} *M*). (ω)$_F$ *i* ∈ *A*} ∈ *sets* (*PiF* {*J*} *M*)
proof
have {ω ∈ *space* (*PiF* {*J*} *M*). (ω)$_F$ *i* ∈ *A*} =
(λω. (ω)$_F$ *i*) −‘ *A* ∩ *space* (*PiF* {*J*} *M*) **by** *auto*
also have ... ∈ *sets* (*PiF* {*J*} *M*)
using *assms A* **by** (*auto intro*: *measurable-sets*[*OF measurable-proj-singleton*] *simp*: *space-PiM*)
finally show *?thesis* **.**
qed *simp*
qed

lemma *space-PiF-singleton-eq-product*:
assumes *finite I*
shows *space* (*PiF* {*I*} *M*) = (Π′ *i*∈*I*. *space* (*M i*))
by (*auto simp*: *product-def space-PiF assms*)

adapted from *sets* (*Pi$_M$?I ?M*) = *sigma-sets* (Π$_E$ *i*∈*?I*. *space* (*?M i*)) {{*f* ∈ Π$_E$ *i*∈*?I*. *space* (*?M i*). *f i* ∈ *A*} |*i A*. *i* ∈ *?I* ∧ *A* ∈ *sets* (*?M i*)}

lemma *sets-PiF-single*:
assumes *finite I I* ≠ {}
shows *sets* (*PiF* {*I*} *M*) =
sigma-sets (Π′ *i*∈*I*. *space* (*M i*))
{{*f*∈Π′ *i*∈*I*. *space* (*M i*). *f i* ∈ *A*} | *i A*. *i* ∈ *I* ∧ *A* ∈ *sets* (*M i*)}
(**is** - = *sigma-sets ?*Ω *?R*)
unfolding *sets-PiF-singleton*
proof (*rule sigma-sets-eqI*)
interpret *R*: *sigma-algebra ?*Ω *sigma-sets ?*Ω *?R* **by** (*rule sigma-algebra-sigma-sets*) *auto*
fix *A* **assume** *A* ∈ {*Pi′ I X* |*X*. *X* ∈ (Π *j*∈*I*. *sets* (*M j*))}
then obtain *X* **where** *X*: *A* = *Pi′ I X X* ∈ (Π *j*∈*I*. *sets* (*M j*)) **by** *auto*
show *A* ∈ *sigma-sets ?*Ω *?R*
proof −
from ‹*I* ≠ {}› *X* **have** *A* = (⋂*j*∈*I*. {*f*∈*space* (*PiF* {*I*} *M*). *f j* ∈ *X j*})
using *sets.sets-into-space*
by (*auto simp*: *space-PiF product-def*) *blast*
also have ... ∈ *sigma-sets ?*Ω *?R*
using *X* ‹*I* ≠ {}› *assms* **by** (*intro R.finite-INT*) (*auto simp*: *space-PiF*)
finally show *A* ∈ *sigma-sets ?*Ω *?R* **.**
qed
next
fix *A* **assume** *A* ∈ *?R*
then obtain *i B* **where** *A*: *A* = {*f*∈Π′ *i*∈*I*. *space* (*M i*). *f i* ∈ *B*} *i* ∈ *I B* ∈

sets (*M i*)
by *auto*
then have $A = (\Pi' j \in I.\ \textit{if } j = i \textit{ then } B \textit{ else space } (M\ j))$
using *sets.sets-into-space*[*OF A*(*3*)]
apply (*auto simp*: *Pi'-iff split*: *if-split-asm*)
apply *blast*
done
also have $\ldots \in$ *sigma-sets ?*Ω $\{Pi'\ I\ X \mid X.\ X \in (\Pi\ j{\in}I.\ sets\ (M\ j))\}$
using *A*
by (*intro sigma-sets.Basic*)
(*auto intro*: *exI*[**where** $x{=}\lambda j.$ *if* $j = i$ *then B else space* $(M\ j)$])
finally show $A \in$ *sigma-sets ?*Ω $\{Pi'\ I\ X \mid X.\ X \in (\Pi\ j{\in}I.\ sets\ (M\ j))\}$ **.**
qed

adapted from $(\bigwedge i.\ i \in ?I \Longrightarrow ?A\ i = ?B\ i) \Longrightarrow Pi_E\ ?I\ ?A = Pi_E\ ?I\ ?B$

lemma *Pi'-cong*:
assumes *finite I*
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i = g\ i$
shows $Pi'\ I\ f = Pi'\ I\ g$
using *assms* **by** (*auto simp*: *Pi'-def*)

adapted from $[\![finite\ ?I;\ \bigwedge i\ n\ m.\ [\![i \in ?I;\ n \le m]\!] \Longrightarrow ?A\ n\ i \subseteq ?A\ m\ i]\!] \Longrightarrow (\bigcup_n Pi\ ?I\ (?A\ n)) = (\Pi\ i{\in}?I.\ \bigcup_n ?A\ n\ i)$

lemma *Pi'-UN*:
fixes $A :: nat \Rightarrow 'i \Rightarrow 'a\ set$
assumes *finite I*
assumes *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \le m \Longrightarrow A\ n\ i \subseteq A\ m\ i$
shows $(\bigcup n.\ Pi'\ I\ (A\ n)) = Pi'\ I\ (\lambda i.\ \bigcup n.\ A\ n\ i)$
proof (*intro set-eqI iffI*)
fix *f* **assume** $f \in Pi'\ I\ (\lambda i.\ \bigcup n.\ A\ n\ i)$
then have $\forall\, i{\in}I.\ \exists\, n.\ f\ i \in A\ n\ i$ *domain f* $= I$ **by** (*auto simp*: ‹*finite I*› *Pi'-def*)
from *bchoice*[*OF this*(*1*)] **obtain** *n* **where** *n*: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in (A\ (n\ i)\ i)$ **by** *auto*
obtain *k* **where** *k*: $\bigwedge i.\ i \in I \Longrightarrow n\ i \le k$
using ‹*finite I*› *finite-nat-set-iff-bounded-le*[*of n'I*] **by** *auto*
have $f \in Pi'\ I\ (\lambda i.\ A\ k\ i)$
proof
fix *i* **assume** $i \in I$
from *mono*[*OF this*, *of n i k*] *k*[*OF this*] *n*[*OF this*] ‹*domain f* $= I$› ‹$i \in I$›
show $f\ i \in A\ k\ i$ **by** (*auto simp*: ‹*finite I*›)
qed (*simp add*: ‹*domain f* $= I$› ‹*finite I*›)
then show $f \in (\bigcup n.\ Pi'\ I\ (A\ n))$ **by** *auto*
qed (*auto simp*: *Pi'-def* ‹*finite I*›)

adapted from $[\![\bigwedge i.\ i \in ?I \Longrightarrow \exists\, S{\subseteq}?E\ i.\ countable\ S \wedge ?\Omega\ i = \bigcup S;\ \bigwedge i.\ i \in ?I \Longrightarrow ?E\ i \subseteq Pow\ (?\Omega\ i);\ \bigwedge j.\ j \in ?J \Longrightarrow finite\ j;\ \bigcup ?J = ?I]\!] \Longrightarrow sets\ (Pi_M\ ?I\ (\lambda i.\ sigma\ (?\Omega\ i)\ (?E\ i))) = sets\ (sigma\ (Pi_E\ ?I\ ?\Omega)\ \{\{f \in Pi_E\ ?I\ ?\Omega.\ \forall\, i{\in}j.\ f\ i \in A\ i\} \mid A\ j.\ j \in ?J \wedge A \in Pi\ j\ ?E\})$

lemma *sigma-fprod-algebra-sigma-eq*:

fixes $E :: 'i \Rightarrow 'a\ set\ set$ **and** $S :: 'i \Rightarrow nat \Rightarrow 'a\ set$
assumes [*simp*]: *finite I I* $\neq \{\}$
and *S-union*: $\bigwedge i.\ i \in I \Longrightarrow (\bigcup j.\ S\ i\ j) = space\ (M\ i)$
and *S-in-E*: $\bigwedge i.\ i \in I \Longrightarrow range\ (S\ i) \subseteq E\ i$
assumes *E-closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (space\ (M\ i))$
and *E-generates*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sigma\text{-}sets\ (space\ (M\ i))\ (E\ i)$
defines $P == \{\ Pi'\ I\ F \mid F.\ \forall i{\in}I.\ F\ i \in E\ i\ \}$
shows *sets* $(PiF\ \{I\}\ M) = sigma\text{-}sets\ (space\ (PiF\ \{I\}\ M))\ P$
proof
let *?P* $= sigma\ (space\ (Pi_F\ \{I\}\ M))\ P$
from ‹*finite I*›[*THEN ex-bij-betw-finite-nat*] **obtain** *T* **where** *bij-betw T I* $\{0..<card\ I\}$ **..**
then have *T*: $\bigwedge i.\ i \in I \Longrightarrow T\ i < card\ I\ \bigwedge i.\ i{\in}I \Longrightarrow the\text{-}inv\text{-}into\ I\ T\ (T\ i) = i$
by (*auto simp add*: *bij-betw-def set-eq-iff image-iff the-inv-into-f-f simp del*: ‹*finite I*›)
have *P-closed*: $P \subseteq Pow\ (space\ (Pi_F\ \{I\}\ M))$
using *E-closed* **by** (*auto simp*: *space-PiF P-def Pi'-iff subset-eq*)
then have *space-P*: *space ?P* $= (\Pi'\ i{\in}I.\ space\ (M\ i))$
by (*simp add*: *space-PiF*)
have *sets* $(PiF\ \{I\}\ M) =$
$sigma\text{-}sets\ (space\ ?P)\ \{\{f \in \Pi'\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
using *sets-PiF-single*[*of I M*] **by** (*simp add*: *space-P*)
also have $\ldots \subseteq$ *sets* $(sigma\ (space\ (PiF\ \{I\}\ M))\ P)$
proof (*safe intro*!: *sets.sigma-sets-subset*)
fix *i A* **assume** $i \in I$ **and** *A*: $A \in sets\ (M\ i)$
have $(\lambda x.\ (x)_F\ i) \in measurable\ ?P\ (sigma\ (space\ (M\ i))\ (E\ i))$
proof (*subst measurable-iff-measure-of*)
show $E\ i \subseteq Pow\ (space\ (M\ i))$ **using** ‹$i \in I$› **by** *fact*
from *space-P* ‹$i \in I$› **show** $(\lambda x.\ (x)_F\ i) \in space\ ?P \to space\ (M\ i)$
by *auto*
show $\forall A{\in}E\ i.\ (\lambda x.\ (x)_F\ i) -` A \cap space\ ?P \in sets\ ?P$
proof
fix *A* **assume** *A*: $A \in E\ i$
from *T* **have** $*$: $\exists xs.\ length\ xs = card\ I \wedge (\forall j{\in}I.\ (g)_F\ j \in (if\ i = j\ then\ A\ else\ S\ j\ (xs\ !\ T\ j)))$
if *domain g* $= I$ **and** $\forall j{\in}I.\ (g)_F\ j \in (if\ i = j\ then\ A\ else\ S\ j\ (f\ j))$ **for** *g f*
using *that* **by** (*auto intro*!: *exI* [*of - map* $(\lambda n.\ f\ (the\text{-}inv\text{-}into\ I\ T\ n))$ $[0..<card\ I]$])
from *A* **have** $(\lambda x.\ (x)_F\ i) -` A \cap space\ ?P = (\Pi'\ j{\in}I.\ if\ i = j\ then\ A\ else\ space\ (M\ j))$
using *E-closed* ‹$i \in I$› **by** (*auto simp*: *space-P Pi-iff subset-eq split*: *if-split-asm*)
also have $\ldots = (\Pi'\ j{\in}I.\ \bigcup n.\ if\ i = j\ then\ A\ else\ S\ j\ n)$
by (*intro Pi'-cong*) (*simp-all add*: *S-union*)
also have $\ldots = \{g.\ domain\ g = I \wedge (\exists f.\ \forall j{\in}I.\ (g)_F\ j \in (if\ i = j\ then\ A\ else\ S\ j\ (f\ j)))\}$
by (*rule set-eqI*) (*simp del*: *if-image-distrib add*: *Pi'-iff bchoice-iff*)

```
      also have ... = (⋃ xs∈{xs. length xs = card I}. Π' j∈I. if i = j then A else
S j (xs ! T j))
        using ∗ by (auto simp add: Pi'-iff split del: if-split)
      also have ... ∈ sets ?P
      proof (safe intro!: sets.countable-UN)
        fix xs show (Π' j∈I. if i = j then A else S j (xs ! T j)) ∈ sets ?P
          using A S-in-E
          by (simp add: P-closed)
             (auto simp: P-def subset-eq intro!: exI[of - λj. if i = j then A else S j
(xs ! T j)])
      qed
      finally show (λx. (x)_F i) -` A ∩ space ?P ∈ sets ?P
        using P-closed by simp
    qed
  qed
  from measurable-sets[OF this, of A] A ‹i ∈ I› E-closed
  have (λx. (x)_F i) -` A ∩ space ?P ∈ sets ?P
    by (simp add: E-generates)
  also have (λx. (x)_F i) -` A ∩ space ?P = {f ∈ Π' i∈I. space (M i). f i ∈ A}
    using P-closed by (auto simp: space-PiF)
  finally show ... ∈ sets ?P .
qed
finally show sets (PiF {I} M) ⊆ sigma-sets (space (PiF {I} M)) P
  by (simp add: P-closed)
show sigma-sets (space (PiF {I} M)) P ⊆ sets (PiF {I} M)
  using ‹finite I› ‹I ≠ {}›
  by (auto intro!: sets.sigma-sets-subset product-in-sets-PiFI simp: E-generates
P-def)
qed

lemma product-open-generates-sets-PiF-single:
  assumes I ≠ {}
  assumes [simp]: finite I
  shows sets (PiF {I} (λ-. borel::'b::second-countable-topology measure)) =
    sigma-sets (space (PiF {I} (λ-. borel))) {Pi' I F |F. (∀ i∈I. F i ∈ Collect
open)}
proof -
  from open-countable-basisE[OF open-UNIV] obtain S::'b set set
    where S: S ⊆ (SOME B. countable B ∧ topological-basis B) UNIV = ⋃ S
    by auto
  show ?thesis
  proof (rule sigma-fprod-algebra-sigma-eq)
    show finite I by simp
    show I ≠ {} by fact
    define S' where S' = from-nat-into S
    show (⋃ j. S' j) = space borel
      using S
     apply (auto simp add: from-nat-into countable-basis-proj S'-def basis-proj-def)
       apply (metis (lifting, mono-tags) UNIV-I UnionE basis-proj-def count-
```

able-basis-proj countable-subset from-nat-into-surj)
done
show *range* $S' \subseteq$ *Collect open*
using S
apply (*auto simp add*: *from-nat-into countable-basis-proj* S'*-def*)
apply (*metis UNIV-not-empty Union-empty from-nat-into subsetD topological-basis-open*[*OF basis-proj*] *basis-proj-def*)
done
show *Collect open* $\subseteq$ *Pow* (*space borel*) **by** *simp*
show *sets borel* = *sigma-sets* (*space borel*) (*Collect open*)
by (*simp add*: *borel-def*)
qed
qed

lemma *finmap-UNIV*[*simp*]: $(\bigcup J{\in}$*Collect finite*. Π' $j{\in}J$. *UNIV*$) =$ *UNIV* **by** *auto*

lemma *borel-eq-PiF-borel*:
shows (*borel* :: ('*i*::*countable* $\Rightarrow_F$ '*a*::*polish-space*) *measure*) =
PiF (*Collect finite*) (λ-. *borel* :: '*a measure*)
unfolding *borel-def PiF-def*
proof (*rule measure-eqI*, *clarsimp*, *rule sigma-sets-eqI*)
fix *a*::('*i* $\Rightarrow_F$ '*a*) *set* **assume** *a* $\in$ *Collect open* **hence** *open a* **by** *simp*
then obtain B' **where** B': $B' \subseteq$*basis-finmap a* $= \bigcup B'$
using *finmap-topological-basis* **by** (*force simp add*: *topological-basis-def*)
have *a* $\in$ *sigma UNIV* {*Pi'* *J X* |*X J*. *finite J* $\wedge$ *X* $\in$ *J* $\rightarrow$ *sigma-sets UNIV* (*Collect open*)}
unfolding ‹*a* $= \bigcup B'$›
proof (*rule sets.countable-Union*)
from B' *countable-basis-finmap* **show** *countable* B' **by** (*metis countable-subset*)
next
show $B' \subseteq$ *sets* (*sigma UNIV*
{*Pi'* *J X* |*X J*. *finite J* $\wedge$ *X* $\in$ *J* $\rightarrow$ *sigma-sets UNIV* (*Collect open*)}) (**is** - $\subseteq$ *sets ?s*)
proof
fix *x* **assume** *x* $\in$ B' **with** B' **have** *x* $\in$ *basis-finmap* **by** *auto*
then obtain *J X* **where** *x* = *Pi'* *J X finite J X* $\in$ *J* $\rightarrow$ *sigma-sets UNIV* (*Collect open*)
by (*auto simp*: *basis-finmap-def topological-basis-open*[*OF basis-proj*])
thus *x* $\in$ *sets ?s* **by** *auto*
qed
qed
thus *a* $\in$ *sigma-sets UNIV* {*Pi'* *J X* |*X J*. *finite J* $\wedge$ *X* $\in$ *J* $\rightarrow$ *sigma-sets UNIV* (*Collect open*)}
by *simp*
next
fix *b*::('*i* $\Rightarrow_F$ '*a*) *set*
assume *b* $\in$ {*Pi'* *J X* |*X J*. *finite J* $\wedge$ *X* $\in$ *J* $\rightarrow$ *sigma-sets UNIV* (*Collect open*)}
hence *b'*: *b* $\in$ *sets* (*Pi*$_F$ (*Collect finite*) (λ-. *borel*)) **by** (*auto simp*: *sets-PiF*

borel-def)
let *?b* = λ*J*. *b* ∩ {*x*. *domain x* = *J*}
have *b* = ⋃((λ*J*. *?b J*) ' *Collect finite*) **by** *auto*
also have ... ∈ *sets borel*
proof (*rule sets.countable-Union*, *safe*)
fix *J*::'*i set* **assume** *finite J*
{ assume *ef*: *J* = {}
have *?b J* ∈ *sets borel*
proof *cases*
assume *?b J* ≠ {}
then obtain *f* **where** *f* ∈ *b domain f* = {} **using** *ef* **by** *auto*
hence *?b J* = {*f*} **using** ‹*J* = {}›
by (*auto simp*: *finmap-eq-iff*)
also have {*f*} ∈ *sets borel* **by** *simp*
finally show *?thesis* **.**
qed *simp*
} moreover {
assume *J* ≠ ({}::'*i set*)
have (*?b J*) = *b* ∩ {*m*. *domain m* ∈ {*J*}} **by** *auto*
also have ... ∈ *sets* (*PiF* {*J*} (λ-. *borel*))
using *b'* **by** (*rule restrict-sets-measurable*) (*auto simp*: ‹*finite J*›)
also have ... = *sigma-sets* (*space* (*PiF* {*J*} (λ-. *borel*)))
{*Pi'* (*J*) *F* |*F*. (∀ *j*∈*J*. *F j* ∈ *Collect open*)}
(**is** - = *sigma-sets* - *?P*)
by (*rule product-open-generates-sets-PiF-single*[*OF* ‹*J* ≠ {}› ‹*finite J*›])
also have ... ⊆ *sigma-sets UNIV* (*Collect open*)
by (*intro sigma-sets-mono''*) (*auto intro*!: *open-Pi'I simp*: *space-PiF*)
finally have (*?b J*) ∈ *sets borel* **by** (*simp add*: *borel-def*)
} ultimately show (*?b J*) ∈ *sets borel* **by** *blast*
qed (*simp add*: *countable-Collect-finite*)
finally show *b* ∈ *sigma-sets UNIV* (*Collect open*) **by** (*simp add*: *borel-def*)
qed (*simp add*: *emeasure-sigma borel-def PiF-def*)

22.10 Isomorphism between Functions and Finite Maps

lemma *measurable-finmap-compose*:
shows (λ*m*. *compose J m f*) ∈ *measurable* (*PiM* (*f* ' *J*) (λ-. *M*)) (*PiM J* (λ-. *M*))
unfolding *compose-def* **by** *measurable*

lemma *measurable-compose-inv*:
assumes *inj*: ⋀*j*. *j* ∈ *J* ⟹ *f'* (*f j*) = *j*
shows (λ*m*. *compose* (*f* ' *J*) *m f'*) ∈ *measurable* (*PiM J* (λ-. *M*)) (*PiM* (*f* ' *J*) (λ-. *M*))
unfolding *compose-def* **by** (*rule measurable-restrict*) (*auto simp*: *inj*)

locale *function-to-finmap* =
fixes *J*::'*a set* **and** *f* :: '*a* ⇒ '*b*::*countable* **and** *f'*
assumes [*simp*]: *finite J*

assumes *inv*: $i \in J \Longrightarrow f'\ (f\ i) = i$
begin

to measure finmaps

definition $fm = (\mathit{finmap\text{-}of}\ (f\ `\ J))\ o\ (\lambda g.\ \mathit{compose}\ (f\ `\ J)\ g\ f')$

lemma *domain-fm*[*simp*]: $\mathit{domain}\ (\mathit{fm}\ x) = f\ `\ J$
unfolding *fm-def* **by** *simp*

lemma *fm-restrict*[*simp*]: $\mathit{fm}\ (\mathit{restrict}\ y\ J) = \mathit{fm}\ y$
unfolding *fm-def* **by** (*auto simp*: *compose-def inv intro*: *restrict-ext*)

lemma *fm-product*:
assumes $\bigwedge i.\ \mathit{space}\ (M\ i) = \mathit{UNIV}$
shows $\mathit{fm} -` \mathit{Pi'}\ (f\ `\ J)\ S \cap \mathit{space}\ (\mathit{Pi}_M\ J\ M) = (\Pi_E\ j \in J.\ S\ (f\ j))$
using *assms*
by (*auto simp*: *inv fm-def compose-def space-PiM Pi'-def*)

lemma *fm-measurable*:
assumes $f\ `\ J \in N$
shows $\mathit{fm} \in \mathit{measurable}\ (\mathit{Pi}_M\ J\ (\lambda\text{-}.\ M))\ (\mathit{Pi}_F\ N\ (\lambda\text{-}.\ M))$
unfolding *fm-def*
proof (*rule measurable-comp*, *rule measurable-compose-inv*)
show $\mathit{finmap\text{-}of}\ (f\ `\ J) \in \mathit{measurable}\ (\mathit{Pi}_M\ (f\ `\ J)\ (\lambda\text{-}.\ M))\ (\mathit{PiF}\ N\ (\lambda\text{-}.\ M))$
using *assms* **by** (*intro measurable-finmap-of measurable-component-singleton*)
auto
qed (*simp-all add*: *inv*)

lemma *proj-fm*:
assumes $x \in J$
shows $\mathit{fm}\ m\ (f\ x) = m\ x$
using *assms* **by** (*auto simp*: *fm-def compose-def o-def inv*)

lemma *inj-on-compose-f'*: $\mathit{inj\text{-}on}\ (\lambda g.\ \mathit{compose}\ (f\ `\ J)\ g\ f')\ (\mathit{extensional}\ J)$
proof (*rule inj-on-inverseI*)
fix $x :: 'a \Rightarrow 'c$ **assume** $x \in \mathit{extensional}\ J$
thus $(\lambda x.\ \mathit{compose}\ J\ x\ f)\ (\mathit{compose}\ (f\ `\ J)\ x\ f') = x$
by (*auto simp*: *compose-def inv extensional-def*)
qed

lemma *inj-on-fm*:
assumes $\bigwedge i.\ \mathit{space}\ (M\ i) = \mathit{UNIV}$
shows $\mathit{inj\text{-}on}\ \mathit{fm}\ (\mathit{space}\ (\mathit{Pi}_M\ J\ M))$
using *assms*
apply (*auto simp*: *fm-def space-PiM PiE-def*)
apply (*rule comp-inj-on*)
apply (*rule inj-on-compose-f'*)
apply (*rule finmap-of-inj-on-extensional-finite*)
apply *simp*

apply (*auto*)
done

to measure functions

definition *mf* = (λ*g*. *compose J g f*) *o proj*

lemma *mf-fm*:
assumes *x* ∈ *space* (*Pi*$_M$ *J* (λ-. *M*))
shows *mf* (*fm x*) = *x*
proof −
have *mf* (*fm x*) ∈ *extensional J*
by (*auto simp*: *mf-def extensional-def compose-def*)
moreover
have *x* ∈ *extensional J* **using** *assms sets.sets-into-space*
by (*force simp*: *space-PiM PiE-def*)
moreover
{ fix *i* **assume** *i* ∈ *J*
hence *mf* (*fm x*) *i* = *x i*
by (*auto simp*: *inv mf-def compose-def fm-def*)
}
ultimately
show *?thesis* **by** (*rule extensionalityI*)
qed

lemma *mf-measurable*:
assumes *space M* = *UNIV*
shows *mf* ∈ *measurable* (*PiF* {*f* ‘ *J*} (λ-. *M*)) (*PiM J* (λ-. *M*))
unfolding *mf-def*
proof (*rule measurable-comp*, *rule measurable-proj-PiM*)
show (λ*g*. *compose J g f*) ∈ *measurable* (*Pi*$_M$ (*f* ‘ *J*) (λ*x*. *M*)) (*Pi*$_M$ *J* (λ-. *M*))
by (*rule measurable-finmap-compose*)
qed (*auto simp add*: *space-PiM extensional-def assms*)

lemma *fm-image-measurable*:
assumes *space M* = *UNIV*
assumes *X* ∈ *sets* (*Pi*$_M$ *J* (λ-. *M*))
shows *fm* ‘ *X* ∈ *sets* (*PiF* {*f* ‘ *J*} (λ-. *M*))
proof −
have *fm* ‘ *X* = (*mf*) −‘ *X* ∩ *space* (*PiF* {*f* ‘ *J*} (λ-. *M*))
proof *safe*
fix *x* **assume** *x* ∈ *X*
with *mf-fm*[*of x*] *sets.sets-into-space*[*OF assms*(*2*)] **show** *fm x* ∈ *mf* −‘ *X* **by** *auto*
show *fm x* ∈ *space* (*PiF* {*f* ‘ *J*} (λ-. *M*)) **by** (*simp add*: *space-PiF assms*)
next
fix *y x*
assume *x*: *mf y* ∈ *X*
assume *y*: *y* ∈ *space* (*PiF* {*f* ‘ *J*} (λ-. *M*))
thus *y* ∈ *fm* ‘ *X*

by (*intro image-eqI*[*OF - x*], *unfold finmap-eq-iff*)
(*auto simp*: *space-PiF fm-def mf-def compose-def inv Pi'-def*)
qed
also have ... ∈ *sets* (*PiF* {*f* ‘ *J*} (λ-. *M*))
using *assms*
by (*intro measurable-sets*[*OF mf-measurable*]) *auto*
finally show *?thesis* **.**
qed

lemma *fm-image-measurable-finite*:
assumes *space M* = *UNIV*
assumes *X* ∈ *sets* (*Pi*$_M$ *J* (λ-. *M*::'*c measure*))
shows *fm* ‘ *X* ∈ *sets* (*PiF* (*Collect finite*) (λ-. *M*::'*c measure*))
using *fm-image-measurable*[*OF assms*]
by (*rule subspace-set-in-sets*) (*auto simp*: *finite-subset*)

measure on finmaps

definition *mapmeasure M N* = *distr M* (*PiF* (*Collect finite*) *N*) (*fm*)

lemma *sets-mapmeasure*[*simp*]: *sets* (*mapmeasure M N*) = *sets* (*PiF* (*Collect finite*) *N*)
unfolding *mapmeasure-def* **by** *simp*

lemma *space-mapmeasure*[*simp*]: *space* (*mapmeasure M N*) = *space* (*PiF* (*Collect finite*) *N*)
unfolding *mapmeasure-def* **by** *simp*

lemma *mapmeasure-PiF*:
assumes *s1*: *space M* = *space* (*Pi*$_M$ *J* (λ-. *N*))
assumes *s2*: *sets M* = *sets* (*Pi*$_M$ *J* (λ-. *N*))
assumes *space N* = *UNIV*
assumes *X* ∈ *sets* (*PiF* (*Collect finite*) (λ-. *N*))
shows *emeasure* (*mapmeasure M* (λ-. *N*)) *X* = *emeasure M* ((*fm* −‘ *X* ∩ *extensional J*))
using *assms*
by (*auto simp*: *measurable-cong-sets*[*OF s2 refl*] *mapmeasure-def emeasure-distr fm-measurable space-PiM PiE-def*)

lemma *mapmeasure-PiM*:
fixes *N*::'*c measure*
assumes *s1*: *space M* = *space* (*Pi*$_M$ *J* (λ-. *N*))
assumes *s2*: *sets M* = (*Pi*$_M$ *J* (λ-. *N*))
assumes *N*: *space N* = *UNIV*
assumes *X*: *X* ∈ *sets M*
shows *emeasure M X* = *emeasure* (*mapmeasure M* (λ-. *N*)) (*fm* ‘ *X*)
unfolding *mapmeasure-def*
proof (*subst emeasure-distr*, *subst measurable-cong-sets*[*OF s2 refl*], *rule fm-measurable*)
have *X* ⊆ *space* (*Pi*$_M$ *J* (λ-. *N*)) **using** *assms* **by** (*simp add*: *sets.sets-into-space*)
from *assms inj-on-fm*[*of* λ-. *N*] *subsetD*[*OF this*] **have** *fm* −‘ *fm* ‘ *X* ∩ *space*

$(Pi_M\ J\ (\lambda\text{-}.\ N)) = X$
by (*auto simp*: *vimage-image-eq inj-on-def*)
thus *emeasure M X* = *emeasure M* (*fm* −‘ *fm* ‘ *X* ∩ *space M*) **using** *s1*
by *simp*
show *fm* ‘ *X* ∈ *sets* (*PiF* (*Collect finite*) (λ-. *N*))
by (*rule fm-image-measurable-finite*[*OF N X*[*simplified s2*]])
qed *simp*

end

end

23 Projective Limit

theory *Projective-Limit*
imports
Fin-Map
Infinite-Product-Measure
HOL−Library.Diagonal-Subsequence
begin

23.1 Sequences of Finite Maps in Compact Sets

locale *finmap-seqs-into-compact* =
fixes *K*::*nat* ⇒ (*nat* ⇒$_F$ *'a*::*metric-space*) *set* **and** *f*::*nat* ⇒ (*nat* ⇒$_F$ *'a*) **and** *M*
assumes *compact*: ⋀*n*. *compact* (*K n*)
assumes *f-in-K*: ⋀*n*. *K n* ≠ {}
assumes *domain-K*: ⋀*n*. *k* ∈ *K n* ⟹ *domain k* = *domain* (*f n*)
assumes *proj-in-K*:
⋀*t n m*. *m* ≥ *n* ⟹ *t* ∈ *domain* (*f n*) ⟹ (*f m*)$_F$ *t* ∈ (λ*k*. (*k*)$_F$ *t*) ‘ *K n*
begin

lemma *proj-in-K'*: (∃ *n*. ∀ *m* ≥ *n*. (*f m*)$_F$ *t* ∈ (λ*k*. (*k*)$_F$ *t*) ‘ *K n*)
using *proj-in-K f-in-K*
proof *cases*
obtain *k* **where** *k* ∈ *K* (*Suc 0*) **using** *f-in-K* **by** *auto*
assume ∀ *n*. *t* ∉ *domain* (*f n*)
thus *?thesis*
by (*auto intro*!: *exI*[**where** *x*=*1*] *image-eqI*[*OF* - ‹*k* ∈ *K* (*Suc 0*)›]
simp: *domain-K*[*OF* ‹*k* ∈ *K* (*Suc 0*)›])
qed *blast*

lemma *proj-in-KE*:
obtains *n* **where** ⋀*m*. *m* ≥ *n* ⟹ (*f m*)$_F$ *t* ∈ (λ*k*. (*k*)$_F$ *t*) ‘ *K n*
using *proj-in-K'* **by** *blast*

lemma *compact-projset*:
shows *compact* ((λ*k*. (*k*)$_F$ *i*) ‘ *K n*)

using *continuous-proj compact* **by** (*rule compact-continuous-image*)

end

lemma *compactE'*:
fixes $S :: 'a :: metric\text{-}space\ set$
assumes *compact* S $\forall n{\geq}m.\ f\ n \in S$
obtains $l\ r$ **where** $l \in S$ *strict-mono* $(r{::}nat{\Rightarrow}nat)$ $((f \circ r) \longrightarrow l)$ *sequentially*
proof *atomize-elim*
have *strict-mono* $((+)\ m)$ **by** (*simp add*: *strict-mono-def*)
have $\forall n.\ (f\ o\ (\lambda i.\ m + i))\ n \in S$ **using** *assms* **by** *auto*
from *seq-compactE*[*OF* ‹*compact S*›[*unfolded compact-eq-seq-compact-metric*] *this*]
obtain $l\ r$ **where** $l \in S$ *strict-mono* r $(f \circ (+)\ m \circ r) \longrightarrow l$ **by** *blast*
hence $l \in S$ *strict-mono* $((\lambda i.\ m + i)\ o\ r) \wedge (f \circ ((\lambda i.\ m + i)\ o\ r)) \longrightarrow l$
using *strict-mono-o*[*OF* ‹*strict-mono* $((+)\ m)$› ‹*strict-mono r*›] **by** (*auto simp*: *o-def*)
thus $\exists l\ r.\ l \in S \wedge$ *strict-mono* $r \wedge (f \circ r) \longrightarrow l$ **by** *blast*
qed

sublocale *finmap-seqs-into-compact* $\subseteq$ *subseqs* $\lambda n\ s.\ (\exists l.\ (\lambda i.\ ((f\ o\ s)\ i)_F\ n) \longrightarrow l)$
proof
fix n **and** $s :: nat \Rightarrow nat$
assume *strict-mono s*
from *proj-in-KE*[*of n*] **obtain** *n0* **where** *n0*: $\bigwedge m.\ n0 \leq m \Longrightarrow (f\ m)_F\ n \in (\lambda k.\ (k)_F\ n)\ `\ K\ n0$
by *blast*
have $\forall i \geq n0.\ ((f \circ s)\ i)_F\ n \in (\lambda k.\ (k)_F\ n)\ `\ K\ n0$
proof *safe*
fix i **assume** $n0 \leq i$
also have $\ldots \leq s\ i$ **by** (*rule seq-suble*) *fact*
finally have $n0 \leq s\ i$ **.**
with *n0* **show** $((f \circ s)\ i)_F\ n \in (\lambda k.\ (k)_F\ n)\ `\ K\ n0$
by *auto*
qed
then obtain *ls rs*
where $ls \in (\lambda k.\ (k)_F\ n)\ `\ K\ n0$ *strict-mono rs* $((\lambda i.\ ((f \circ s)\ i)_F\ n) \circ rs) \longrightarrow ls$
by (*rule compactE'*[*OF compact-projset*])
thus $\exists r'.$ *strict-mono* $r' \wedge (\exists l.\ (\lambda i.\ ((f \circ (s \circ r'))\ i)_F\ n) \longrightarrow l)$ **by** (*auto simp*: *o-def*)
qed

lemma (**in** *finmap-seqs-into-compact*) *diagonal-tendsto*: $\exists l.\ (\lambda i.\ (f\ (diagseq\ i))_F\ n) \longrightarrow l$
proof −
obtain l **where** $(\lambda i.\ ((f\ o\ (diagseq\ o\ (+)\ (Suc\ n)))\ i)_F\ n) \longrightarrow l$
proof (*atomize-elim*, *rule diagseq-holds*)
fix $r\ s\ n$

assume *strict-mono* $(r :: nat \Rightarrow nat)$
assume $\exists\, l.\ (\lambda i.\ ((f \circ s)\ i)_F\ n) \longrightarrow l$
then obtain l **where** $((\lambda i.\ (f\ i)_F\ n)\ o\ s) \longrightarrow l$
by (*auto simp*: *o-def*)
hence $((\lambda i.\ (f\ i)_F\ n)\ o\ s\ o\ r) \longrightarrow l$ **using** ‹*strict-mono r*›
by (*rule LIMSEQ-subseq-LIMSEQ*)
thus $\exists\, l.\ (\lambda i.\ ((f \circ (s \circ r))\ i)_F\ n) \longrightarrow l$ **by** (*auto simp add*: *o-def*)
qed
hence $(\lambda i.\ ((f\ (diagseq\ (i + Suc\ n))))_F\ n) \longrightarrow l$ **by** (*simp add*: *ac-simps*)
hence $(\lambda i.\ (f\ (diagseq\ i))_F\ n) \longrightarrow l$ **by** (*rule LIMSEQ-offset*)
thus *?thesis* **..**
qed

23.2 Daniell-Kolmogorov Theorem

Existence of Projective Limit

locale *polish-projective* = *projective-family I P* λ*-. borel*::*'a*::*polish-space measure*
for *I*::*'i set* **and** *P*
begin

lemma *emeasure-lim-emb*:
assumes *X*: $J \subseteq I$ *finite J X* ∈ *sets* $(\Pi_M\ i{\in}J.\ borel)$
shows *lim* $(emb\ I\ J\ X) = P\ J\ X$
proof (*rule emeasure-lim*)
write *mu-G* (‹μG›)
interpret *generator*: *algebra space* $(PiM\ I\ (\lambda i.\ borel))$ *generator*
by (*rule algebra-generator*)

fix *J* **and** *B* :: *nat* ⇒ (*'i* ⇒ *'a*) *set*
assume *J*: $\bigwedge n.$ *finite* $(J\ n)$ $\bigwedge n.$ $J\ n \subseteq I$ $\bigwedge n.$ $B\ n \in sets\ (\Pi_M\ i{\in}J\ n.\ borel)$ *incseq J*
and *B*: *decseq* $(\lambda n.\ emb\ I\ (J\ n)\ (B\ n))$
and $0 < (INF\ i.\ P\ (J\ i)\ (B\ i))$ (**is** $0 < ?a$)
moreover have $?a \leq 1$
using *J* **by** (*auto intro*!: *INF-lower2*[*of 0*] *prob-space-P*[*THEN prob-space.measure-le-1*])
ultimately obtain *r* **where** *r*: $?a = ennreal\ r$ $0 < r$ $r \leq 1$
by (*cases ?a*) (*auto simp*: *top-unique*)
define *Z* **where** $Z\ n = emb\ I\ (J\ n)\ (B\ n)$ **for** *n*
have *Z-mono*: $n \leq m \Longrightarrow Z\ m \subseteq Z\ n$ **for** *n m*
unfolding *Z-def* **using** *B*[*THEN antimonoD*, *of n m*] **.**
have *J-mono*: $\bigwedge n\ m.\ n \leq m \Longrightarrow J\ n \subseteq J\ m$
using ‹*incseq J*› **by** (*force simp*: *incseq-def*)
note [*simp*] = ‹$\bigwedge n.$ *finite* $(J\ n)$›
interpret *prob-space P* $(J\ i)$ **for** *i* **using** *J prob-space-P* **by** *simp*

have *P-eq*[*simp*]:
$sets\ (P\ (J\ i)) = sets\ (\Pi_M\ i{\in}J\ i.\ borel)$ $space\ (P\ (J\ i)) = space\ (\Pi_M\ i{\in}J\ i.\ borel)$ **for** *i*
using *J* **by** (*auto simp*: *sets-P space-P*)

have $Z\ i \in generator$ **for** i
unfolding *Z-def* **by** (*auto intro*!: *generator.intros J*)

have *countable-UN-J*: *countable* ($\bigcup n.\ J\ n$) **by** (*simp add*: *countable-finite*)
define *Utn* **where** $Utn = to\text{-}nat\text{-}on\ (\bigcup n.\ J\ n)$
interpret *function-to-finmap J n Utn from-nat-into* ($\bigcup n.\ J\ n$) **for** n
by *unfold-locales* (*auto simp*: *Utn-def intro*: *from-nat-into-to-nat-on*[*OF countable-UN-J*])
have *inj-on-Utn*: *inj-on Utn* ($\bigcup n.\ J\ n$)
unfolding *Utn-def* **using** *countable-UN-J* **by** (*rule inj-on-to-nat-on*)
hence *inj-on-Utn-J*: $\bigwedge n.$ *inj-on Utn* (*J n*) **by** (*rule inj-on-subset*) *auto*
define P' **where** $P'\ n = mapmeasure\ n\ (P\ (J\ n))\ (\lambda\text{-}.\ borel)$ **for** n
interpret P': *prob-space* $P'\ n$ **for** n
unfolding P'*-def mapmeasure-def* **using** J
by (*auto intro*!: *prob-space-distr fm-measurable simp*: *measurable-cong-sets*[*OF sets-P*])

let *?SUP* = $\lambda n.\ SUP\ K \in \{K.\ K \subseteq fm\ n\ `\ (B\ n) \wedge compact\ K\}.\ emeasure\ (P'\ n)\ K$
{ fix n
have *emeasure* (*P* (*J n*)) (*B n*) = *emeasure* ($P'\ n$) (*fm n* ` (*B n*))
using J **by** (*auto simp*: P'*-def mapmeasure-PiM space-P sets-P*)
also
have ... = *?SUP n*
proof (*rule inner-regular*)
show *sets* ($P'\ n$) = *sets borel* **by** (*simp add*: *borel-eq-PiF-borel* P'*-def*)
next
show *fm n* ` *B n* ∈ *sets borel*
unfolding *borel-eq-PiF-borel* **by** (*auto simp*: P'*-def fm-image-measurable-finite sets-P J*(*3*))
qed *simp*
finally have ∗: *emeasure* (*P* (*J n*)) (*B n*) = *?SUP n* **.**
have *?SUP n* ≠ ∞
unfolding ∗[*symmetric*] **by** *simp*
note ∗ *this*
} note R = *this*
have ∀ *n*. ∃ *K*. *emeasure* (*P* (*J n*)) (*B n*) − *emeasure* ($P'\ n$) *K* ≤ *2 powr* (−*n*) ∗ *?a* ∧ *compact K* ∧ *K* ⊆ *fm n* ` *B n*
proof
fix n **show** ∃ *K*. *emeasure* (*P* (*J n*)) (*B n*) − *emeasure* ($P'\ n$) *K* ≤ *ennreal* (*2 powr* − *real n*) ∗ *?a* ∧
compact K ∧ *K* ⊆ *fm n* ` *B n*
unfolding *R*[*of n*]
proof (*rule ccontr*)
assume *H*: ∄ K'. *?SUP n* − *emeasure* ($P'\ n$) K' ≤ *ennreal* (*2 powr* − *real n*) ∗ *?a* ∧
compact K' ∧ K' ⊆ *fm n* ` *B n*
have *?SUP n* + *0* < *?SUP n* + *2 powr* (−*n*) ∗ *?a*

using *R*[*of n*] **unfolding** *ennreal-add-left-cancel-less ennreal-zero-less-mult-iff*
by (*auto intro*: ‹*0 < ?a*›)
also have ... = (*SUP K*∈{*K*. *K* ⊆ *fm n* ' *B n* ∧ *compact K*}. *emeasure* (*P'* *n*) *K* + *2 powr* (−*n*) ∗ *?a*)
by (*rule ennreal-SUP-add-left*[*symmetric*]) *auto*
also have ... ≤ *?SUP n*
proof (*intro SUP-least*)
fix *K* **assume** *K* ∈ {*K*. *K* ⊆ *fm n* ' *B n* ∧ *compact K*}
with *H* **have** *2 powr* (−*n*) ∗ *?a* < *?SUP n* − *emeasure* (*P'* *n*) *K*
by *auto*
then show *emeasure* (*P'* *n*) *K* + (*2 powr* (−*n*)) ∗ *?a* ≤ *?SUP n*
by (*subst* (*asm*) *less-diff-eq-ennreal*) (*auto simp*: *less-top*[*symmetric*] *R*(*1*)[*symmetric*] *ac-simps*)
qed
finally show *False* **by** *simp*
qed
qed
then obtain *K'* **where** *K'*:
⋀*n*. *emeasure* (*P* (*J n*)) (*B n*) − *emeasure* (*P'* *n*) (*K'* *n*) ≤ *ennreal* (*2 powr* − *real n*) ∗ *?a*
⋀*n*. *compact* (*K'* *n*) ⋀*n*. *K'* *n* ⊆ *fm n* ' *B n*
unfolding *choice-iff* **by** *blast*
define *K* **where** *K n* = *fm n* −' *K'* *n* ∩ *space* (*Pi*$_M$ (*J n*) (λ-. *borel*)) **for** *n*
have *K-sets*: ⋀*n*. *K n* ∈ *sets* (*Pi*$_M$ (*J n*) (λ-. *borel*))
unfolding *K-def*
using *compact-imp-closed*[*OF* ‹*compact* (*K'* -)›]
by (*intro measurable-sets*[*OF fm-measurable*, *of* - *Collect finite*])
(*auto simp*: *borel-eq-PiF-borel*[*symmetric*])
have *K-B*: ⋀*n*. *K n* ⊆ *B n*
proof
fix *x n* **assume** *x* ∈ *K n*
then have *fm-in*: *fm n x* ∈ *fm n* ' *B n*
using *K'* **by** (*force simp*: *K-def*)
show *x* ∈ *B n*
using ‹*x* ∈ *K n*› *K-sets sets.sets-into-space J*(*1*,*2*,*3*)[*of n*] *inj-on-image-mem-iff*[*OF inj-on-fm*]
by (*metis* (*no-types*) *Int-iff K-def fm-in space-borel*)
qed
define *Z'* **where** *Z'* *n* = *emb I* (*J n*) (*K n*) **for** *n*
have *Z'*: ⋀*n*. *Z'* *n* ⊆ *Z n*
unfolding *Z'-def Z-def*
proof (*rule prod-emb-mono*, *safe*)
fix *n x* **assume** *x* ∈ *K n*
hence *fm n x* ∈ *K'* *n* *x* ∈ *space* (*Pi*$_M$ (*J n*) (λ-. *borel*))
by (*simp-all add*: *K-def space-P*)
note *this*(*1*)
also have *K'* *n* ⊆ *fm n* ' *B n* **by** (*simp add*: *K'*)
finally have *fm n x* ∈ *fm n* ' *B n* **.**
thus *x* ∈ *B n*

proof *safe*
fix y **assume** y: $y \in B\ n$
hence $y \in space\ (Pi_M\ (J\ n)\ (\lambda\text{-}.\ borel))$ **using** J *sets.sets-into-space*[*of* $B\ n$ $P\ (J\ n)$]
by (*auto simp add*: *space-P sets-P*)
assume $fm\ n\ x = fm\ n\ y$
note *inj-onD*[*OF inj-on-fm*[*OF space-borel*], *OF* ‹$fm\ n\ x = fm\ n\ y$› ‹$x \in space$ -› ‹$y \in space$ -›]
with y **show** $x \in B\ n$ **by** *simp*
qed
qed
have $\bigwedge n.\ Z'\ n \in generator$ **using** J $K'(2)$ **unfolding** Z'-*def*
by (*auto intro*!: *generator.intros measurable-sets*[*OF fm-measurable*[*of* - *Collect finite*]]
simp: *K-def borel-eq-PiF-borel*[*symmetric*] *compact-imp-closed*)
define Y **where** $Y\ n = (\bigcap i{\in}\{1..n\}.\ Z'\ i)$ **for** n
hence $\bigwedge n\ k.\ Y\ (n + k) \subseteq Y\ n$ **by** (*induct-tac* k) (*auto simp*: *Y-def*)
hence *Y-mono*: $\bigwedge n\ m.\ n \leq m \Longrightarrow Y\ m \subseteq Y\ n$ **by** (*auto simp*: *le-iff-add*)
have *Y-Z'*: $\bigwedge n.\ n \geq 1 \Longrightarrow Y\ n \subseteq Z'\ n$ **by** (*auto simp*: *Y-def*)
hence *Y-Z*: $\bigwedge n.\ n \geq 1 \Longrightarrow Y\ n \subseteq Z\ n$ **using** Z' **by** *auto*

have *Y-notempty*: $\bigwedge n.\ n \geq 1 \Longrightarrow (Y\ n) \neq \{\}$
proof −
fix n::*nat* **assume** $n \geq 1$ **hence** $Y\ n \subseteq Z\ n$ **by** *fact*
have $Y\ n = (\bigcap i{\in}\{1..n\}.\ emb\ I\ (J\ n)\ (emb\ (J\ n)\ (J\ i)\ (K\ i)))$ **using** J *J-mono*
by (*auto simp*: *Y-def Z'-def*)
also have $\ldots = prod\text{-}emb\ I\ (\lambda\text{-}.\ borel)\ (J\ n)\ (\bigcap i{\in}\{1..n\}.\ emb\ (J\ n)\ (J\ i)\ (K\ i))$
using ‹$n \geq 1$›
by (*subst prod-emb-INT*) *auto*
finally
have *Y-emb*:
$Y\ n = prod\text{-}emb\ I\ (\lambda\text{-}.\ borel)\ (J\ n)\ (\bigcap i{\in}\{1..n\}.\ prod\text{-}emb\ (J\ n)\ (\lambda\text{-}.\ borel)\ (J\ i)\ (K\ i))$ **.**
hence $Y\ n \in generator$ **using** J *J-mono K-sets* ‹$n \geq 1$›
by (*auto simp del*: *prod-emb-INT intro*!: *generator.intros*)
have $*$: $\mu G\ (Z\ n) = P\ (J\ n)\ (B\ n)$
unfolding *Z-def* **using** J **by** (*intro mu-G-spec*) *auto*
then have $\mu G\ (Z\ n) \neq \infty$ **by** *auto*
note $*$
moreover have $*$: $\mu G\ (Y\ n) = P\ (J\ n)\ (\bigcap i{\in}\{Suc\ 0..n\}.\ prod\text{-}emb\ (J\ n)\ (\lambda\text{-}.\ borel)\ (J\ i)\ (K\ i))$
unfolding *Y-emb* **using** J *J-mono K-sets* ‹$n \geq 1$› **by** (*subst mu-G-spec*) *auto*
then have $\mu G\ (Y\ n) \neq \infty$ **by** *auto*
note $*$
moreover have $\mu G\ (Z\ n - Y\ n) =$
$P\ (J\ n)\ (B\ n - (\bigcap i{\in}\{Suc\ 0..n\}.\ prod\text{-}emb\ (J\ n)\ (\lambda\text{-}.\ borel)\ (J\ i)\ (K\ i)))$
unfolding *Z-def Y-emb prod-emb-Diff*[*symmetric*] **using** J *J-mono K-sets* ‹$n \geq 1$›

by (*subst mu-G-spec*) (*auto intro*!: *sets.Diff*)
ultimately
have $\mu G\ (Z\ n) - \mu G\ (Y\ n) = \mu G\ (Z\ n - Y\ n)$
using *J J-mono K-sets* ‹$n \geq 1$›
by (*simp only*: *emeasure-eq-measure Z-def*)
(*auto dest*!: *bspec*[**where** *x*=*n*] *intro*!: *measure-Diff*[*symmetric*] *subsetD*[*OF K-B*]
intro!: *arg-cong*[**where** *f*=*ennreal*]
simp: *extensional-restrict emeasure-eq-measure prod-emb-iff sets-P space-P*
ennreal-minus measure-nonneg)
also have *subs*: $Z\ n - Y\ n \subseteq (\bigcup i{\in}\{1..n\}.\ (Z\ i - Z'\ i))$
using ‹$n \geq 1$› **unfolding** *Y-def UN-extend-simps*(7) **by** (*intro UN-mono Diff-mono Z-mono order-refl*) *auto*
have $Z\ n - Y\ n \in \mathit{generator}\ (\bigcup i{\in}\{1..n\}.\ (Z\ i - Z'\ i)) \in \mathit{generator}$
using ‹Z' - ∈ *generator*› ‹Z - ∈ *generator*› ‹Y - ∈ *generator*› **by** *auto*
hence $\mu G\ (Z\ n - Y\ n) \leq \mu G\ (\bigcup i{\in}\{1..n\}.\ (Z\ i - Z'\ i))$
using *subs generator.additive-increasing*[*OF positive-mu-G additive-mu-G*]
unfolding *increasing-def* **by** *auto*
also have $\ldots \leq (\sum\ i{\in}\{1..n\}.\ \mu G\ (Z\ i - Z'\ i))$ **using** ‹Z - ∈ *generator*› ‹Z' - ∈ *generator*›
by (*intro generator.subadditive*[*OF positive-mu-G additive-mu-G*]) *auto*
also have $\ldots \leq (\sum\ i{\in}\{1..n\}.\ 2\ powr\ -real\ i * ?a)$
proof (*rule sum-mono*)
fix *i* **assume** $i \in \{1..n\}$ **hence** $i \leq n$ **by** *simp*
have $\mu G\ (Z\ i - Z'\ i) = \mu G\ (\mathit{prod\text{-}emb}\ I\ (\lambda\text{-}.\ \mathit{borel})\ (J\ i)\ (B\ i - K\ i))$
unfolding *Z'-def Z-def* **by** *simp*
also have $\ldots = P\ (J\ i)\ (B\ i - K\ i)$
using *J K-sets* **by** (*subst mu-G-spec*) *auto*
also have $\ldots = P\ (J\ i)\ (B\ i) - P\ (J\ i)\ (K\ i)$
using *K-sets J* ‹K - ⊆ B -› **by** (*simp add*: *emeasure-Diff*)
also have $\ldots = P\ (J\ i)\ (B\ i) - P'\ i\ (K'\ i)$
unfolding *K-def P'-def*
by (*auto simp*: *mapmeasure-PiF borel-eq-PiF-borel*[*symmetric*]
compact-imp-closed[*OF* ‹*compact* (K' -)›] *space-PiM PiE-def*)
also have $\ldots \leq \mathit{ennreal}\ (2\ powr - real\ i) * ?a$ **using** $K'(1)$[*of i*] **.**
finally show $\mu G\ (Z\ i - Z'\ i) \leq (2\ powr - real\ i) * ?a$ **.**
qed
also have $\ldots = \mathit{ennreal}\ ((\sum\ i{\in}\{1..n\}.\ (2\ powr\ -enn2real\ i)) * enn2real\ ?a)$
using *r* **by** (*simp add*: *sum-distrib-right ennreal-mult*[*symmetric*])
also have $\ldots < \mathit{ennreal}\ (1 * enn2real\ ?a)$
proof (*intro ennreal-lessI mult-strict-right-mono*)
have $(\sum i = 1..n.\ 2\ powr - real\ i) = (\sum i = 1..{<}Suc\ n.\ (1/2)\ \hat{}\ i)$
by (*rule sum.cong*) (*auto simp*: *powr-realpow powr-divide power-divide powr-minus-divide*)
also have $\{1..{<}Suc\ n\} = \{..{<}Suc\ n\} - \{0\}$ **by** *auto*
also have $\mathit{sum}\ ((\hat{})\ (1\ /\ 2{::}real))\ (\{..{<}Suc\ n\} - \{0\}) =$
$\mathit{sum}\ ((\hat{})\ (1\ /\ 2))\ (\{..{<}Suc\ n\}) - 1$ **by** (*auto simp*: *sum-diff1*)
also have $\ldots < 1$ **by** (*subst geometric-sum*) *auto*

finally show $(\sum i = 1..n.\ 2\ powr\ -\ enn2real\ i) < 1$ **by** *simp*
qed (*auto simp*: *r enn2real-positive-iff*)
also have $\ldots = ?a$ **by** (*auto simp*: *r*)
also have $\ldots \leq \mu G\ (Z\ n)$
using *J* **by** (*auto intro*: *INF-lower simp*: *Z-def mu-G-spec*)
finally have $\mu G\ (Z\ n) - \mu G\ (Y\ n) < \mu G\ (Z\ n)$ **.**
hence *R*: $\mu G\ (Z\ n) < \mu G\ (Z\ n) + \mu G\ (Y\ n)$
using $\langle \mu G\ (Y\ n) \neq \infty \rangle$ **by** (*auto simp*: *zero-less-iff-neq-zero*)
then have $\mu G\ (Y\ n) > 0$
by *simp*
thus $Y\ n \neq \{\}$ **using** *positive-mu-G* **by** (*auto simp add*: *positive-def*)
qed
hence $\forall\, n{\in}\{1..\}.\ \exists\, y.\ y \in Y\ n$ **by** *auto*
then obtain *y* **where** *y*: $\bigwedge n.\ n \geq 1 \Longrightarrow y\ n \in Y\ n$ **unfolding** *bchoice-iff* **by** *force*
{
fix *t* **and** *n m*::*nat*
assume $1 \leq n\ n \leq m$ **hence** $1 \leq m$ **by** *simp*
from *Y-mono*[*OF* $\langle m \geq n \rangle$] *y*[*OF* $\langle 1 \leq m \rangle$] **have** $y\ m \in Y\ n$ **by** *auto*
also have $\ldots \subseteq Z'\ n$ **using** *Y-Z'*[*OF* $\langle 1 \leq n \rangle$] **.**
finally
have *fm n* (*restrict* (*y m*) (*J n*)) $\in K'\ n$
unfolding *Z'-def K-def prod-emb-iff* **by** (*simp add*: *Z'-def K-def prod-emb-iff*)
moreover have *finmap-of* (*J n*) (*restrict* (*y m*) (*J n*)) = *finmap-of* (*J n*) (*y m*)
using *J* **by** (*simp add*: *fm-def*)
ultimately have *fm n* (*y m*) $\in K'\ n$ **by** *simp*
} **note** *fm-in-K'* = *this*
interpret *finmap-seqs-into-compact* $\lambda n.\ K'\ (Suc\ n)$ $\lambda k.\ fm\ (Suc\ k)\ (y\ (Suc\ k))$ *borel*
proof
fix *n* **show** *compact* (*K' n*) **by** *fact*
next
fix *n*
from *Y-mono*[*of n Suc n*] *y*[*of Suc n*] **have** $y\ (Suc\ n) \in Y\ (Suc\ n)$ **by** *auto*
also have $\ldots \subseteq Z'\ (Suc\ n)$ **using** *Y-Z'* **by** *auto*
finally
have *fm* (*Suc n*) (*restrict* (*y* (*Suc n*)) (*J* (*Suc n*))) $\in K'\ (Suc\ n)$
unfolding *Z'-def K-def prod-emb-iff* **by** (*simp add*: *Z'-def K-def prod-emb-iff*)
thus $K'\ (Suc\ n) \neq \{\}$ **by** *auto*
fix *k*
assume $k \in K'\ (Suc\ n)$
with *K'*[*of Suc n*] *sets.sets-into-space* **have** $k \in fm\ (Suc\ n)\ `\ B\ (Suc\ n)$ **by** *auto*
then obtain *b* **where** $k = fm\ (Suc\ n)\ b$ **by** *auto*
thus *domain k* = *domain* (*fm* (*Suc n*) (*y* (*Suc n*)))
by (*simp-all add*: *fm-def*)
next
fix *t* **and** *n m*::*nat*

```
    assume n ≤ m hence Suc n ≤ Suc m by simp
    assume t ∈ domain (fm (Suc n) (y (Suc n)))
    then obtain j where j: t = Utn j j ∈ J (Suc n) by auto
    hence j ∈ J (Suc m) using J-mono[OF ‹Suc n ≤ Suc m›] by auto
    have img: fm (Suc n) (y (Suc m)) ∈ K' (Suc n) using ‹n ≤ m›
      by (intro fm-in-K') simp-all
    show (fm (Suc m) (y (Suc m)))_F t ∈ (λk. (k)_F t) ‘ K' (Suc n)
      apply (rule image-eqI[OF - img])
      using ‹j ∈ J (Suc n)› ‹j ∈ J (Suc m)›
      unfolding j by (subst proj-fm, auto)+
  qed
  have ∀t. ∃z. (λi. (fm (Suc (diagseq i)) (y (Suc (diagseq i))))_F t) ⟶ z
    using diagonal-tendsto ..
  then obtain z where z:
    ⋀t. (λi. (fm (Suc (diagseq i)) (y (Suc (diagseq i))))_F t) ⟶ z t
    unfolding choice-iff by blast
  {
    fix n :: nat assume n ≥ 1
    have ⋀i. domain (fm n (y (Suc (diagseq i)))) = domain (finmap-of (Utn ‘ J
n) z)
      by simp
    moreover
    {
      fix t
      assume t: t ∈ domain (finmap-of (Utn ‘ J n) z)
      hence t ∈ Utn ‘ J n by simp
      then obtain j where j: t = Utn j j ∈ J n by auto
      have (λi. (fm n (y (Suc (diagseq i))))_F t) ⟶ z t
        apply (subst (2) tendsto-iff, subst eventually-sequentially)
      proof safe
        fix e :: real assume 0 < e
        { fix i and x :: 'i ⇒ 'a assume i: i ≥ n
          assume t ∈ domain (fm n x)
          hence t ∈ domain (fm i x) using J-mono[OF ‹i ≥ n›] by auto
          with i have (fm i x)_F t = (fm n x)_F t
            using j by (auto simp: proj-fm dest!: inj-onD[OF inj-on-Utn])
        } note index-shift = this
        have I: ⋀i. i ≥ n ⟹ Suc (diagseq i) ≥ n
          apply (rule le-SucI)
          apply (rule order-trans) apply simp
          apply (rule seq-suble[OF subseq-diagseq])
          done
        from z
        have ∃N. ∀i≥N. dist ((fm (Suc (diagseq i)) (y (Suc (diagseq i))))_F t) (z
t) < e
          unfolding tendsto-iff eventually-sequentially using ‹0 < e› by auto
        then obtain N where N: ⋀i. i ≥ N ⟹
          dist ((fm (Suc (diagseq i)) (y (Suc (diagseq i))))_F t) (z t) < e by auto
        show ∃N. ∀na≥N. dist ((fm n (y (Suc (diagseq na))))_F t) (z t) < e
```

```
      proof (rule exI[where x=max N n], safe)
        fix na assume max N n ≤ na
        hence  dist ((fm n (y (Suc (diagseq na))))_F t) (z t) =
              dist ((fm (Suc (diagseq na)) (y (Suc (diagseq na))))_F t) (z t) using t
          by (subst index-shift[OF I]) auto
        also have ... < e using ‹max N n ≤ na› by (intro N) simp
        finally show dist ((fm n (y (Suc (diagseq na))))_F t) (z t) < e .
      qed
    qed
     hence (λi. (fm n (y (Suc (diagseq i))))_F t) ⟶ (finmap-of (Utn ‘ J n)
z)_F t
      by (simp add: tendsto-intros)
  } ultimately
  have (λi. fm n (y (Suc (diagseq i)))) ⟶ finmap-of (Utn ‘ J n) z
    by (rule tendsto-finmap)
  hence ((λi. fm n (y (Suc (diagseq i)))) o (λi. i + n)) ⟶ finmap-of (Utn
‘ J n) z
    by (rule LIMSEQ-subseq-LIMSEQ) (simp add: strict-mono-def)
  moreover
  have (∀ i. ((λi. fm n (y (Suc (diagseq i)))) o (λi. i + n)) i ∈ K' n)
    apply (auto simp add: o-def intro!: fm-in-K' ‹1 ≤ n› le-SucI)
    apply (rule le-trans)
    apply (rule le-add2)
    using seq-suble[OF subseq-diagseq]
    apply auto
    done
  moreover
  from ‹compact (K' n)› have closed (K' n) by (rule compact-imp-closed)
  ultimately
  have finmap-of (Utn ‘ J n) z ∈ K' n
    unfolding closed-sequential-limits by blast
  also have finmap-of (Utn ‘ J n) z  = fm n (λi. z (Utn i))
    unfolding finmap-eq-iff
  proof clarsimp
    fix i assume i: i ∈ J n
    hence from-nat-into (⋃ n. J n) (Utn i) = i
      unfolding Utn-def
      by (subst from-nat-into-to-nat-on[OF countable-UN-J]) auto
    with i show z (Utn i) = (fm n (λi. z (Utn i)))_F (Utn i)
      by (simp add: finmap-eq-iff fm-def compose-def)
  qed
  finally have fm n (λi. z (Utn i)) ∈ K' n .
  moreover
  let ?J = ⋃ n. J n
  have (?J ∩ J n) = J n by auto
  ultimately have restrict (λi. z (Utn i)) (?J ∩ J n) ∈ K n
    unfolding K-def by (auto simp: space-P space-PiM)
  hence restrict (λi. z (Utn i)) ?J ∈ Z' n unfolding Z'-def
    using J by (auto simp: prod-emb-def PiE-def extensional-def)
```

also have ... $\subseteq$ *Z n* **using** *Z'* **by** *simp*
finally have *restrict* ($\lambda i.\ z$ (*Utn i*)) *?J* $\in$ *Z n* **.**
} note *in-Z = this*
hence ($\bigcap i \in \{1..\}.\ Z\ i$) $\neq$ {} **by** *auto*
thus ($\bigcap i.\ Z\ i$) $\neq$ {}
using *INT-decseq-offset*[*OF antimonoI*[*OF Z-mono*]] **by** *simp*
qed *fact+*

lemma *measure-lim-emb*:
J $\subseteq$ *I* $\Longrightarrow$ *finite J* $\Longrightarrow$ *X* $\in$ *sets* (Π_M *i*$\in$*J. borel*) $\Longrightarrow$ *measure lim* (*emb I J X*) = *measure* (*P J*) *X*
unfolding *measure-def* **by** (*subst emeasure-lim-emb*) *auto*

end

hide-const (**open**) *PiF*
hide-const (**open**) Pi_F
hide-const (**open**) *Pi'*
hide-const (**open**) *finmap-of*
hide-const (**open**) *proj*
hide-const (**open**) *domain*
hide-const (**open**) *basis-finmap*

sublocale *polish-projective* $\subseteq$ *P*: *prob-space lim*
proof
have $*$: *emb I* {} {$\lambda x.$ *undefined*} = *space* (Π_M *i*$\in$*I. borel*)
by (*auto simp*: *prod-emb-def space-PiM*)
interpret *prob-space P* {}
using *prob-space-P* **by** *simp*
show *emeasure lim* (*space lim*) = *1*
using *emeasure-lim-emb*[*of* {} {$\lambda x.$ *undefined*}] *emeasure-space-1*
by (*simp add*: $*$ *PiM-empty space-P*)
qed

locale *polish-product-prob-space* =
product-prob-space λ-. *borel*::('*a*::*polish-space*) *measure I* **for** *I*::'*i set*

sublocale *polish-product-prob-space* $\subseteq$ *P*: *polish-projective I* $\lambda J.$ *PiM J* (λ-. *borel*::('*a*) *measure*)
..

lemma (**in** *polish-product-prob-space*) *limP-eq-PiM*: *lim* = *PiM I* (λ-. *borel*)
by (*rule PiM-eq*) (*auto simp*: *emeasure-PiM emeasure-lim-emb*)

end

24 Random Permutations

theory *Random-Permutations*

imports
HOL−Combinatorics.Multiset-Permutations
Probability-Mass-Function
begin

Choosing a set permutation (i.e. a distinct list with the same elements as the set) uniformly at random is the same as first choosing the first element of the list and then choosing the rest of the list as a permutation of the remaining set.

lemma *random-permutation-of-set*:
assumes *finite A* $A \neq \{\}$
shows *pmf-of-set* (*permutations-of-set A*) =
do {
$x \leftarrow$ *pmf-of-set A*;
xs $\leftarrow$ *pmf-of-set* (*permutations-of-set* $(A - \{x\})$);
return-pmf $(x \# xs)$
} (**is** *?lhs* = *?rhs*)
proof −
from *assms* **have** *permutations-of-set A* = $(\bigcup x \in A.\ (\#)\ x$ ' *permutations-of-set* $(A - \{x\}))$
by (*simp add*: *permutations-of-set-nonempty*)
also from *assms* **have** *pmf-of-set* ... = *?rhs*
by (*subst pmf-of-set-UN*[**where** *n* = *fact* (*card A* − *1*)])
(*auto simp*: *card-image disjoint-family-on-def map-pmf-def* [*symmetric*] *map-pmf-of-set-inj*)
finally show *?thesis* **.**
qed

A generic fold function that takes a function, an initial state, and a set and chooses a random order in which it then traverses the set in the same fashion as a left fold over a list. We first give a recursive definition.

function *fold-random-permutation* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a$ *set* $\Rightarrow 'b$ *pmf*
where
fold-random-permutation f x $\{\}$ = *return-pmf x*
| ¬*finite A* $\Longrightarrow$ *fold-random-permutation f x A* = *return-pmf x*
| *finite A* $\Longrightarrow$ $A \neq \{\} \Longrightarrow$
fold-random-permutation f x A =
pmf-of-set A $\gg\!=$ $(\lambda a.$ *fold-random-permutation f* $(f\ a\ x)$ $(A - \{a\}))$
by *simp-all fastforce*
termination proof (*relation Wellfounded.measure* $(\lambda(\text{-},\text{-},A).$ *card A*))
fix *A* :: $'a$ *set* **and** *f* :: $'a \Rightarrow 'b \Rightarrow 'b$ **and** *x* :: $'b$ **and** *y* :: $'a$
assume *A*: *finite A* $A \neq \{\}$ $y \in$ *set-pmf* (*pmf-of-set A*)
then have *card A* > *0* **by** (*simp add*: *card-gt-0-iff*)
with *A* **show** $((f, f\ y\ x, A - \{y\}), f, x, A) \in$ *Wellfounded.measure* $(\lambda(\text{-}, \text{-}, A).$ *card A*)
by *simp*
qed *simp-all*

We can now show that the above recursive definition is equivalent to choosing

a random set permutation and folding over it (in any direction).

lemma *fold-random-permutation-foldl*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (*foldl* (λ*x y*. *f y x*) *x*) (*pmf-of-set* (*permutations-of-set A*))
using *assms*
proof (*induction f x A rule*: *fold-random-permutation.induct* [*case-names empty infinite remove*])
case (*remove A f x*)
from *remove*
have *fold-random-permutation f x A* =
pmf-of-set A ⤜ (λ*a*. *fold-random-permutation f* (*f a x*) (*A* − {*a*})) **by** *simp*
also from *remove*
have ... = *pmf-of-set A* ⤜ (λ*a*. *map-pmf* (*foldl* (λ*x y*. *f y x*) *x*)
(*map-pmf* ((#) *a*) (*pmf-of-set* (*permutations-of-set* (*A* − {*a*})))))
by (*intro bind-pmf-cong*) (*simp-all add*: *pmf.map-comp o-def*)
also from *remove* **have** ... = *map-pmf* (*foldl* (λ*x y*. *f y x*) *x*) (*pmf-of-set* (*permutations-of-set A*))
by (*simp-all add*: *random-permutation-of-set map-bind-pmf map-pmf-def* [*symmetric*])
finally show *?case* **.**
qed (*simp-all add*: *pmf-of-set-singleton*)

lemma *fold-random-permutation-foldr*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (λ*xs*. *foldr f xs x*) (*pmf-of-set* (*permutations-of-set A*))
proof −
have *fold-random-permutation f x A* =
map-pmf (*foldl* (λ*x y*. *f y x*) *x* ∘ *rev*) (*pmf-of-set* (*permutations-of-set A*))
using *assms* **by** (*subst fold-random-permutation-foldl* [*OF assms*])
(*simp-all add*: *pmf.map-comp* [*symmetric*] *map-pmf-of-set-inj*)
also have *foldl* (λ*x y*. *f y x*) *x* ∘ *rev* = (λ*xs*. *foldr f xs x*)
by (*intro ext*) (*simp add*: *foldl-conv-foldr*)
finally show *?thesis* **.**
qed

lemma *fold-random-permutation-fold*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (λ*xs*. *fold f xs x*) (*pmf-of-set* (*permutations-of-set A*))
by (*subst fold-random-permutation-foldl* [*OF assms*], *intro map-pmf-cong*)
(*simp-all add*: *foldl-conv-fold*)

lemma *fold-random-permutation-code* [*code*]:
fold-random-permutation f x (*set xs*) =
map-pmf (*foldl* (λ*x y*. *f y x*) *x*) (*pmf-of-set* (*permutations-of-set* (*set xs*)))
by (*simp add*: *fold-random-permutation-foldl*)

We now introduce a slightly generalised version of the above fold operation that does not simply return the result in the end, but applies a monadic bind to it. This may seem somewhat arbitrary, but it is a common use case, e.g. in the Social Decision Scheme of Random Serial Dictatorship, where voters narrow down a set of possible winners in a random order and the winner is chosen from the remaining set uniformly at random.

function *fold-bind-random-permutation*
  :: ('*a* ⇒ '*b* ⇒ '*b*) ⇒ ('*b* ⇒ '*c pmf*) ⇒ '*b* ⇒ '*a set* ⇒ '*c pmf* **where**
 fold-bind-random-permutation f g x {} = *g x*
| ¬*finite A* ⟹ *fold-bind-random-permutation f g x A* = *g x*
| *finite A* ⟹ *A* ≠ {} ⟹
   fold-bind-random-permutation f g x A =
    pmf-of-set A ⤜ (λ*a*. *fold-bind-random-permutation f g* (*f a x*) (*A* − {*a*}))
 by *simp-all fastforce*
termination proof (*relation Wellfounded.measure* (λ(-,-,-,*A*). *card A*))
 fix *A* :: '*a set* **and** *f* :: '*a* ⇒ '*b* ⇒ '*b* **and** *x* :: '*b*
  and *y* :: '*a* **and** *g* :: '*b* ⇒ '*c pmf*
 assume *A*: *finite A A* ≠ {} *y* ∈ *set-pmf* (*pmf-of-set A*)
 then have *card A* > *0* **by** (*simp add*: *card-gt-0-iff*)
 with *A* **show** ((*f*, *g*, *f y x*, *A* − {*y*}), *f*, *g*, *x*, *A*) ∈ *Wellfounded.measure* (λ(-, -, -, *A*). *card A*)
  by *simp*
qed *simp-all*

We now show that the recursive definition is equivalent to a random fold followed by a monadic bind.

lemma *fold-bind-random-permutation-altdef* [*code*]:
 fold-bind-random-permutation f g x A = *fold-random-permutation f x A* ⤜ *g*
proof (*induction f x A rule*: *fold-random-permutation.induct* [*case-names empty infinite remove*])
 case (*remove A f x*)
 from *remove* **have** *pmf-of-set A* ⤜ (λ*a*. *fold-bind-random-permutation f g* (*f a x*) (*A* − {*a*})) =
                pmf-of-set A ⤜ (λ*a*. *fold-random-permutation f* (*f a x*) (*A* − {*a*}) ⤜ *g*)
  by (*intro bind-pmf-cong*) *simp-all*
 with *remove* **show** *?case* **by** (*simp add*: *bind-return-pmf bind-assoc-pmf*)
qed (*simp-all add*: *bind-return-pmf*)

We can now derive the following nice monadic representations of the combined fold-and-bind:

lemma *fold-bind-random-permutation-foldl*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
      do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldl* (λ*x y*. *f y x*) *x xs*)}
 using *assms* **by** (*simp add*: *fold-bind-random-permutation-altdef bind-assoc-pmf*
          fold-random-permutation-foldl bind-return-pmf map-pmf-def)

lemma *fold-bind-random-permutation-foldr*:
assumes *finite A*
shows *fold-bind-random-permutation f g x A* =
do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldr f xs x*)}
using *assms* **by** (*simp add*: *fold-bind-random-permutation-altdef bind-assoc-pmf fold-random-permutation-foldr bind-return-pmf map-pmf-def*)

lemma *fold-bind-random-permutation-fold*:
assumes *finite A*
shows *fold-bind-random-permutation f g x A* =
do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*fold f xs x*)}
using *assms* **by** (*simp add*: *fold-bind-random-permutation-altdef bind-assoc-pmf fold-random-permutation-fold bind-return-pmf map-pmf-def*)

The following useful lemma allows us to swap partitioning a set w. r. t. a predicate and drawing a random permutation of that set.

lemma *partition-random-permutations*:
assumes *finite A*
shows *map-pmf* (*partition P*) (*pmf-of-set* (*permutations-of-set A*)) =
pair-pmf (*pmf-of-set* (*permutations-of-set* {*x*∈*A*. *P x*}))
(*pmf-of-set* (*permutations-of-set* {*x*∈*A*. ¬*P x*})) (**is** *?lhs* = *?rhs*)
proof (*rule pmf-eqI*, *clarify*, *goal-cases*)
case (*1 xs ys*)
show *?case*
proof (*cases xs* ∈ *permutations-of-set* {*x*∈*A*. *P x*} ∧ *ys* ∈ *permutations-of-set* {*x*∈*A*. ¬*P x*})
case *True*
let *?n1* = *card* {*x*∈*A*. *P x*} **and** *?n2* = *card* {*x*∈*A*. ¬*P x*}
have *card-eq*: *card A* = *?n1* + *?n2*
proof −
have *?n1* + *?n2* = *card* ({*x*∈*A*. *P x*} ∪ {*x*∈*A*. ¬*P x*})
using *assms* **by** (*intro card-Un-disjoint* [*symmetric*]) *auto*
also have {*x*∈*A*. *P x*} ∪ {*x*∈*A*. ¬*P x*} = *A* **by** *blast*
finally show *?thesis* **..**
qed

from *True* **have** *lengths* [*simp*]: *length xs* = *?n1 length ys* = *?n2*
by (*auto intro*!: *length-finite-permutations-of-set*)
have *pmf ?lhs* (*xs*, *ys*) =
real (*card* (*permutations-of-set A* ∩ *partition P* −‘ {(*xs*, *ys*)})) / *fact* (*card A*)
using *assms* **by** (*auto simp*: *pmf-map measure-pmf-of-set*)
also have *partition P* −‘ {(*xs*, *ys*)} = *shuffles xs ys*
using *True* **by** (*intro inv-image-partition*) (*auto simp*: *permutations-of-set-def*)
also have *permutations-of-set A* ∩ *shuffles xs ys* = *shuffles xs ys*
using *True distinct-disjoint-shuffles*[*of xs ys*]
by (*auto simp*: *permutations-of-set-def dest*: *set-shuffles*)
also have *card* (*shuffles xs ys*) = *length xs* + *length ys choose length xs*
using *True* **by** (*intro card-disjoint-shuffles*) (*auto simp*: *permutations-of-set-def*)

also have *length xs + length ys = card A* **by** (*simp add*: *card-eq*)
also have *real* (*card A choose length xs*) = *fact* (*card A*) / (*fact ?n1* ∗ *fact* (*card A* − *?n1*))
by (*subst binomial-fact*) (*auto intro*!: *card-mono assms*)
also have ... / *fact* (*card A*) = *1* / (*fact ?n1* ∗ *fact ?n2*)
by (*simp add*: *field-split-simps card-eq*)
also have ... = *pmf ?rhs* (*xs*, *ys*) **using** *True assms* **by** (*simp add*: *pmf-pair*)
finally show *?thesis* **.**
next
case *False*
hence ∗: *xs* ∉ *permutations-of-set* {*x*∈*A*. *P x*} ∨ *ys* ∉ *permutations-of-set* {*x*∈*A*. ¬*P x*} **by** *blast*
hence *eq*: *permutations-of-set A* ∩ (*partition P* −‘ {(*xs*, *ys*)}) = {}
by (*auto simp*: *o-def permutations-of-set-def*)
from ∗ **show** *?thesis*
by (*elim disjE*) (*insert assms eq*, *simp-all add*: *pmf-pair pmf-map measure-pmf-of-set*)
qed
qed

end

25 Discrete subprobability distribution

theory *SPMF* **imports**
Probability-Mass-Function
HOL−Library.Complete-Partial-Order2
HOL−Library.Rewrite
begin

25.1 Auxiliary material

lemma *cSUP-singleton* [*simp*]: (*SUP x*∈{*x*}. *f x* :: - :: *conditionally-complete-lattice*) = *f x*
by (*metis cSup-singleton image-empty image-insert*)

25.1.1 More about extended reals

lemma [*simp*]:
shows *ennreal-max-0*: *ennreal* (*max 0 x*) = *ennreal x*
and *ennreal-max-0'*: *ennreal* (*max x 0*) = *ennreal x*
by(*simp-all add*: *max-def ennreal-eq-0-iff*)

lemma *e2ennreal-0* [*simp*]: *e2ennreal 0* = *0*
by(*simp add*: *zero-ennreal-def*)

lemma *enn2real-bot* [*simp*]: *enn2real* ⊥ = *0*
by(*simp add*: *bot-ennreal-def*)

lemma *continuous-at-ennreal*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *ennreal* (*f x*))
unfolding *continuous-def* **by** *auto*

lemma *ennreal-Sup*:
assumes ∗: (*SUP a*∈*A*. *ennreal a*) ≠ ⊤
and *A* ≠ {}
shows *ennreal* (*Sup A*) = (*SUP a*∈*A*. *ennreal a*)
proof (*rule continuous-at-Sup-mono*)
obtain *r* **where** *r*: *ennreal r* = (*SUP a*∈*A*. *ennreal a*) *r* ≥ *0*
using ∗ **by**(*cases* (*SUP a*∈*A*. *ennreal a*)) *simp-all*
then show *bdd-above A*
by(*auto intro*!: *SUP-upper bdd-aboveI*[*of* - *r*] *simp add*: *ennreal-le-iff*[*symmetric*])
qed (*auto simp*: *mono-def continuous-at-imp-continuous-at-within continuous-at-ennreal ennreal-leI assms*)

lemma *ennreal-SUP*:
⟦ (*SUP a*∈*A*. *ennreal* (*f a*)) ≠ ⊤; *A* ≠ {} ⟧ ⟹ *ennreal* (*SUP a*∈*A*. *f a*) = (*SUP a*∈*A*. *ennreal* (*f a*))
using *ennreal-Sup*[*of f* ‘ *A*] **by** (*auto simp*: *image-comp*)

lemma *ennreal-lt-0*: *x* < *0* ⟹ *ennreal x* = *0*
by(*simp add*: *ennreal-eq-0-iff*)

25.1.2 More about *'a option*

lemma *None-in-map-option-image* [*simp*]: *None* ∈ *map-option f* ‘ *A* ⟷ *None* ∈ *A*
by *auto*

lemma *Some-in-map-option-image* [*simp*]: *Some x* ∈ *map-option f* ‘ *A* ⟷ (∃ *y*. *x* = *f y* ∧ *Some y* ∈ *A*)
by (*smt* (*verit*, *best*) *imageE imageI map-option-eq-Some*)

lemma *case-option-collapse*: *case-option x* (λ-. *x*) = (λ-. *x*)
by(*simp add*: *fun-eq-iff split*: *option.split*)

lemma *case-option-id*: *case-option None Some* = *id*
by(*rule ext*)(*simp split*: *option.split*)

inductive *ord-option* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a option* ⇒ *'b option* ⇒ *bool*
for *ord* :: *'a* ⇒ *'b* ⇒ *bool*
where
None: *ord-option ord None x*
| *Some*: *ord x y* ⟹ *ord-option ord* (*Some x*) (*Some y*)

inductive-simps *ord-option-simps* [*simp*]:
ord-option ord None x
ord-option ord x None

ord-option ord (*Some x*) (*Some y*)
ord-option ord (*Some x*) *None*

inductive-simps *ord-option-eq-simps* [*simp*]:
ord-option (=) *None y*
ord-option (=) (*Some x*) *y*

lemma *ord-option-reflI*: (⋀*y*. *y* ∈ *set-option x* ⟹ *ord y y*) ⟹ *ord-option ord x x*
by(*cases x*) *simp-all*

lemma *reflp-ord-option*: *reflp ord* ⟹ *reflp* (*ord-option ord*)
by(*simp add*: *reflp-def ord-option-reflI*)

lemma *ord-option-trans*:
⟦ *ord-option ord x y*; *ord-option ord y z*;
⋀*a b c*. ⟦ *a* ∈ *set-option x*; *b* ∈ *set-option y*; *c* ∈ *set-option z*; *ord a b*; *ord b c* ⟧ ⟹ *ord a c* ⟧
⟹ *ord-option ord x z*
by(*auto elim*!: *ord-option.cases*)

lemma *transp-ord-option*: *transp ord* ⟹ *transp* (*ord-option ord*)
unfolding *transp-def* **by**(*blast intro*: *ord-option-trans*)

lemma *antisymp-ord-option*: *antisymp ord* ⟹ *antisymp* (*ord-option ord*)
by(*auto intro*!: *antisympI elim*!: *ord-option.cases dest*: *antisympD*)

lemma *ord-option-chainD*:
Complete-Partial-Order.chain (*ord-option ord*) *Y*
⟹ *Complete-Partial-Order.chain ord* {*x*. *Some x* ∈ *Y*}
by(*rule chainI*)(*auto dest*: *chainD*)

definition *lub-option* :: ('*a set* ⇒ '*b*) ⇒ '*a option set* ⇒ '*b option*
where *lub-option lub Y* = (*if Y* ⊆ {*None*} *then None else Some* (*lub* {*x*. *Some x* ∈ *Y*}))

lemma *map-lub-option*: *map-option f* (*lub-option lub Y*) = *lub-option* (*f* ∘ *lub*) *Y*
by(*simp add*: *lub-option-def*)

lemma *lub-option-upper*:
assumes *Complete-Partial-Order.chain* (*ord-option ord*) *Y x* ∈ *Y*
and *lub-upper*: ⋀*Y x*. ⟦ *Complete-Partial-Order.chain ord Y*; *x* ∈ *Y* ⟧ ⟹ *ord x* (*lub Y*)
shows *ord-option ord x* (*lub-option lub Y*)
using *assms*(*1*−*2*)
by(*cases x*)(*auto simp*: *lub-option-def intro*: *lub-upper*[*OF ord-option-chainD*])

lemma *lub-option-least*:
assumes *Y*: *Complete-Partial-Order.chain* (*ord-option ord*) *Y*

and *upper*: $\bigwedge x.\ x \in Y \Longrightarrow$ *ord-option ord x y*
assumes *lub-least*: $\bigwedge Y\ y.\ [\![$ *Complete-Partial-Order.chain ord Y*; $\bigwedge x.\ x \in Y \Longrightarrow$ *ord x y* $]\!] \Longrightarrow$ *ord* (*lub Y*) *y*
shows *ord-option ord* (*lub-option lub Y*) *y*
using *Y*
by(*cases y*)(*auto 4 3 simp add*: *lub-option-def intro*: *lub-least*[*OF ord-option-chainD*] *dest*: *upper*)

lemma *lub-map-option*: *lub-option lub* (*map-option f ' Y*) = *lub-option* (*lub* ∘ (') *f*) *Y*
proof −
have $\bigwedge u\ y.\ [\![$*Some u* ∈ *Y*; *y* ∈ *Y*$]\!] \Longrightarrow$ {*f y* |*y*. *Some y* ∈ *Y*} = *f* ' {*x*. *Some x* ∈ *Y*}
by *blast*
then show *?thesis*
by (*auto simp*: *lub-option-def*)
qed

lemma *ord-option-mono*: $[\![$ *ord-option A x y*; $\bigwedge x\ y.$ *A x y* $\Longrightarrow$ *B x y* $]\!] \Longrightarrow$ *ord-option B x y*
by(*auto elim*: *ord-option.cases*)

lemma *ord-option-mono'* [*mono*]:
($\bigwedge x\ y.$ *A x y* ⟶ *B x y*) $\Longrightarrow$ *ord-option A x y* ⟶ *ord-option B x y*
by(*blast intro*: *ord-option-mono*)

lemma *ord-option-compp*: *ord-option* (*A OO B*) = *ord-option A OO ord-option B*
by(*auto simp*: *fun-eq-iff elim*!: *ord-option.cases intro*: *ord-option.intros*)

lemma *ord-option-inf*: *inf* (*ord-option A*) (*ord-option B*) = *ord-option* (*inf A B*) (**is** *?lhs* = *?rhs*)
proof(*rule antisym*)
show *?lhs* ≤ *?rhs* **by**(*auto elim*!: *ord-option.cases*)
qed(*auto elim*: *ord-option-mono*)

lemma *ord-option-map2*: *ord-option ord x* (*map-option f y*) = *ord-option* (λ*x y*. *ord x* (*f y*)) *x y*
by(*auto elim*: *ord-option.cases*)

lemma *ord-option-map1*: *ord-option ord* (*map-option f x*) *y* = *ord-option* (λ*x y*. *ord* (*f x*) *y*) *x y*
by(*auto elim*: *ord-option.cases*)

lemma *option-ord-Some1-iff*: *option-ord* (*Some x*) *y* ⟷ *y* = *Some x*
by(*auto simp*: *flat-ord-def*)

25.1.3 A relator for sets that treats sets like predicates

context includes *lifting-syntax*

begin

definition *rel-pred* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a set* ⇒ '*b set* ⇒ *bool*
where *rel-pred R A B* = (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*)

lemma *rel-predI*: (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*) ⟹ *rel-pred R A B*
by(*simp add*: *rel-pred-def*)

lemma *rel-predD*: ⟦ *rel-pred R A B*; *R x y* ⟧ ⟹ *x* ∈ *A* ⟷ *y* ∈ *B*
by(*simp add*: *rel-pred-def rel-fun-def*)

lemma *Collect-parametric*: ((*A* ===> (=)) ===> *rel-pred A*) *Collect Collect*
— Declare this rule as *transfer-rule* only locally because it blows up the search space for *transfer* (in combination with *Collect-transfer*)
by(*simp add*: *rel-funI rel-predI*)

end

25.1.4 Monotonicity rules

lemma *monotone-gfp-eadd1*: *monotone* (≥) (≥) (λ*x*. *x* + *y* :: *enat*)
by(*auto intro*!: *monotoneI*)

lemma *monotone-gfp-eadd2*: *monotone* (≥) (≥) (λ*y*. *x* + *y* :: *enat*)
by(*auto intro*!: *monotoneI*)

lemma *mono2mono-gfp-eadd*[*THEN gfp.mono2mono2*, *cont-intro*, *simp*]:
shows *monotone-eadd*: *monotone* (*rel-prod* (≥) (≥)) (≥) (λ(*x*, *y*). *x* + *y* :: *enat*)
by(*simp add*: *monotone-gfp-eadd1 monotone-gfp-eadd2*)

lemma *eadd-gfp-partial-function-mono* [*partial-function-mono*]:
⟦ *monotone* (*fun-ord* (≥)) (≥) *f*; *monotone* (*fun-ord* (≥)) (≥) *g* ⟧
⟹ *monotone* (*fun-ord* (≥)) (≥) (λ*x*. *f x* + *g x* :: *enat*)
by(*rule mono2mono-gfp-eadd*)

lemma *mono2mono-ereal*[*THEN lfp.mono2mono*]:
shows *monotone-ereal*: *monotone* (≤) (≤) *ereal*
by(*rule monotoneI*) *simp*

lemma *mono2mono-ennreal*[*THEN lfp.mono2mono*]:
shows *monotone-ennreal*: *monotone* (≤) (≤) *ennreal*
by(*rule monotoneI*)(*simp add*: *ennreal-leI*)

25.1.5 Bijections

lemma *bi-unique-rel-set-bij-betw*:
assumes *unique*: *bi-unique R*
and *rel*: *rel-set R A B*
shows ∃*f*. *bij-betw f A B* ∧ (∀ *x*∈*A*. *R x* (*f x*))
proof −

from *assms* **obtain** *f* **where** *f*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$ **and** *B*: $\bigwedge x.\ x \in A \Longrightarrow f\ x \in B$
by (*metis bi-unique-rel-set-lemma image-eqI*)
have *inj-on f A*
by (*metis* (*no-types*, *lifting*) *bi-unique-def f inj-on-def unique*)
moreover have *f* ' *A* = *B* **using** *rel*
by (*smt* (*verit*) *bi-unique-def bi-unique-rel-set-lemma f image-cong unique*)
ultimately have *bij-betw f A B* **unfolding** *bij-betw-def* **..**
thus *?thesis* **using** *f* **by** *blast*
qed

lemma *bij-betw-rel-setD*: *bij-betw f A B* $\Longrightarrow$ *rel-set* ($\lambda x\ y.\ y = f\ x$) *A B*
by(*rule rel-setI*)(*auto dest*: *bij-betwE bij-betw-imp-surj-on*[*symmetric*])

25.2 Subprobability mass function

type-synonym *'a spmf* = *'a option pmf*
translations (*type*) *'a spmf* $\leftharpoonup$ (*type*) *'a option pmf*

definition *measure-spmf* :: *'a spmf* $\Rightarrow$ *'a measure*
where *measure-spmf p* = *distr* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*) *the*

abbreviation *spmf* :: *'a spmf* $\Rightarrow$ *'a* $\Rightarrow$ *real*
where *spmf p x* $\equiv$ *pmf p* (*Some x*)

lemma *space-measure-spmf*: *space* (*measure-spmf p*) = *UNIV*
by(*simp add*: *measure-spmf-def*)

lemma *sets-measure-spmf* [*simp*, *measurable-cong*]: *sets* (*measure-spmf p*) = *sets* (*count-space UNIV*)
by(*simp add*: *measure-spmf-def*)

lemma *measure-spmf-not-bot* [*simp*]: *measure-spmf p* $\neq \bot$
by (*metis empty-not-UNIV space-bot space-measure-spmf*)

lemma *measurable-the-measure-pmf-Some* [*measurable*, *simp*]:
the $\in$ *measurable* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*)
by(*auto simp*: *measurable-def sets-restrict-space space-restrict-space integral-restrict-space*)

lemma *measurable-spmf-measure1*[*simp*]: *measurable* (*measure-spmf M*) *N* = *UNIV* $\rightarrow$ *space N*
by(*auto simp*: *measurable-def space-measure-spmf*)

lemma *measurable-spmf-measure2*[*simp*]: *measurable N* (*measure-spmf M*) = *measurable N* (*count-space UNIV*)
by(*intro measurable-cong-sets*) *simp-all*

lemma *subprob-space-measure-spmf* [*simp*, *intro*!]: *subprob-space* (*measure-spmf p*)
proof
show *emeasure* (*measure-spmf p*) (*space* (*measure-spmf p*)) ≤ *1*
by(*simp add*: *measure-spmf-def emeasure-distr emeasure-restrict-space space-restrict-space measure-pmf.measure-le-1*)
qed(*simp add*: *space-measure-spmf*)

interpretation *measure-spmf*: *subprob-space measure-spmf p* **for** *p*
by(*rule subprob-space-measure-spmf*)

lemma *finite-measure-spmf* [*simp*]: *finite-measure* (*measure-spmf p*)
by *unfold-locales*

lemma *spmf-conv-measure-spmf*: *spmf p x* = *measure* (*measure-spmf p*) {*x*}
by(*auto simp*: *measure-spmf-def measure-distr measure-restrict-space pmf.rep-eq space-restrict-space intro*: *arg-cong2*[**where** *f*=*measure*])

lemma *emeasure-measure-spmf-conv-measure-pmf*:
emeasure (*measure-spmf p*) *A* = *emeasure* (*measure-pmf p*) (*Some* ‘ *A*)
by(*auto simp*: *measure-spmf-def emeasure-distr emeasure-restrict-space space-restrict-space intro*: *arg-cong2*[**where** *f*=*emeasure*])

lemma *measure-measure-spmf-conv-measure-pmf*:
measure (*measure-spmf p*) *A* = *measure* (*measure-pmf p*) (*Some* ‘ *A*)
using *emeasure-measure-spmf-conv-measure-pmf*[*of p A*]
by(*simp add*: *measure-spmf.emeasure-eq-measure measure-pmf.emeasure-eq-measure*)

lemma *emeasure-spmf-map-pmf-Some* [*simp*]:
emeasure (*measure-spmf* (*map-pmf Some p*)) *A* = *emeasure* (*measure-pmf p*) *A*
by(*auto simp*: *measure-spmf-def emeasure-distr emeasure-restrict-space space-restrict-space intro*: *arg-cong2*[**where** *f*=*emeasure*])

lemma *measure-spmf-map-pmf-Some* [*simp*]:
measure (*measure-spmf* (*map-pmf Some p*)) *A* = *measure* (*measure-pmf p*) *A*
using *emeasure-spmf-map-pmf-Some*[*of p A*] **by**(*simp add*: *measure-spmf.emeasure-eq-measure measure-pmf.emeasure-eq-measure*)

lemma *nn-integral-measure-spmf*: (∫⁺ *x*. *f x* ∂*measure-spmf p*) = ∫⁺ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space UNIV*
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = ∫⁺ *x*. *pmf p x* ∗ *f* (*the x*) ∂*count-space* (*range Some*)
by(*simp add*: *measure-spmf-def nn-integral-distr nn-integral-restrict-space nn-integral-measure-pmf nn-integral-count-space-indicator ac-simps*
flip: *times-ereal.simps* [*symmetric*])
also have ... = ∫⁺ *x*. *ennreal* (*spmf p* (*the x*)) ∗ *f* (*the x*) ∂*count-space* (*range Some*)
by(*rule nn-integral-cong*) *auto*
also have ... = ∫⁺ *x*. *spmf p* (*the* (*Some x*)) ∗ *f* (*the* (*Some x*)) ∂*count-space*

UNIV
by(*rule nn-integral-bij-count-space*[*symmetric*])(*simp add*: *bij-betw-def*)
also have ... = *?rhs* **by** *simp*
finally show *?thesis* **.**
qed

lemma *integral-measure-spmf*:
assumes *integrable* (*measure-spmf p*) *f*
shows ($\int$ *x*. *f x* ∂*measure-spmf p*) = $\int$ *x*. *spmf p x* ∗ *f x* ∂*count-space UNIV*
proof −
have *integrable* (*count-space UNIV*) (λ*x*. *spmf p x* ∗ *f x*)
using *assms* **by**(*simp add*: *integrable-iff-bounded nn-integral-measure-spmf abs-mult ennreal-mult''*)
then show *?thesis* **using** *assms*
by(*simp add*: *real-lebesgue-integral-def nn-integral-measure-spmf ennreal-mult'*[*symmetric*])
qed

lemma *emeasure-spmf-single*: *emeasure* (*measure-spmf p*) {*x*} = *spmf p x*
by(*simp add*: *measure-spmf.emeasure-eq-measure spmf-conv-measure-spmf*)

lemma *measurable-measure-spmf*[*measurable*]:
(λ*x*. *measure-spmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
by (*auto simp*: *space-subprob-algebra*)

lemma *nn-integral-measure-spmf-conv-measure-pmf*:
assumes [*measurable*]: *f* ∈ *borel-measurable* (*count-space UNIV*)
shows *nn-integral* (*measure-spmf p*) *f* = *nn-integral* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*f* ∘ *the*)
by(*simp add*: *measure-spmf-def nn-integral-distr o-def*)

lemma *measure-spmf-in-space-subprob-algebra* [*simp*]:
measure-spmf p ∈ *space* (*subprob-algebra* (*count-space UNIV*))
by(*simp add*: *space-subprob-algebra*)

lemma *nn-integral-spmf-neq-top*: ($\int^+$ *x*. *spmf p x* ∂*count-space UNIV*) ≠ ⊤
using *nn-integral-measure-spmf*[**where** *f*=λ-. *1*, *of p*, *symmetric*]
by *simp*

lemma *SUP-spmf-neq-top'*: (*SUP p*∈*Y*. *ennreal* (*spmf p x*)) ≠ ⊤
by (*metis SUP-least ennreal-le-1 ennreal-one-neq-top neq-top-trans pmf-le-1*)

lemma *SUP-spmf-neq-top*: (*SUP i*. *ennreal* (*spmf* (*Y i*) *x*)) ≠ ⊤
by (*meson SUP-eq-top-iff ennreal-le-1 ennreal-one-less-top linorder-not-le pmf-le-1*)

lemma *SUP-emeasure-spmf-neq-top*: (*SUP p*∈*Y*. *emeasure* (*measure-spmf p*) *A*) ≠ ⊤
by (*metis ennreal-one-less-top less-SUP-iff linorder-not-le measure-spmf.subprob-emeasure-le-1*)

25.3 Support

definition *set-spmf* :: *'a spmf* ⇒ *'a set*
where *set-spmf p* = *set-pmf p* ⤜ *set-option*

lemma *set-spmf-rep-eq*: *set-spmf p* = {*x*. *measure* (*measure-spmf p*) {*x*} ≠ *0*}
proof −
have ⋀*x* :: *'a*. *the* −‘ {*x*} ∩ *range Some* = {*Some x*} **by** *auto*
then show *?thesis*
unfolding *set-spmf-def measure-spmf-def*
by(*auto simp*: *set-pmf.rep-eq measure-distr measure-restrict-space space-restrict-space*)
qed

lemma *in-set-spmf*: *x* ∈ *set-spmf p* ⟷ *Some x* ∈ *set-pmf p*
by(*simp add*: *set-spmf-def*)

lemma *AE-measure-spmf-iff* [*simp*]: (*AE x in measure-spmf p*. *P x*) ⟷ (∀ *x*∈*set-spmf p*. *P x*)
unfolding *set-spmf-def measure-spmf-def*
by(*force simp*: *AE-distr-iff AE-restrict-space-iff AE-measure-pmf-iff cong del*: *AE-cong*)

lemma *spmf-eq-0-set-spmf*: *spmf p x* = *0* ⟷ *x* ∉ *set-spmf p*
by(*auto simp*: *pmf-eq-0-set-pmf set-spmf-def*)

lemma *in-set-spmf-iff-spmf*: *x* ∈ *set-spmf p* ⟷ *spmf p x* ≠ *0*
by(*auto simp*: *set-spmf-def set-pmf-iff*)

lemma *set-spmf-return-pmf-None* [*simp*]: *set-spmf* (*return-pmf None*) = {}
by(*auto simp*: *set-spmf-def*)

lemma *countable-set-spmf* [*simp*]: *countable* (*set-spmf p*)
by(*simp add*: *set-spmf-def bind-UNION*)

lemma *spmf-eqI*:
assumes ⋀*i*. *spmf p i* = *spmf q i*
shows *p* = *q*
proof(*rule pmf-eqI*)
fix *i*
show *pmf p i* = *pmf q i*
proof(*cases i*)
case (*Some i'*)
thus *?thesis* **by**(*simp add*: *assms*)
next
case *None*
have *ennreal* (*pmf p i*) = *measure* (*measure-pmf p*) {*i*} **by**(*simp add*: *pmf-def*)
also have {*i*} = *space* (*measure-pmf p*) − *range Some*
by(*auto simp*: *None intro*: *ccontr*)
also have *measure* (*measure-pmf p*) ... = *ennreal 1* − *measure* (*measure-pmf p*) (*range Some*)

by(*simp add*: *measure-pmf.prob-compl ennreal-minus*[*symmetric*] *del*: *space-measure-pmf*)
also have *range Some* = (⋃ *x*∈*set-spmf p*. {*Some x*}) ∪ *Some* ' (− *set-spmf p*)
by *auto*
also have *measure* (*measure-pmf p*) ... = *measure* (*measure-pmf p*) (⋃ *x*∈*set-spmf p*. {*Some x*})
by(*rule measure-pmf.measure-zero-union*)(*auto simp*: *measure-pmf.prob-eq-0 AE-measure-pmf-iff in-set-spmf-iff-spmf set-pmf-iff*)
also have *ennreal* ... = ∫⁺ *x*. *measure* (*measure-pmf p*) {*Some x*} ∂*count-space* (*set-spmf p*)
unfolding *measure-pmf.emeasure-eq-measure*[*symmetric*]
by(*simp-all add*: *emeasure-UN-countable disjoint-family-on-def*)
also have ... = ∫⁺ *x*. *spmf p x* ∂*count-space* (*set-spmf p*) **by**(*simp add*: *pmf-def*)
also have ... = ∫⁺ *x*. *spmf q x* ∂*count-space* (*set-spmf p*) **by**(*simp add*: *assms*)
also have *set-spmf p* = *set-spmf q* **by**(*auto simp*: *in-set-spmf-iff-spmf assms*)
also have (∫⁺ *x*. *spmf q x* ∂*count-space* (*set-spmf q*)) = ∫⁺ *x*. *measure* (*measure-pmf q*) {*Some x*} ∂*count-space* (*set-spmf q*)
by(*simp add*: *pmf-def*)
also have ... = *measure* (*measure-pmf q*) (⋃ *x*∈*set-spmf q*. {*Some x*})
unfolding *measure-pmf.emeasure-eq-measure*[*symmetric*]
by(*simp-all add*: *emeasure-UN-countable disjoint-family-on-def*)
also have ... = *measure* (*measure-pmf q*) ((⋃ *x*∈*set-spmf q*. {*Some x*}) ∪ *Some* ' (− *set-spmf q*))
by(*rule ennreal-cong measure-pmf.measure-zero-union*[*symmetric*])+(*auto simp*: *measure-pmf.prob-eq-0 AE-measure-pmf-iff in-set-spmf-iff-spmf set-pmf-iff*)
also have ((⋃ *x*∈*set-spmf q*. {*Some x*}) ∪ *Some* ' (− *set-spmf q*)) = *range Some* **by** *auto*
also have *ennreal 1* − *measure* (*measure-pmf q*) ... = *measure* (*measure-pmf q*) (*space* (*measure-pmf q*) − *range Some*)
by(*simp add*: *one-ereal-def measure-pmf.prob-compl ennreal-minus*[*symmetric*] *del*: *space-measure-pmf*)
also have *space* (*measure-pmf q*) − *range Some* = {*i*}
by(*auto simp*: *None intro*: *ccontr*)
also have *measure* (*measure-pmf q*) ... = *pmf q i* **by**(*simp add*: *pmf-def*)
finally show *?thesis* **by** *simp*
qed
qed

lemma *integral-measure-spmf-restrict*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *second-countable-topology*}
shows (∫ *x*. *f x* ∂*measure-spmf M*) = (∫ *x*. *f x* ∂*restrict-space* (*measure-spmf M*) (*set-spmf M*))
by(*auto intro*!: *integral-cong-AE simp add*: *integral-restrict-space*)

lemma *nn-integral-measure-spmf'*:
(∫⁺ *x*. *f x* ∂*measure-spmf p*) = ∫⁺ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space* (*set-spmf p*)
by(*auto simp*: *nn-integral-measure-spmf nn-integral-count-space-indicator in-set-spmf-iff-spmf intro*!: *nn-integral-cong split*: *split-indicator*)

25.4 Functorial structure

abbreviation *map-spmf* :: $('a \Rightarrow 'b) \Rightarrow 'a\ spmf \Rightarrow 'b\ spmf$
where *map-spmf f* $\equiv$ *map-pmf* (*map-option f*)

context begin
local-setup ‹*Local-Theory.map-background-naming* (*Name-Space.mandatory-path spmf*)›

lemma *map-comp*: *map-spmf f* (*map-spmf g p*) = *map-spmf* (*f* ∘ *g*) *p*
by(*simp add*: *pmf.map-comp o-def option.map-comp*)

lemma *map-id0*: *map-spmf id* = *id*
by(*simp add*: *pmf.map-id option.map-id0*)

lemma *map-id* [*simp*]: *map-spmf id p* = *p*
by(*simp add*: *map-id0*)

lemma *map-ident* [*simp*]: *map-spmf* (λ*x*. *x*) *p* = *p*
by(*simp add*: *id-def*[*symmetric*])

end

lemma *set-map-spmf* [*simp*]: *set-spmf* (*map-spmf f p*) = *f* ‘ *set-spmf p*
by(*simp add*: *set-spmf-def image-bind bind-image o-def Option.option.set-map*)

lemma *map-spmf-cong*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* ⟹ *f x* = *g x* ⟧ ⟹ *map-spmf f p* = *map-spmf g q*
by(*auto intro*: *pmf.map-cong option.map-cong simp add*: *in-set-spmf*)

lemma *map-spmf-cong-simp*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* =*simp*=> *f x* = *g x* ⟧
⟹ *map-spmf f p* = *map-spmf g q*
unfolding *simp-implies-def* **by**(*rule map-spmf-cong*)

lemma *map-spmf-idI*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *f x* = *x*) ⟹ *map-spmf f p* = *p*
by(*rule map-pmf-idI map-option-idI*)+(*simp add*: *in-set-spmf*)

lemma *emeasure-map-spmf*:
emeasure (*measure-spmf* (*map-spmf f p*)) *A* = *emeasure* (*measure-spmf p*) (*f* −‘ *A*)
by(*auto simp*: *measure-spmf-def emeasure-distr measurable-restrict-space1 space-restrict-space emeasure-restrict-space intro*: *arg-cong2*[**where** *f*=*emeasure*])

lemma *measure-map-spmf*: *measure* (*measure-spmf* (*map-spmf f p*)) *A* = *measure* (*measure-spmf p*) (*f* −‘ *A*)
using *emeasure-map-spmf*[*of f p A*] **by**(*simp add*: *measure-spmf.emeasure-eq-measure*)

lemma *measure-map-spmf-conv-distr*:
measure-spmf (*map-spmf f p*) = *distr* (*measure-spmf p*) (*count-space UNIV*) *f*

by(*rule measure-eqI*)(*simp-all add*: *emeasure-map-spmf emeasure-distr*)

lemma *spmf-map-pmf-Some* [*simp*]: *spmf* (*map-pmf Some p*) *i* = *pmf p i*
by(*simp add*: *pmf-map-inj'*)

lemma *spmf-map-inj*: ⟦ *inj-on f* (*set-spmf M*); *x* ∈ *set-spmf M* ⟧ ⟹ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
by (*smt* (*verit*) *elem-set in-set-spmf inj-on-def option.inj-map-strong option.map*(*2*) *pmf-map-inj*)

lemma *spmf-map-inj'*: *inj f* ⟹ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
by(*subst option.map*(*2*)[*symmetric*, **where** *f*=*f*])(*rule pmf-map-inj'*[*OF option.inj-map*])

lemma *spmf-map-outside*: *x* ∉ *f* ‘ *set-spmf M* ⟹ *spmf* (*map-spmf f M*) *x* = *0*
unfolding *spmf-eq-0-set-spmf* **by** *simp*

lemma *ennreal-spmf-map*: *ennreal* (*spmf* (*map-spmf f p*) *x*) = *emeasure* (*measure-spmf p*) (*f* −‘ {*x*})
by (*metis emeasure-map-spmf emeasure-spmf-single*)

lemma *spmf-map*: *spmf* (*map-spmf f p*) *x* = *measure* (*measure-spmf p*) (*f* −‘ {*x*})
using *ennreal-spmf-map*[*of f p x*] **by**(*simp add*: *measure-spmf.emeasure-eq-measure*)

lemma *ennreal-spmf-map-conv-nn-integral*:
ennreal (*spmf* (*map-spmf f p*) *x*) = *integral*^{N} (*measure-spmf p*) (*indicator* (*f* −‘ {*x*}))
by (*simp add*: *ennreal-spmf-map*)

25.5 Monad operations

25.5.1 Return

abbreviation *return-spmf* :: *'a* ⇒ *'a spmf*
where *return-spmf x* ≡ *return-pmf* (*Some x*)

lemma *pmf-return-spmf*: *pmf* (*return-spmf x*) *y* = *indicator* {*y*} (*Some x*)
by(*fact pmf-return*)

lemma *measure-spmf-return-spmf*: *measure-spmf* (*return-spmf x*) = *Giry-Monad.return* (*count-space UNIV*) *x*
by(*rule measure-eqI*)(*simp-all add*: *measure-spmf-def emeasure-distr space-restrict-space emeasure-restrict-space indicator-def*)

lemma *measure-spmf-return-pmf-None* [*simp*]: *measure-spmf* (*return-pmf None*) = *null-measure* (*count-space UNIV*)
by (*simp add*: *emeasure-measure-spmf-conv-measure-pmf measure-eqI*)

lemma *set-return-spmf* [*simp*]: *set-spmf* (*return-spmf x*) = {*x*}
by(*auto simp*: *set-spmf-def*)

25.5.2 Bind

definition *bind-spmf* :: *'a spmf ⇒ ('a ⇒ 'b spmf) ⇒ 'b spmf*
where *bind-spmf x f = bind-pmf x (λa. case a of None ⇒ return-pmf None | Some a' ⇒ f a')*

adhoc-overloading *Monad-Syntax.bind ⇌ bind-spmf*

lemma *return-None-bind-spmf* [*simp*]: *return-pmf None ⤜ (f :: 'a ⇒ -) = return-pmf None*
by(*simp add*: *bind-spmf-def bind-return-pmf*)

lemma *return-bind-spmf* [*simp*]: *return-spmf x ⤜ f = f x*
by(*simp add*: *bind-spmf-def bind-return-pmf*)

lemma *bind-return-spmf* [*simp*]: *x ⤜ return-spmf = x*
proof −
have *⋀a :: 'a option. (case a of None ⇒ return-pmf None | Some a' ⇒ return-spmf a') = return-pmf a*
by(*simp split*: *option.split*)
then show *?thesis*
by(*simp add*: *bind-spmf-def bind-return-pmf'*)
qed

lemma *bind-spmf-assoc* [*simp*]:
fixes *x :: 'a spmf* **and** *f :: 'a ⇒ 'b spmf* **and** *g :: 'b ⇒ 'c spmf*
shows *(x ⤜ f) ⤜ g = x ⤜ (λy. f y ⤜ g)*
unfolding *bind-spmf-def*
by (*smt* (*verit*, *best*) *bind-assoc-pmf bind-pmf-cong bind-return-pmf option.case-eq-if*)

lemma *pmf-bind-spmf-None*: *pmf (p ⤜ f) None = pmf p None + ∫ x. pmf (f x) None ∂measure-spmf p*
(**is** *?lhs = ?rhs*)
proof −
let *?f = λx. pmf (case x of None ⇒ return-pmf None | Some x ⇒ f x) None*
have *?lhs = ∫ x. ?f x ∂measure-pmf p*
by(*simp add*: *bind-spmf-def pmf-bind*)
also have *... = ∫ x. ?f None * indicator {None} x + ?f x * indicator (range Some) x ∂measure-pmf p*
by(*rule Bochner-Integration.integral-cong*)(*auto simp*: *indicator-def*)
also have *... = (∫ x. ?f None * indicator {None} x ∂measure-pmf p) + (∫ x. ?f x * indicator (range Some) x ∂measure-pmf p)*
by(*rule Bochner-Integration.integral-add*)(*auto 4 3 intro*: *integrable-real-mult-indicator measure-pmf.integrable-const-bound*[**where** *B=1*] *simp add*: *AE-measure-pmf-iff pmf-le-1*)
also have *... = pmf p None + ∫ x. indicator (range Some) x * pmf (f (the x)) None ∂measure-pmf p*
by(*auto simp*: *measure-measure-pmf-finite indicator-eq-0-iff intro*!: *Bochner-Integration.integral-cong*)
also have *... = ?rhs*
unfolding *measure-spmf-def*

by(*subst integral-distr*)(*auto simp*: *integral-restrict-space*)
finally show *?thesis* **.**
qed

lemma *spmf-bind*: *spmf* (*p* ⤜ *f*) *y* = ∫ *x*. *spmf* (*f x*) *y* ∂*measure-spmf p*
proof −
have ⋀*x*. *spmf* (*case x of None* ⇒ *return-pmf None* | *Some x* ⇒ *f x*) *y* =
indicat-real (*range Some*) *x* ∗ *spmf* (*f* (*the x*)) *y*
by (*simp add*: *split*: *option.split*)
then show *?thesis*
by (*simp add*: *measure-spmf-def integral-distr bind-spmf-def pmf-bind integral-restrict-space*)
qed

lemma *ennreal-spmf-bind*: *ennreal* (*spmf* (*p* ⤜ *f*) *x*) = ∫$^+$ *y*. *spmf* (*f y*) *x* ∂*measure-spmf p*
proof −
have ⋀*y*. *ennreal* (*spmf* (*case y of None* ⇒ *return-pmf None* | *Some x* ⇒ *f x*) *x*) =
ennreal (*spmf* (*f* (*the y*)) *x*) ∗ *indicator* (*range Some*) *y*
by (*simp add*: *split*: *option.split*)
then show *?thesis*
by (*simp add*: *bind-spmf-def ennreal-pmf-bind nn-integral-measure-spmf-conv-measure-pmf nn-integral-restrict-space*)
qed

lemma *measure-spmf-bind-pmf*: *measure-spmf* (*p* ⤜ *f*) = *measure-pmf p* ⤜ *measure-spmf* ∘ *f*
(**is** *?lhs* = *?rhs*)
proof(*rule measure-eqI*)
show *sets ?lhs* = *sets ?rhs*
by (*simp add*: *Giry-Monad.bind-def*)
next
fix *A* :: *'a set*
have *emeasure ?lhs A* = ∫$^+$ *x*. *emeasure* (*measure-spmf* (*f x*)) *A* ∂*measure-pmf p*
by(*simp add*: *measure-spmf-def emeasure-distr space-restrict-space emeasure-restrict-space bind-spmf-def*)
also have ... = *emeasure ?rhs A*
by(*simp add*: *emeasure-bind*[**where** *N*=*count-space UNIV*] *space-measure-spmf space-subprob-algebra*)
finally show *emeasure ?lhs A* = *emeasure ?rhs A* **.**
qed

lemma *measure-spmf-bind*: *measure-spmf* (*p* ⤜ *f*) = *measure-spmf p* ⤜ *measure-spmf* ∘ *f*
(**is** *?lhs* = *?rhs*)
proof(*rule measure-eqI*)
show *sets ?lhs* = *sets ?rhs*

by(*simp add*: *sets-bind*[**where** *N*=*count-space UNIV*] *space-measure-spmf*)
next
fix *A* :: *'a set*
let *?A* = *the* −‘ *A* ∩ *range Some*
have *emeasure ?lhs A* = ∫⁺ *x*. *emeasure* (*measure-pmf* (*case x of None* ⇒ *return-pmf None* | *Some x* ⇒ *f x*)) *?A* ∂*measure-pmf p*
by(*simp add*: *measure-spmf-def emeasure-distr space-restrict-space emeasure-restrict-space bind-spmf-def*)
also have ... = ∫⁺ *x*. *emeasure* (*measure-pmf* (*f* (*the x*))) *?A* ∗ *indicator* (*range Some*) *x* ∂*measure-pmf p*
by(*rule nn-integral-cong*)(*auto split*: *option.split simp add*: *indicator-def*)
also have ... = ∫⁺ *x*. *emeasure* (*measure-spmf* (*f x*)) *A* ∂*measure-spmf p*
by(*simp add*: *measure-spmf-def nn-integral-distr nn-integral-restrict-space emeasure-distr space-restrict-space emeasure-restrict-space*)
also have ... = *emeasure ?rhs A*
by(*simp add*: *emeasure-bind*[**where** *N*=*count-space UNIV*] *space-measure-spmf space-subprob-algebra*)
finally show *emeasure ?lhs A* = *emeasure ?rhs A* **.**
qed

lemma *map-spmf-bind-spmf*: *map-spmf f* (*bind-spmf p g*) = *bind-spmf p* (*map-spmf f* ∘ *g*)
by(*auto simp*: *bind-spmf-def map-bind-pmf fun-eq-iff split*: *option.split intro*: *arg-cong2*[**where** *f*=*bind-pmf*])

lemma *bind-map-spmf*: *map-spmf f p* ⤜ *g* = *p* ⤜ *g* ∘ *f*
by(*simp add*: *bind-spmf-def bind-map-pmf o-def cong del*: *option.case-cong-weak*)

lemma *spmf-bind-leI*:
assumes ⋀*y*. *y* ∈ *set-spmf p* ⟹ *spmf* (*f y*) *x* ≤ *r*
and *0* ≤ *r*
shows *spmf* (*bind-spmf p f*) *x* ≤ *r*
proof −
have *ennreal* (*spmf* (*bind-spmf p f*) *x*) = ∫⁺ *y*. *spmf* (*f y*) *x* ∂*measure-spmf p*
by(*rule ennreal-spmf-bind*)
also have ... ≤ ∫⁺ *y*. *r* ∂*measure-spmf p*
by(*rule nn-integral-mono-AE*)(*simp add*: *assms*)
also have ... ≤ *r*
using *assms measure-spmf.emeasure-space-le-1*
by(*auto simp*: *measure-spmf.emeasure-eq-measure intro*!: *mult-left-le*)
finally show *?thesis* **using** *assms*(*2*) **by**(*simp*)
qed

lemma *map-spmf-conv-bind-spmf*: *map-spmf f p* = (*p* ⤜ (λ*x*. *return-spmf* (*f x*)))
by(*simp add*: *map-pmf-def bind-spmf-def*)(*rule bind-pmf-cong*, *simp-all split*: *option.split*)

lemma *bind-spmf-cong*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* ⟹ *f x* = *g x* ⟧ ⟹ *bind-spmf p f* = *bind-spmf q g*

by(*auto simp*: *bind-spmf-def in-set-spmf intro*: *bind-pmf-cong option.case-cong*)

lemma *bind-spmf-cong-simp*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* =*simp*=> *f x* = *g x* ⟧
⟹ *bind-spmf p f* = *bind-spmf q g*
by(*simp add*: *simp-implies-def cong*: *bind-spmf-cong*)

lemma *set-bind-spmf*: *set-spmf* (*M* ⤜ *f*) = *set-spmf M* ⤜ (*set-spmf* ∘ *f*)
by(*auto simp*: *set-spmf-def bind-spmf-def bind-UNION split*: *option.splits*)

lemma *bind-spmf-const-return-None* [*simp*]: *bind-spmf p* (λ-. *return-pmf None*) = *return-pmf None*
by(*simp add*: *bind-spmf-def case-option-collapse*)

lemma *bind-commute-spmf*:
bind-spmf p (λ*x*. *bind-spmf q* (*f x*)) = *bind-spmf q* (λ*y*. *bind-spmf p* (λ*x*. *f x y*))
(**is** *?lhs* = *?rhs*)
proof −
let *?f* = λ*x y*. *case x of None* ⇒ *return-pmf None* | *Some a* ⇒ (*case y of None* ⇒ *return-pmf None* | *Some b* ⇒ *f a b*)
have *?lhs* = *p* ⤜ (λ*x*. *q* ⤜ *?f x*)
unfolding *bind-spmf-def* **by**(*rule bind-pmf-cong*[*OF refl*])(*simp split*: *option.split*)
also have ... = *q* ⤜ (λ*y*. *p* ⤜ (λ*x*. *?f x y*)) **by**(*rule bind-commute-pmf*)
also have ... = *?rhs* **unfolding** *bind-spmf-def*
by(*rule bind-pmf-cong*[*OF refl*])(*auto split*: *option.split*, *metis bind-spmf-const-return-None bind-spmf-def*)
finally show *?thesis* **.**
qed

25.6 Relator

abbreviation *rel-spmf* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a spmf* ⇒ '*b spmf* ⇒ *bool*
where *rel-spmf R* ≡ *rel-pmf* (*rel-option R*)

lemma *rel-spmf-mono*:
⟦*rel-spmf A f g*; ⋀*x y*. *A x y* ⟹ *B x y* ⟧ ⟹ *rel-spmf B f g*
by (*metis option.rel-sel pmf.rel-mono-strong*)

lemma *rel-spmf-mono-strong*:
⟦ *rel-spmf A f g*; ⋀*x y*. ⟦ *A x y*; *x* ∈ *set-spmf f*; *y* ∈ *set-spmf g* ⟧ ⟹ *B x y* ⟧ ⟹ *rel-spmf B f g*
by (*metis elem-set in-set-spmf option.rel-mono-strong pmf.rel-mono-strong*)

lemma *rel-spmf-reflI*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *P x x*) ⟹ *rel-spmf P p p*
by (*metis* (*mono-tags*, *lifting*) *option.rel-eq pmf.rel-eq rel-spmf-mono-strong*)

lemma *rel-spmfI* [*intro?*]:
⟦ ⋀*x y*. (*x*, *y*) ∈ *set-spmf pq* ⟹ *P x y*; *map-spmf fst pq* = *p*; *map-spmf snd pq* = *q* ⟧

$\Longrightarrow$ *rel-spmf P p q*
by(*rule rel-pmf.intros*[**where** *pq*=*map-pmf* (λ*x*. *case x of None* $\Rightarrow$ (*None*, *None*)
| *Some* (*a*, *b*) $\Rightarrow$ (*Some a*, *Some b*)) *pq*])
(*auto simp*: *pmf.map-comp o-def in-set-spmf split*: *option.splits intro*: *pmf.map-cong*)

lemma *rel-spmfE* [*elim?*, *consumes 1*, *case-names rel-spmf*]:
assumes *rel-spmf P p q*
obtains *pq* **where**
$\bigwedge$*x y*. (*x*, *y*) $\in$ *set-spmf pq* $\Longrightarrow$ *P x y*
p = *map-spmf fst pq*
q = *map-spmf snd pq*
using *assms*
proof(*cases rule*: *rel-pmf.cases*[*consumes 1*, *case-names rel-pmf*])
case (*rel-pmf pq*)
let *?pq* = *map-pmf* (λ(*a*, *b*). *case* (*a*, *b*) *of* (*Some x*, *Some y*) $\Rightarrow$ *Some* (*x*, *y*) | -
$\Rightarrow$ *None*) *pq*
have $\bigwedge$*x y*. (*x*, *y*) $\in$ *set-spmf ?pq* $\Longrightarrow$ *P x y*
by(*auto simp*: *in-set-spmf split*: *option.split-asm dest*: *rel-pmf*(*1*))
moreover
have $\bigwedge$*x*. (*x*, *None*) $\in$ *set-pmf pq* $\Longrightarrow$ *x* = *None* **by**(*auto dest*!: *rel-pmf*(*1*))
then have *p* = *map-spmf fst ?pq* **using** *rel-pmf*(*2*)
by(*auto simp*: *pmf.map-comp split-beta intro*!: *pmf.map-cong split*: *option.split*)
moreover
have $\bigwedge$*y*. (*None*, *y*) $\in$ *set-pmf pq* $\Longrightarrow$ *y* = *None* **by**(*auto dest*!: *rel-pmf*(*1*))
then have *q* = *map-spmf snd ?pq* **using** *rel-pmf*(*3*)
by(*auto simp*: *pmf.map-comp split-beta intro*!: *pmf.map-cong split*: *option.split*)
ultimately show *thesis* **..**
qed

lemma *rel-spmf-simps*:
rel-spmf R p q $\longleftrightarrow$ ($\exists$ *pq*. ($\forall$ (*x*, *y*)$\in$*set-spmf pq*. *R x y*) $\wedge$ *map-spmf fst pq* = *p* $\wedge$
map-spmf snd pq = *q*)
by(*auto intro*: *rel-spmfI elim*!: *rel-spmfE*)

lemma *spmf-rel-map*:
shows *spmf-rel-map1*: $\bigwedge$*R f x*. *rel-spmf R* (*map-spmf f x*) = *rel-spmf* (λ*x*. *R* (*f*
x)) *x*
and *spmf-rel-map2*: $\bigwedge$*R x g y*. *rel-spmf R x* (*map-spmf g y*) = *rel-spmf* (λ*x y*.
R x (*g y*)) *x y*
by(*simp-all add*: *fun-eq-iff pmf.rel-map option.rel-map*[*abs-def*])

lemma *spmf-rel-conversep*: *rel-spmf* R^{-1-1} = (*rel-spmf R*)$^{-1-1}$
by(*simp add*: *option.rel-conversep pmf.rel-conversep*)

lemma *spmf-rel-eq*: *rel-spmf* (=) = (=)
by(*simp add*: *pmf.rel-eq option.rel-eq*)

context includes *lifting-syntax*
begin

lemma *bind-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> (*A* ===> *rel-spmf B*) ===> *rel-spmf B*) *bind-spmf bind-spmf*
unfolding *bind-spmf-def*[*abs-def*] **by** *transfer-prover*

lemma *return-spmf-parametric*: (*A* ===> *rel-spmf A*) *return-spmf return-spmf*
by *transfer-prover*

lemma *map-spmf-parametric*: ((*A* ===> *B*) ===> *rel-spmf A* ===> *rel-spmf B*) *map-spmf map-spmf*
by *transfer-prover*

lemma *rel-spmf-parametric*:
((*A* ===> *B* ===> (=)) ===> *rel-spmf A* ===> *rel-spmf B* ===> (=)) *rel-spmf rel-spmf*
by *transfer-prover*

lemma *set-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> *rel-set A*) *set-spmf set-spmf*
unfolding *set-spmf-def*[*abs-def*] **by** *transfer-prover*

lemma *return-spmf-None-parametric*:
(*rel-spmf A*) (*return-pmf None*) (*return-pmf None*)
by *simp*

end

lemma *rel-spmf-bindI*:
⟦ *rel-spmf R p q*; ⋀*x y*. *R x y* ⟹ *rel-spmf P* (*f x*) (*g y*) ⟧
⟹ *rel-spmf P* (*p* ⤜ *f*) (*q* ⤜ *g*)
by(*fact bind-spmf-parametric*[*THEN rel-funD*, *THEN rel-funD*, *OF - rel-funI*])

lemma *rel-spmf-bind-reflI*:
(⋀*x*. *x* ∈ *set-spmf p* ⟹ *rel-spmf P* (*f x*) (*g x*)) ⟹ *rel-spmf P* (*p* ⤜ *f*) (*p* ⤜ *g*)
by(*rule rel-spmf-bindI*[**where** *R*=λ*x y*. *x* = *y* ∧ *x* ∈ *set-spmf p*])(*auto intro*: *rel-spmf-reflI*)

lemma *rel-pmf-return-pmfI*: *P x y* ⟹ *rel-pmf P* (*return-pmf x*) (*return-pmf y*)
by *simp*

context includes *lifting-syntax*
begin

We do not yet have a relator for *'a measure*, so we combine *Sigma-Algebra.measure* and *measure-pmf*

lemma *measure-pmf-parametric*:
(*rel-pmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-pmf p*)) (λ*q*. *measure* (*measure-pmf q*))

proof(*rule rel-funI*)+
fix *p q X Y*
assume *rel-pmf A p q* **and** *rel-pred A X Y*
from *this*(*1*) **obtain** *pq* **where** *A*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *A x y*
and *p*: *p* = *map-pmf fst pq* **and** *q*: *q* = *map-pmf snd pq* **by** *cases auto*
show *measure p X* = *measure q Y* **unfolding** *p q measure-map-pmf*
by(*rule measure-pmf.finite-measure-eq-AE*)(*auto simp*: *AE-measure-pmf-iff dest*!: *A rel-predD*[*OF* ‹*rel-pred* - - -›])
qed

lemma *measure-spmf-parametric*:
(*rel-spmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-spmf p*)) (λ*q*. *measure* (*measure-spmf q*))
proof −
have ⋀*x y xa ya*. *rel-pred A xa ya* ⟹ *rel-pred* (*rel-option A*) (*Some* ‘ *xa*) (*Some* ‘ *ya*)
by(*auto simp*: *rel-pred-def rel-fun-def elim*: *option.rel-cases*)
then show *?thesis*
unfolding *measure-measure-spmf-conv-measure-pmf*[*abs-def*]
by (*intro rel-funI*) (*force elim*!: *measure-pmf-parametric*[*THEN rel-funD*, *THEN rel-funD*])
qed

end

25.7 From *'a pmf* to *'a spmf*

definition *spmf-of-pmf* :: *'a pmf* ⇒ *'a spmf*
where *spmf-of-pmf* = *map-pmf Some*

lemma *set-spmf-spmf-of-pmf* [*simp*]: *set-spmf* (*spmf-of-pmf p*) = *set-pmf p*
by(*auto simp*: *spmf-of-pmf-def set-spmf-def bind-image o-def*)

lemma *spmf-spmf-of-pmf* [*simp*]: *spmf* (*spmf-of-pmf p*) *x* = *pmf p x*
by(*simp add*: *spmf-of-pmf-def*)

lemma *pmf-spmf-of-pmf-None* [*simp*]: *pmf* (*spmf-of-pmf p*) *None* = *0*
using *ennreal-pmf-map*[*of Some p None*] **by**(*simp add*: *spmf-of-pmf-def*)

lemma *emeasure-spmf-of-pmf* [*simp*]: *emeasure* (*measure-spmf* (*spmf-of-pmf p*)) *A* = *emeasure* (*measure-pmf p*) *A*
by(*simp add*: *emeasure-measure-spmf-conv-measure-pmf spmf-of-pmf-def inj-vimage-image-eq*)

lemma *measure-spmf-spmf-of-pmf* [*simp*]: *measure-spmf* (*spmf-of-pmf p*) = *measure-pmf p*
by(*rule measure-eqI*) *simp-all*

lemma *map-spmf-of-pmf* [*simp*]: *map-spmf f* (*spmf-of-pmf p*) = *spmf-of-pmf* (*map-pmf f p*)

by(*simp add*: *spmf-of-pmf-def pmf.map-comp o-def*)

lemma *rel-spmf-spmf-of-pmf* [*simp*]: *rel-spmf R* (*spmf-of-pmf p*) (*spmf-of-pmf q*) = *rel-pmf R p q*
by(*simp add*: *spmf-of-pmf-def pmf.rel-map*)

lemma *spmf-of-pmf-return-pmf* [*simp*]: *spmf-of-pmf* (*return-pmf x*) = *return-spmf x*
by(*simp add*: *spmf-of-pmf-def*)

lemma *bind-spmf-of-pmf* [*simp*]: *bind-spmf* (*spmf-of-pmf p*) *f* = *bind-pmf p f*
by(*simp add*: *spmf-of-pmf-def bind-spmf-def bind-map-pmf*)

lemma *set-spmf-bind-pmf*: *set-spmf* (*bind-pmf p f*) = *Set.bind* (*set-pmf p*) (*set-spmf* ∘ *f*)
unfolding *bind-spmf-of-pmf*[*symmetric*] **by**(*subst set-bind-spmf*) *simp*

lemma *spmf-of-pmf-bind*: *spmf-of-pmf* (*bind-pmf p f*) = *bind-pmf p* (λ*x*. *spmf-of-pmf* (*f x*))
by(*simp add*: *spmf-of-pmf-def map-bind-pmf*)

lemma *bind-pmf-return-spmf*: *p* ⤜ (λ*x*. *return-spmf* (*f x*)) = *spmf-of-pmf* (*map-pmf f p*)
by(*simp add*: *map-pmf-def spmf-of-pmf-bind*)

25.8 Weight of a subprobability

abbreviation *weight-spmf* :: *'a spmf* ⇒ *real*
where *weight-spmf p* ≡ *measure* (*measure-spmf p*) (*space* (*measure-spmf p*))

lemma *weight-spmf-def*: *weight-spmf p* = *measure* (*measure-spmf p*) *UNIV*
by(*simp add*: *space-measure-spmf*)

lemma *weight-spmf-le-1*: *weight-spmf p* ≤ *1*
by(*rule measure-spmf.subprob-measure-le-1*)

lemma *weight-return-spmf* [*simp*]: *weight-spmf* (*return-spmf x*) = *1*
by(*simp add*: *measure-spmf-return-spmf measure-return*)

lemma *weight-return-pmf-None* [*simp*]: *weight-spmf* (*return-pmf None*) = *0*
by(*simp*)

lemma *weight-map-spmf* [*simp*]: *weight-spmf* (*map-spmf f p*) = *weight-spmf p*
by(*simp add*: *weight-spmf-def measure-map-spmf*)

lemma *weight-spmf-of-pmf* [*simp*]: *weight-spmf* (*spmf-of-pmf p*) = *1*
by *simp*

lemma *weight-spmf-nonneg*: *weight-spmf p* ≥ *0*

by(*fact measure-nonneg*)

lemma (**in** *finite-measure*) *integrable-weight-spmf* [*simp*]:
(λ*x*. *weight-spmf* (*f x*)) ∈ *borel-measurable M* ⟹ *integrable M* (λ*x*. *weight-spmf* (*f x*))
by(*rule integrable-const-bound*[**where** *B=1*])(*simp-all add*: *weight-spmf-nonneg weight-spmf-le-1*)

lemma *weight-spmf-eq-nn-integral-spmf*: *weight-spmf p* = ∫⁺ *x*. *spmf p x* ∂*count-space UNIV*
by (*metis NO-MATCH-def measure-spmf.emeasure-eq-measure nn-integral-count-space-indicator nn-integral-indicator nn-integral-measure-spmf sets-UNIV sets-measure-spmf space-measure-spmf*)

lemma *weight-spmf-eq-nn-integral-support*:
weight-spmf p = ∫⁺ *x*. *spmf p x* ∂*count-space* (*set-spmf p*)
unfolding *weight-spmf-eq-nn-integral-spmf*
by(*auto simp*: *nn-integral-count-space-indicator in-set-spmf-iff-spmf intro*!: *nn-integral-cong split*: *split-indicator*)

lemma *pmf-None-eq-weight-spmf*: *pmf p None* = *1* − *weight-spmf p*
proof −
have *weight-spmf p* = ∫⁺ *x*. *spmf p x* ∂*count-space UNIV* **by**(*rule weight-spmf-eq-nn-integral-spmf*)
also have ... = ∫⁺ *x*. *ennreal* (*pmf p x*) ∗ *indicator* (*range Some*) *x* ∂*count-space UNIV*
by(*simp add*: *nn-integral-count-space-indicator*[*symmetric*] *embed-measure-count-space*[*symmetric*] *nn-integral-embed-measure measurable-embed-measure1*)
also have ... + *pmf p None* = ∫⁺ *x*. *ennreal* (*pmf p x*) ∗ *indicator* (*range Some*) *x* + *ennreal* (*pmf p None*) ∗ *indicator* {*None*} *x* ∂*count-space UNIV*
by(*subst nn-integral-add*)(*simp-all add*: *max-def*)
also have ... = ∫⁺ *x*. *pmf p x* ∂*count-space UNIV*
by(*rule nn-integral-cong*)(*auto split*: *split-indicator*)
also have ... = *1* **by** (*simp add*: *nn-integral-pmf*)
finally show *?thesis* **by**(*simp add*: *ennreal-plus*[*symmetric*] *del*: *ennreal-plus*)
qed

lemma *weight-spmf-conv-pmf-None*: *weight-spmf p* = *1* − *pmf p None*
by(*simp add*: *pmf-None-eq-weight-spmf*)

lemma *weight-spmf-lt-0*: ¬ *weight-spmf p* < *0*
by(*simp add*: *not-less weight-spmf-nonneg*)

lemma *spmf-le-weight*: *spmf p x* ≤ *weight-spmf p*
by (*simp add*: *measure-spmf.bounded-measure spmf-conv-measure-spmf*)

lemma *weight-spmf-eq-0*: *weight-spmf p* = *0* ⟷ *p* = *return-pmf None*
by (*metis measure-le-0-iff measure-spmf.bounded-measure spmf-conv-measure-spmf spmf-eqI weight-return-pmf-None*)

lemma *weight-bind-spmf*: *weight-spmf* (*x* ⤜ *f*) = *lebesgue-integral* (*measure-spmf*

x) (*weight-spmf* ∘ *f*)
unfolding *weight-spmf-def*
by(*simp add*: *measure-spmf-bind o-def measure-spmf.measure-bind*[**where** *N*=*count-space UNIV*])

lemma *rel-spmf-weightD*: *rel-spmf A p q* ⟹ *weight-spmf p* = *weight-spmf q*
by(*erule rel-spmfE*) *simp*

lemma *rel-spmf-bij-betw*:
assumes *f*: *bij-betw f* (*set-spmf p*) (*set-spmf q*)
and *eq*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *spmf p x* = *spmf q* (*f x*)
shows *rel-spmf* (λ*x y*. *f x* = *y*) *p q*
proof −
let *?f* = *map-option f*

have *weq*: *ennreal* (*weight-spmf p*) = *ennreal* (*weight-spmf q*)
unfolding *weight-spmf-eq-nn-integral-support*
by(*subst nn-integral-bij-count-space*[*OF f*, *symmetric*])(*rule nn-integral-cong-AE*, *simp add*: *eq AE-count-space*)
then have *None* ∈ *set-pmf p* ⟷ *None* ∈ *set-pmf q*
by(*simp add*: *pmf-None-eq-weight-spmf set-pmf-iff*)
with *f* **have** *bij-betw* (*map-option f*) (*set-pmf p*) (*set-pmf q*)
apply(*auto simp*: *bij-betw-def in-set-spmf inj-on-def intro*: *option.expand split*: *option.split*)
apply(*rename-tac* [!] *x*)
apply(*case-tac* [!] *x*)
apply(*auto iff*: *in-set-spmf*)
done
then have *rel-pmf* (λ*x y*. *?f x* = *y*) *p q*
proof (*rule rel-pmf-bij-betw*)
show *pmf p x* = *pmf q* (*map-option f x*) **if** *x* ∈ *set-pmf p* **for** *x*
proof (*cases x*)
case *None*
then show *?thesis*
by (*metis ennreal-inj measure-nonneg option.map-disc-iff pmf-None-eq-weight-spmf weq*)
qed (*use eq in-set-spmf that* **in** *force*)
qed
thus *?thesis*
by (*smt* (*verit*, *ccfv-SIG*) *None-eq-map-option-iff option.map-sel option.rel-sel pmf.rel-mono-strong*)
qed

25.9 From density to spmfs

context fixes *f* :: *'a* ⇒ *real* **begin**

definition *embed-spmf* :: *'a spmf*
where *embed-spmf* = *embed-pmf* (λ*x*. *case x of None* ⇒ *1* − *enn2real* (∫⁺ *x*.

ennreal (f x) ∂count-space UNIV) | Some x′ ⇒ max 0 (f x′))

context
assumes *prob*: $(\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV) \leq 1$
begin

lemma *nn-integral-embed-spmf-eq-1*:
$(\int^+ x.\ ennreal\ (case\ x\ of\ None \Rightarrow 1 - enn2real\ (\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV) \mid Some\ x' \Rightarrow max\ 0\ (f\ x'))\ \partial count\text{-}space\ UNIV) = 1$
(**is** *?lhs* = - **is** $(\int^+ x.\ ?f\ x\ \partial ?M) = -$)
proof −
have $?lhs = \int^+ x.\ ?f\ x * indicator\ \{None\}\ x + ?f\ x * indicator\ (range\ Some)\ x\ \partial ?M$
by(*rule nn-integral-cong*)(*auto split*: *split-indicator*)
also have $\ldots = (1 - enn2real\ (\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV)) + \int^+ x.\ ?f\ x * indicator\ (range\ Some)\ x\ \partial ?M$
(**is** - = *?None* + *?Some*)
by(*subst nn-integral-add*)(*simp-all add*: *AE-count-space max-def le-diff-eq real-le-ereal-iff one-ereal-def*[*symmetric*] *prob split*: *option.split*)
also have $?Some = \int^+ x.\ ?f\ x\ \partial count\text{-}space\ (range\ Some)$
by(*simp add*: *nn-integral-count-space-indicator*)
also have *count-space* (*range Some*) = *embed-measure* (*count-space UNIV*) *Some*
by(*simp add*: *embed-measure-count-space*)
also have $(\int^+ x.\ ?f\ x\ \partial \ldots) = \int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV$
by(*subst nn-integral-embed-measure*)(*simp-all add*: *measurable-embed-measure1*)
also have *?None* + … = *1* **using** *prob*
by(*auto simp*: *ennreal-minus*[*symmetric*] *ennreal-1*[*symmetric*] *ennreal-enn2real-if top-unique simp del*: *ennreal-1*)(*simp add*: *diff-add-self-ennreal*)
finally show *?thesis* **.**
qed

lemma *pmf-embed-spmf-None*: *pmf embed-spmf None* = *1* − *enn2real* $(\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV)$
unfolding *embed-spmf-def*
by (*smt* (*verit*, *del-insts*) *enn2real-leI ennreal-1 nn-integral-cong nn-integral-embed-spmf-eq-1 option.case-eq-if pmf-embed-pmf prob*)

lemma *spmf-embed-spmf* [*simp*]: *spmf embed-spmf x* = *max 0* (*f x*)
unfolding *embed-spmf-def*
by (*smt* (*verit*, *best*) *enn2real-leI ennreal-1 nn-integral-cong nn-integral-embed-spmf-eq-1 option.case-eq-if option.simps*(*5*) *pmf-embed-pmf prob*)

end

end

lemma *embed-spmf-K-0*[*simp*]: *embed-spmf* (λ-. *0*) = *return-pmf None*
by(*rule spmf-eqI*)(*simp add*: *zero-ereal-def*[*symmetric*])

25.10 Ordering on spmfs

rel-pmf does not preserve a ccpo structure. Counterexample by Saheb-Djahromi: Take prefix order over *bool llist* and the set *range* (λn :: *nat*. *uniform* (*llist-n n*)) where *llist-n* is the set of all *llist*s of length n and *uniform* returns a uniform distribution over the given set. The set forms a chain in *ord-pmf lprefix*, but it has not an upper bound. Any upper bound may contain only infinite lists in its support because otherwise it is not greater than the *n+1*-st element in the chain where n is the length of the finite list. Moreover its support must contain all infinite lists, because otherwise there is a finite list all of whose finite extensions are not in the support - a contradiction to the upper bound property. Hence, the support is uncountable, but pmf's only have countable support.

However, if all chains in the ccpo are finite, then it should preserve the ccpo structure.

abbreviation *ord-spmf* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a spmf* ⇒ '*a spmf* ⇒ *bool*
  where *ord-spmf ord* ≡ *rel-pmf* (*ord-option ord*)

locale *ord-spmf-syntax* **begin**
notation *ord-spmf* (**infix** ‹⊑₁› *60*)
end

lemma *ord-spmf-map-spmf1*: *ord-spmf R* (*map-spmf f p*) = *ord-spmf* (λ*x*. *R* (*f x*)) *p*
  by(*simp add*: *pmf.rel-map*[*abs-def*] *ord-option-map1*[*abs-def*])

lemma *ord-spmf-map-spmf2*: *ord-spmf R p* (*map-spmf f q*) = *ord-spmf* (λ*x y*. *R x* (*f y*)) *p q*
  by(*simp add*: *pmf.rel-map ord-option-map2*)

lemma *ord-spmf-map-spmf12*: *ord-spmf R* (*map-spmf f p*) (*map-spmf f q*) = *ord-spmf* (λ*x y*. *R* (*f x*) (*f y*)) *p q*
  by(*simp add*: *pmf.rel-map ord-option-map1*[*abs-def*] *ord-option-map2*)

lemmas *ord-spmf-map-spmf* = *ord-spmf-map-spmf1 ord-spmf-map-spmf2 ord-spmf-map-spmf12*

context fixes *ord* :: '*a* ⇒ '*a* ⇒ *bool* (**structure**) **begin**
interpretation *ord-spmf-syntax* **.**

lemma *ord-spmfI*:
  ⟦ ⋀*x y*. (*x*, *y*) ∈ *set-spmf pq* ⟹ *ord x y*; *map-spmf fst pq* = *p*; *map-spmf snd pq* = *q* ⟧
  ⟹ *p* ⊑ *q*
  by(*rule rel-pmf.intros*[**where** *pq*=*map-pmf* (λ*x*. *case x of None* ⇒ (*None*, *None*) | *Some* (*a*, *b*) ⇒ (*Some a*, *Some b*)) *pq*])
    (*auto simp*: *pmf.map-comp o-def in-set-spmf split*: *option.splits intro*: *pmf.map-cong*)

lemma *ord-spmf-None* [*simp*]: *return-pmf None* $\sqsubseteq$ *x*
by(*rule rel-pmf.intros*[**where** *pq*=*map-pmf* (*Pair None*) *x*])(*auto simp*: *pmf.map-comp o-def*)

lemma *ord-spmf-reflI*: ($\bigwedge$*x*. *x* $\in$ *set-spmf p* $\Longrightarrow$ *ord x x*) $\Longrightarrow$ *p* $\sqsubseteq$ *p*
by (*metis elem-set in-set-spmf ord-option-reflI pmf.rel-refl-strong*)

lemma *rel-spmf-inf*:
assumes *p* $\sqsubseteq$ *q*
and *q* $\sqsubseteq$ *p*
and *refl*: *reflp ord*
and *trans*: *transp ord*
shows *rel-spmf* (*inf ord ord*$^{-1-1}$) *p q*
proof −
from ‹*p* $\sqsubseteq$ *q*› ‹*q* $\sqsubseteq$ *p*›
have *rel-pmf* (*inf* (*ord-option ord*) (*ord-option ord*)$^{-1-1}$) *p q*
using *local.refl local.trans reflp-ord-option rel-pmf-inf transp-ord-option* **by** *blast*
also have *inf* (*ord-option ord*) (*ord-option ord*)$^{-1-1}$ = *rel-option* (*inf ord ord*$^{-1-1}$)
by(*auto simp*: *fun-eq-iff elim*: *ord-option.cases option.rel-cases*)
finally show *?thesis* **.**
qed

end

lemma *ord-spmf-return-spmf2*: *ord-spmf R p* (*return-spmf y*) $\longleftrightarrow$ ($\forall$ *x*$\in$*set-spmf p*. *R x y*)
by(*auto simp*: *rel-pmf-return-pmf2 in-set-spmf ord-option.simps intro*: *ccontr*)

lemma *ord-spmf-mono*: ⟦ *ord-spmf A p q*; $\bigwedge$*x y*. *A x y* $\Longrightarrow$ *B x y* ⟧ $\Longrightarrow$ *ord-spmf B p q*
by(*erule pmf.rel-mono-strong*)(*erule ord-option-mono*)

lemma *ord-spmf-compp*: *ord-spmf* (*A OO B*) = *ord-spmf A OO ord-spmf B*
by(*simp add*: *ord-option-compp pmf.rel-compp*)

lemma *ord-spmf-bindI*:
assumes *pq*: *ord-spmf R p q*
and *fg*: $\bigwedge$*x y*. *R x y* $\Longrightarrow$ *ord-spmf P* (*f x*) (*g y*)
shows *ord-spmf P* (*p* $\gg=$ *f*) (*q* $\gg=$ *g*)
unfolding *bind-spmf-def* **using** *pq*
by(*rule rel-pmf-bindI*)(*auto split*: *option.split intro*: *fg*)

lemma *ord-spmf-bind-reflI*:
($\bigwedge$*x*. *x* $\in$ *set-spmf p* $\Longrightarrow$ *ord-spmf R* (*f x*) (*g x*)) $\Longrightarrow$ *ord-spmf R* (*p* $\gg=$ *f*) (*p* $\gg=$ *g*)
by(*rule ord-spmf-bindI*[**where** *R*=λ*x y*. *x* = *y* $\wedge$ *x* $\in$ *set-spmf p*])(*auto intro*: *ord-spmf-reflI*)

lemma *ord-pmf-increaseI*:

assumes *le*: $\bigwedge x.\ spmf\ p\ x \leq spmf\ q\ x$
and *refl*: $\bigwedge x.\ x \in set\text{-}spmf\ p \Longrightarrow R\ x\ x$
shows *ord-spmf R p q*
proof(*rule rel-pmf.intros*)
define *pq* **where** *pq* = *embed-pmf*
($\lambda(x,\ y).$ *case x of Some x'* ⇒ (*case y of Some y'* ⇒ *if x' = y' then spmf p x' else 0* | *None* ⇒ *0*)
| *None* ⇒ (*case y of None* ⇒ *pmf q None* | *Some y'* ⇒ *spmf q y' − spmf p y'*))
(**is** - = *embed-pmf ?f*)
have *nonneg*: $\bigwedge xy.$ *?f xy* ≥ *0*
by(*clarsimp simp add*: *le field-simps split*: *option.split*)
have *integral*: $(\int^+ xy.$ *?f xy* ∂*count-space UNIV*$) = 1$ (**is** *nn-integral ?M* - = -)
proof −
have $(\int^+ xy.$ *?f xy* ∂*count-space UNIV*$) =$
$\int^+ xy.$ *ennreal* (*?f xy*) ∗ *indicator* {(*None*, *None*)} *xy* +
ennreal (*?f xy*) ∗ *indicator* (*range* ($\lambda x.$ (*None*, *Some x*))) *xy* +
ennreal (*?f xy*) ∗ *indicator* (*range* ($\lambda x.$ (*Some x*, *Some x*))) *xy* ∂*?M*
by(*rule nn-integral-cong*)(*auto split*: *split-indicator option.splits if-split-asm*)
also have ... = $(\int^+ xy.$ *?f xy* ∗ *indicator* {(*None*, *None*)} *xy* ∂*?M*) +
$(\int^+ xy.$ *ennreal* (*?f xy*) ∗ *indicator* (*range* ($\lambda x.$ (*None*, *Some x*))) *xy* ∂*?M*)
+
$(\int^+ xy.$ *ennreal* (*?f xy*) ∗ *indicator* (*range* ($\lambda x.$ (*Some x*, *Some x*))) *xy* ∂*?M*)
(**is** - = *?None* + *?Some2* + *?Some*)
by(*subst nn-integral-add*)(*simp-all add*: *nn-integral-add AE-count-space le-diff-eq le split*: *option.split*)
also have *?None* = *pmf q None* **by** *simp*
also have *?Some2* = $\int^+ x.$ *ennreal* (*spmf q x*) − *spmf p x* ∂*count-space UNIV*
by(*simp add*: *nn-integral-count-space-indicator*[*symmetric*] *embed-measure-count-space*[*symmetric*] *inj-on-def nn-integral-embed-measure measurable-embed-measure1 ennreal-minus*)
also have ... = $(\int^+ x.$ *spmf q x* ∂*count-space UNIV*$)$ − $(\int^+ x.$ *spmf p x* ∂*count-space UNIV*$)$
(**is** - = *?Some2'* − *?Some2''*)
by(*subst nn-integral-diff*)(*simp-all add*: *le nn-integral-spmf-neq-top*)
also have *?Some* = $\int^+ x.$ *spmf p x* ∂*count-space UNIV*
by(*simp add*: *nn-integral-count-space-indicator*[*symmetric*] *embed-measure-count-space*[*symmetric*] *inj-on-def nn-integral-embed-measure measurable-embed-measure1*)
also have *pmf q None* + (*?Some2'* − *?Some2''*) + ... = *pmf q None* + *?Some2'*
by(*auto simp*: *diff-add-self-ennreal le intro*!: *nn-integral-mono*)
also have ... = $\int^+ x.$ *ennreal* (*pmf q x*) ∗ *indicator* {*None*} *x* + *ennreal* (*pmf q x*) ∗ *indicator* (*range Some*) *x* ∂*count-space UNIV*
by(*subst nn-integral-add*)(*simp-all add*: *nn-integral-count-space-indicator*[*symmetric*] *embed-measure-count-space*[*symmetric*] *nn-integral-embed-measure measurable-embed-measure1*)
also have ... = $\int^+ x.$ *pmf q x* ∂*count-space UNIV*
by(*rule nn-integral-cong*)(*auto split*: *split-indicator*)
also have ... = *1*
by(*simp add*: *nn-integral-pmf*)
finally show *?thesis* **.**

qed
note f = *nonneg integral*

{ fix x y
assume $(x, y) \in$ *set-pmf pq*
hence *?f* $(x, y) \neq 0$ **unfolding** *pq-def* **by**(*simp add*: *set-embed-pmf*[*OF f*])
then show *ord-option R x y*
by(*simp add*: *spmf-eq-0-set-spmf refl split*: *option.split-asm if-split-asm*) **}**

have *weight-le*: *weight-spmf p* $\leq$ *weight-spmf q*
by(*subst ennreal-le-iff*[*symmetric*])(*auto simp*: *weight-spmf-eq-nn-integral-spmf intro*!: *nn-integral-mono le*)

show *map-pmf fst pq* = *p*
proof(*rule pmf-eqI*)
fix *i* :: *'a option*
have *bi*: *bij-betw* (*Pair i*) *UNIV* (*fst* $-$ ‘ $\{i\}$)
by(*auto simp*: *bij-betw-def inj-on-def*)
have *ennreal* (*pmf* (*map-pmf fst pq*) *i*) = $(\int^+$ *y*. *pmf pq* (*i*, *y*) ∂*count-space UNIV*)
unfolding *pq-def ennreal-pmf-map*
apply (*simp add*: *embed-pmf.rep-eq*[*OF f*] *o-def emeasure-density flip*: *nn-integral-count-space-indicator*)
by (*smt* (*verit*, *best*) *nn-integral-bij-count-space* [*OF bi*] *integral nn-integral-cong nonneg pmf-embed-pmf*)
also have ... = *pmf p i*
proof(*cases i*)
case (*Some x*)
have $(\int^+$ *y*. *pmf pq* (*Some x*, *y*) ∂*count-space UNIV*) = $\int^+$ *y*. *pmf p* (*Some x*) $*$ *indicator* $\{$*Some x*$\}$ *y* ∂*count-space UNIV*
by(*rule nn-integral-cong*)(*simp add*: *pq-def pmf-embed-pmf*[*OF f*] *split*: *option.split*)
then show *?thesis* **using** *Some* **by** *simp*
next
case *None*
have $(\int^+$ *y*. *pmf pq* (*None*, *y*) ∂*count-space UNIV*) =
$(\int^+$ *y*. *ennreal* (*pmf pq* (*None*, *Some* (*the y*))) $*$ *indicator* (*range Some*) *y* +
ennreal (*pmf pq* (*None*, *None*)) $*$ *indicator* $\{$*None*$\}$ *y* ∂*count-space UNIV*)
by(*rule nn-integral-cong*)(*auto split*: *split-indicator*)
also have ... = $(\int^+$ *y*. *ennreal* (*pmf pq* (*None*, *Some* (*the y*))) ∂*count-space* (*range Some*)) + *pmf pq* (*None*, *None*)
by(*subst nn-integral-add*)(*simp-all add*: *nn-integral-count-space-indicator*)
also have ... = $(\int^+$ *y*. *ennreal* (*spmf q y*) $-$ *ennreal* (*spmf p y*) ∂*count-space UNIV*) + *pmf q None*
by(*simp add*: *pq-def pmf-embed-pmf*[*OF f*] *embed-measure-count-space*[*symmetric*] *nn-integral-embed-measure measurable-embed-measure1 ennreal-minus*)
also have $(\int^+$ *y*. *ennreal* (*spmf q y*) $-$ *ennreal* (*spmf p y*) ∂*count-space UNIV*) =

$(\int^+ y.\ spmf\ q\ y\ \partial count\text{-}space\ UNIV) - (\int^+ y.\ spmf\ p\ y\ \partial count\text{-}space\ UNIV)$
by(*subst nn-integral-diff*)(*simp-all add*: *AE-count-space le nn-integral-spmf-neq-top split*: *split-indicator*)
also have $\ldots = pmf\ p\ None - pmf\ q\ None$
by(*simp add*: *pmf-None-eq-weight-spmf weight-spmf-eq-nn-integral-spmf*[*symmetric*] *ennreal-minus*)
also have $\ldots = ennreal\ (pmf\ p\ None) - ennreal\ (pmf\ q\ None)$ **by**(*simp add*: *ennreal-minus*)
finally show *?thesis* **using** *None weight-le*
by(*auto simp*: *diff-add-self-ennreal pmf-None-eq-weight-spmf intro*: *ennreal-leI*)
qed
finally show $pmf\ (map\text{-}pmf\ fst\ pq)\ i = pmf\ p\ i$ **by** *simp*
qed

show $map\text{-}pmf\ snd\ pq = q$
proof(*rule pmf-eqI*)
fix $i :: 'a\ option$
have *bi*: *bij-betw* $(\lambda x.\ (x, i))\ UNIV\ (snd -` \{i\})$
by (*auto simp*: *bij-betw-def inj-on-def*)
have $ennreal\ (pmf\ (map\text{-}pmf\ snd\ pq)\ i) = (\int^+ x.\ pmf\ pq\ (x, i)\ \partial count\text{-}space\ UNIV)$
unfolding *pq-def ennreal-pmf-map*
apply(*simp add*: *embed-pmf.rep-eq*[*OF f*] *o-def emeasure-density nn-integral-count-space-indicator*[*symmet*
by (*smt* (*verit*, *best*) *nn-integral-bij-count-space* [*OF bi*] *integral nn-integral-cong nonneg pmf-embed-pmf*)
also have $\ldots = ennreal\ (pmf\ q\ i)$
proof(*cases i*)
case *None*
have $(\int^+ x.\ pmf\ pq\ (x, None)\ \partial count\text{-}space\ UNIV) = \int^+ x.\ pmf\ q\ None * indicator\ \{None :: 'a\ option\}\ x\ \partial count\text{-}space\ UNIV$
by(*rule nn-integral-cong*)(*simp add*: *pq-def pmf-embed-pmf*[*OF f*] *split*: *option.split*)
then show *?thesis* **using** *None* **by** *simp*
next
case (*Some y*)
have $(\int^+ x.\ pmf\ pq\ (x, Some\ y)\ \partial count\text{-}space\ UNIV) =$
$(\int^+ x.\ ennreal\ (pmf\ pq\ (x, Some\ y)) * indicator\ (range\ Some)\ x +$
$ennreal\ (pmf\ pq\ (None, Some\ y)) * indicator\ \{None\}\ x\ \partial count\text{-}space\ UNIV)$
by(*rule nn-integral-cong*)(*auto split*: *split-indicator*)
also have $\ldots = (\int^+ x.\ ennreal\ (pmf\ pq\ (x, Some\ y)) * indicator\ (range\ Some)\ x\ \partial count\text{-}space\ UNIV) + pmf\ pq\ (None, Some\ y)$
by(*subst nn-integral-add*)(*simp-all*)
also have $\ldots = (\int^+ x.\ ennreal\ (spmf\ p\ y) * indicator\ \{Some\ y\}\ x\ \partial count\text{-}space\ UNIV) + (spmf\ q\ y - spmf\ p\ y)$
by(*auto simp*: *pq-def pmf-embed-pmf*[*OF f*] *one-ereal-def*[*symmetric*] *simp del*: *nn-integral-indicator-singleton intro*!: *arg-cong2*[**where** *f*=(+)] *nn-integral-cong split*:

option.split)
also have ... = *spmf q y* **by**(*simp add*: *ennreal-minus*[*symmetric*] *le*)
finally show *?thesis* **using** *Some* **by** *simp*
qed
finally show *pmf* (*map-pmf snd pq*) *i* = *pmf q i* **by** *simp*
qed
qed

lemma *ord-spmf-eq-leD*:
assumes *ord-spmf* (=) *p q*
shows *spmf p x* ≤ *spmf q x*
proof(*cases x* ∈ *set-spmf p*)
case *False*
thus *?thesis* **by**(*simp add*: *in-set-spmf-iff-spmf*)
next
case *True*
from *assms* **obtain** *pq*
where *pq*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *ord-option* (=) *x y*
and *p*: *p* = *map-pmf fst pq*
and *q*: *q* = *map-pmf snd pq* **by** *cases auto*
have *ennreal* (*spmf p x*) = *integral*^{N} *pq* (*indicator* (*fst* −‘ {*Some x*}))
using *p* **by**(*simp add*: *ennreal-pmf-map*)
also have ... = *integral*^{N} *pq* (*indicator* {(*Some x*, *Some x*)})
by(*rule nn-integral-cong-AE*)(*auto simp*: *AE-measure-pmf-iff split*: *split-indicator dest*: *pq*)
also have ... ≤ *integral*^{N} *pq* (*indicator* (*snd* −‘ {*Some x*}))
by(*rule nn-integral-mono*) *simp*
also have ... = *ennreal* (*spmf q x*) **using** *q* **by**(*simp add*: *ennreal-pmf-map*)
finally show *?thesis* **by** *simp*
qed

lemma *ord-spmf-eqD-set-spmf*: *ord-spmf* (=) *p q* ⟹ *set-spmf p* ⊆ *set-spmf q*
by (*metis ord-spmf-eq-leD pmf-le-0-iff spmf-eq-0-set-spmf subsetI*)

lemma *ord-spmf-eqD-emeasure*:
ord-spmf (=) *p q* ⟹ *emeasure* (*measure-spmf p*) *A* ≤ *emeasure* (*measure-spmf q*) *A*
by(*auto intro*!: *nn-integral-mono split*: *split-indicator dest*: *ord-spmf-eq-leD simp add*: *nn-integral-measure-spmf nn-integral-indicator*[*symmetric*])

lemma *ord-spmf-eqD-measure-spmf*: *ord-spmf* (=) *p q* ⟹ *measure-spmf p* ≤ *measure-spmf q*
by (*subst le-measure*) (*auto simp*: *ord-spmf-eqD-emeasure*)

25.11 CCPO structure for the flat ccpo *ord-option* (=)

context fixes *Y* :: *'a spmf set* **begin**

definition *lub-spmf* :: *'a spmf*

where *lub-spmf* = *embed-spmf* (λ*x*. *enn2real* (*SUP p* ∈ *Y*. *ennreal* (*spmf p x*)))
— We go through *ennreal* to have a sensible definition even if *Y* is empty.

lemma *lub-spmf-empty* [*simp*]: *SPMF.lub-spmf* {} = *return-pmf None*
by(*simp add*: *SPMF.lub-spmf-def bot-ereal-def*)

context assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
begin

lemma *chain-ord-spmf-eqD*: *Complete-Partial-Order.chain* (≤) ((λ*p x*. *ennreal* (*spmf p x*)) ‘ *Y*)
(**is** *Complete-Partial-Order.chain* - (*?f* ‘ -))
proof(*rule chainI*)
fix *f g*
assume *f* ∈ *?f* ‘ *Y g* ∈ *?f* ‘ *Y*
then obtain *p q* **where** *f*: *f* = *?f p p* ∈ *Y* **and** *g*: *g* = *?f q q* ∈ *Y* **by** *blast*
from *chain* ‹*p* ∈ *Y*› ‹*q* ∈ *Y*› **have** *ord-spmf* (=) *p q* ∨ *ord-spmf* (=) *q p* **by**(*rule chainD*)
thus *f* ≤ *g* ∨ *g* ≤ *f*
by (*metis ennreal-leI f*(*1*) *g*(*1*) *le-funI ord-spmf-eq-leD*)
qed

lemma *ord-spmf-eq-pmf-None-eq*:
assumes *le*: *ord-spmf* (=) *p q*
and *None*: *pmf p None* = *pmf q None*
shows *p* = *q*
proof(*rule spmf-eqI*)
fix *i*
from *le* **have** *le′*: ⋀*x*. *spmf p x* ≤ *spmf q x* **by**(*rule ord-spmf-eq-leD*)
have (∫⁺ *x*. *ennreal* (*spmf q x*) − *spmf p x* ∂*count-space UNIV*) =
(∫⁺ *x*. *spmf q x* ∂*count-space UNIV*) − (∫⁺ *x*. *spmf p x* ∂*count-space UNIV*)
by(*subst nn-integral-diff*)(*simp-all add*: *AE-count-space le′ nn-integral-spmf-neq-top*)
also have ... = (*1* − *pmf q None*) − (*1* − *pmf p None*) **unfolding** *pmf-None-eq-weight-spmf*
by(*simp add*: *weight-spmf-eq-nn-integral-spmf*[*symmetric*] *ennreal-minus*)
also have ... = *0* **using** *None* **by** *simp*
finally have ⋀*x*. *spmf q x* ≤ *spmf p x*
by(*simp add*: *nn-integral-0-iff-AE AE-count-space ennreal-minus ennreal-eq-0-iff*)
with *le′* **show** *spmf p i* = *spmf q i* **by**(*rule antisym*)
qed

lemma *ord-spmf-eqD-pmf-None*:
assumes *ord-spmf* (=) *x y*
shows *pmf x None* ≥ *pmf y None*
using *assms*
apply *cases*
apply(*clarsimp simp only*: *ennreal-le-iff*[*symmetric*, *OF pmf-nonneg*] *ennreal-pmf-map*)
apply(*fastforce simp*: *AE-measure-pmf-iff intro*!: *nn-integral-mono-AE*)
done

Chains on *′a spmf* maintain countable support. Thanks to Johannes Hölzl

for the proof idea.

lemma *spmf-chain-countable*: *countable* $(\bigcup p{\in}Y.\ set\text{-}spmf\ p)$
proof(*cases* $Y = \{\}$)
case Y: *False*
show *?thesis*
proof(*cases* $\exists\, x{\in}Y.\ \forall\, y{\in}Y.\ ord\text{-}spmf\ (=)\ y\ x$)
case *True*
then obtain x **where** x: $x \in Y$ **and** *upper*: $\bigwedge y.\ y \in Y \Longrightarrow ord\text{-}spmf\ (=)\ y\ x$
by *blast*
hence $(\bigcup x{\in}Y.\ set\text{-}spmf\ x) \subseteq set\text{-}spmf\ x$ **by**(*auto dest*: *ord-spmf-eqD-set-spmf*)
thus *?thesis* **by**(*rule countable-subset*) *simp*
next
case *False*
define N :: *'a option pmf* $\Rightarrow$ *real* **where** $N\ p = pmf\ p\ None$ **for** p

have *N-less-imp-le-spmf*: $[\![\ x \in Y;\ y \in Y;\ N\ y < N\ x\ ]\!] \Longrightarrow ord\text{-}spmf\ (=)\ x\ y$
for $x\ y$
using *chainD*[*OF chain*, *of x y*] *ord-spmf-eqD-pmf-None*[*of x y*] *ord-spmf-eqD-pmf-None*[*of y x*]
by (*auto simp*: *N-def*)
have *N-eq-imp-eq*: $[\![\ x \in Y;\ y \in Y;\ N\ y = N\ x\ ]\!] \Longrightarrow x = y$ **for** $x\ y$
using *chainD*[*OF chain*, *of x y*] **by**(*auto simp*: *N-def dest*: *ord-spmf-eq-pmf-None-eq*)

have *NC*: $N\ `\ Y \neq \{\}$ *bdd-below* $(N\ `\ Y)$
using ‹$Y \neq \{\}$› **by**(*auto intro*!: *bdd-belowI*[*of - 0*] *simp*: *N-def*)
have *NC-less*: *Inf* $(N\ `\ Y) < N\ x$ **if** $x \in Y$ **for** x **unfolding** *cInf-less-iff*[*OF NC*]
proof(*rule ccontr*)
assume $**$: $\neg\ (\exists\, y{\in}N\ `\ Y.\ y < N\ x)$
{ fix y
assume $y \in Y$
with $**$ **consider** $N\ x < N\ y \mid N\ x = N\ y$ **by**(*auto simp*: *not-less le-less*)
hence *ord-spmf* $(=)\ y\ x$ **using** ‹$y \in Y$› ‹$x \in Y$›
by *cases*(*auto dest*: *N-less-imp-le-spmf N-eq-imp-eq intro*: *ord-spmf-reflI*) **}**
with *False* ‹$x \in Y$› **show** *False* **by** *blast*
qed

from *NC* **have** *Inf* $(N\ `\ Y) \in closure\ (N\ `\ Y)$ **by** (*intro closure-contains-Inf*)
then obtain X' **where** $\bigwedge n.\ X'\ n \in N\ `\ Y$ **and** $X'$: $X' \longrightarrow Inf\ (N\ `\ Y)$
unfolding *closure-sequential* **by** *auto*
then obtain X **where** X: $\bigwedge n.\ X\ n \in Y$ **and** $X' = (\lambda n.\ N\ (X\ n))$ **unfolding** *image-iff Bex-def* **by** *metis*

with X' **have** *seq*: $(\lambda n.\ N\ (X\ n)) \longrightarrow Inf\ (N\ `\ Y)$ **by** *simp*
have $(\bigcup x \in Y.\ set\text{-}spmf\ x) \subseteq (\bigcup n.\ set\text{-}spmf\ (X\ n))$
proof(*rule UN-least*)
fix x
assume $x \in Y$
from *order-tendstoD*(*2*)[*OF seq NC-less*[*OF* ‹$x \in Y$›]]

```
    obtain i where N (X i) < N x by (auto simp: eventually-sequentially)
    thus set-spmf x ⊆ (⋃n. set-spmf (X n)) using X ‹x ∈ Y›
      by(blast dest: N-less-imp-le-spmf ord-spmf-eqD-set-spmf)
  qed
  thus ?thesis by(rule countable-subset) simp
 qed
qed simp

lemma lub-spmf-subprob: (∫⁺ x. (SUP p ∈ Y. ennreal (spmf p x)) ∂count-space
UNIV) ≤ 1
proof(cases Y = {})
 case True
 thus ?thesis by(simp add: bot-ennreal)
next
 case False
 let ?B = ⋃p∈Y. set-spmf p
 have countable: countable ?B by(rule spmf-chain-countable)

 have (∫⁺ x. (SUP p∈Y. ennreal (spmf p x)) ∂count-space UNIV) =
     (∫⁺ x. (SUP p∈Y. ennreal (spmf p x) * indicator ?B x) ∂count-space UNIV)
   by (intro nn-integral-cong arg-cong [of - - Sup]) (auto split: split-indicator simp
add: spmf-eq-0-set-spmf)
 also have ... = (∫⁺ x. (SUP p∈Y. ennreal (spmf p x)) ∂count-space ?B)
   unfolding ennreal-indicator[symmetric] using False
  by(subst SUP-mult-right-ennreal[symmetric])(simp add: ennreal-indicator nn-integral-count-space-indicator)
 also have ... = (SUP p∈Y. ∫⁺ x. spmf p x ∂count-space ?B) using False -
countable
  by(rule nn-integral-monotone-convergence-SUP-countable)(rule chain-ord-spmf-eqD)
 also have ... ≤ 1
 proof(rule SUP-least)
   fix p
   assume p ∈ Y
  have (∫⁺ x. spmf p x ∂count-space ?B) = ∫⁺ x. ennreal (spmf p x) * indicator
?B x ∂count-space UNIV
    by(simp add: nn-integral-count-space-indicator)
   also have ... = ∫⁺ x. spmf p x ∂count-space UNIV
   by(rule nn-integral-cong)(auto split: split-indicator simp add: spmf-eq-0-set-spmf
‹p ∈ Y›)
   also have ... ≤ 1
    by(simp add: weight-spmf-eq-nn-integral-spmf[symmetric] weight-spmf-le-1)
   finally show (∫⁺ x. spmf p x ∂count-space ?B) ≤ 1 .
 qed
 finally show ?thesis .
qed

lemma spmf-lub-spmf:
 assumes Y ≠ {}
 shows spmf lub-spmf x = (SUP p ∈ Y. spmf p x)
proof −
```

from *assms* **obtain** *p* **where** $p \in Y$ **by** *auto*
have *spmf lub-spmf x = max 0 (enn2real (SUP p∈Y. ennreal (spmf p x)))*
unfolding *lub-spmf-def*
by(*rule spmf-embed-spmf*)(*simp del*: *SUP-eq-top-iff Sup-eq-top-iff add*: *ennreal-enn2real-if SUP-spmf-neq-top' lub-spmf-subprob*)
also have ... = *enn2real (SUP p∈Y. ennreal (spmf p x))*
by(*rule max-absorb2*)(*simp*)
also have ... = *enn2real (ennreal (SUP p ∈ Y. spmf p x))* **using** *assms*
by(*subst ennreal-SUP*[*symmetric*])(*simp-all add*: *SUP-spmf-neq-top' del*: *SUP-eq-top-iff Sup-eq-top-iff*)
also have $0 \leq (\bigsqcup p \in Y.\ spmf\ p\ x)$ **using** *assms*
by(*auto intro*!: *cSUP-upper2 bdd-aboveI*[**where** *M=1*] *simp add*: *pmf-le-1*)
then have *enn2real (ennreal (SUP p ∈ Y. spmf p x)) = (SUP p ∈ Y. spmf p x)*
by(*rule enn2real-ennreal*)
finally show *?thesis* **.**
qed

lemma *ennreal-spmf-lub-spmf*: $Y \neq \{\} \Longrightarrow$ *ennreal (spmf lub-spmf x) = (SUP p∈Y. ennreal (spmf p x))*
by (*metis SUP-spmf-neq-top' ennreal-SUP spmf-lub-spmf*)

lemma *lub-spmf-upper*:
assumes *p*: $p \in Y$
shows *ord-spmf (=) p lub-spmf*
proof(*rule ord-pmf-increaseI*)
fix *x*
from *p* **have** [*simp*]: $Y \neq \{\}$ **by** *auto*
from *p* **have** *ennreal (spmf p x) ≤ (SUP p∈Y. ennreal (spmf p x))* **by**(*rule SUP-upper*)
also have ... = *ennreal (spmf lub-spmf x)* **using** *p*
by(*subst spmf-lub-spmf*)(*auto simp*: *ennreal-SUP SUP-spmf-neq-top' simp del*: *SUP-eq-top-iff Sup-eq-top-iff*)
finally show *spmf p x ≤ spmf lub-spmf x* **by** *simp*
qed *simp*

lemma *lub-spmf-least*:
assumes *z*: $\bigwedge x.\ x \in Y \Longrightarrow$ *ord-spmf (=) x z*
shows *ord-spmf (=) lub-spmf z*
proof(*cases* $Y = \{\}$)
case *nonempty*: *False*
show *?thesis*
proof(*rule ord-pmf-increaseI*)
fix *x*
from *nonempty* **obtain** *p* **where** *p*: $p \in Y$ **by** *auto*
have *ennreal (spmf lub-spmf x) = (SUP p∈Y. ennreal (spmf p x))*
by(*subst spmf-lub-spmf*)(*auto simp*: *ennreal-SUP SUP-spmf-neq-top' nonempty simp del*: *SUP-eq-top-iff Sup-eq-top-iff*)
also have ... ≤ *ennreal (spmf z x)* **by**(*rule SUP-least*)(*simp add*: *ord-spmf-eq-leD z*)

finally show *spmf lub-spmf x ≤ spmf z x* **by** *simp*
qed *simp*
qed *simp*

lemma *set-lub-spmf*: *set-spmf lub-spmf* = (⋃ *p*∈*Y*. *set-spmf p*) (**is** *?lhs* = *?rhs*)
proof(*cases Y* = {})
case [*simp*]: *False*
show *?thesis*
proof(*rule set-eqI*)
fix *x*
have *x* ∈ *?lhs* ⟷ *ennreal* (*spmf lub-spmf x*) > *0*
by(*simp-all add*: *in-set-spmf-iff-spmf less-le*)
also have ... ⟷ (∃ *p*∈*Y*. *ennreal* (*spmf p x*) > *0*)
by(*simp add*: *ennreal-spmf-lub-spmf less-SUP-iff*)
also have ... ⟷ *x* ∈ *?rhs*
by(*auto simp*: *in-set-spmf-iff-spmf less-le*)
finally show *x* ∈ *?lhs* ⟷ *x* ∈ *?rhs* **.**
qed
qed *simp*

lemma *emeasure-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *emeasure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *emeasure* (*measure-spmf y*) *A*)
(**is** *?lhs* = *?rhs*)
proof −
let *?M* = *count-space* (*set-spmf lub-spmf*)
have *?lhs* = ∫⁺ *x*. *ennreal* (*spmf lub-spmf x*) ∗ *indicator A x* ∂*?M*
by(*auto simp*: *nn-integral-indicator*[*symmetric*] *nn-integral-measure-spmf*′)
also have ... = ∫⁺ *x*. (*SUP y*∈*Y*. *ennreal* (*spmf y x*) ∗ *indicator A x*) ∂*?M*
unfolding *ennreal-indicator*[*symmetric*]
by(*simp add*: *spmf-lub-spmf assms ennreal-SUP*[*OF SUP-spmf-neq-top*′] *SUP-mult-right-ennreal*)
also from *assms* **have** ... = (*SUP y*∈*Y*. ∫⁺ *x*. *ennreal* (*spmf y x*) ∗ *indicator A x* ∂*?M*)
proof(*rule nn-integral-monotone-convergence-SUP-countable*)
have (λ*i x*. *ennreal* (*spmf i x*) ∗ *indicator A x*) ‘ *Y* = (λ*f x*. *f x* ∗ *indicator A x*) ‘ (λ*p x*. *ennreal* (*spmf p x*)) ‘ *Y*
by(*simp add*: *image-image*)
also have *Complete-Partial-Order.chain* (≤) ... **using** *chain-ord-spmf-eqD*
by(*rule chain-imageI*)(*auto simp*: *le-fun-def split*: *split-indicator*)
finally show *Complete-Partial-Order.chain* (≤) ((λ*i x*. *ennreal* (*spmf i x*) ∗ *indicator A x*) ‘ *Y*) **.**
qed *simp*
also have ... = (*SUP y*∈*Y*. ∫⁺ *x*. *ennreal* (*spmf y x*) ∗ *indicator A x* ∂*count-space UNIV*)
by(*auto simp*: *nn-integral-count-space-indicator set-lub-spmf spmf-eq-0-set-spmf split*: *split-indicator intro*!: *arg-cong* [*of - - Sup*] *image-cong nn-integral-cong*)
also have ... = *?rhs*
by(*auto simp*: *nn-integral-indicator*[*symmetric*] *nn-integral-measure-spmf*)

finally show *?thesis* **.**
qed

lemma *measure-lub-spmf*:
assumes *Y*: $Y \neq \{\}$
shows *measure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *measure* (*measure-spmf y*) *A*) (**is** *?lhs* = *?rhs*)
proof −
have *ennreal ?lhs* = *ennreal ?rhs*
using *emeasure-lub-spmf*[*OF assms*] *SUP-emeasure-spmf-neq-top*[*of A Y*] *Y*
unfolding *measure-spmf.emeasure-eq-measure* **by**(*subst ennreal-SUP*)
moreover have *0* ≤ *?rhs* **using** *Y*
by(*auto intro*!: *cSUP-upper2 bdd-aboveI*[**where** *M=1*] *measure-spmf.subprob-measure-le-1*)
ultimately show *?thesis* **by**(*simp*)
qed

lemma *weight-lub-spmf*:
assumes *Y*: $Y \neq \{\}$
shows *weight-spmf lub-spmf* = (*SUP y*∈*Y*. *weight-spmf y*)
by (*smt* (*verit*, *best*) *SUP-cong assms measure-lub-spmf space-measure-spmf*)

lemma *measure-spmf-lub-spmf*:
assumes *Y*: $Y \neq \{\}$
shows *measure-spmf lub-spmf* = (*SUP p*∈*Y*. *measure-spmf p*) (**is** *?lhs* = *?rhs*)
proof(*rule measure-eqI*)
from *assms* **obtain** *p* **where** *p*: $p \in Y$ **by** *auto*
from *chain* **have** *chain'*: *Complete-Partial-Order.chain* (≤) (*measure-spmf* ‘ *Y*)
by(*rule chain-imageI*)(*rule ord-spmf-eqD-measure-spmf*)
show *sets ?lhs* = *sets ?rhs*
using *Y* **by** (*subst sets-SUP*) *auto*
show *emeasure ?lhs A* = *emeasure ?rhs A* **for** *A*
using *chain' Y p* **by** (*subst emeasure-SUP-chain*) (*auto simp*: *emeasure-lub-spmf*)
qed

end

end

lemma *partial-function-definitions-spmf*: *partial-function-definitions* (*ord-spmf* (=)) *lub-spmf*
(**is** *partial-function-definitions ?R* -)
proof
fix *x* **show** *?R x x* **by**(*simp add*: *ord-spmf-reflI*)
next
fix *x y z*
assume *?R x y ?R y z*
with *transp-ord-option*[*OF transp-on-equality*] **show** *?R x z* **by**(*rule transp-rel-pmf*[*THEN transpD*])
next

fix *x y*
assume *?R x y ?R y x*
thus *x = y*
by(*rule rel-pmf-antisym*)(*simp-all add*: *reflp-ord-option transp-ord-option antisymp-ord-option*)
next
fix *Y x*
assume *Complete-Partial-Order.chain ?R Y x ∈ Y*
then show *?R x* (*lub-spmf Y*)
by(*rule lub-spmf-upper*)
next
fix *Y z*
assume *Complete-Partial-Order.chain ?R Y* ⋀*x. x ∈ Y* ⟹ *?R x z*
then show *?R* (*lub-spmf Y*) *z*
by(*cases Y = {}*)(*simp-all add*: *lub-spmf-least*)
qed

lemma *ccpo-spmf*: *class.ccpo lub-spmf* (*ord-spmf* (=)) (*mk-less* (*ord-spmf* (=)))
by(*metis ccpo partial-function-definitions-spmf*)

interpretation *spmf*: *partial-function-definitions ord-spmf* (=) *lub-spmf*
rewrites *lub-spmf {}* ≡ *return-pmf None*
by(*rule partial-function-definitions-spmf*) *simp*

declaration ‹*Partial-Function.init spmf* ***term*** ‹*spmf.fixp-fun*›
term ‹*spmf.mono-body*› @{*thm spmf.fixp-rule-uc*} @{*thm spmf.fixp-induct-uc*}
NONE›

declare *spmf.leq-refl*[*simp*]
declare *admissible-leI*[*OF ccpo-spmf*, *cont-intro*]

abbreviation *mono-spmf* ≡ *monotone* (*fun-ord* (*ord-spmf* (=))) (*ord-spmf* (=))

lemma *lub-spmf-const* [*simp*]: *lub-spmf* {*p*} = *p*
by(*rule spmf-eqI*)(*simp add*: *spmf-lub-spmf*[*OF ccpo.chain-singleton*[*OF ccpo-spmf*]])

lemma *bind-spmf-mono′*:
assumes *fg*: *ord-spmf* (=) *f g*
and *hk*: ⋀*x* :: *′a. ord-spmf* (=) (*h x*) (*k x*)
shows *ord-spmf* (=) (*f* ⤜ *h*) (*g* ⤜ *k*)
unfolding *bind-spmf-def* **using** *assms*(*1*)
by(*rule rel-pmf-bindI*)(*auto split*: *option.split simp add*: *hk*)

lemma *bind-spmf-mono* [*partial-function-mono*]:
assumes *mf*: *mono-spmf B* **and** *mg*: ⋀*y. mono-spmf* (λ*f. C y f*)
shows *mono-spmf* (λ*f. bind-spmf* (*B f*) (λ*y. C y f*))
proof (*rule monotoneI*)
fix *f g* :: *′a* ⇒ *′b spmf*
assume *fg*: *fun-ord* (*ord-spmf* (=)) *f g*

with *mf* **have** *ord-spmf* (=) (*B f*) (*B g*) **by** (*rule monotoneD*[*of - - - f g*])
moreover from *mg* **have** $\bigwedge y'$. *ord-spmf* (=) (*C y' f*) (*C y' g*)
by (*rule monotoneD*) (*rule fg*)
ultimately show *ord-spmf* (=) (*bind-spmf* (*B f*) (λ*y*. *C y f*)) (*bind-spmf* (*B g*) (λ*y'*. *C y' g*))
by(*rule bind-spmf-mono'*)
qed

lemma *monotone-bind-spmf1*: *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) (λ*y*. *bind-spmf y g*)
by(*rule monotoneI*)(*simp add*: *bind-spmf-mono' ord-spmf-reflI*)

lemma *monotone-bind-spmf2*:
assumes *g*: $\bigwedge x$. *monotone ord* (*ord-spmf* (=)) (λ*y*. *g y x*)
shows *monotone ord* (*ord-spmf* (=)) (λ*y*. *bind-spmf p* (*g y*))
by(*rule monotoneI*)(*auto intro*: *bind-spmf-mono' monotoneD*[*OF g*] *ord-spmf-reflI*)

lemma *bind-lub-spmf*:
assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
shows *bind-spmf* (*lub-spmf Y*) *f* = *lub-spmf* ((λ*p*. *bind-spmf p f*) ‘ *Y*) (**is** *?lhs* = *?rhs*)
proof(*cases Y* = {})
case *Y*: *False*
show *?thesis*
proof(*rule spmf-eqI*)
fix *i*
have *chain'*: *Complete-Partial-Order.chain* (≤) ((λ*p x*. *ennreal* (*spmf p x* ∗ *spmf* (*f x*) *i*)) ‘ *Y*)
using *chain* **by**(*rule chain-imageI*)(*auto simp*: *le-fun-def dest*: *ord-spmf-eq-leD intro*: *mult-right-mono*)
have *chain''*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) ((λ*p*. *p* ⤜ *f*) ‘ *Y*)
using *chain* **by**(*rule chain-imageI*)(*auto intro*!: *monotoneI bind-spmf-mono' ord-spmf-reflI*)
let *?M* = *count-space* (*set-spmf* (*lub-spmf Y*))
have *ennreal* (*spmf ?lhs i*) = $\int^+ x$. *ennreal* (*spmf* (*lub-spmf Y*) *x*) ∗ *ennreal* (*spmf* (*f x*) *i*) ∂*?M*
by(*auto simp*: *ennreal-spmf-lub-spmf ennreal-spmf-bind nn-integral-measure-spmf'*)
also have ... = $\int^+ x$. (*SUP p*∈*Y*. *ennreal* (*spmf p x* ∗ *spmf* (*f x*) *i*)) ∂*?M*
by(*subst ennreal-spmf-lub-spmf*[*OF chain Y*])(*subst SUP-mult-right-ennreal*, *simp-all add*: *ennreal-mult Y*)
also have ... = (*SUP p*∈*Y*. $\int^+ x$. *ennreal* (*spmf p x* ∗ *spmf* (*f x*) *i*) ∂*?M*)
using *Y chain'* **by**(*rule nn-integral-monotone-convergence-SUP-countable*) *simp*
also have ... = (*SUP p*∈*Y*. *ennreal* (*spmf* (*bind-spmf p f*) *i*))
by(*auto simp*: *ennreal-spmf-bind nn-integral-measure-spmf nn-integral-count-space-indicator set-lub-spmf*[*OF chain*] *in-set-spmf-iff-spmf ennreal-mult intro*!: *arg-cong* [*of - - Sup*] *image-cong nn-integral-cong split*: *split-indicator*)
also have ... = *ennreal* (*spmf ?rhs i*) **using** *chain''* **by**(*simp add*: *ennreal-spmf-lub-spmf Y image-comp*)

finally show *spmf ?lhs i = spmf ?rhs i* **by** *simp*
qed
qed *simp*

lemma *map-lub-spmf*:
Complete-Partial-Order.chain (*ord-spmf* (=)) *Y*
⟹ *map-spmf f* (*lub-spmf Y*) = *lub-spmf* (*map-spmf f* ‘ *Y*)
unfolding *map-spmf-conv-bind-spmf*[*abs-def*] **by**(*simp add*: *bind-lub-spmf o-def*)

lemma *mcont-bind-spmf1*: *mcont lub-spmf* (*ord-spmf* (=)) *lub-spmf* (*ord-spmf* (=)) (λ*y*. *bind-spmf y f*)
using *monotone-bind-spmf1*
by(*intro contI mcontI*) (*auto simp*: *bind-lub-spmf*)

lemma *bind-lub-spmf2*:
assumes *chain*: *Complete-Partial-Order.chain ord Y*
and *g*: ⋀*y*. *monotone ord* (*ord-spmf* (=)) (*g y*)
shows *bind-spmf x* (λ*y*. *lub-spmf* (*g y* ‘ *Y*)) = *lub-spmf* ((λ*p*. *bind-spmf x* (λ*y*. *g y p*)) ‘ *Y*)
(**is** *?lhs = ?rhs*)
proof(*cases Y* = {})
case *Y*: *False*
show *?thesis*
proof(*rule spmf-eqI*)
fix *i*
have *chain′*: ⋀*y*. *Complete-Partial-Order.chain* (*ord-spmf* (=)) (*g y* ‘ *Y*)
using *chain g*[*THEN monotoneD*] **by**(*rule chain-imageI*)
have *chain″*: *Complete-Partial-Order.chain* (≤) ((λ*p y*. *ennreal* (*spmf x y* ∗ *spmf* (*g y p*) *i*)) ‘ *Y*)
using *chain* **by**(*rule chain-imageI*)(*auto simp*: *le-fun-def dest*: *ord-spmf-eq-leD monotoneD*[*OF g*] *intro*!: *mult-left-mono*)
have *chain‴*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) ((λ*p*. *bind-spmf x* (λ*y*. *g y p*)) ‘ *Y*)
using *chain* **by**(*rule chain-imageI*)(*rule monotone-bind-spmf2*[*OF g*, *THEN monotoneD*])

have *ennreal* (*spmf ?lhs i*) = ∫⁺ *y*. (*SUP p*∈*Y*. *ennreal* (*spmf x y* ∗ *spmf* (*g y p*) *i*)) ∂*count-space* (*set-spmf x*)
by(*simp add*: *ennreal-spmf-bind ennreal-spmf-lub-spmf*[*OF chain′*] *Y nn-integral-measure-spmf′ SUP-mult-left-ennreal ennreal-mult image-comp*)
also have ... = (*SUP p*∈*Y*. ∫⁺ *y*. *ennreal* (*spmf x y* ∗ *spmf* (*g y p*) *i*) ∂*count-space* (*set-spmf x*))
unfolding *nn-integral-measure-spmf′* **using** *Y chain″*
by(*rule nn-integral-monotone-convergence-SUP-countable*) *simp*
also have ... = (*SUP p*∈*Y*. *ennreal* (*spmf* (*bind-spmf x* (λ*y*. *g y p*)) *i*))
by(*simp add*: *ennreal-spmf-bind nn-integral-measure-spmf′ ennreal-mult*)
also have ... = *ennreal* (*spmf ?rhs i*) **using** *chain‴*
by(*auto simp*: *ennreal-spmf-lub-spmf Y image-comp*)
finally show *spmf ?lhs i = spmf ?rhs i* **by** *simp*

qed
qed *simp*

lemma *mcont-bind-spmf* [*cont-intro*]:
assumes *f*: *mcont luba orda lub-spmf* (*ord-spmf* (=)) *f*
and *g*: ⋀*y*. *mcont luba orda lub-spmf* (*ord-spmf* (=)) (*g y*)
shows *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *bind-spmf* (*f x*) (λ*y*. *g y x*))
proof(*rule spmf.mcont2mcont'*[*OF - - f*])
fix *z*
show *mcont lub-spmf* (*ord-spmf* (=)) *lub-spmf* (*ord-spmf* (=)) (λ*x*. *bind-spmf x* (λ*y*. *g y z*))
by(*rule mcont-bind-spmf1*)
next
fix *x*
let *?f* = λ*z*. *bind-spmf x* (λ*y*. *g y z*)
have *monotone orda* (*ord-spmf* (=)) *?f* **using** *mcont-mono*[*OF g*] **by**(*rule monotone-bind-spmf2*)
moreover have *cont luba orda lub-spmf* (*ord-spmf* (=)) *?f*
proof(*rule contI*)
fix *Y*
assume *chain*: *Complete-Partial-Order.chain orda Y* **and** *Y*: *Y* ≠ {}
have *bind-spmf x* (λ*y*. *g y* (*luba Y*)) = *bind-spmf x* (λ*y*. *lub-spmf* (*g y* ‘ *Y*))
by(*rule bind-spmf-cong*)(*simp-all add*: *mcont-contD*[*OF g chain Y*])
also have ... = *lub-spmf* ((λ*p*. *x* ⤜ (λ*y*. *g y p*)) ‘ *Y*) **using** *chain*
by(*rule bind-lub-spmf2*)(*rule mcont-mono*[*OF g*])
finally show *bind-spmf x* (λ*y*. *g y* (*luba Y*)) = ... **.**
qed
ultimately show *mcont luba orda lub-spmf* (*ord-spmf* (=)) *?f* **by**(*rule mcontI*)
qed

lemma *bind-pmf-mono* [*partial-function-mono*]:
(⋀*y*. *mono-spmf* (λ*f*. *C y f*)) ⟹ *mono-spmf* (λ*f*. *bind-pmf p* (λ*x*. *C x f*))
using *bind-spmf-mono*[*of* λ-. *spmf-of-pmf p C*] **by** *simp*

lemma *map-spmf-mono* [*partial-function-mono*]: *mono-spmf B* ⟹ *mono-spmf* (λ*g*. *map-spmf f* (*B g*))
unfolding *map-spmf-conv-bind-spmf* **by**(*rule bind-spmf-mono*) *simp-all*

lemma *mcont-map-spmf* [*cont-intro*]:
mcont luba orda lub-spmf (*ord-spmf* (=)) *g*
⟹ *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *map-spmf f* (*g x*))
unfolding *map-spmf-conv-bind-spmf* **by**(*rule mcont-bind-spmf*) *simp-all*

lemma *monotone-set-spmf*: *monotone* (*ord-spmf* (=)) (⊆) *set-spmf*
by(*rule monotoneI*)(*rule ord-spmf-eqD-set-spmf*)

lemma *cont-set-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
by(*rule contI*)(*subst set-lub-spmf*; *simp*)

lemma *mcont2mcont-set-spmf*[*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-set-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
by(*rule mcontI monotone-set-spmf cont-set-spmf*)+

lemma *monotone-spmf*: *monotone* (*ord-spmf* (=)) (≤) (λ*p*. *spmf p x*)
by(*rule monotoneI*)(*simp add*: *ord-spmf-eq-leD*)

lemma *cont-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
by(*rule contI*)(*simp add*: *spmf-lub-spmf*)

lemma *mcont-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
by(*metis mcontI monotone-spmf cont-spmf*)

lemma *cont-ennreal-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
by(*rule contI*)(*simp add*: *ennreal-spmf-lub-spmf*)

lemma *mcont2mcont-ennreal-spmf* [*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-ennreal-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
by(*metis mcontI mono2mono-ennreal monotone-spmf cont-ennreal-spmf*)

lemma *nn-integral-map-spmf* [*simp*]: *nn-integral* (*measure-spmf* (*map-spmf f p*)) *g* = *nn-integral* (*measure-spmf p*) (*g* ∘ *f*)
by(*force simp*: *measure-spmf-def nn-integral-distr nn-integral-restrict-space intro*: *nn-integral-cong split*: *split-indicator*)

25.11.1 Admissibility of *rel-spmf*

lemma *rel-spmf-measureD*:
assumes *rel-spmf R p q*
shows *measure* (*measure-spmf p*) *A* ≤ *measure* (*measure-spmf q*) {*y*. ∃ *x*∈*A*. *R x y*} (**is** *?lhs* ≤ *?rhs*)
proof −
have *?lhs* = *measure* (*measure-pmf p*) (*Some* ‘ *A*) **by**(*simp add*: *measure-measure-spmf-conv-measure-pmf*)
also have ... ≤ *measure* (*measure-pmf q*) {*y*. ∃ *x*∈*Some* ‘ *A*. *rel-option R x y*}
using *assms* **by**(*rule rel-pmf-measureD*)
also have ... = *?rhs* **unfolding** *measure-measure-spmf-conv-measure-pmf*
by(*rule arg-cong2*[**where** *f*=*measure*])(*auto simp*: *option-rel-Some1*)
finally show *?thesis* **.**
qed

locale *rel-spmf-characterisation* =
assumes *rel-pmf-measureI*:
⋀(*R* :: *'a option* ⇒ *'b option* ⇒ *bool*) *p q*.
(⋀*A*. *measure* (*measure-pmf p*) *A* ≤ *measure* (*measure-pmf q*) {*y*. ∃ *x*∈*A*. *R x y*})
⟹ *rel-pmf R p q*

— This assumption is shown to hold in general in the AFP entry *MFMC-Countable*.
begin

context fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **begin**

lemma *rel-spmf-measureI*:
  assumes *eq1*: $\bigwedge A.$ *measure* (*measure-spmf p*) $A \leq$ *measure* (*measure-spmf q*) $\{y. \exists x{\in}A.\ R\ x\ y\}$
  assumes *eq2*: *weight-spmf q* $\leq$ *weight-spmf p*
  shows *rel-spmf R p q*
proof(*rule rel-pmf-measureI*)
  fix $A :: 'a$ *option set*
  define A' **where** $A' =$ *the* ‘ $(A \cap$ *range Some*)
  define A'' **where** $A'' = A \cap \{None\}$
  have *A*: $A =$ *Some* ‘ $A' \cup A''$ *Some* ‘ $A' \cap A'' = \{\}$
    unfolding A'*-def* A''*-def* **by**(*auto simp*: *image-iff*)
  have *measure* (*measure-pmf p*) $A =$ *measure* (*measure-pmf p*) (*Some* ‘ A') + *measure* (*measure-pmf p*) A''
    by(*simp add*: *A measure-pmf.finite-measure-Union*)
  also have *measure* (*measure-pmf p*) (*Some* ‘ A') = *measure* (*measure-spmf p*) A'
    by(*simp add*: *measure-measure-spmf-conv-measure-pmf*)
  also have ... $\leq$ *measure* (*measure-spmf q*) $\{y. \exists x{\in}A'.\ R\ x\ y\}$ **by**(*rule eq1*)
  also (*ord-eq-le-trans*[*OF - add-right-mono*])
  have ... = *measure* (*measure-pmf q*) $\{y. \exists x{\in}A'.$ *rel-option R* (*Some x*) $y\}$
    unfolding *measure-measure-spmf-conv-measure-pmf*
    by(*rule arg-cong2*[**where** *f*=*measure*])(*auto simp*: A'*-def option-rel-Some1*)
  also
  { have *weight-spmf p* $\leq$ *measure* (*measure-spmf q*) $\{y. \exists x.\ R\ x\ y\}$
      using *eq1*[*of UNIV*] **unfolding** *weight-spmf-def* **by** *simp*
    also have ... $\leq$ *weight-spmf q* **unfolding** *weight-spmf-def*
      by(*rule measure-spmf.finite-measure-mono*) *simp-all*
    finally have *weight-spmf p* = *weight-spmf q* **using** *eq2* **by** *simp* **}**
  then have *measure* (*measure-pmf p*) $A'' =$ *measure* (*measure-pmf q*) (*if None* $\in$ *A then* $\{None\}$ *else* $\{\}$)
    unfolding A''*-def* **by**(*simp add*: *pmf-None-eq-weight-spmf measure-pmf-single*)
  also have *measure* (*measure-pmf q*) $\{y. \exists x{\in}A'.$ *rel-option R* (*Some x*) $y\}$ + ... = *measure* (*measure-pmf q*) $\{y. \exists x{\in}A.$ *rel-option R x y*$\}$
    by(*subst measure-pmf.finite-measure-Union*[*symmetric*])
      (*auto 4 3 intro*!: *arg-cong2*[**where** *f*=*measure*] *simp add*: *option-rel-Some1 option-rel-Some2* A'*-def intro*: *rev-bexI elim*: *option.rel-cases*)
  finally show *measure* (*measure-pmf p*) $A \leq$ ... **.**
qed

lemma *admissible-rel-spmf*:
  ccpo.admissible (*prod-lub lub-spmf lub-spmf*) (*rel-prod* (*ord-spmf* (=)) (*ord-spmf* (=))) (*case-prod* (*rel-spmf R*))
  (**is** *ccpo.admissible ?lub ?ord ?P*)
proof(*rule ccpo.admissibleI*)
  fix *Y*

assume *chain*: *Complete-Partial-Order.chain ?ord Y*
and *Y*: $Y \neq \{\}$
and *R*: $\forall (p, q) \in Y.$ *rel-spmf R p q*
from *R* **have** *R*: $\bigwedge p\ q.\ (p, q) \in Y \Longrightarrow$ *rel-spmf R p q* **by** *auto*
have *chain1*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) (*fst* ‘ *Y*)
and *chain2*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) (*snd* ‘ *Y*)
using *chain* **by**(*rule chain-imageI*; *clarsimp*)+
from *Y* **have** *Y1*: *fst* ‘ *Y* $\neq \{\}$ **and** *Y2*: *snd* ‘ *Y* $\neq \{\}$ **by** *auto*

have *rel-spmf R* (*lub-spmf* (*fst* ‘ *Y*)) (*lub-spmf* (*snd* ‘ *Y*))
proof(*rule rel-spmf-measureI*)
show *weight-spmf* (*lub-spmf* (*snd* ‘ *Y*)) $\leq$ *weight-spmf* (*lub-spmf* (*fst* ‘ *Y*))
by(*auto simp*: *weight-lub-spmf chain1 chain2 Y rel-spmf-weightD*[*OF R*, *symmetric*] *intro*!: *cSUP-least intro*: *cSUP-upper2*[*OF bdd-aboveI2*[*OF weight-spmf-le-1*]])

fix *A*
have *measure* (*measure-spmf* (*lub-spmf* (*fst* ‘ *Y*))) *A* = (*SUP y*∈*fst* ‘ *Y*. *measure* (*measure-spmf y*) *A*)
using *chain1 Y1* **by**(*rule measure-lub-spmf*)
also have ... $\leq$ (*SUP y*∈*snd* ‘ *Y*. *measure* (*measure-spmf y*) {*y*. ∃ *x*∈*A*. *R x y*}) **using** *Y1*
by(*rule cSUP-least*)(*auto intro*!: *cSUP-upper2*[*OF bdd-aboveI2*[*OF measure-spmf.subprob-measure-le-1*]] *rel-spmf-measureD R*)
also have ... = *measure* (*measure-spmf* (*lub-spmf* (*snd* ‘ *Y*))) {*y*. ∃ *x*∈*A*. *R x y*}
using *chain2 Y2* **by**(*rule measure-lub-spmf*[*symmetric*])
finally show *measure* (*measure-spmf* (*lub-spmf* (*fst* ‘ *Y*))) *A* $\leq$ ... **.**
qed
then show *?P* (*?lub Y*) **by**(*simp add*: *prod-lub-def*)
qed

lemma *admissible-rel-spmf-mcont* [*cont-intro*]:
⟦ *mcont lub ord lub-spmf* (*ord-spmf* (=)) *f*; *mcont lub ord lub-spmf* (*ord-spmf* (=)) *g* ⟧
$\Longrightarrow$ *ccpo.admissible lub ord* (λ*x*. *rel-spmf R* (*f x*) (*g x*))
by(*rule admissible-subst*[*OF admissible-rel-spmf*, **where** *f*=λ*x*. (*f x*, *g x*), *simplified*])(*rule mcont-Pair*)

context includes *lifting-syntax*
begin

lemma *fixp-spmf-parametric'*:
assumes *f*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *F*
and *g*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *G*
and *param*: (*rel-spmf R* ===> *rel-spmf R*) *F G*
shows (*rel-spmf R*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *F*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *G*)
by(*rule parallel-fixp-induct*[*OF ccpo-spmf ccpo-spmf - f g*])(*auto intro*: *param*[*THEN rel-funD*])

lemma *fixp-spmf-parametric*:
assumes *f*: ⋀*x*. *mono-spmf* (λ*f*. *F f x*)
and *g*: ⋀*x*. *mono-spmf* (λ*f*. *G f x*)
and *param*: ((*A* ===> *rel-spmf R*) ===> *A* ===> *rel-spmf R*) *F G*
shows (*A* ===> *rel-spmf R*) (*spmf.fixp-fun F*) (*spmf.fixp-fun G*)
using *f g*
proof(*rule parallel-fixp-induct-1-1*[*OF partial-function-definitions-spmf partial-function-definitions-spmf - - reflexive reflexive*, **where** *P*=(*A* ===> *rel-spmf R*)])
show *ccpo.admissible* (*prod-lub* (*fun-lub lub-spmf*) (*fun-lub lub-spmf*)) (*rel-prod* (*fun-ord* (*ord-spmf* (=))) (*fun-ord* (*ord-spmf* (=)))) (λ*x*. (*A* ===> *rel-spmf R*) (*fst x*) (*snd x*))
unfolding *rel-fun-def*
by(*fastforce intro*: *admissible-all admissible-imp admissible-rel-spmf-mcont*)
show (*A* ===> *rel-spmf R*) (λ-. *lub-spmf* {}) (λ-. *lub-spmf* {})
by *auto*
show (*A* ===> *rel-spmf R*) (*F f*) (*G g*) **if** (*A* ===> *rel-spmf R*) *f g* **for** *f g*
using *that* **by**(*rule rel-funD*[*OF param*])
qed

end

end

end

25.12 Restrictions on spmfs

definition *restrict-spmf* :: *'a spmf* ⇒ *'a set* ⇒ *'a spmf* (**infixl** ‹⨾› *110*)
where *p* ⨾ *A* = *map-pmf* (λ*x*. *x* ⤜ (λ*y*. *if y* ∈ *A then Some y else None*)) *p*

lemma *set-restrict-spmf* [*simp*]: *set-spmf* (*p* ⨾ *A*) = *set-spmf p* ∩ *A*
by(*fastforce simp*: *restrict-spmf-def set-spmf-def split*: *bind-splits if-split-asm*)

lemma *restrict-map-spmf*: *map-spmf f p* ⨾ *A* = *map-spmf f* (*p* ⨾ (*f* −‘ *A*))
by(*simp add*: *restrict-spmf-def pmf.map-comp o-def map-option-bind bind-map-option if-distrib cong del*: *if-weak-cong*)

lemma *restrict-restrict-spmf* [*simp*]: *p* ⨾ *A* ⨾ *B* = *p* ⨾ (*A* ∩ *B*)
by(*auto simp*: *restrict-spmf-def pmf.map-comp o-def intro*!: *pmf.map-cong bind-option-cong*)

lemma *restrict-spmf-empty* [*simp*]: *p* ⨾ {} = *return-pmf None*
by(*simp add*: *restrict-spmf-def*)

lemma *restrict-spmf-UNIV* [*simp*]: *p* ⨾ *UNIV* = *p*
by(*simp add*: *restrict-spmf-def*)

lemma *spmf-restrict-spmf-outside* [*simp*]: *x* ∉ *A* ⟹ *spmf* (*p* ⨾ *A*) *x* = *0*
by(*simp add*: *spmf-eq-0-set-spmf*)

lemma *emeasure-restrict-spmf* [*simp*]: *emeasure* (*measure-spmf* (*p* ⌈ *A*)) *X* = *emeasure* (*measure-spmf p*) (*X* ∩ *A*)
proof −
have (λ*x*. *x* ⤜ (λ*y*. *if y* ∈ *A then Some y else None*)) −‘ *the* −‘ *X* ∩ (λ*x*. *x* ⤜ (λ*y*. *if y* ∈ *A then Some y else None*)) −‘ *range Some* = *the* −‘ *X* ∩ *the* −‘ *A* ∩ *range Some*
by(*auto split*: *bind-splits if-split-asm*)
then show *?thesis*
by (*simp add*: *restrict-spmf-def measure-spmf-def emeasure-distr emeasure-restrict-space*)
qed

lemma *measure-restrict-spmf* [*simp*]:
measure (*measure-spmf* (*p* ⌈ *A*)) *X* = *measure* (*measure-spmf p*) (*X* ∩ *A*)
using *emeasure-restrict-spmf*[*of p A X*]
by(*simp only*: *measure-spmf.emeasure-eq-measure ennreal-inj measure-nonneg*)

lemma *spmf-restrict-spmf*: *spmf* (*p* ⌈ *A*) *x* = (*if x* ∈ *A then spmf p x else 0*)
by(*simp add*: *spmf-conv-measure-spmf*)

lemma *spmf-restrict-spmf-inside* [*simp*]: *x* ∈ *A* ⟹ *spmf* (*p* ⌈ *A*) *x* = *spmf p x*
by(*simp add*: *spmf-restrict-spmf*)

lemma *pmf-restrict-spmf-None*: *pmf* (*p* ⌈ *A*) *None* = *pmf p None* + *measure* (*measure-spmf p*) (− *A*)
proof −
have [*simp*]: *None* ∉ *Some* ‘ (− *A*) **by** *auto*
have (λ*x*. *x* ⤜ (λ*y*. *if y* ∈ *A then Some y else None*)) −‘ {*None*} = {*None*} ∪ (*Some* ‘ (− *A*))
by(*auto split*: *bind-splits if-split-asm*)
then show *?thesis* **unfolding** *ereal.inject*[*symmetric*]
by(*simp add*: *restrict-spmf-def ennreal-pmf-map emeasure-pmf-single del*: *ereal.inject*)
(*simp add*: *pmf.rep-eq measure-pmf.finite-measure-Union*[*symmetric*] *measure-measure-spmf-conv-measure-pmf measure-pmf.emeasure-eq-measure*)
qed

lemma *restrict-spmf-trivial*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *x* ∈ *A*) ⟹ *p* ⌈ *A* = *p*
by(*rule spmf-eqI*)(*auto simp*: *spmf-restrict-spmf spmf-eq-0-set-spmf*)

lemma *restrict-spmf-trivial′*: *set-spmf p* ⊆ *A* ⟹ *p* ⌈ *A* = *p*
by(*rule restrict-spmf-trivial*) *blast*

lemma *restrict-return-spmf*: *return-spmf x* ⌈ *A* = (*if x* ∈ *A then return-spmf x else return-pmf None*)
by(*simp add*: *restrict-spmf-def*)

lemma *restrict-return-spmf-inside* [*simp*]: *x* ∈ *A* ⟹ *return-spmf x* ⌈ *A* = *return-spmf x*
by(*simp add*: *restrict-return-spmf*)

lemma *restrict-return-spmf-outside* [*simp*]: $x \notin A \Longrightarrow$ *return-spmf* $x \upharpoonright A =$ *return-pmf None*
by(*simp add*: *restrict-return-spmf*)

lemma *restrict-spmf-return-pmf-None* [*simp*]: *return-pmf None* $\upharpoonright A =$ *return-pmf None*
by(*simp add*: *restrict-spmf-def*)

lemma *restrict-bind-pmf*: *bind-pmf p g* $\upharpoonright A = p \gg\!= (\lambda x.\ g\ x \upharpoonright A)$
by(*simp add*: *restrict-spmf-def map-bind-pmf o-def*)

lemma *restrict-bind-spmf*: *bind-spmf p g* $\upharpoonright A = p \gg\!= (\lambda x.\ g\ x \upharpoonright A)$
by(*auto simp*: *bind-spmf-def restrict-bind-pmf cong del*: *option.case-cong-weak cong*: *option.case-cong intro*!: *bind-pmf-cong split*: *option.split*)

lemma *bind-restrict-pmf*: *bind-pmf* $(p \upharpoonright A)\ g = p \gg\!= (\lambda x.$ *if* $x \in$ *Some* ` A *then g x else g None*)
by(*auto simp*: *restrict-spmf-def bind-map-pmf fun-eq-iff split*: *bind-split intro*: *arg-cong2*[**where** *f*=*bind-pmf*])

lemma *bind-restrict-spmf*: *bind-spmf* $(p \upharpoonright A)\ g = p \gg\!= (\lambda x.$ *if* $x \in A$ *then g x else return-pmf None*)
by(*auto simp*: *bind-spmf-def bind-restrict-pmf fun-eq-iff intro*: *arg-cong2*[**where** *f*=*bind-pmf*] *split*: *option.split*)

lemma *spmf-map-restrict*: *spmf* (*map-spmf fst* $(p \upharpoonright (snd -` \{y\}))) x =$ *spmf p* (x, y)
by(*subst spmf-map*)(*auto intro*: *arg-cong2*[**where** *f*=*measure*] *simp add*: *spmf-conv-measure-spmf*)

lemma *measure-eqI-restrict-spmf*:
assumes *rel-spmf R* (*restrict-spmf p A*) (*restrict-spmf q B*)
shows *measure* (*measure-spmf p*) *A* = *measure* (*measure-spmf q*) *B*
proof −
from *assms* **have** *weight-spmf* (*restrict-spmf p A*) = *weight-spmf* (*restrict-spmf q B*) **by**(*rule rel-spmf-weightD*)
thus *?thesis* **by**(*simp add*: *weight-spmf-def*)
qed

25.13 Subprobability distributions of sets

definition *spmf-of-set* :: *'a set* ⇒ *'a spmf*
where
spmf-of-set A = (*if finite A* ∧ *A* ≠ {} *then spmf-of-pmf* (*pmf-of-set A*) *else return-pmf None*)

lemma *spmf-of-set*: *spmf* (*spmf-of-set A*) *x* = *indicator A x* / *card A*
by(*auto simp*: *spmf-of-set-def*)

lemma *pmf-spmf-of-set-None* [*simp*]: *pmf* (*spmf-of-set A*) *None* = *indicator* {*A*. *infinite A* ∨ *A* = {}} *A*
by(*simp add*: *spmf-of-set-def*)

lemma *set-spmf-of-set*: *set-spmf* (*spmf-of-set A*) = (*if finite A then A else* {})
by(*simp add*: *spmf-of-set-def*)

lemma *set-spmf-of-set-finite* [*simp*]: *finite A* ⟹ *set-spmf* (*spmf-of-set A*) = *A*
by(*simp add*: *set-spmf-of-set*)

lemma *spmf-of-set-singleton*: *spmf-of-set* {*x*} = *return-spmf x*
by(*simp add*: *spmf-of-set-def pmf-of-set-singleton*)

lemma *map-spmf-of-set-inj-on* [*simp*]:
inj-on f A ⟹ *map-spmf f* (*spmf-of-set A*) = *spmf-of-set* (*f* ‘ *A*)
by(*auto simp*: *spmf-of-set-def map-pmf-of-set-inj dest*: *finite-imageD*)

lemma *spmf-of-pmf-pmf-of-set* [*simp*]:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *spmf-of-pmf* (*pmf-of-set A*) = *spmf-of-set A*
by(*simp add*: *spmf-of-set-def*)

lemma *weight-spmf-of-set*:
weight-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then 1 else 0*)
by(*auto simp only*: *spmf-of-set-def weight-spmf-of-pmf weight-return-pmf-None split*: *if-split*)

lemma *weight-spmf-of-set-finite* [*simp*]: ⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *weight-spmf* (*spmf-of-set A*) = *1*
by(*simp add*: *weight-spmf-of-set*)

lemma *weight-spmf-of-set-infinite* [*simp*]: *infinite A* ⟹ *weight-spmf* (*spmf-of-set A*) = *0*
by(*simp add*: *weight-spmf-of-set*)

lemma *measure-spmf-spmf-of-set*:
measure-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then measure-pmf* (*pmf-of-set A*) *else null-measure* (*count-space UNIV*))
by(*simp add*: *spmf-of-set-def del*: *spmf-of-pmf-pmf-of-set*)

lemma *emeasure-spmf-of-set*:
emeasure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
by(*auto simp*: *measure-spmf-spmf-of-set emeasure-pmf-of-set*)

lemma *measure-spmf-of-set*:
measure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
by(*auto simp*: *measure-spmf-spmf-of-set measure-pmf-of-set*)

lemma *nn-integral-spmf-of-set*: *nn-integral* (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*

by(*cases finite A*)(*auto simp*: *spmf-of-set-def nn-integral-pmf-of-set card-gt-0-iff simp del*: *spmf-of-pmf-pmf-of-set*)

lemma *integral-spmf-of-set*: $integral^L$ (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*
by (*metis card.infinite div-0 division-ring-divide-zero integral-null-measure integral-pmf-of-set measure-spmf-spmf-of-set of-nat-0 sum.empty*)

notepad begin — *pmf-of-set* is not fully parametric.
define *R* :: *nat* ⇒ *nat* ⇒ *bool* **where** *R x y* ⟷ (*x* ≠ *0* ⟶ *y* = *0*) **for** *x y*
define *A* :: *nat set* **where** *A* = {*0*, *1*}
define *B* :: *nat set* **where** *B* = {*0*, *1*, *2*}
have *rel-set R A B* **unfolding** *R-def*[*abs-def*] *A-def B-def rel-set-def* **by** *auto*
have ¬ *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
proof
assume *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
then obtain *pq* **where** *pq*: ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *R x y*
and *1*: *map-pmf fst pq* = *pmf-of-set A*
and *2*: *map-pmf snd pq* = *pmf-of-set B*
by *cases auto*
have *pmf* (*pmf-of-set B*) *1* = *1* / *3* **by**(*simp add*: *B-def*)
have *pmf* (*pmf-of-set B*) *2* = *1* / *3* **by**(*simp add*: *B-def*)

have *2* / *3* = *pmf* (*pmf-of-set B*) *1* + *pmf* (*pmf-of-set B*) *2* **by**(*simp add*: *B-def*)
also have ... = *measure* (*measure-pmf* (*pmf-of-set B*)) ({*1*} ∪ {*2*})
by(*subst measure-pmf.finite-measure-Union*)(*simp-all add*: *measure-pmf-single*)
also have ... = *emeasure* (*measure-pmf pq*) (*snd* −‘ {*2*, *1*})
unfolding *2*[*symmetric*] *measure-pmf.emeasure-eq-measure*[*symmetric*] **by**(*simp*)
also have ... = *emeasure* (*measure-pmf pq*) {(*0*, *2*), (*0*, *1*)}
by(*rule emeasure-eq-AE*)(*auto simp*: *AE-measure-pmf-iff R-def dest*!: *pq*)
also have ... ≤ *emeasure* (*measure-pmf pq*) (*fst* −‘ {*0*})
by(*rule emeasure-mono*) *auto*
also have ... = *emeasure* (*measure-pmf* (*pmf-of-set A*)) {*0*}
unfolding *1*[*symmetric*] **by** *simp*
also have ... = *pmf* (*pmf-of-set A*) *0*
by(*simp add*: *measure-pmf-single measure-pmf.emeasure-eq-measure*)
also have *pmf* (*pmf-of-set A*) *0* = *1* / *2* **by**(*simp add*: *A-def*)
finally show *False* **by**(*subst* (*asm*) *ennreal-le-iff*; *simp*)
qed
end

lemma *rel-pmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *A*: *A* ≠ {} *finite A*
and *B*: *B* ≠ {} *finite B*
and *R*: ⋀*x*. *x* ∈ *A* ⟹ *R x* (*f x*)
shows *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
proof(*rule pmf.rel-mono-strong*)

define AB **where** $AB = (\lambda x.\ (x,\ f\ x))\ `\ A$
define R' **where** $R'\ x\ y \longleftrightarrow (x,\ y) \in AB$ **for** $x\ y$
have $(x,\ y) \in AB$ **if** $(x,\ y) \in$ *set-pmf* (*pmf-of-set* AB) **for** $x\ y$
using *that* **by**(*auto simp*: *AB-def A*)
moreover have *map-pmf fst* (*pmf-of-set* AB) = *pmf-of-set A*
by(*simp add*: *AB-def map-pmf-of-set-inj*[*symmetric*] *inj-on-def A pmf.map-comp o-def*)
moreover
from *f* **have** [*simp*]: *inj-on f A* **by**(*rule bij-betw-imp-inj-on*)
from *f* **have** [*simp*]: *f* ` *A* = *B* **by**(*rule bij-betw-imp-surj-on*)
have *map-pmf snd* (*pmf-of-set* AB) = *pmf-of-set B*
by(*simp add*: *AB-def map-pmf-of-set-inj*[*symmetric*] *inj-on-def A pmf.map-comp o-def*)
(*simp add*: *map-pmf-of-set-inj A*)
ultimately show *rel-pmf* ($\lambda x\ y.\ (x,\ y) \in AB$) (*pmf-of-set A*) (*pmf-of-set B*) **..**
qed(*auto intro*: *R*)

lemma *rel-spmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *R*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
shows *rel-spmf R* (*spmf-of-set A*) (*spmf-of-set B*)
proof −
obtain *finite A* $\longleftrightarrow$ *finite B A* = {} $\longleftrightarrow$ *B* = {}
using *bij-betw-empty1 bij-betw-empty2 bij-betw-finite f* **by** *blast*
then show *?thesis*
using *assms*
by (*metis rel-pmf-of-set-bij rel-spmf-spmf-of-pmf return-spmf-None-parametric spmf-of-set-def*)
qed

context includes *lifting-syntax*
begin

lemma *rel-spmf-of-set*:
assumes *bi-unique R*
shows (*rel-set R* ===> *rel-spmf R*) *spmf-of-set spmf-of-set*
proof
fix *A B*
assume *R*: *rel-set R A B*
with *assms* **obtain** *f* **where** *bij-betw f A B* **and** *f*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
by(*auto dest*: *bi-unique-rel-set-bij-betw*)
then show *rel-spmf R* (*spmf-of-set A*) (*spmf-of-set B*)
by(*rule rel-spmf-of-set-bij*)
qed

end

lemma *map-mem-spmf-of-set*:
assumes *finite B B* ≠ {}

shows *map-spmf* (λ*x*. *x* ∈ *A*) (*spmf-of-set B*) = *spmf-of-pmf* (*bernoulli-pmf* (*card* (*A* ∩ *B*) / *card B*))
(**is** *?lhs* = *?rhs*)
proof(*rule spmf-eqI*)
fix *i*
have *ennreal* (*spmf ?lhs i*) = *card* (*B* ∩ (λ*x*. *x* ∈ *A*) −‘ {*i*}) / (*card B*)
by(*subst ennreal-spmf-map*)(*simp add*: *measure-spmf-spmf-of-set assms emeasure-pmf-of-set*)
also have ... = (*if i then card* (*B* ∩ *A*) / *card B else card* (*B* − *A*) / *card B*)
by(*auto intro*: *arg-cong*[**where** *f*=*card*])
also have ... = (*if i then card* (*B* ∩ *A*) / *card B else* (*card B* − *card* (*B* ∩ *A*)) / *card B*)
by(*auto simp*: *card-Diff-subset-Int assms*)
also have ... = *ennreal* (*spmf ?rhs i*)
by(*simp add*: *assms card-gt-0-iff field-simps card-mono Int-commute of-nat-diff*)
finally show *spmf ?lhs i* = *spmf ?rhs i* **by** *simp*
qed

abbreviation *coin-spmf* :: *bool spmf*
where *coin-spmf* ≡ *spmf-of-set UNIV*

lemma *map-eq-const-coin-spmf*: *map-spmf* ((=) *c*) *coin-spmf* = *coin-spmf*
proof −
have *inj* ((⟷) *c*) *range* ((⟷) *c*) = *UNIV* **by**(*auto intro*: *inj-onI*)
then show *?thesis* **by** *simp*
qed

lemma *bind-coin-spmf-eq-const*: *coin-spmf* ⤜ (λ*x* :: *bool*. *return-spmf* (*b* = *x*)) = *coin-spmf*
using *map-eq-const-coin-spmf* **unfolding** *map-spmf-conv-bind-spmf* **by** *simp*

lemma *bind-coin-spmf-eq-const'*: *coin-spmf* ⤜ (λ*x* :: *bool*. *return-spmf* (*x* = *b*)) = *coin-spmf*
by(*rewrite* **in** - = ⌑ *bind-coin-spmf-eq-const*[*symmetric*, *of b*])(*auto intro*: *bind-spmf-cong*)

25.14 Losslessness

definition *lossless-spmf* :: *'a spmf* ⇒ *bool*
where *lossless-spmf p* ⟷ *weight-spmf p* = *1*

lemma *lossless-iff-pmf-None*: *lossless-spmf p* ⟷ *pmf p None* = *0*
by(*simp add*: *lossless-spmf-def pmf-None-eq-weight-spmf*)

lemma *lossless-return-spmf* [*iff*]: *lossless-spmf* (*return-spmf x*)
by(*simp add*: *lossless-iff-pmf-None*)

lemma *lossless-return-pmf-None* [*iff*]: ¬ *lossless-spmf* (*return-pmf None*)
by(*simp add*: *lossless-iff-pmf-None*)

lemma *lossless-map-spmf* [*simp*]: *lossless-spmf* (*map-spmf f p*) ⟷ *lossless-spmf p*
by(*auto simp*: *lossless-iff-pmf-None pmf-eq-0-set-pmf*)

lemma *lossless-bind-spmf* [*simp*]:
lossless-spmf (*p* ⤜ *f*) ⟷ *lossless-spmf p* ∧ (∀ *x*∈*set-spmf p*. *lossless-spmf* (*f x*))
by(*simp add*: *lossless-iff-pmf-None pmf-bind-spmf-None add-nonneg-eq-0-iff integral-nonneg-AE integral-nonneg-eq-0-iff-AE measure-spmf.integrable-const-bound*[**where** *B=1*] *pmf-le-1*)

lemma *lossless-weight-spmfD*: *lossless-spmf p* ⟹ *weight-spmf p* = *1*
by(*simp add*: *lossless-spmf-def*)

lemma *lossless-iff-set-pmf-None*:
lossless-spmf p ⟷ *None* ∉ *set-pmf p*
by (*simp add*: *lossless-iff-pmf-None pmf-eq-0-set-pmf*)

lemma *lossless-spmf-of-set* [*simp*]: *lossless-spmf* (*spmf-of-set A*) ⟷ *finite A* ∧ *A* ≠ {}
by(*auto simp*: *lossless-spmf-def weight-spmf-of-set*)

lemma *lossless-spmf-spmf-of-spmf* [*simp*]: *lossless-spmf* (*spmf-of-pmf p*)
by(*simp add*: *lossless-spmf-def*)

lemma *lossless-spmf-bind-pmf* [*simp*]:
lossless-spmf (*bind-pmf p f*) ⟷ (∀ *x*∈*set-pmf p*. *lossless-spmf* (*f x*))
by(*simp add*: *lossless-iff-pmf-None pmf-bind integral-nonneg-AE integral-nonneg-eq-0-iff-AE measure-pmf.integrable-const-bound*[**where** *B=1*] *AE-measure-pmf-iff pmf-le-1*)

lemma *lossless-spmf-conv-spmf-of-pmf*: *lossless-spmf p* ⟷ (∃ *p'*. *p* = *spmf-of-pmf p'*)
proof
assume *lossless-spmf p*
hence ∗: ⋀*y*. *y* ∈ *set-pmf p* ⟹ ∃ *x*. *y* = *Some x*
by(*case-tac y*)(*simp-all add*: *lossless-iff-set-pmf-None*)

let *?p* = *map-pmf the p*
have *p* = *spmf-of-pmf ?p*
proof(*rule spmf-eqI*)
fix *i*
have *ennreal* (*pmf* (*map-pmf the p*) *i*) = ∫⁺ *x*. *indicator* (*the* −‘ {*i*}) *x* ∂*p*
by(*simp add*: *ennreal-pmf-map*)
also have ... = ∫⁺ *x*. *indicator* {*i*} *x* ∂*measure-spmf p* **unfolding** *measure-spmf-def*
by(*subst nn-integral-distr*)(*auto simp*: *nn-integral-restrict-space AE-measure-pmf-iff simp del*: *nn-integral-indicator intro*!: *nn-integral-cong-AE split*: *split-indicator dest*!: ∗)
also have ... = *spmf p i* **by**(*simp add*: *emeasure-spmf-single*)

finally show *spmf p i = spmf (spmf-of-pmf ?p) i* **by** *simp*
qed
thus $\exists p'.\ p = spmf\text{-}of\text{-}pmf\ p'$ **..**
qed *auto*

lemma *spmf-False-conv-True*: *lossless-spmf p* $\Longrightarrow$ *spmf p False = 1 − spmf p True*
by(*clarsimp simp add*: *lossless-spmf-conv-spmf-of-pmf pmf-False-conv-True*)

lemma *spmf-True-conv-False*: *lossless-spmf p* $\Longrightarrow$ *spmf p True = 1 − spmf p False*
by(*simp add*: *spmf-False-conv-True*)

lemma *bind-eq-return-spmf*:
bind-spmf p f = return-spmf x $\longleftrightarrow$ $(\forall y{\in}$*set-spmf p. f y = return-spmf x*$) \wedge$ *lossless-spmf p*
apply (*simp add*: *bind-spmf-def bind-eq-return-pmf split*: *option.split*)
by (*metis in-set-spmf lossless-iff-set-pmf-None not-None-eq*)

lemma *rel-spmf-return-spmf2*:
rel-spmf R p (*return-spmf x*) $\longleftrightarrow$ *lossless-spmf p* $\wedge$ $(\forall a{\in}$*set-spmf p. R a x*$)$
apply (*simp add*: *lossless-iff-set-pmf-None rel-pmf-return-pmf2 option-rel-Some2 in-set-spmf*)
by (*metis in-set-spmf not-None-eq option.sel*)

lemma *rel-spmf-return-spmf1*:
rel-spmf R (*return-spmf x*) *p* $\longleftrightarrow$ *lossless-spmf p* $\wedge$ $(\forall a{\in}$*set-spmf p. R x a*$)$
using *rel-spmf-return-spmf2*[*of* R^{-1-1}] **by**(*simp add*: *spmf-rel-conversep*)

lemma *rel-spmf-bindI1*:
assumes *f*: $\bigwedge x.\ x \in$ *set-spmf p* $\Longrightarrow$ *rel-spmf R* (*f x*) *q*
and *p*: *lossless-spmf p*
shows *rel-spmf R* (*bind-spmf p f*) *q*
proof −
fix *x* :: *'a*
have *rel-spmf R* (*bind-spmf p f*) (*bind-spmf* (*return-spmf x*) (λ-. *q*))
by(*rule rel-spmf-bindI*[**where** *R*=λ*x* -. *x* ∈ *set-spmf p*])(*simp-all add*: *rel-spmf-return-spmf2 p f*)
then show *?thesis* **by** *simp*
qed

lemma *rel-spmf-bindI2*:
⟦ $\bigwedge x.\ x \in$ *set-spmf q* $\Longrightarrow$ *rel-spmf R p* (*f x*); *lossless-spmf q* ⟧
$\Longrightarrow$ *rel-spmf R p* (*bind-spmf q f*)
using *rel-spmf-bindI1*[*of q conversep R f p*] **by**(*simp add*: *spmf-rel-conversep*)

25.15 Scaling

definition *scale-spmf* :: *real* ⇒ *'a spmf* ⇒ *'a spmf*
where
scale-spmf r p = embed-spmf (λ*x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗

spmf p x)

lemma *scale-spmf-le-1*:
(∫⁺ *x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ *spmf p x* ∂*count-space UNIV*)
≤ *1* (**is** *?lhs* ≤ -)
proof −
have *?lhs* = *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ ∫⁺ *x*. *spmf p x* ∂*count-space UNIV*
by(*subst nn-integral-cmult*[*symmetric*])(*simp-all add*: *weight-spmf-nonneg max-def min-def ennreal-mult*)
also have ... ≤ *1* **unfolding** *weight-spmf-eq-nn-integral-spmf*[*symmetric*]
by(*simp add*: *min-def max-def weight-spmf-nonneg order.strict-iff-order field-simps ennreal-mult*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *spmf-scale-spmf*: *spmf* (*scale-spmf r p*) *x* = *max 0* (*min* (*inverse* (*weight-spmf p*)) *r*) ∗ *spmf p x* (**is** *?lhs* = *?rhs*)
unfolding *scale-spmf-def*
apply(*subst spmf-embed-spmf*[*OF scale-spmf-le-1*])
apply(*simp add*: *max-def min-def measure-le-0-iff field-simps weight-spmf-nonneg not-le order.strict-iff-order*)
apply(*metis antisym-conv order-trans weight-spmf-nonneg zero-le-mult-iff zero-le-one*)
done

lemma *real-inverse-le-1-iff*: **fixes** *x* :: *real*
shows ⟦ *0* ≤ *x*; *x* ≤ *1* ⟧ ⟹ *1* / *x* ≤ *1* ⟷ *x* = *1* ∨ *x* = *0*
by *auto*

lemma *spmf-scale-spmf'*: *r* ≤ *1* ⟹ *spmf* (*scale-spmf r p*) *x* = *max 0 r* ∗ *spmf p x*
using *real-inverse-le-1-iff*[*OF weight-spmf-nonneg weight-spmf-le-1*, *of p*]
by(*auto simp*: *spmf-scale-spmf max-def min-def field-simps*)(*metis pmf-le-0-iff spmf-le-weight*)

lemma *scale-spmf-neg*: *r* ≤ *0* ⟹ *scale-spmf r p* = *return-pmf None*
by(*rule spmf-eqI*)(*simp add*: *spmf-scale-spmf' max-def*)

lemma *scale-spmf-return-None* [*simp*]: *scale-spmf r* (*return-pmf None*) = *return-pmf None*
by(*rule spmf-eqI*)(*simp add*: *spmf-scale-spmf*)

lemma *scale-spmf-conv-bind-bernoulli*:
assumes *r* ≤ *1*
shows *scale-spmf r p* = *bind-pmf* (*bernoulli-pmf r*) (λ*b*. *if b then p else return-pmf None*) (**is** *?lhs* = *?rhs*)
proof(*rule spmf-eqI*)
fix *x*
have ⟦*weight-spmf p* = *0*⟧ ⟹ *spmf p x* = *0*

by (*metis pmf-le-0-iff spmf-le-weight*)
moreover have ⟦*weight-spmf p* ≠ *0*; *1* / *weight-spmf p* < *1*⟧ ⟹ *weight-spmf p* = *1*
by (*smt* (*verit*) *divide-less-eq-1 measure-spmf.subprob-measure-le-1 weight-spmf-lt-0*)
ultimately have *ennreal* (*spmf ?lhs x*) = *ennreal* (*spmf ?rhs x*)
using *assms*
unfolding *spmf-scale-spmf ennreal-pmf-bind nn-integral-measure-pmf UNIV-bool bernoulli-pmf.rep-eq*
by(*auto simp*: *nn-integral-count-space-finite max-def min-def field-simps real-inverse-le-1-iff*[*OF weight-spmf-nonneg weight-spmf-le-1*] *ennreal-mult*[*symmetric*])
thus *spmf ?lhs x* = *spmf ?rhs x* **by** *simp*
qed

lemma *nn-integral-spmf*: (∫⁺ *x*. *spmf p x* ∂*count-space A*) = *emeasure* (*measure-spmf p*) *A*
proof −
have *bij-betw Some A* (*the* −‘ *A* ∩ *range Some*)
by(*auto simp*: *bij-betw-def*)
then show *?thesis*
by (*metis bij-betw-def emeasure-measure-spmf-conv-measure-pmf nn-integral-pmf'*)
qed

lemma *measure-spmf-scale-spmf*: *measure-spmf* (*scale-spmf r p*) = *scale-measure* (*min* (*inverse* (*weight-spmf p*)) *r*) (*measure-spmf p*)
by(*rule measure-eqI*; *simp add*: *spmf-scale-spmf ennreal-mult' flip*: *nn-integral-spmf nn-integral-cmult*)

lemma *measure-spmf-scale-spmf'*:
assumes *r* ≤ *1*
shows *measure-spmf* (*scale-spmf r p*) = *scale-measure r* (*measure-spmf p*)
proof(*cases weight-spmf p* > *0*)
case *True*
with *assms* **show** *?thesis*
by(*simp add*: *measure-spmf-scale-spmf field-simps weight-spmf-le-1 mult-le-one*)
next
case *False*
then show *?thesis*
by (*simp add*: *order-less-le weight-spmf-eq-0*)
qed

lemma *scale-spmf-1* [*simp*]: *scale-spmf 1 p* = *p*
by (*simp add*: *spmf-eqI spmf-scale-spmf'*)

lemma *scale-spmf-0* [*simp*]: *scale-spmf 0 p* = *return-pmf None*
by (*simp add*: *scale-spmf-neg*)

lemma *bind-scale-spmf*:
assumes *r*: *r* ≤ *1*
shows *bind-spmf* (*scale-spmf r p*) *f* = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))

(**is** *?lhs = ?rhs*)
proof(*rule spmf-eqI*)
fix *x*
have *ennreal* (*spmf ?lhs x*) = *ennreal* (*spmf ?rhs x*)
using *r*
by(*simp add*: *ennreal-spmf-bind measure-spmf-scale-spmf' nn-integral-scale-measure spmf-scale-spmf' ennreal-mult nn-integral-cmult*)
thus *spmf ?lhs x = spmf ?rhs x* **by** *simp*
qed

lemma *scale-bind-spmf*:
assumes *r* ≤ *1*
shows *scale-spmf r* (*bind-spmf p f*) = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))
(**is** *?lhs = ?rhs*)
proof(*rule spmf-eqI*)
fix *x*
have *ennreal* (*spmf ?lhs x*) = *ennreal* (*spmf ?rhs x*) **using** *assms*
unfolding *spmf-scale-spmf'*[*OF assms*]
by(*simp add*: *ennreal-mult ennreal-spmf-bind spmf-scale-spmf' nn-integral-cmult max-def min-def*)
thus *spmf ?lhs x = spmf ?rhs x* **by** *simp*
qed

lemma *bind-spmf-const*: *bind-spmf p* (λ*x*. *q*) = *scale-spmf* (*weight-spmf p*) *q* (**is** *?lhs = ?rhs*)
proof(*rule spmf-eqI*)
fix *x*
have *ennreal* (*spmf ?lhs x*) = *ennreal* (*spmf ?rhs x*)
using *measure-spmf.subprob-measure-le-1*[*of p space* (*measure-spmf p*)]
by(*subst ennreal-spmf-bind*)(*simp add*: *spmf-scale-spmf' weight-spmf-le-1 ennreal-mult mult.commute max-def min-def measure-spmf.emeasure-eq-measure*)
thus *spmf ?lhs x = spmf ?rhs x* **by** *simp*
qed

lemma *map-scale-spmf*: *map-spmf f* (*scale-spmf r p*) = *scale-spmf r* (*map-spmf f p*) (**is** *?lhs = ?rhs*)
proof(*rule spmf-eqI*)
fix *i*
show *spmf ?lhs i = spmf ?rhs i* **unfolding** *spmf-scale-spmf*
by(*subst* (*1 2*) *spmf-map*)(*auto simp*: *measure-spmf-scale-spmf max-def min-def ennreal-lt-0*)
qed

lemma *set-scale-spmf*: *set-spmf* (*scale-spmf r p*) = (*if r > 0 then set-spmf p else* {})
apply(*auto simp*: *in-set-spmf-iff-spmf spmf-scale-spmf*)
apply(*simp add*: *min-def weight-spmf-eq-0 split*: *if-split-asm*)
done

lemma *set-scale-spmf'* [*simp*]: *0 < r ⟹ set-spmf (scale-spmf r p) = set-spmf p*
by(*simp add*: *set-scale-spmf*)

lemma *rel-spmf-scaleI*:
assumes *r > 0 ⟹ rel-spmf A p q*
shows *rel-spmf A* (*scale-spmf r p*) (*scale-spmf r q*)
proof(*cases r > 0*)
case *True*
from *assms*[*OF True*] **show** *?thesis*
by(*rule rel-spmfE*)(*auto simp*: *map-scale-spmf*[*symmetric*] *spmf-rel-map True intro*: *rel-spmf-reflI*)
qed(*simp add*: *not-less scale-spmf-neg*)

lemma *weight-scale-spmf*: *weight-spmf* (*scale-spmf r p*) = *min 1* (*max 0 r ∗ weight-spmf p*)
proof −
have ⟦*1 / weight-spmf p ≤ r*; *ennreal r ∗ ennreal* (*weight-spmf p*) *< 1*⟧ ⟹ *weight-spmf p = 0*
by (*smt* (*verit*) *ennreal-less-one-iff ennreal-mult'' measure-le-0-iff mult-imp-less-div-pos*)
moreover
have ⟦*r < 1 / weight-spmf p*; *1 ≤ ennreal r ∗ ennreal* (*weight-spmf p*)⟧ ⟹ *weight-spmf p = 0*
by (*smt* (*verit*, *ccfv-threshold*) *ennreal-ge-1 ennreal-mult'' mult-imp-div-pos-le weight-spmf-lt-0*)
ultimately
have *ennreal* (*weight-spmf* (*scale-spmf r p*)) = *min 1* (*max 0 r ∗ ennreal* (*weight-spmf p*))
unfolding *weight-spmf-eq-nn-integral-spmf*
apply(*simp add*: *spmf-scale-spmf ennreal-mult zero-ereal-def*[*symmetric*] *nn-integral-cmult*)
apply(*auto simp*: *weight-spmf-eq-nn-integral-spmf*[*symmetric*] *field-simps min-def max-def not-le weight-spmf-lt-0 ennreal-mult*[*symmetric*])
done
thus *?thesis*
by(*auto simp*: *min-def max-def ennreal-mult*[*symmetric*] *split*: *if-split-asm*)
qed

lemma *weight-scale-spmf'* [*simp*]:
⟦ *0 ≤ r*; *r ≤ 1* ⟧ ⟹ *weight-spmf* (*scale-spmf r p*) = *r ∗ weight-spmf p*
by(*simp add*: *weight-scale-spmf max-def min-def*)(*metis antisym-conv mult-left-le order-trans weight-spmf-le-1*)

lemma *pmf-scale-spmf-None*:
pmf (*scale-spmf k p*) *None* = *1 − min 1* (*max 0 k ∗* (*1 − pmf p None*))
unfolding *pmf-None-eq-weight-spmf* **by**(*simp add*: *weight-scale-spmf*)

lemma *scale-scale-spmf*:
scale-spmf r (*scale-spmf r' p*) = *scale-spmf* (*r ∗ max 0* (*min* (*inverse* (*weight-spmf p*)) *r'*)) *p*

(**is** *?lhs = ?rhs*)
proof(*cases weight-spmf p > 0*)
case *False*
thus *?thesis*
by (*simp add*: *weight-spmf-eq-0 zero-less-measure-iff*)
next
case *True*
show *?thesis*
proof(*rule spmf-eqI*)
fix *i*
have ∗: *max 0* (*min* (*1 / weight-spmf p*) *r′*) ∗ *max 0* (*min* (*1 / min 1* (*weight-spmf p* ∗ *max 0 r′*)) *r*) =
max 0 (*min* (*1 / weight-spmf p*) (*r* ∗ *max 0* (*min* (*1 / weight-spmf p*) *r′*)))
using *True*
by (*simp add*: *max-def*) (*auto simp*: *min-def field-simps zero-le-mult-iff*)
show *spmf ?lhs i = spmf ?rhs i*
by (*simp add*: *spmf-scale-spmf*) (*metis* ∗ *inverse-eq-divide mult.commute weight-scale-spmf*)
qed
qed

lemma *scale-scale-spmf′* [*simp*]:
assumes *0 ≤ r r ≤ 1 0 ≤ r′ r′ ≤ 1*
shows *scale-spmf r* (*scale-spmf r′ p*) = *scale-spmf* (*r* ∗ *r′*) *p*
proof(*cases weight-spmf p > 0*)
case *True*
with *assms* **have** *r′ = 1* **if** *1 ≤ r′ ∗ weight-spmf p*
by (*smt* (*verit*, *best*) *measure-spmf.subprob-measure-le-1 mult-eq-1 mult-le-one that*)
with *assms True* **show** *?thesis*
by (*smt* (*verit*, *best*) *eq-divide-imp measure-le-0-iff mult.assoc mult-nonneg-nonneg scale-scale-spmf weight-scale-spmf′*)
next
case *False*
with *assms* **show** *?thesis*
by (*simp add*: *weight-spmf-eq-0 zero-less-measure-iff*)
qed

lemma *scale-spmf-eq-same*: *scale-spmf r p = p* ⟷ *weight-spmf p = 0* ∨ *r = 1* ∨ *r ≥ 1* ∧ *weight-spmf p = 1*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
hence *weight-spmf* (*scale-spmf r p*) = *weight-spmf p* **by** *simp*
hence ∗: *min 1* (*max 0 r* ∗ *weight-spmf p*) = *weight-spmf p* **by**(*simp add*: *weight-scale-spmf*)
hence ∗∗: *weight-spmf p = 0* ∨ *r ≥ 1* **by**(*auto simp*: *min-def max-def split*: *if-split-asm*)
show *?rhs*

proof(*cases weight-spmf p = 0*)
case *False*
with ∗∗ **have** *r ≥ 1*
by *simp*
with ∗ *False* **have** *r = 1 ∨ weight-spmf p = 1*
by(*simp add*: *max-def min-def not-le split*: *if-split-asm*)
with ‹*r ≥ 1*› **show** *?thesis*
by *simp*
qed *simp*
next
show *weight-spmf p = 0 ∨ r = 1 ∨ 1 ≤ r ∧ weight-spmf p = 1 ⟹ scale-spmf r p = p*
by (*smt* (*verit*) *div-by-1 inverse-eq-divide inverse-positive-iff-positive scale-scale-spmf scale-spmf-1*)
qed

lemma *map-const-spmf-of-set*:
⟦ *finite A*; *A ≠ {}* ⟧ ⟹ *map-spmf* (λ-. *c*) (*spmf-of-set A*) = *return-spmf c*
by(*simp add*: *map-spmf-conv-bind-spmf bind-spmf-const*)

25.16 Conditional spmfs

lemma *set-pmf-Int-Some*: *set-pmf p ∩ Some ‘ A = {} ⟷ set-spmf p ∩ A = {}*
by(*auto simp*: *in-set-spmf*)

lemma *measure-spmf-zero-iff*: *measure* (*measure-spmf p*) *A = 0 ⟷ set-spmf p ∩ A = {}*
unfolding *measure-measure-spmf-conv-measure-pmf* **by**(*simp add*: *measure-pmf-zero-iff set-pmf-Int-Some*)

definition *cond-spmf* :: *'a spmf ⇒ 'a set ⇒ 'a spmf*
where *cond-spmf p A* = (*if set-spmf p ∩ A = {} then return-pmf None else cond-pmf p* (*Some ‘ A*))

lemma *set-cond-spmf* [*simp*]: *set-spmf* (*cond-spmf p A*) = *set-spmf p ∩ A*
by(*auto 4 4 simp add*: *cond-spmf-def in-set-spmf iff*: *set-cond-pmf*[*THEN set-eq-iff*[*THEN iffD1*], *THEN spec*, *rotated*])

lemma *cond-map-spmf* [*simp*]: *cond-spmf* (*map-spmf f p*) *A* = *map-spmf f* (*cond-spmf p* (*f −‘ A*))
proof −
have *map-option f −‘ Some ‘ A = Some ‘ f −‘ A* **by** *auto*
moreover have *set-pmf p ∩ map-option f −‘ Some ‘ A ≠ {}* **if** *Some x ∈ set-pmf p f x ∈ A* **for** *x*
using *that* **by** *auto*
ultimately show *?thesis* **by**(*auto simp*: *cond-spmf-def in-set-spmf cond-map-pmf*)
qed

lemma *spmf-cond-spmf* [*simp*]:

spmf (*cond-spmf p A*) *x* = (*if x* ∈ *A then spmf p x* / *measure* (*measure-spmf p*) *A else 0*)
by(*auto simp*: *cond-spmf-def pmf-cond set-pmf-Int-Some*[*symmetric*] *measure-measure-spmf-conv-measure-pm measure-pmf-zero-iff*)

lemma *bind-eq-return-pmf-None*:
bind-spmf p f = *return-pmf None* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)
by(*auto simp*: *bind-spmf-def bind-eq-return-pmf in-set-spmf split*: *option.splits*)

lemma *return-pmf-None-eq-bind*:
return-pmf None = *bind-spmf p f* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)
using *bind-eq-return-pmf-None*[*of p f*] **by** *auto*

25.17 Product spmf

definition *pair-spmf* :: *'a spmf* ⇒ *'b spmf* ⇒ (*'a* × *'b*) *spmf*
where *pair-spmf p q* = *bind-pmf* (*pair-pmf p q*) (λ*xy*. *case xy of* (*Some x*, *Some y*) ⇒ *return-spmf* (*x*, *y*) | - ⇒ *return-pmf None*)

lemma *map-fst-pair-spmf* [*simp*]: *map-spmf fst* (*pair-spmf p q*) = *scale-spmf* (*weight-spmf q*) *p*
unfolding *bind-spmf-const*[*symmetric*]
apply(*simp add*: *pair-spmf-def map-bind-pmf pair-pmf-def bind-assoc-pmf option.case-distrib*)
apply(*subst bind-commute-pmf*)
apply(*force intro*!: *bind-pmf-cong*[*OF refl*] *simp add*: *bind-return-pmf bind-spmf-def bind-return-pmf' case-option-collapse*
option.case-distrib[**where** *h*=*map-spmf* -] *option.case-distrib*[*symmetric*] *case-option-id split*: *option.split cong*: *option.case-cong*)
done

lemma *map-snd-pair-spmf* [*simp*]: *map-spmf snd* (*pair-spmf p q*) = *scale-spmf* (*weight-spmf p*) *q*
unfolding *bind-spmf-const*[*symmetric*]
apply(*simp add*: *pair-spmf-def map-bind-pmf pair-pmf-def bind-assoc-pmf option.case-distrib*
cong del: *option.case-cong-weak*)
apply(*auto intro*!: *bind-pmf-cong*[*OF refl*] *simp add*: *bind-return-pmf bind-spmf-def bind-return-pmf' case-option-collapse*
option.case-distrib[**where** *h*=*map-spmf* -] *option.case-distrib*[*symmetric*] *case-option-id split*: *option.split cong del*: *option.case-cong-weak*)
done

lemma *set-pair-spmf* [*simp*]: *set-spmf* (*pair-spmf p q*) = *set-spmf p* × *set-spmf q*
by(*force simp add*: *pair-spmf-def set-spmf-bind-pmf bind-UNION in-set-spmf split*: *option.splits*)

lemma *spmf-pair* [*simp*]: *spmf* (*pair-spmf p q*) (*x*, *y*) = *spmf p x* ∗ *spmf q y* (**is** *?lhs* = *?rhs*)

proof −
have *ennreal ?lhs* = $\int^+$ *a*. $\int^+$ *b*. *indicator* {(*x*, *y*)} (*a*, *b*) ∂*measure-spmf q* ∂*measure-spmf p*
unfolding *measure-spmf-def pair-spmf-def ennreal-pmf-bind nn-integral-pair-pmf'*
by(*auto simp*: *zero-ereal-def*[*symmetric*] *nn-integral-distr nn-integral-restrict-space nn-integral-multc*[*symmetric*] *intro*!: *nn-integral-cong split*: *option.split split-indicator*)
also have ... = $\int^+$ *a*. ($\int^+$ *b*. *indicator* {*y*} *b* ∂*measure-spmf q*) ∗ *indicator* {*x*} *a* ∂*measure-spmf p*
by(*subst nn-integral-multc*[*symmetric*])(*auto intro*!: *nn-integral-cong split*: *split-indicator*)
also have ... = *ennreal ?rhs* **by**(*simp add*: *emeasure-spmf-single max-def ennreal-mult mult.commute*)
finally show *?thesis* **by** *simp*
qed

lemma *pair-map-spmf2*: *pair-spmf p* (*map-spmf f q*) = *map-spmf* (*apsnd f*) (*pair-spmf p q*)
unfolding *pair-spmf-def pair-map-pmf2 bind-map-pmf map-bind-pmf*
by (*intro bind-pmf-cong refl*) (*auto split*: *option.split*)

lemma *pair-map-spmf1*: *pair-spmf* (*map-spmf f p*) *q* = *map-spmf* (*apfst f*) (*pair-spmf p q*)
unfolding *pair-spmf-def pair-map-pmf1 bind-map-pmf map-bind-pmf*
by (*intro bind-pmf-cong refl*) (*auto split*: *option.split*)

lemma *pair-map-spmf*: *pair-spmf* (*map-spmf f p*) (*map-spmf g q*) = *map-spmf* (*map-prod f g*) (*pair-spmf p q*)
unfolding *pair-map-spmf2 pair-map-spmf1 spmf.map-comp*
by(*simp add*: *apfst-def apsnd-def o-def prod.map-comp*)

lemma *pair-spmf-alt-def*: *pair-spmf p q* = *bind-spmf p* (λ*x*. *bind-spmf q* (λ*y*. *return-spmf* (*x*, *y*)))
unfolding *pair-spmf-def pair-pmf-def bind-spmf-def bind-assoc-pmf bind-return-pmf*
by (*intro bind-pmf-cong refl*) (*auto split*: *option.split*)

lemma *weight-pair-spmf* [*simp*]: *weight-spmf* (*pair-spmf p q*) = *weight-spmf p* ∗ *weight-spmf q*
unfolding *pair-spmf-alt-def* **by**(*simp add*: *weight-bind-spmf o-def*)

lemma *pair-scale-spmf1*:
r ≤ *1* ⟹ *pair-spmf* (*scale-spmf r p*) *q* = *scale-spmf r* (*pair-spmf p q*)
by(*simp add*: *pair-spmf-alt-def scale-bind-spmf bind-scale-spmf*)

lemma *pair-scale-spmf2*:
r ≤ *1* ⟹ *pair-spmf p* (*scale-spmf r q*) = *scale-spmf r* (*pair-spmf p q*)
by(*simp add*: *pair-spmf-alt-def scale-bind-spmf bind-scale-spmf*)

lemma *pair-spmf-return-None1* [*simp*]: *pair-spmf* (*return-pmf None*) *p* = *return-pmf None*
by(*rule spmf-eqI*)(*clarsimp*)

lemma *pair-spmf-return-None2* [*simp*]: *pair-spmf p* (*return-pmf None*) = *return-pmf None*
by(*rule spmf-eqI*)(*clarsimp*)

lemma *pair-spmf-return-spmf1*: *pair-spmf* (*return-spmf x*) *q* = *map-spmf* (*Pair x*) *q*
by(*rule spmf-eqI*)(*auto split*: *split-indicator simp add*: *spmf-map-inj' inj-on-def intro*: *spmf-map-outside*)

lemma *pair-spmf-return-spmf2*: *pair-spmf p* (*return-spmf y*) = *map-spmf* (λ*x*. (*x*, *y*)) *p*
by(*rule spmf-eqI*)(*auto split*: *split-indicator simp add*: *inj-on-def intro*!: *spmf-map-outside spmf-map-inj'*[*symmetric*])

lemma *pair-spmf-return-spmf* [*simp*]: *pair-spmf* (*return-spmf x*) (*return-spmf y*) = *return-spmf* (*x*, *y*)
by(*simp add*: *pair-spmf-return-spmf1*)

lemma *rel-pair-spmf-prod*:
rel-spmf (*rel-prod A B*) (*pair-spmf p q*) (*pair-spmf p' q'*) ⟷
rel-spmf A (*scale-spmf* (*weight-spmf q*) *p*) (*scale-spmf* (*weight-spmf q'*) *p'*) ∧
rel-spmf B (*scale-spmf* (*weight-spmf p*) *q*) (*scale-spmf* (*weight-spmf p'*) *q'*)
(**is** *?lhs* ⟷ *?rhs* **is** - ⟷ *?A* ∧ *?B* **is** - ⟷ *rel-spmf* - *?p ?p'* ∧ *rel-spmf* - *?q ?q'*)
proof(*intro iffI conjI*)
assume *?rhs*
then obtain *pq pq'* **where** *p*: *map-spmf fst pq* = *?p* **and** *p'*: *map-spmf snd pq* = *?p'*
and *q*: *map-spmf fst pq'* = *?q* **and** *q'*: *map-spmf snd pq'* = *?q'*
and ∗: ⋀*x x'*. (*x*, *x'*) ∈ *set-spmf pq* ⟹ *A x x'*
and ∗∗: ⋀*y y'*. (*y*, *y'*) ∈ *set-spmf pq'* ⟹ *B y y'* **by**(*auto elim*!: *rel-spmfE*)
let *?f* = λ((*x*, *x'*), (*y*, *y'*)). ((*x*, *y*), (*x'*, *y'*))
let *?r* = *1* / (*weight-spmf p* ∗ *weight-spmf q*)
let *?pq* = *scale-spmf ?r* (*map-spmf ?f* (*pair-spmf pq pq'*))

{ fix *p* :: *'x spmf* **and** *q* :: *'y spmf*
assume *weight-spmf q* ≠ *0*
and *weight-spmf p* ≠ *0*
and *1* / (*weight-spmf p* ∗ *weight-spmf q*) ≤ *weight-spmf p* ∗ *weight-spmf q*
hence *1* ≤ (*weight-spmf p* ∗ *weight-spmf q*) ∗ (*weight-spmf p* ∗ *weight-spmf q*)
by(*simp add*: *pos-divide-le-eq order.strict-iff-order weight-spmf-nonneg*)
moreover have (*weight-spmf p* ∗ *weight-spmf q*) ∗ (*weight-spmf p* ∗ *weight-spmf q*) ≤ (*1* ∗ *1*) ∗ (*1* ∗ *1*)
by(*intro mult-mono*)(*simp-all add*: *weight-spmf-nonneg weight-spmf-le-1*)
ultimately have (*weight-spmf p* ∗ *weight-spmf q*) ∗ (*weight-spmf p* ∗ *weight-spmf q*) = *1* **by** *simp*
hence ∗: *weight-spmf p* ∗ *weight-spmf q* = *1*
by(*metis antisym-conv less-le mult-less-cancel-left1 weight-pair-spmf weight-spmf-le-1*

weight-spmf-nonneg)
hence $**$: *weight-spmf p = 1* **by**(*metis antisym-conv mult-left-le weight-spmf-le-1 weight-spmf-nonneg*)
moreover from $*$ $**$ **have** *weight-spmf q = 1* **by** *simp*
moreover note *calculation* **}**
note *full = this*

show *?lhs*
proof
have [*simp*]: *fst ∘ ?f = map-prod fst fst* **by**(*simp add*: *fun-eq-iff*)
have *map-spmf fst ?pq = scale-spmf ?r* (*pair-spmf ?p ?q*)
by(*simp add*: *pair-map-spmf*[*symmetric*] *p q map-scale-spmf spmf.map-comp*)
also have ... = *pair-spmf p q* **using** *full*[*of p q*]
by(*simp add*: *pair-scale-spmf1 pair-scale-spmf2 weight-spmf-le-1 weight-spmf-nonneg*)
(*auto simp*: *scale-scale-spmf max-def min-def field-simps weight-spmf-nonneg weight-spmf-eq-0*)
finally show *map-spmf fst ?pq* = ... **.**

have [*simp*]: *snd ∘ ?f = map-prod snd snd* **by**(*simp add*: *fun-eq-iff*)
from ‹*?rhs*› **have** *eq*: *weight-spmf p* $*$ *weight-spmf q = weight-spmf p′* $*$ *weight-spmf q′*
by(*auto dest*!: *rel-spmf-weightD simp add*: *weight-spmf-le-1 weight-spmf-nonneg*)

have *map-spmf snd ?pq = scale-spmf ?r* (*pair-spmf ?p′ ?q′*)
by(*simp add*: *pair-map-spmf*[*symmetric*] *p′ q′ map-scale-spmf spmf.map-comp*)
also have ... = *pair-spmf p′ q′* **using** *full*[*of p′ q′*] *eq*
by(*simp add*: *pair-scale-spmf1 pair-scale-spmf2 weight-spmf-le-1 weight-spmf-nonneg*)
(*auto simp*: *scale-scale-spmf max-def min-def field-simps weight-spmf-nonneg weight-spmf-eq-0*)
finally show *map-spmf snd ?pq* = ... **.**
qed(*auto simp*: *set-scale-spmf split*: *if-split-asm dest*: $*$ $**$)
next
assume *?lhs*
then obtain *pq* **where** *pq*: *map-spmf fst pq = pair-spmf p q*
and *pq′*: *map-spmf snd pq = pair-spmf p′ q′*
and $*$: $\bigwedge x\ y\ x'\ y'.\ ((x, y), (x', y')) \in$ *set-spmf pq* $\Longrightarrow$ *A x x′* $\wedge$ *B y y′*
by(*auto elim*: *rel-spmfE*)

show *?A*
proof
let *?f* = $(\lambda((x, y), (x', y')).\ (x, x'))$
let *?pq = map-spmf ?f pq*
have [*simp*]: *fst ∘ ?f = fst ∘ fst* **by**(*simp add*: *split-def o-def*)
show *map-spmf fst ?pq = scale-spmf* (*weight-spmf q*) *p* **using** *pq*
by(*simp add*: *spmf.map-comp*)(*simp add*: *spmf.map-comp*[*symmetric*])

have [*simp*]: *snd ∘ ?f = fst ∘ snd* **by**(*simp add*: *split-def o-def*)
show *map-spmf snd ?pq = scale-spmf* (*weight-spmf q′*) *p′* **using** *pq′*
by(*simp add*: *spmf.map-comp*)(*simp add*: *spmf.map-comp*[*symmetric*])

qed(*auto dest*: ∗)

show *?B*
proof
let *?f* = (λ((*x*, *y*), (*x′*, *y′*)). (*y*, *y′*))
let *?pq* = *map-spmf ?f pq*
have [*simp*]: *fst* ∘ *?f* = *snd* ∘ *fst* **by**(*simp add*: *split-def o-def*)
show *map-spmf fst ?pq* = *scale-spmf* (*weight-spmf p*) *q* **using** *pq*
by(*simp add*: *spmf.map-comp*)(*simp add*: *spmf.map-comp*[*symmetric*])

have [*simp*]: *snd* ∘ *?f* = *snd* ∘ *snd* **by**(*simp add*: *split-def o-def*)
show *map-spmf snd ?pq* = *scale-spmf* (*weight-spmf p′*) *q′* **using** *pq′*
by(*simp add*: *spmf.map-comp*)(*simp add*: *spmf.map-comp*[*symmetric*])
qed(*auto dest*: ∗)
qed

lemma *pair-pair-spmf*:
pair-spmf (*pair-spmf p q*) *r* = *map-spmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-spmf p* (*pair-spmf q r*))
by(*simp add*: *pair-spmf-alt-def map-spmf-conv-bind-spmf*)

lemma *pair-commute-spmf*:
pair-spmf p q = *map-spmf* (λ(*y*, *x*). (*x*, *y*)) (*pair-spmf q p*)
unfolding *pair-spmf-alt-def* **by**(*subst bind-commute-spmf*)(*simp add*: *map-spmf-conv-bind-spmf*)

25.18 Assertions

definition *assert-spmf* :: *bool* ⇒ *unit spmf*
where *assert-spmf b* = (*if b then return-spmf* () *else return-pmf None*)

lemma *assert-spmf-simps* [*simp*]:
assert-spmf True = *return-spmf* ()
assert-spmf False = *return-pmf None*
by(*simp-all add*: *assert-spmf-def*)

lemma *in-set-assert-spmf* [*simp*]: *x* ∈ *set-spmf* (*assert-spmf p*) ⟷ *p*
by(*cases p*) *simp-all*

lemma *set-spmf-assert-spmf-eq-empty* [*simp*]: *set-spmf* (*assert-spmf b*) = {} ⟷ ¬ *b*
by *auto*

lemma *lossless-assert-spmf* [*iff*]: *lossless-spmf* (*assert-spmf b*) ⟷ *b*
by(*cases b*) *simp-all*

25.19 Try

definition *try-spmf* :: *'a spmf* ⇒ *'a spmf* ⇒ *'a spmf*
(‹(‹*open-block notation*=‹*mixfix try-spmf*›› *TRY - ELSE -*)› [*0*,*60*] *59*)

where *TRY p ELSE q = bind-pmf p* (λ*x*. *case x of None* ⇒ *q* | *Some y* ⇒ *return-spmf y*)

lemma *try-spmf-lossless* [*simp*]:
assumes *lossless-spmf p*
shows *TRY p ELSE q = p*
proof −
have *TRY p ELSE q = bind-pmf p return-pmf* **unfolding** *try-spmf-def* **using** *assms*
by(*auto simp*: *lossless-iff-set-pmf-None split*: *option.split intro*: *bind-pmf-cong*)
thus *?thesis* **by**(*simp add*: *bind-return-pmf'*)
qed

lemma *try-spmf-return-spmf1*: *TRY return-spmf x ELSE q = return-spmf x*
by *simp*

lemma *try-spmf-return-None* [*simp*]: *TRY return-pmf None ELSE q = q*
by(*simp add*: *try-spmf-def bind-return-pmf*)

lemma *try-spmf-return-pmf-None2* [*simp*]: *TRY p ELSE return-pmf None = p*
by(*simp add*: *try-spmf-def option.case-distrib*[*symmetric*] *bind-return-pmf' case-option-id*)

lemma *map-try-spmf*: *map-spmf f* (*try-spmf p q*) = *try-spmf* (*map-spmf f p*) (*map-spmf f q*)
by(*simp add*: *try-spmf-def map-bind-pmf bind-map-pmf option.case-distrib*[**where** *h*=*map-spmf f*] *o-def cong del*: *option.case-cong-weak*)

lemma *try-spmf-bind-pmf*: *TRY* (*bind-pmf p f*) *ELSE q = bind-pmf p* (λ*x*. *TRY* (*f x*) *ELSE q*)
by(*simp add*: *try-spmf-def bind-assoc-pmf*)

lemma *try-spmf-bind-spmf-lossless*:
lossless-spmf p ⟹ *TRY* (*bind-spmf p f*) *ELSE q = bind-spmf p* (λ*x*. *TRY* (*f x*) *ELSE q*)
by (*metis* (*mono-tags*, *lifting*) *bind-spmf-of-pmf lossless-spmf-conv-spmf-of-pmf try-spmf-bind-pmf*)

lemma *try-spmf-bind-out*:
lossless-spmf p ⟹ *bind-spmf p* (λ*x*. *TRY* (*f x*) *ELSE q*) = *TRY* (*bind-spmf p f*) *ELSE q*
by(*simp add*: *try-spmf-bind-spmf-lossless*)

lemma *lossless-try-spmf* [*simp*]:
lossless-spmf (*TRY p ELSE q*) ⟷ *lossless-spmf p* ∨ *lossless-spmf q*
by(*auto simp*: *try-spmf-def in-set-spmf lossless-iff-set-pmf-None split*: *option.splits*)

context includes *lifting-syntax*
begin

lemma *try-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A ===> rel-spmf A ===> rel-spmf A*) *try-spmf try-spmf*
unfolding *try-spmf-def*[*abs-def*] **by** *transfer-prover*

end

lemma *try-spmf-cong*:
⟦ *p = p'*; ¬ *lossless-spmf p'* ⟹ *q = q'* ⟧ ⟹ *TRY p ELSE q = TRY p' ELSE q'*
unfolding *try-spmf-def*
by(*rule bind-pmf-cong*)(*auto split*: *option.split simp add*: *lossless-iff-set-pmf-None*)

lemma *rel-spmf-try-spmf*:
⟦ *rel-spmf R p p'*; ¬ *lossless-spmf p'* ⟹ *rel-spmf R q q'* ⟧
⟹ *rel-spmf R* (*TRY p ELSE q*) (*TRY p' ELSE q'*)
unfolding *try-spmf-def*
apply(*rule rel-pmf-bindI*[**where** *R*=λ*x y*. *rel-option R x y* ∧ *x* ∈ *set-pmf p* ∧ *y* ∈ *set-pmf p'*])
apply (*simp add*: *pmf.rel-mono-strong*)
apply(*auto split*: *option.split simp add*: *lossless-iff-set-pmf-None*)
done

lemma *spmf-try-spmf*:
spmf (*TRY p ELSE q*) *x* = *spmf p x* + *pmf p None* ∗ *spmf q x*
proof −
have *ennreal* (*spmf* (*TRY p ELSE q*) *x*) = ∫⁺ *y*. *ennreal* (*spmf q x*) ∗ *indicator* {*None*} *y* + *indicator* {*Some x*} *y* ∂*measure-pmf p*
unfolding *try-spmf-def ennreal-pmf-bind* **by**(*rule nn-integral-cong*)(*simp split*: *option.split split-indicator*)
also have ... = (∫⁺ *y*. *ennreal* (*spmf q x*) ∗ *indicator* {*None*} *y* ∂*measure-pmf p*) + ∫⁺ *y*. *indicator* {*Some x*} *y* ∂*measure-pmf p*
by(*simp add*: *nn-integral-add*)
also have ... = *ennreal* (*spmf q x*) ∗ *pmf p None* + *spmf p x* **by**(*simp add*: *emeasure-pmf-single*)
finally show *?thesis* **by**(*simp flip*: *ennreal-plus ennreal-mult*)
qed

lemma *try-scale-spmf-same* [*simp*]: *lossless-spmf p* ⟹ *TRY scale-spmf k p ELSE p = p*
by(*rule spmf-eqI*)(*auto simp*: *spmf-try-spmf spmf-scale-spmf pmf-scale-spmf-None lossless-iff-pmf-None weight-spmf-conv-pmf-None min-def max-def field-simps*)

lemma *pmf-try-spmf-None* [*simp*]: *pmf* (*TRY p ELSE q*) *None* = *pmf p None* ∗ *pmf q None* (**is** *?lhs = ?rhs*)
proof −
have *?lhs* = ∫ *x*. *pmf q None* ∗ *indicator* {*None*} *x* ∂*measure-pmf p*
unfolding *try-spmf-def pmf-bind* **by**(*rule Bochner-Integration.integral-cong*)(*simp-all split*: *option.split*)
also have ... = *?rhs* **by**(*simp add*: *measure-pmf-single*)
finally show *?thesis* **.**

qed

lemma *try-bind-spmf-lossless2*:
lossless-spmf q $\Longrightarrow$ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* $\gg=$ (λ*x*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
by(*rule spmf-eqI*)(*simp add*: *spmf-try-spmf pmf-bind-spmf-None spmf-bind field-simps measure-spmf.integrable-const-bound*[**where** *B=1*] *pmf-le-1 lossless-iff-pmf-None*)

lemma *try-bind-spmf-lossless2'*:
fixes *f* :: *'a* $\Rightarrow$ *'b spmf* **shows**
$[\![$ *NO-MATCH* (λ*x* :: *'a*. *try-spmf* (*g x* :: *'b spmf*) (*h x*)) *f*; *lossless-spmf q* $]\!]$
$\Longrightarrow$ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* $\gg=$ (λ*x* :: *'a*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
by(*rule try-bind-spmf-lossless2*)

lemma *try-bind-assert-spmf*:
TRY (*assert-spmf b* $\gg=$ *f*) *ELSE q* = (*if b then TRY* (*f* ()) *ELSE q else q*)
by *simp*

25.20 Miscellaneous

lemma assumes *rel-spmf* (λ*x y*. *bad1 x* = *bad2 y* $\wedge$ ($\neg$ *bad2 y* $\longrightarrow$ *A x* $\longleftrightarrow$ *B y*)) *p q* (**is** *rel-spmf ?A - -*)
shows *fundamental-lemma-bad*: *measure* (*measure-spmf p*) {*x*. *bad1 x*} = *measure* (*measure-spmf q*) {*y*. *bad2 y*} (**is** *?bad*)
and *fundamental-lemma*: |*measure* (*measure-spmf p*) {*x*. *A x*} − *measure* (*measure-spmf q*) {*y*. *B y*}| $\leq$
measure (*measure-spmf p*) {*x*. *bad1 x*} (**is** *?fundamental*)
proof −
have *good*: *rel-fun ?A* (=) (λ*x*. *A x* $\wedge$ $\neg$ *bad1 x*) (λ*y*. *B y* $\wedge$ $\neg$ *bad2 y*) **by**(*auto simp*: *rel-fun-def*)
from *assms* **have** *1*: *measure* (*measure-spmf p*) {*x*. *A x* $\wedge$ $\neg$ *bad1 x*} = *measure* (*measure-spmf q*) {*y*. *B y* $\wedge$ $\neg$ *bad2 y*}
by(*rule measure-spmf-parametric*[*THEN rel-funD*, *THEN rel-funD*])(*rule Collect-parametric*[*THEN rel-funD*, *OF good*])

have *bad*: *rel-fun ?A* (=) *bad1 bad2* **by**(*simp add*: *rel-fun-def*)
show *2*: *?bad* **using** *assms*
by(*rule measure-spmf-parametric*[*THEN rel-funD*, *THEN rel-funD*])(*rule Collect-parametric*[*THEN rel-funD*, *OF bad*])

let *?μp* = *measure* (*measure-spmf p*) **and** *?μq* = *measure* (*measure-spmf q*)
have {*x*. *A x* $\wedge$ *bad1 x*} $\cup$ {*x*. *A x* $\wedge$ $\neg$ *bad1 x*} = {*x*. *A x*}
and {*y*. *B y* $\wedge$ *bad2 y*} $\cup$ {*y*. *B y* $\wedge$ $\neg$ *bad2 y*} = {*y*. *B y*} **by** *auto*
then have |*?μp* {*x*. *A x*} − *?μq* {*x*. *B x*}| = |*?μp* ({*x*. *A x* $\wedge$ *bad1 x*} $\cup$ {*x*. *A x* $\wedge$ $\neg$ *bad1 x*}) − *?μq* ({*y*. *B y* $\wedge$ *bad2 y*} $\cup$ {*y*. *B y* $\wedge$ $\neg$ *bad2 y*})|
by *simp*
also have ... = |*?μp* {*x*. *A x* $\wedge$ *bad1 x*} + *?μp* {*x*. *A x* $\wedge$ $\neg$ *bad1 x*} − *?μq* {*y*. *B y* $\wedge$ *bad2 y*} − *?μq* {*y*. *B y* $\wedge$ $\neg$ *bad2 y*}|

by(*subst* (*1 2*) *measure-Union*)(*auto*)
also have ... = |*?μp* {*x*. *A x* ∧ *bad1 x*} − *?μq* {*y*. *B y* ∧ *bad2 y*}| **using** *1* **by** *simp*
also have ... ≤ *max* (*?μp* {*x*. *A x* ∧ *bad1 x*}) (*?μq* {*y*. *B y* ∧ *bad2 y*})
by(*rule abs-leI*)(*auto simp*: *max-def not-le*, *simp-all only*: *add-increasing measure-nonneg mult-2*)
also have ... ≤ *max* (*?μp* {*x*. *bad1 x*}) (*?μq* {*y*. *bad2 y*})
by(*rule max.mono*; *rule measure-spmf.finite-measure-mono*; *auto*)
also note *2*[*symmetric*]
finally show *?fundamental* **by** *simp*
qed

end

26 Indexed products of PMFs

theory *Product-PMF*
imports *Probability-Mass-Function Independent-Family*
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** ‹*abs'-summable'-on*› *46*)

26.1 Preliminaries

lemma *pmf-expectation-eq-infsetsum*: *measure-pmf.expectation p f* = *infsetsum* (λ*x*. *pmf p x* ∗ *f x*) *UNIV*
unfolding *infsetsum-def measure-pmf-eq-density* **by** (*subst integral-density*) *simp-all*

lemma *measure-pmf-prob-product*:
assumes *countable A countable B*
shows *measure-pmf.prob* (*pair-pmf M N*) (*A* × *B*) = *measure-pmf.prob M A* ∗ *measure-pmf.prob N B*
proof −
have *measure-pmf.prob* (*pair-pmf M N*) (*A* × *B*) = (∑ *a*(*a*, *b*)∈*A* × *B*. *pmf M a* ∗ *pmf N b*)
by (*auto intro*!: *infsetsum-cong simp add*: *measure-pmf-conv-infsetsum pmf-pair*)
also have ... = *measure-pmf.prob M A* ∗ *measure-pmf.prob N B*
using *assms* **by** (*subst infsetsum-product*) (*auto simp add*: *measure-pmf-conv-infsetsum*)
finally show *?thesis*
by *simp*
qed

26.2 Definition

In analogy to Pi_M, we define an indexed product of PMFs. In the literature, this is typically called taking a vector of independent random variables. Note that the components do not have to be identically distributed.

The operation takes an explicit index set A and a function f that maps each element from A to a PMF and defines the product measure $\bigotimes_{i \in A} f(i)$, which is represented as a $('a \Rightarrow 'b)$ *pmf*.

Note that unlike Pi_M, this only works for *finite* index sets. It could be extended to countable sets and beyond, but the construction becomes somewhat more involved.

definition *Pi-pmf* :: *'a set* ⇒ *'b* ⇒ (*'a* ⇒ *'b pmf*) ⇒ (*'a* ⇒ *'b*) *pmf* **where**
Pi-pmf A dflt p =
embed-pmf (λ*f*. *if* (∀ *x*. *x* ∉ *A* ⟶ *f x* = *dflt*) *then* ∏ *x*∈*A*. *pmf* (*p x*) (*f x*) *else 0*)

A technical subtlety that needs to be addressed is this: Intuitively, the functions in the support of a product distribution have domain A. However, since HOL is a total logic, these functions must still return *some* value for inputs outside A. The product measure Pi_M simply lets these functions return *undefined* in these cases. We chose a different solution here, which is to supply a default value *dflt* that is returned in these cases.

As one possible application, one could model the result of n different independent coin tosses as *Pi-pmf* {*0*..<*n*} *False* (λ-. *bernoulli-pmf* (*1* / *2*)). This returns a function of type *nat* ⇒ *bool* that maps every natural number below n to the result of the corresponding coin toss, and every other natural number to *False*.

lemma *pmf-Pi*:
assumes *A*: *finite A*
shows *pmf* (*Pi-pmf A dflt p*) *f* =
(*if* (∀ *x*. *x* ∉ *A* ⟶ *f x* = *dflt*) *then* ∏ *x*∈*A*. *pmf* (*p x*) (*f x*) *else 0*)
unfolding *Pi-pmf-def*
proof (*rule pmf-embed-pmf*, *goal-cases*)
case *2*
define *S* **where** *S* = {*f*. ∀ *x*. *x* ∉ *A* ⟶ *f x* = *dflt*}
define *B* **where** *B* = (λ*x*. *set-pmf* (*p x*))

have *neutral-left*: (∏ *xa*∈*A*. *pmf* (*p xa*) (*f xa*)) = *0*
if *f* ∈ *PiE A B* − (λ*f*. *restrict f A*) ‘ *S* **for** *f*
proof −
have *restrict* (λ*x*. *if x* ∈ *A then f x else dflt*) *A* ∈ (λ*f*. *restrict f A*) ‘ *S*
by (*intro imageI*) (*auto simp*: *S-def*)
also have *restrict* (λ*x*. *if x* ∈ *A then f x else dflt*) *A* = *f*
using *that* **by** (*auto simp*: *PiE-def Pi-def extensional-def fun-eq-iff*)
finally show *?thesis* **using** *that* **by** *blast*
qed
have *neutral-right*: (∏ *xa*∈*A*. *pmf* (*p xa*) (*f xa*)) = *0*
if *f* ∈ (λ*f*. *restrict f A*) ‘ *S* − *PiE A B* **for** *f*
proof −
from *that* **obtain** *f'* **where** *f'*: *f* = *restrict f' A f'* ∈ *S* **by** *auto*
moreover from *this* **and** *that* **have** *restrict f' A* ∉ *PiE A B* **by** *simp*

then obtain x **where** $x \in A$ *pmf* $(p\ x)$ $(f'\ x) = 0$ **by** (*auto simp*: *B-def set-pmf-eq*)
with f' **and** A **show** *?thesis* **by** *auto*
qed

have $(\lambda f.\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$ *abs-summable-on PiE A B*
by (*intro abs-summable-on-prod-PiE A*) (*auto simp*: *B-def*)
also have *?this* $\longleftrightarrow$ $(\lambda f.\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$ *abs-summable-on* $(\lambda f.\ restrict\ f\ A)$ ' *S*
by (*intro abs-summable-on-cong-neutral neutral-left neutral-right*) *auto*
also have ... $\longleftrightarrow$ $(\lambda f.\ \prod x{\in}A.\ pmf\ (p\ x)\ (restrict\ f\ A\ x))$ *abs-summable-on S*
by (*rule abs-summable-on-reindex-iff* [*symmetric*]) (*force simp*: *inj-on-def fun-eq-iff S-def*)
also have ... $\longleftrightarrow$ $(\lambda f.\ if\ \forall x.\ x \notin A \longrightarrow f\ x = dflt\ then\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x)\ else\ 0)$
abs-summable-on UNIV
by (*intro abs-summable-on-cong-neutral*) (*auto simp*: *S-def*)
finally have *summable*: ... **.**

have $1 = (\prod x{\in}A.\ 1{::}real)$ **by** *simp*
also have $(\prod x{\in}A.\ 1) = (\prod x{\in}A.\ \sum_a y{\in}B\ x.\ pmf\ (p\ x)\ y)$
unfolding *B-def* **by** (*subst infsetsum-pmf-eq-1*) *auto*
also have $(\prod x{\in}A.\ \sum_a y{\in}B\ x.\ pmf\ (p\ x)\ y) = (\sum_a f{\in}Pi_E\ A\ B.\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$
by (*intro infsetsum-prod-PiE* [*symmetric*] *A*) (*auto simp*: *B-def*)
also have ... $= (\sum_a f{\in}(\lambda f.\ restrict\ f\ A)\ `\ S.\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$ **using** *A*
by (*intro infsetsum-cong-neutral neutral-left neutral-right refl*)
also have ... $= (\sum_a f{\in}S.\ \prod x{\in}A.\ pmf\ (p\ x)\ (restrict\ f\ A\ x))$
by (*rule infsetsum-reindex*) (*force simp*: *inj-on-def fun-eq-iff S-def*)
also have ... $= (\sum_a f{\in}S.\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$
by (*intro infsetsum-cong*) (*auto simp*: *S-def*)
also have ... $= (\sum_a f.\ if\ \forall x.\ x \notin A \longrightarrow f\ x = dflt\ then\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x)\ else\ 0)$
by (*intro infsetsum-cong-neutral*) (*auto simp*: *S-def*)
also have *ennreal* ... $= (\int^+ f.\ ennreal\ (if\ \forall x.\ x \notin A \longrightarrow f\ x = dflt\ then\ \prod x{\in}A.\ pmf\ (p\ x)\ (f\ x)\ else\ 0)\ \partial count\text{-}space\ UNIV)$
by (*intro nn-integral-conv-infsetsum* [*symmetric*] *summable*) (*auto simp*: *prod-nonneg*)
finally show *?case* **by** *simp*
qed (*auto simp*: *prod-nonneg*)

lemma *Pi-pmf-cong*:
assumes $A = A'$ $dflt = dflt'$ $\bigwedge x.\ x \in A \Longrightarrow f\ x = f'\ x$
shows *Pi-pmf A dflt f* = *Pi-pmf A' dflt' f'*
proof −
have $(\lambda fa.\ if\ \forall x.\ x \notin A \longrightarrow fa\ x = dflt\ then\ \prod x{\in}A.\ pmf\ (f\ x)\ (fa\ x)\ else\ 0) =$
$(\lambda f.\ if\ \forall x.\ x \notin A' \longrightarrow f\ x = dflt'\ then\ \prod x{\in}A'.\ pmf\ (f'\ x)\ (f\ x)\ else\ 0)$
using *assms* **by** (*intro ext*) (*auto intro*!: *prod.cong*)
thus *?thesis*
by (*simp only*: *Pi-pmf-def*)

qed

lemma *pmf-Pi'*:
assumes *finite A* $\bigwedge x.\ x \notin A \Longrightarrow f\ x = dflt$
shows *pmf* (*Pi-pmf A dflt p*) *f* = ($\prod x \in A.$ *pmf* (*p x*) (*f x*))
using *assms* **by** (*subst pmf-Pi*) *auto*

lemma *pmf-Pi-outside*:
assumes *finite A* $\exists x.\ x \notin A \wedge f\ x \neq dflt$
shows *pmf* (*Pi-pmf A dflt p*) *f* = *0*
using *assms* **by** (*subst pmf-Pi*) *auto*

lemma *pmf-Pi-empty* [*simp*]: *Pi-pmf* {} *dflt p* = *return-pmf* (λ-. *dflt*)
by (*intro pmf-eqI*, *subst pmf-Pi*) (*auto simp*: *indicator-def*)

lemma *set-Pi-pmf-subset*: *finite A* ⟹ *set-pmf* (*Pi-pmf A dflt p*) ⊆ {*f*. ∀ *x*. *x* ∉ *A* ⟶ *f x* = *dflt*}
by (*auto simp*: *set-pmf-eq pmf-Pi*)

26.3 Dependent product sets with a default

The following describes a dependent product of sets where the functions are required to return the default value *dflt* outside their domain, in analogy to Pi_E, which uses *undefined*.

definition *PiE-dflt*
where *PiE-dflt A dflt B* = {*f*. ∀ *x*. (*x* ∈ *A* ⟶ *f x* ∈ *B x*) ∧ (*x* ∉ *A* ⟶ *f x* = *dflt*)}

lemma *restrict-PiE-dflt*: (λ*h*. *restrict h A*) ' *PiE-dflt A dflt B* = *PiE A B*
proof (*intro equalityI subsetI*)
fix *h* **assume** *h* ∈ (λ*h*. *restrict h A*) ' *PiE-dflt A dflt B*
thus *h* ∈ *PiE A B*
by (*auto simp*: *PiE-dflt-def*)
next
fix *h* **assume** *h*: *h* ∈ *PiE A B*
hence *restrict* (λ*x*. *if x* ∈ *A then h x else dflt*) *A* ∈ (λ*h*. *restrict h A*) ' *PiE-dflt A dflt B*
by (*intro imageI*) (*auto simp*: *PiE-def extensional-def PiE-dflt-def*)
also have *restrict* (λ*x*. *if x* ∈ *A then h x else dflt*) *A* = *h*
using *h* **by** (*auto simp*: *fun-eq-iff*)
finally show *h* ∈ (λ*h*. *restrict h A*) ' *PiE-dflt A dflt B* **.**
qed

lemma *dflt-image-PiE*: (λ*h x*. *if x* ∈ *A then h x else dflt*) ' *PiE A B* = *PiE-dflt A dflt B*
(**is** *?f* ' *?X* = *?Y*)
proof (*intro equalityI subsetI*)
fix *h* **assume** *h* ∈ *?f* ' *?X*
thus *h* ∈ *?Y*

```
    by (auto simp: PiE-dflt-def PiE-def)
next
  fix h assume h: h ∈ ?Y
  hence ?f (restrict h A) ∈ ?f ‘ ?X
    by (intro imageI) (auto simp: PiE-def extensional-def PiE-dflt-def)
  also have ?f (restrict h A) = h
    using h by (auto simp: fun-eq-iff PiE-dflt-def)
  finally show h ∈ ?f ‘ ?X .
qed

lemma finite-PiE-dflt [intro]:
  assumes finite A ⋀x. x ∈ A ⟹ finite (B x)
  shows   finite (PiE-dflt A d B)
proof −
  have PiE-dflt A d B = (λf x. if x ∈ A then f x else d) ‘ PiE A B
    by (rule dflt-image-PiE [symmetric])
  also have finite ...
    by (intro finite-imageI finite-PiE assms)
  finally show ?thesis .
qed

lemma card-PiE-dflt:
  assumes finite A ⋀x. x ∈ A ⟹ finite (B x)
  shows   card (PiE-dflt A d B) = (∏ x∈A. card (B x))
proof −
  from assms have (∏ x∈A. card (B x)) = card (PiE A B)
    by (intro card-PiE [symmetric]) auto
  also have PiE A B = (λf. restrict f A) ‘ PiE-dflt A d B
    by (rule restrict-PiE-dflt [symmetric])
  also have card ... = card (PiE-dflt A d B)
   by (intro card-image) (force simp: inj-on-def restrict-def fun-eq-iff PiE-dflt-def)
  finally show ?thesis ..
qed

lemma PiE-dflt-empty-iff [simp]: PiE-dflt A dflt B = {} ⟷ (∃ x∈A. B x = {})
  by (simp add: dflt-image-PiE [symmetric] PiE-eq-empty-iff)

lemma set-Pi-pmf-subset':
  assumes finite A
  shows   set-pmf (Pi-pmf A dflt p) ⊆ PiE-dflt A dflt (set-pmf ∘ p)
  using assms by (auto simp: set-pmf-eq pmf-Pi PiE-dflt-def)

lemma set-Pi-pmf:
  assumes finite A
  shows   set-pmf (Pi-pmf A dflt p) = PiE-dflt A dflt (set-pmf ∘ p)
proof (rule equalityI)
  show PiE-dflt A dflt (set-pmf ∘ p) ⊆ set-pmf (Pi-pmf A dflt p)
  proof safe
    fix f assume f: f ∈ PiE-dflt A dflt (set-pmf ∘ p)
```

```
    hence pmf (Pi-pmf A dflt p) f = (∏ x∈A. pmf (p x) (f x))
      using assms by (auto simp: pmf-Pi PiE-dflt-def)
    also have ... > 0
      using f by (intro prod-pos) (auto simp: PiE-dflt-def set-pmf-eq)
    finally show f ∈ set-pmf (Pi-pmf A dflt p)
      by (auto simp: set-pmf-eq)
  qed
qed (use set-Pi-pmf-subset'[OF assms, of dflt p] in auto)
```

The probability of an independent combination of events is precisely the product of the probabilities of each individual event.

```
lemma measure-Pi-pmf-PiE-dflt:
  assumes [simp]: finite A
  shows   measure-pmf.prob (Pi-pmf A dflt p) (PiE-dflt A dflt B) =
            (∏ x∈A. measure-pmf.prob (p x) (B x))
proof −
  define B' where B' = (λx. B x ∩ set-pmf (p x))
  have measure-pmf.prob (Pi-pmf A dflt p) (PiE-dflt A dflt B) =
         (∑ a h∈PiE-dflt A dflt B. pmf (Pi-pmf A dflt p) h)
    by (rule measure-pmf-conv-infsetsum)
  also have ... = (∑ a h∈PiE-dflt A dflt B. ∏ x∈A. pmf (p x) (h x))
    by (intro infsetsum-cong, subst pmf-Pi') (auto simp: PiE-dflt-def)
  also have ... = (∑ a h∈(λh. restrict h A) ' PiE-dflt A dflt B. ∏ x∈A. pmf (p
x) (h x))
    by (subst infsetsum-reindex) (force simp: inj-on-def PiE-dflt-def fun-eq-iff)+
  also have (λh. restrict h A) ' PiE-dflt A dflt B = PiE A B
    by (rule restrict-PiE-dflt)
  also have (∑ a h∈PiE A B. ∏ x∈A. pmf (p x) (h x)) = (∑ a h∈PiE A B'. ∏ x∈A.
pmf (p x) (h x))
    by (intro infsetsum-cong-neutral) (auto simp: B'-def set-pmf-eq)
  also have (∑ a h∈PiE A B'. ∏ x∈A. pmf (p x) (h x)) = (∏ x∈A. infsetsum (pmf
(p x)) (B' x))
    by (intro infsetsum-prod-PiE) (auto simp: B'-def)
  also have ... = (∏ x∈A. infsetsum (pmf (p x)) (B x))
    by (intro prod.cong infsetsum-cong-neutral) (auto simp: B'-def set-pmf-eq)
  also have ... = (∏ x∈A. measure-pmf.prob (p x) (B x))
    by (subst measure-pmf-conv-infsetsum) (rule refl)
  finally show ?thesis .
qed

lemma measure-Pi-pmf-Pi:
  fixes t::nat
  assumes [simp]: finite A
  shows   measure-pmf.prob (Pi-pmf A dflt p) (Pi A B) =
            (∏ x∈A. measure-pmf.prob (p x) (B x)) (is ?lhs = ?rhs)
proof −
  have ?lhs = measure-pmf.prob (Pi-pmf A dflt p) (PiE-dflt A dflt B)
    by (intro measure-prob-cong-0)
       (auto simp: PiE-dflt-def PiE-def intro!: pmf-Pi-outside)+
```

also have ... = *?rhs*
using *assms* **by** (*simp add*: *measure-Pi-pmf-PiE-dflt*)
finally show *?thesis*
by *simp*
qed

26.4 Common PMF operations on products

Pi-pmf distributes over the 'bind' operation in the Giry monad:

lemma *Pi-pmf-bind*:
assumes *finite A*
shows *Pi-pmf A d* (λ*x*. *bind-pmf* (*p x*) (*q x*)) =
do {*f* ← *Pi-pmf A d'* *p*; *Pi-pmf A d* (λ*x*. *q x* (*f x*))} (**is** *?lhs* = *?rhs*)
proof (*rule pmf-eqI*, *goal-cases*)
case (*1 f*)
show *?case*
proof (*cases* ∃ *x*∈−*A*. *f x* ≠ *d*)
case *False*
define *B* **where** *B* = (λ*x*. *set-pmf* (*p x*))
have [*simp*]: *countable* (*B x*) **for** *x* **by** (*auto simp*: *B-def*)

{
fix *x* :: *'a*
have (λ*a*. *pmf* (*p x*) *a* ∗ *1*) *abs-summable-on B x*
by (*simp add*: *pmf-abs-summable*)
moreover have *norm* (*pmf* (*p x*) *a* ∗ *1*) ≥ *norm* (*pmf* (*p x*) *a* ∗ *pmf* (*q x a*) (*f x*)) **for** *a*
unfolding *norm-mult* **by** (*intro mult-left-mono*) (*auto simp*: *pmf-le-1*)
ultimately have (λ*a*. *pmf* (*p x*) *a* ∗ *pmf* (*q x a*) (*f x*)) *abs-summable-on B x*
by (*rule abs-summable-on-comparison-test*)
} **note** *summable* = *this*

have *pmf ?rhs f* = (∑ *a g*. *pmf* (*Pi-pmf A d'* *p*) *g* ∗ (∏ *x*∈*A*. *pmf* (*q x* (*g x*)) (*f x*)))
by (*subst pmf-bind*, *subst pmf-Pi'*)
(*insert assms False*, *simp-all add*: *pmf-expectation-eq-infsetsum*)
also have ... = (∑ *a g*∈*PiE-dflt A d'* *B*.
pmf (*Pi-pmf A d'* *p*) *g* ∗ (∏ *x*∈*A*. *pmf* (*q x* (*g x*)) (*f x*)))
unfolding *B-def*
using *assms* **by** (*intro infsetsum-cong-neutral*) (*auto simp*: *pmf-Pi PiE-dflt-def set-pmf-eq*)
also have ... = (∑ *a g*∈*PiE-dflt A d'* *B*.
(∏ *x*∈*A*. *pmf* (*p x*) (*g x*) ∗ *pmf* (*q x* (*g x*)) (*f x*)))
using *assms* **by** (*intro infsetsum-cong*) (*auto simp*: *pmf-Pi PiE-dflt-def prod.distrib*)
also have ... = (∑ *a g*∈(λ*g*. *restrict g A*) ' *PiE-dflt A d'* *B*.
(∏ *x*∈*A*. *pmf* (*p x*) (*g x*) ∗ *pmf* (*q x* (*g x*)) (*f x*)))
by (*subst infsetsum-reindex*) (*force simp*: *PiE-dflt-def inj-on-def fun-eq-iff*)+
also have (λ*g*. *restrict g A*) ' *PiE-dflt A d'* *B* = *PiE A B*

```
    by (rule restrict-PiE-dflt)
  also have (∑a g∈.... (∏ x∈A. pmf (p x) (g x) * pmf (q x (g x)) (f x))) =
            (∏ x∈A. ∑a a∈B x. pmf (p x) a * pmf (q x a) (f x))
    using assms summable by (subst infsetsum-prod-PiE) simp-all
  also have ... = (∏ x∈A. ∑a a. pmf (p x) a * pmf (q x a) (f x))
    by (intro prod.cong infsetsum-cong-neutral) (auto simp: B-def set-pmf-eq)
  also have ... = pmf ?lhs f
   using False assms by (subst pmf-Pi') (simp-all add: pmf-bind pmf-expectation-eq-infsetsum)
  finally show ?thesis ..
 next
  case True
  have pmf ?rhs f =
        measure-pmf.expectation (Pi-pmf A d' p) (λx. pmf (Pi-pmf A d (λxa. q
xa (x xa))) f)
    using assms by (simp add: pmf-bind)
  also have ... = measure-pmf.expectation (Pi-pmf A d' p) (λx. 0)
   using assms True by (intro Bochner-Integration.integral-cong pmf-Pi-outside)
auto
  also have ... = pmf ?lhs f
    using assms True by (subst pmf-Pi-outside) auto
  finally show ?thesis ..
 qed
qed

lemma Pi-pmf-return-pmf [simp]:
 assumes finite A
 shows  Pi-pmf A dflt (λx. return-pmf (f x)) = return-pmf (λx. if x ∈ A then f
x else dflt)
  using assms by (intro pmf-eqI) (auto simp: pmf-Pi simp: indicator-def split:
if-splits)
```

Analogously any componentwise mapping can be pulled outside the product:

```
lemma Pi-pmf-map:
 assumes [simp]: finite A and f dflt = dflt'
 shows  Pi-pmf A dflt' (λx. map-pmf f (g x)) = map-pmf (λh. f ∘ h) (Pi-pmf A
dflt g)
proof −
 have Pi-pmf A dflt' (λx. map-pmf f (g x)) =
       Pi-pmf A dflt' (λx. g x ⤜ (λx. return-pmf (f x)))
   using assms by (simp add: map-pmf-def Pi-pmf-bind)
 also have ... = Pi-pmf A dflt g ⤜ (λh. return-pmf (λx. if x ∈ A then f (h x)
else dflt'))
  by (subst Pi-pmf-bind[where d' = dflt]) auto
 also have ... = map-pmf (λh. f ∘ h) (Pi-pmf A dflt g)
   unfolding map-pmf-def using set-Pi-pmf-subset'[of A dflt g]
   by (intro bind-pmf-cong refl arg-cong[of - - return-pmf])
      (auto dest: simp: fun-eq-iff PiE-dflt-def assms(2))
 finally show ?thesis .
qed
```

We can exchange the default value in a product of PMFs like this:

lemma *Pi-pmf-default-swap*:
assumes *finite A*
shows *map-pmf* ($\lambda f\ x.$ *if* $x \in A$ *then* $f\ x$ *else dflt'*) (*Pi-pmf A dflt p*) =
Pi-pmf A dflt' p (**is** *?lhs* = *?rhs*)
proof (*rule pmf-eqI*, *goal-cases*)
case (*1 f*)
let *?B* = ($\lambda f\ x.$ *if* $x \in A$ *then* $f\ x$ *else dflt'*) $-$ ‘ $\{f\} \cap$ *PiE-dflt A dflt* (λ-. *UNIV*)
show *?case*
proof (*cases* $\exists\, x \in -A.\ f\ x \neq$ *dflt'*)
case *False*
let *?f'* = $\lambda x.$ *if* $x \in A$ *then* $f\ x$ *else dflt*
from *False* **have** *pmf ?lhs f* = *measure-pmf.prob* (*Pi-pmf A dflt p*) *?B*
using *assms* **unfolding** *pmf-map*
by (*intro measure-prob-cong-0*) (*auto simp*: *PiE-dflt-def pmf-Pi-outside*)
also from *False* **have** *?B* = {*?f'*}
by (*auto simp*: *fun-eq-iff PiE-dflt-def*)
also have *measure-pmf.prob* (*Pi-pmf A dflt p*) {*?f'*} = *pmf* (*Pi-pmf A dflt p*) *?f'*
by (*simp add*: *measure-pmf-single*)
also have $\ldots$ = *pmf ?rhs f*
using *False assms* **by** (*subst* (*1 2*) *pmf-Pi*) *auto*
finally show *?thesis* **.**
next
case *True*
have *pmf ?lhs f* = *measure-pmf.prob* (*Pi-pmf A dflt p*) *?B*
using *assms* **unfolding** *pmf-map*
by (*intro measure-prob-cong-0*) (*auto simp*: *PiE-dflt-def pmf-Pi-outside*)
also from *True* **have** *?B* = {} **by** *auto*
also have *measure-pmf.prob* (*Pi-pmf A dflt p*) $\ldots$ = *0*
by *simp*
also have *0* = *pmf ?rhs f*
using *True assms* **by** (*intro pmf-Pi-outside* [*symmetric*]) *auto*
finally show *?thesis* **.**
qed
qed

The following rule allows reindexing the product:

lemma *Pi-pmf-bij-betw*:
assumes *finite A bij-betw h A B* $\bigwedge x.\ x \notin A \Longrightarrow h\ x \notin B$
shows *Pi-pmf A dflt* (λ-. *f*) = *map-pmf* ($\lambda g.\ g \circ h$) (*Pi-pmf B dflt* (λ-. *f*))
(**is** *?lhs* = *?rhs*)
proof $-$
have *B*: *finite B*
using *assms bij-betw-finite* **by** *auto*
have *pmf ?lhs g* = *pmf ?rhs g* **for** *g*
proof (*cases* $\forall\, a.\ a \notin A \longrightarrow g\ a =$ *dflt*)
case *True*
define *h'* **where** *h'* = *the-inv-into A h*

```
    have h': h' (h x) = x if x ∈ A for x
    unfolding h'-def using that assms by (auto simp add: bij-betw-def the-inv-into-f-f)
    have h: h (h' x) = x if x ∈ B for x
      unfolding h'-def using that assms f-the-inv-into-f-bij-betw by fastforce
    have pmf ?rhs g = measure-pmf.prob (Pi-pmf B dflt (λ-. f)) ((λg. g ∘ h) -` {g})
      unfolding pmf-map by simp
    also have ... = measure-pmf.prob (Pi-pmf B dflt (λ-. f))
                                     (((λg. g ∘ h) -` {g}) ∩ PiE-dflt B dflt (λ-. UNIV))
    using B by (intro measure-prob-cong-0) (auto simp: PiE-dflt-def pmf-Pi-outside)
    also have ... = pmf (Pi-pmf B dflt (λ-. f)) (λx. if x ∈ B then g (h' x) else dflt)
    proof -
      have (if h x ∈ B then g (h' (h x)) else dflt) = g x for x
        using h' assms True by (cases x ∈ A) (auto simp add: bij-betwE)
      then have (λg. g ∘ h) -` {g} ∩ PiE-dflt B dflt (λ-. UNIV) =
            {(λx. if x ∈ B then g (h' x) else dflt)}
        using assms h' h True unfolding PiE-dflt-def by auto
      then show ?thesis
        by (simp add: measure-pmf-single)
    qed
    also have ... = pmf (Pi-pmf A dflt (λ-. f)) g
      using B assms True  h'-def
      by (auto simp add: pmf-Pi intro!: prod.reindex-bij-betw bij-betw-the-inv-into)
    finally show ?thesis
      by simp
  next
    case False
    have pmf ?rhs g = infsetsum (pmf (Pi-pmf B dflt (λ-. f))) ((λg. g ∘ h) -` {g})
      using assms by (auto simp add: measure-pmf-conv-infsetsum pmf-map)
    also have ... = infsetsum (λ-. 0) ((λg x. g (h x)) -` {g})
      using B False assms by (intro infsetsum-cong pmf-Pi-outside) fastforce+
    also have ... = 0
      by simp
    finally show ?thesis
      using assms False by (auto simp add: pmf-Pi pmf-map)
  qed
  then show ?thesis
    by (rule pmf-eqI)
qed
```

A product of uniform random choices is again a uniform distribution.

```
lemma Pi-pmf-of-set:
  assumes finite A ⋀x. x ∈ A ⟹ finite (B x) ⋀x. x ∈ A ⟹ B x ≠ {}
  shows   Pi-pmf A d (λx. pmf-of-set (B x)) = pmf-of-set (PiE-dflt A d B) (is ?lhs = ?rhs)
proof (rule pmf-eqI, goal-cases)
  case (1 f)
  show ?case
```

proof (*cases* $\exists x.\ x \notin A \wedge f\ x \neq d$)
case *True*
hence *pmf ?lhs f = 0*
using *assms* **by** (*intro pmf-Pi-outside*) (*auto simp*: *PiE-dflt-def*)
also from *True* **have** *f* $\notin$ *PiE-dflt A d B*
by (*auto simp*: *PiE-dflt-def*)
hence *0 = pmf ?rhs f*
using *assms* **by** (*subst pmf-of-set*) *auto*
finally show *?thesis* **.**
next
case *False*
hence *pmf ?lhs f* = ($\prod x{\in}A.$ *pmf* (*pmf-of-set* (*B x*)) (*f x*))
using *assms* **by** (*subst pmf-Pi'*) *auto*
also have ... = ($\prod x{\in}A.$ *indicator* (*B x*) (*f x*) / *real* (*card* (*B x*)))
by (*intro prod.cong refl*, *subst pmf-of-set*) (*use assms False* **in** *auto*)
also have ... = ($\prod x{\in}A.$ *indicator* (*B x*) (*f x*)) / *real* ($\prod x{\in}A.$ *card* (*B x*))
by (*subst prod-divide*f) *simp-all*
also have ($\prod x{\in}A.$ *indicator* (*B x*) (*f x*) :: *real*) = *indicator* (*PiE-dflt A d B*) *f*
using *assms False* **by** (*auto simp*: *indicator-def PiE-dflt-def*)
also have ($\prod x{\in}A.$ *card* (*B x*)) = *card* (*PiE-dflt A d B*)
using *assms* **by** (*intro card-PiE-dflt* [*symmetric*]) *auto*
also have *indicator* (*PiE-dflt A d B*) *f* / ... = *pmf ?rhs f*
using *assms* **by** (*intro pmf-of-set* [*symmetric*]) *auto*
finally show *?thesis* **.**
qed
qed

26.5 Merging and splitting PMF products

The following lemma shows that we can add a single PMF to a product:

lemma *Pi-pmf-insert*:
assumes *finite A x* $\notin$ *A*
shows *Pi-pmf* (*insert x A*) *dflt p = map-pmf* ($\lambda(y,f).$ *f*(*x*:=*y*)) (*pair-pmf* (*p x*) (*Pi-pmf A dflt p*))
proof (*intro pmf-eqI*)
fix *f*
let *?M = pair-pmf* (*p x*) (*Pi-pmf A dflt p*)
have *pmf* (*map-pmf* ($\lambda(y, f).$ *f*(*x* := *y*)) *?M*) *f* =
measure-pmf.prob ?M (($\lambda(y, f).$ *f*(*x* := *y*)) −‘ {*f*})
by (*subst pmf-map*) *auto*
also have (($\lambda(y, f).$ *f*(*x* := *y*)) −‘ {*f*}) = ($\bigcup y'.$ {(*f x*, *f*(*x* := *y*′))})
by (*auto simp*: *fun-upd-def fun-eq-iff*)
also have *measure-pmf.prob ?M* ... = *measure-pmf.prob ?M* {(*f x*, *f*(*x* := *dflt*))}
using *assms* **by** (*intro measure-prob-cong-0*) (*auto simp*: *pmf-pair pmf-Pi split*: *if-splits*)
also have ... = *pmf* (*p x*) (*f x*) * *pmf* (*Pi-pmf A dflt p*) (*f*(*x* := *dflt*))
by (*simp add*: *measure-pmf-single pmf-pair pmf-Pi*)
also have ... = *pmf* (*Pi-pmf* (*insert x A*) *dflt p*) *f*
proof (*cases* $\forall y.\ y \notin$ *insert x A* $\longrightarrow$ *f y = dflt*)

```
    case True
    with assms have pmf (p x) (f x) * pmf (Pi-pmf A dflt p) (f(x := dflt)) =
                    pmf (p x) (f x) * (∏ xa∈A. pmf (p xa) ((f(x := dflt)) xa))
      by (subst pmf-Pi') auto
    also have (∏ xa∈A. pmf (p xa) ((f(x := dflt)) xa)) = (∏ xa∈A. pmf (p xa) (f
xa))
      using assms by (intro prod.cong) auto
    also have pmf (p x) (f x) * ... = pmf (Pi-pmf (insert x A) dflt p) f
      using assms True by (subst pmf-Pi') auto
    finally show ?thesis .
  qed (insert assms, auto simp: pmf-Pi)
  finally show ... = pmf (map-pmf (λ(y, f). f(x := y)) ?M) f ..
qed

lemma Pi-pmf-insert':
  assumes finite A  x ∉ A
  shows   Pi-pmf (insert x A) dflt p =
          do {y ← p x; f ← Pi-pmf A dflt p; return-pmf (f(x := y))}
  using assms
  by (subst Pi-pmf-insert)
      (auto simp add: map-pmf-def pair-pmf-def case-prod-beta' bind-return-pmf
bind-assoc-pmf)

lemma Pi-pmf-singleton:
  Pi-pmf {x} dflt p = map-pmf (λa b. if b = x then a else dflt) (p x)
proof −
  have Pi-pmf {x} dflt p = map-pmf (fun-upd (λ-. dflt) x) (p x)
    by (subst Pi-pmf-insert) (simp-all add: pair-return-pmf2 pmf.map-comp o-def)
  also have fun-upd (λ-. dflt) x = (λz y. if y = x then z else dflt)
    by (simp add: fun-upd-def fun-eq-iff)
  finally show ?thesis .
qed
```

Projecting a product of PMFs onto a component yields the expected result:

```
lemma Pi-pmf-component:
  assumes finite A
  shows   map-pmf (λf. f x) (Pi-pmf A dflt p) = (if x ∈ A then p x else return-pmf
dflt)
proof (cases x ∈ A)
  case True
  define A' where A' = A − {x}
  from assms and True have A': A = insert x A'
    by (auto simp: A'-def)
  from assms have map-pmf (λf. f x) (Pi-pmf A dflt p) = p x unfolding A'
    by (subst Pi-pmf-insert)
       (auto simp: A'-def pmf.map-comp o-def case-prod-unfold map-fst-pair-pmf)
  with True show ?thesis by simp
next
  case False
```

have *map-pmf* (λ*f*. *f x*) (*Pi-pmf A dflt p*) = *map-pmf* (λ-. *dflt*) (*Pi-pmf A dflt p*)
using *assms False set-Pi-pmf-subset*[*of A dflt p*]
by (*intro pmf.map-cong refl*) (*auto simp*: *set-pmf-eq pmf-Pi-outside*)
with *False* **show** *?thesis* **by** *simp*
qed

We can take merge two PMF products on disjoint sets like this:

lemma *Pi-pmf-union*:
assumes *finite A finite B A* ∩ *B* = {}
shows *Pi-pmf* (*A* ∪ *B*) *dflt p* =
map-pmf (λ(*f*,*g*) *x*. *if x* ∈ *A then f x else g x*)
(*pair-pmf* (*Pi-pmf A dflt p*) (*Pi-pmf B dflt p*)) (**is** - = *map-pmf* (*?h A*) (*?q A*))
using *assms*(*1*,*3*)
proof (*induction rule*: *finite-induct*)
case (*insert x A*)
have *map-pmf* (*?h* (*insert x A*)) (*?q* (*insert x A*)) =
do {*v* ← *p x*; (*f*, *g*) ← *pair-pmf* (*Pi-pmf A dflt p*) (*Pi-pmf B dflt p*);
return-pmf (λ*y*. *if y* ∈ *insert x A then* (*f*(*x* := *v*)) *y else g y*)}
by (*subst Pi-pmf-insert*)
(*insert insert.hyps insert.prems*,
simp-all add: *pair-pmf-def map-bind-pmf bind-map-pmf bind-assoc-pmf bind-return-pmf*)
also have ... = *do* {*v* ← *p x*; (*f*, *g*) ← *?q A*; *return-pmf* ((*?h A* (*f*,*g*))(*x* := *v*))}
by (*intro bind-pmf-cong refl*) (*auto simp*: *fun-eq-iff*)
also have ... = *do* {*v* ← *p x*; *f* ← *map-pmf* (*?h A*) (*?q A*); *return-pmf* (*f*(*x* := *v*))}
by (*simp add*: *bind-map-pmf map-bind-pmf case-prod-unfold cong*: *if-cong*)
also have ... = *do* {*v* ← *p x*; *f* ← *Pi-pmf* (*A* ∪ *B*) *dflt p*; *return-pmf* (*f*(*x* := *v*))}
using *insert.hyps* **and** *insert.prems* **by** (*intro bind-pmf-cong insert.IH* [*symmetric*] *refl*) *auto*
also have ... = *Pi-pmf* (*insert x* (*A* ∪ *B*)) *dflt p*
by (*subst Pi-pmf-insert*)
(*insert assms insert.hyps insert.prems*, *auto simp*: *pair-pmf-def map-bind-pmf*)
also have *insert x* (*A* ∪ *B*) = *insert x A* ∪ *B*
by *simp*
finally show *?case* **..**
qed (*simp-all add*: *case-prod-unfold map-snd-pair-pmf*)

We can also project a product to a subset of the indices by mapping all the other indices to the default value:

lemma *Pi-pmf-subset*:
assumes *finite A A′* ⊆ *A*
shows *Pi-pmf A′ dflt p = map-pmf* (λ*f x*. *if x* ∈ *A′ then f x else dflt*) (*Pi-pmf A dflt p*)
proof −
let *?P* = *pair-pmf* (*Pi-pmf A′ dflt p*) (*Pi-pmf* (*A* − *A′*) *dflt p*)

from *assms* **have** [*simp*]: *finite A′*
by (*blast dest*: *finite-subset*)
from *assms* **have** $A = A' \cup (A - A')$
by *blast*
also have *Pi-pmf ... dflt p = map-pmf* (λ(*f*,*g*) *x*. *if x ∈ A′ then f x else g x*) *?P*
using *assms* **by** (*intro Pi-pmf-union*) *auto*
also have *map-pmf* (λ*f x*. *if x ∈ A′ then f x else dflt*) *... = map-pmf fst ?P*
unfolding *map-pmf-comp o-def case-prod-unfold*
using *set-Pi-pmf-subset*[*of A′ dflt p*] **by** (*intro map-pmf-cong refl*) (*auto simp*: *fun-eq-iff*)
also have *... = Pi-pmf A′ dflt p*
by (*simp add*: *map-fst-pair-pmf*)
finally show *?thesis* **..**
qed

lemma *Pi-pmf-subset′*:
fixes *f* :: *′a* ⇒ *′b pmf*
assumes *finite A B* ⊆ *A* ⋀*x*. *x* ∈ *A* − *B* ⟹ *f x* = *return-pmf dflt*
shows *Pi-pmf A dflt f* = *Pi-pmf B dflt f*
proof −
have *Pi-pmf* (*B* ∪ (*A* − *B*)) *dflt f* =
map-pmf (λ(*f*, *g*) *x*. *if x ∈ B then f x else g x*)
(*pair-pmf* (*Pi-pmf B dflt f*) (*Pi-pmf* (*A* − *B*) *dflt f*))
using *assms* **by** (*intro Pi-pmf-union*) (*auto dest*: *finite-subset*)
also have *Pi-pmf* (*A* − *B*) *dflt f* = *Pi-pmf* (*A* − *B*) *dflt* (λ-. *return-pmf dflt*)
using *assms* **by** (*intro Pi-pmf-cong*) *auto*
also have *... = return-pmf* (λ-. *dflt*)
using *assms* **by** *simp*
also have *map-pmf* (λ(*f*, *g*) *x*. *if x ∈ B then f x else g x*)
(*pair-pmf* (*Pi-pmf B dflt f*) (*return-pmf* (λ-. *dflt*))) =
map-pmf (λ*f x*. *if x ∈ B then f x else dflt*) (*Pi-pmf B dflt f*)
by (*simp add*: *map-pmf-def pair-pmf-def bind-assoc-pmf bind-return-pmf bind-return-pmf′*)
also have *... = Pi-pmf B dflt f*
using *assms* **by** (*intro Pi-pmf-default-swap*) (*auto dest*: *finite-subset*)
also have *B* ∪ (*A* − *B*) = *A*
using *assms* **by** *auto*
finally show *?thesis* **.**
qed

lemma *Pi-pmf-if-set*:
assumes *finite A*
shows *Pi-pmf A dflt* (λ*x*. *if b x then f x else return-pmf dflt*) =
Pi-pmf {*x*∈*A*. *b x*} *dflt f*
proof −
have *Pi-pmf A dflt* (λ*x*. *if b x then f x else return-pmf dflt*) =
Pi-pmf {*x*∈*A*. *b x*} *dflt* (λ*x*. *if b x then f x else return-pmf dflt*)
using *assms* **by** (*intro Pi-pmf-subset′*) *auto*
also have *... = Pi-pmf* {*x*∈*A*. *b x*} *dflt f*
by (*intro Pi-pmf-cong*) *auto*

finally show *?thesis* **.**
qed

lemma *Pi-pmf-if-set'*:
assumes *finite A*
shows *Pi-pmf A dflt* (λ*x*. *if b x then return-pmf dflt else f x*) =
Pi-pmf {*x*∈*A*. ¬*b x*} *dflt f*
proof −
have *Pi-pmf A dflt* (λ*x*. *if b x then return-pmf dflt else f x*) =
Pi-pmf {*x*∈*A*. ¬*b x*} *dflt* (λ*x*. *if b x then return-pmf dflt else f x*)
using *assms* **by** (*intro Pi-pmf-subset'*) *auto*
also have ... = *Pi-pmf* {*x*∈*A*. ¬*b x*} *dflt f*
by (*intro Pi-pmf-cong*) *auto*
finally show *?thesis* **.**
qed

Lastly, we can delete a single component from a product:

lemma *Pi-pmf-remove*:
assumes *finite A*
shows *Pi-pmf* (*A* − {*x*}) *dflt p* = *map-pmf* (λ*f*. *f*(*x* := *dflt*)) (*Pi-pmf A dflt p*)
proof −
have *Pi-pmf* (*A* − {*x*}) *dflt p* =
map-pmf (λ*f xa*. *if xa* ∈ *A* − {*x*} *then f xa else dflt*) (*Pi-pmf A dflt p*)
using *assms* **by** (*intro Pi-pmf-subset*) *auto*
also have ... = *map-pmf* (λ*f*. *f*(*x* := *dflt*)) (*Pi-pmf A dflt p*)
using *set-Pi-pmf-subset*[*of A dflt p*] *assms*
by (*intro map-pmf-cong refl*) (*auto simp*: *fun-eq-iff*)
finally show *?thesis* **.**
qed

26.6 Additional properties

lemma *nn-integral-prod-Pi-pmf*:
assumes *finite A*
shows *nn-integral* (*Pi-pmf A dflt p*) (λ*y*. ∏ *x*∈*A*. *f x* (*y x*)) = (∏ *x*∈*A*. *nn-integral* (*p x*) (*f x*))
using *assms*
proof (*induction rule*: *finite-induct*)
case (*insert x A*)
have *nn-integral* (*Pi-pmf* (*insert x A*) *dflt p*) (λ*y*. ∏ *z*∈*insert x A*. *f z* (*y z*)) =
(∫⁺*a*. ∫⁺*b*. *f x a* ∗ (∏ *z*∈*A*. *f z* (*if z* = *x then a else b z*)) ∂*Pi-pmf A dflt p* ∂*p x*)
using *insert* **by** (*auto simp*: *Pi-pmf-insert case-prod-unfold nn-integral-pair-pmf' cong*: *if-cong*)
also have (λ*a b*. ∏ *z*∈*A*. *f z* (*if z* = *x then a else b z*)) = (λ*a b*. ∏ *z*∈*A*. *f z* (*b z*))
by (*intro ext prod.cong*) (*use insert.hyps* **in** *auto*)
also have (∫⁺*a*. ∫⁺*b*. *f x a* ∗ (∏ *z*∈*A*. *f z* (*b z*)) ∂*Pi-pmf A dflt p* ∂*p x*) =

$(\int^+ y.\ f\ x\ y\ \partial(p\ x)) * (\int^+ y.\ (\prod z{\in}A.\ f\ z\ (y\ z))\ \partial(\textit{Pi-pmf}\ A\ \textit{dflt}\ p))$
by (*simp add*: *nn-integral-multc nn-integral-cmult*)
also have $(\int^+ y.\ (\prod z{\in}A.\ f\ z\ (y\ z))\ \partial(\textit{Pi-pmf}\ A\ \textit{dflt}\ p)) = (\prod x{\in}A.\ \textit{nn-integral}\ (p\ x)\ (f\ x))$
by (*rule insert.IH*)
also have $(\int^+ y.\ f\ x\ y\ \partial(p\ x)) * \ldots = (\prod x{\in}\textit{insert}\ x\ A.\ \textit{nn-integral}\ (p\ x)\ (f\ x))$
using *insert.hyps* **by** *simp*
finally show *?case* **.**
qed *auto*

lemma *integrable-prod-Pi-pmf*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{\textit{real-normed-field}, \textit{second-countable-topology}, \textit{banach}\}$
assumes *finite A* **and** $\bigwedge x.\ x \in A \Longrightarrow \textit{integrable}\ (\textit{measure-pmf}\ (p\ x))\ (f\ x)$
shows $\textit{integrable}\ (\textit{measure-pmf}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p))\ (\lambda h.\ \prod x{\in}A.\ f\ x\ (h\ x))$
proof (*intro integrableI-bounded*)
have $(\int^+ x.\ \textit{ennreal}\ (\textit{norm}\ (\prod xa{\in}A.\ f\ xa\ (x\ xa)))\ \partial\textit{measure-pmf}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p)) =$
$(\int^+ x.\ (\prod y{\in}A.\ \textit{ennreal}\ (\textit{norm}\ (f\ y\ (x\ y))))\ \partial\textit{measure-pmf}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p))$
by (*simp flip*: *prod-norm prod-ennreal*)
also have $\ldots = (\prod x{\in}A.\ \int^+ a.\ \textit{ennreal}\ (\textit{norm}\ (f\ x\ a))\ \partial\textit{measure-pmf}\ (p\ x))$
by (*intro nn-integral-prod-Pi-pmf*) *fact*
also have $(\int^+ a.\ \textit{ennreal}\ (\textit{norm}\ (f\ i\ a))\ \partial\textit{measure-pmf}\ (p\ i)) \neq \textit{top}$ **if** *i*: $i \in A$ **for** *i*
using *assms(2)*[*OF i*] **by** (*simp add*: *integrable-iff-bounded*)
hence $(\prod x{\in}A.\ \int^+ a.\ \textit{ennreal}\ (\textit{norm}\ (f\ x\ a))\ \partial\textit{measure-pmf}\ (p\ x)) \neq \textit{top}$
by (*subst ennreal-prod-eq-top*) *auto*
finally show $(\int^+ x.\ \textit{ennreal}\ (\textit{norm}\ (\prod xa{\in}A.\ f\ xa\ (x\ xa)))\ \partial\textit{measure-pmf}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p)) < \infty$
by (*simp add*: *top.not-eq-extremum*)
qed *auto*

lemma *expectation-prod-Pi-pmf*:
fixes $f :: - \Rightarrow - \Rightarrow \textit{real}$
assumes *finite A*
assumes $\bigwedge x.\ x \in A \Longrightarrow \textit{integrable}\ (\textit{measure-pmf}\ (p\ x))\ (f\ x)$
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in \textit{set-pmf}\ (p\ x) \Longrightarrow f\ x\ y \geq 0$
shows $\textit{measure-pmf.expectation}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p)\ (\lambda y.\ \prod x{\in}A.\ f\ x\ (y\ x)) =$
$(\prod x{\in}A.\ \textit{measure-pmf.expectation}\ (p\ x)\ (\lambda v.\ f\ x\ v))$
proof −
have *nonneg*: $\textit{measure-pmf.expectation}\ (p\ x)\ (f\ x) \geq 0$ **if** $x \in A$ **for** x
using *that* **by** (*intro Bochner-Integration.integral-nonneg-AE AE-pmfI assms*)
have *nonneg'*: $0 \leq \textit{measure-pmf.expectation}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p)\ (\lambda y.\ \prod x{\in}A.\ f\ x\ (y\ x))$
by (*intro Bochner-Integration.integral-nonneg-AE AE-pmfI assms prod-nonneg*)
(*use assms* **in** ‹*auto simp*: *set-Pi-pmf PiE-dflt-def*›)

have $\textit{ennreal}\ (\textit{measure-pmf.expectation}\ (\textit{Pi-pmf}\ A\ \textit{dflt}\ p)\ (\lambda y.\ \prod x{\in}A.\ f\ x\ (y\ x)))$
=

```
      nn-integral (Pi-pmf A dflt p) (λy. ennreal (∏ x∈A. f x (y x))) using assms
   by (intro nn-integral-eq-integral [symmetric] assms integrable-prod-Pi-pmf)
     (auto simp: AE-measure-pmf-iff set-Pi-pmf PiE-dflt-def prod-nonneg)
 also have ... = nn-integral (Pi-pmf A dflt p) (λy. (∏ x∈A. ennreal (f x (y x))))
   by (intro nn-integral-cong-AE AE-pmfI prod-ennreal [symmetric])
     (use assms(1) in ‹auto simp: set-Pi-pmf PiE-dflt-def intro!: assms(3)›)
 also have ... = (∏ x∈A. ∫⁺ a. ennreal (f x a) ∂measure-pmf (p x))
   by (rule nn-integral-prod-Pi-pmf) fact+
 also have ... = (∏ x∈A. ennreal (measure-pmf.expectation (p x) (f x)))
   by (intro prod.cong nn-integral-eq-integral assms AE-pmfI) auto
 also have ... = ennreal (∏ x∈A. measure-pmf.expectation (p x) (f x))
   by (intro prod-ennreal nonneg)
 finally show ?thesis
   using nonneg nonneg' by (subst (asm) ennreal-inj) (auto intro!: prod-nonneg)
qed

lemma indep-vars-Pi-pmf:
 assumes fin: finite I
 shows   prob-space.indep-vars (measure-pmf (Pi-pmf I dflt p))
          (λ-. count-space UNIV) (λx f. f x) I
proof (cases I = {})
 case True
 show ?thesis
   by (subst prob-space.indep-vars-def [OF measure-pmf.prob-space-axioms],
     subst prob-space.indep-sets-def [OF measure-pmf.prob-space-axioms]) (simp-all
add: True)
next
 case [simp]: False
 show ?thesis
 proof (subst prob-space.indep-vars-iff-distr-eq-PiM')
   show distr (measure-pmf (Pi-pmf I dflt p)) (Pi_M I (λi. count-space UNIV))
(λx. restrict x I) =
       Pi_M I (λi. distr (measure-pmf (Pi-pmf I dflt p)) (count-space UNIV) (λf.
f i))
   proof (rule product-sigma-finite.PiM-eqI, goal-cases)
     case 1
       interpret product-prob-space λi. distr (measure-pmf (Pi-pmf I dflt p))
(count-space UNIV) (λf. f i)
     by (intro product-prob-spaceI prob-space.prob-space-distr measure-pmf.prob-space-axioms)
         simp-all
     show ?case by unfold-locales
   next
     case 3
      have sets (Pi_M I (λi. distr (measure-pmf (Pi-pmf I dflt p)) (count-space
UNIV) (λf. f i))) =
           sets (Pi_M I (λ-. count-space UNIV))
       by (intro sets-PiM-cong) simp-all
     thus ?case by simp
   next
```

```
    case (4 A)
    have Pi_E I A ∈ sets (Pi_M I (λi. count-space UNIV))
      using 4 by (intro sets-PiM-I-finite fin) auto
  hence emeasure (distr (measure-pmf (Pi-pmf I dflt p)) (Pi_M I (λi. count-space
UNIV))
          (λx. restrict x I)) (Pi_E I A) =
        emeasure (measure-pmf (Pi-pmf I dflt p)) ((λx. restrict x I) -` Pi_E I A)
      using 4 by (subst emeasure-distr) (auto simp: space-PiM)
   also have ... = emeasure (measure-pmf (Pi-pmf I dflt p)) (PiE-dflt I dflt A)
    by (intro emeasure-eq-AE AE-pmfI) (auto simp: PiE-dflt-def set-Pi-pmf fin)
    also have ... = (∏ i∈I. emeasure (measure-pmf (p i)) (A i))
      by (simp add: measure-pmf.emeasure-eq-measure measure-Pi-pmf-PiE-dflt
fin prod-ennreal)
    also have ... = (∏ i∈I. emeasure (measure-pmf (map-pmf (λf. f i) (Pi-pmf
I dflt p))) (A i))
      by (intro prod.cong refl, subst Pi-pmf-component) (auto simp: fin)
    finally show ?case
      by (simp add: map-pmf-rep-eq)
  qed fact+
 qed (simp-all add: measure-pmf.prob-space-axioms)
qed

lemma
 fixes h :: 'a :: comm-monoid-add ⇒ 'b::{banach, second-countable-topology}
 assumes fin: finite I
 assumes integrable: ⋀i. i ∈ I ⟹ integrable (measure-pmf (D i)) h
 shows   integrable-sum-Pi-pmf: integrable (Pi-pmf I dflt D) (λg. ∑ i∈I. h (g i))
  and    expectation-sum-Pi-pmf:
         measure-pmf.expectation (Pi-pmf I dflt D) (λg. ∑ i∈I. h (g i)) =
         (∑ i∈I. measure-pmf.expectation (D i) h)
proof −
 have integrable': integrable (Pi-pmf I dflt D) (λg. h (g i)) if i: i ∈ I for i
 proof −
  have integrable (D i) h
    using i by (rule assms)
  also have D i = map-pmf (λg. g i) (Pi-pmf I dflt D)
    by (subst Pi-pmf-component) (use fin i in auto)
  finally show integrable (measure-pmf (Pi-pmf I dflt D)) (λx. h (x i))
    by simp
 qed
 thus integrable (Pi-pmf I dflt D) (λg. ∑ i∈I. h (g i))
  by (intro Bochner-Integration.integrable-sum)

 have measure-pmf.expectation (Pi-pmf I dflt D) (λx. ∑ i∈I. h (x i)) =
            (∑ i∈I. measure-pmf.expectation (map-pmf (λx. x i) (Pi-pmf I dflt
D)) h)
   using integrable' by (subst Bochner-Integration.integral-sum) auto
 also have ... = (∑ i∈I. measure-pmf.expectation (D i) h)
   by (intro sum.cong refl, subst Pi-pmf-component) (use fin in auto)
```

finally show *measure-pmf.expectation* (*Pi-pmf I dflt D*) (λ*g*. ∑ *i*∈*I*. *h* (*g i*)) =
(∑ *i*∈*I*. *measure-pmf.expectation* (*D i*) *h*) .
qed

26.7 Applications

Choosing a subset of a set uniformly at random is equivalent to tossing a fair coin independently for each element and collecting all the elements that came up heads.

lemma *pmf-of-set-Pow-conv-bernoulli*:
assumes *finite* (*A* :: *'a set*)
shows *map-pmf* (λ*b*. {*x*∈*A*. *b x*}) (*Pi-pmf A P* (λ-. *bernoulli-pmf* (*1/2*))) =
pmf-of-set (*Pow A*)
proof −
have *Pi-pmf A P* (λ-. *bernoulli-pmf* (*1/2*)) = *pmf-of-set* (*PiE-dflt A P* (λ*x*. *UNIV*))
using *assms* **by** (*simp add*: *bernoulli-pmf-half-conv-pmf-of-set Pi-pmf-of-set*)
also have *map-pmf* (λ*b*. {*x*∈*A*. *b x*}) ... = *pmf-of-set* (*Pow A*)
proof −
have *bij-betw* (λ*b*. {*x* ∈ *A*. *b x*}) (*PiE-dflt A P* (λ-. *UNIV*)) (*Pow A*)
by (*rule bij-betwI*[*of - - - λB b. if b* ∈ *A then b* ∈ *B else P*]) (*auto simp add*: *PiE-dflt-def*)
then show *?thesis*
using *assms* **by** (*intro map-pmf-of-set-bij-betw*) *auto*
qed
finally show *?thesis*
by *simp*
qed

A binomial distribution can be seen as the number of successes in n independent coin tosses.

lemma *binomial-pmf-altdef'*:
fixes *A* :: *'a set*
assumes *finite A* **and** *card A* = *n* **and** *p*: *p* ∈ {*0..1*}
shows *binomial-pmf n p* =
map-pmf (λ*f*. *card* {*x*∈*A*. *f x*}) (*Pi-pmf A dflt* (λ-. *bernoulli-pmf p*)) (**is** *?lhs* = *?rhs*)
proof −
from *assms* **have** *?lhs* = *binomial-pmf* (*card A*) *p*
by *simp*
also have ... = *?rhs*
using *assms*(*1*)
proof (*induction rule*: *finite-induct*)
case *empty*
with *p* **show** *?case* **by** (*simp add*: *binomial-pmf-0*)
next
case (*insert x A*)
from *insert.hyps* **have** *card* (*insert x A*) = *Suc* (*card A*)

```
    by simp
  also have binomial-pmf ... p = do {
                                   b ← bernoulli-pmf p;
                                   f ← Pi-pmf A dflt (λ-. bernoulli-pmf p);
                                  return-pmf ((if b then 1 else 0) + card {y ∈ A. f y})
                                  }
    using p by (simp add: binomial-pmf-Suc insert.IH bind-map-pmf)
  also have ... = do {
                    b ← bernoulli-pmf p;
                    f ← Pi-pmf A dflt (λ-. bernoulli-pmf p);
                    return-pmf (card {y ∈ insert x A. (f(x := b)) y})
                  }
  proof (intro bind-pmf-cong refl, goal-cases)
    case (1 b f)
    have (if b then 1 else 0) + card {y∈A. f y} = card ((if b then {x} else {}) ∪
{y∈A. f y})
      using insert.hyps by auto
    also have (if b then {x} else {}) ∪ {y∈A. f y} = {y∈insert x A. (f(x := b))
y}
      using insert.hyps by auto
    finally show ?case by simp
  qed
  also have ... = map-pmf (λf. card {y∈insert x A. f y})
                  (Pi-pmf (insert x A) dflt (λ-. bernoulli-pmf p))
  using insert.hyps by (subst Pi-pmf-insert) (simp-all add: pair-pmf-def map-bind-pmf)
  finally show ?case .
 qed
 finally show ?thesis .
qed

end
```

27 Hoeffding's Lemma and Hoeffding's Inequality

```
theory Hoeffding
 imports Product-PMF Independent-Family
begin
```

Hoeffding's inequality shows that a sum of bounded independent random variables is concentrated around its mean, with an exponential decay of the tail probabilities.

27.1 Hoeffding's Lemma

```
lemma convex-on-exp:
 fixes l :: real
 assumes l ≥ 0
 shows   convex-on UNIV (λx. exp(l*x))
```

```
  using assms
  by (intro convex-on-realI[where f' = λx. l * exp (l * x)])
     (auto intro!: derivative-eq-intros mult-left-mono)

lemma mult-const-minus-self-real-le:
  fixes x :: real
  shows x * (c - x) ≤ c^2 / 4
proof -
  have x * (c - x) = -(x - c / 2)^2 + c^2 / 4
    by (simp add: field-simps power2-eq-square)
  also have ... ≤ 0 + c^2 / 4
    by (intro add-mono) auto
  finally show ?thesis by simp
qed

lemma Hoeffdings-lemma-aux:
  fixes h p :: real
  assumes h ≥ 0 and p ≥ 0
  defines L ≡ (λh. -h * p + ln (1 + p * (exp h - 1)))
  shows   L h ≤ h^2 / 8
proof (cases h = 0)
  case False
  hence h: h > 0
    using ‹h ≥ 0› by simp
  define L' where L' = (λh. -p + p * exp h / (1 + p * (exp h - 1)))
  define L'' where L'' = (λh. -(p^2) * exp h * exp h / (1 + p * (exp h - 1))^2 +
                          p * exp h / (1 + p * (exp h - 1)))
  define Ls where Ls = (λn. [L, L', L''] ! n)

  have [simp]: L 0 = 0 L' 0 = 0
    by (auto simp: L-def L'-def)

  have L': (L has-real-derivative L' x) (at x) if x ∈ {0..h} for x
  proof -
    have 1 + p * (exp x - 1) > 0
      using ‹p ≥ 0› that by (intro add-pos-nonneg mult-nonneg-nonneg) auto
    thus ?thesis
      unfolding L-def L'-def by (auto intro!: derivative-eq-intros)
  qed

  have L'': (L' has-real-derivative L'' x) (at x) if x ∈ {0..h} for x
  proof -
    have *: 1 + p * (exp x - 1) > 0
      using ‹p ≥ 0› that by (intro add-pos-nonneg mult-nonneg-nonneg) auto
    show ?thesis
      unfolding L'-def L''-def
     by (insert *, (rule derivative-eq-intros refl | simp)+) (auto simp: divide-simps;
algebra)
  qed
```

```
  have diff: ∀ m t. m < 2 ∧ 0 ≤ t ∧ t ≤ h ⟶ (Ls m has-real-derivative Ls (Suc
m) t) (at t)
    using L' L'' by (auto simp: Ls-def nth-Cons split: nat.splits)
  from Taylor[of 2 Ls L 0 h 0 h, OF - - diff]
    obtain t where t: t ∈ {0<..<h} L h = L'' t * h² / 2
      using ‹h > 0› by (auto simp: Ls-def lessThan-nat-numeral)
  define u where u = p * exp t / (1 + p * (exp t - 1))

  have L'' t = u * (1 - u)
    by (simp add: L''-def u-def divide-simps; algebra)
  also have ... ≤ 1 / 4
    using mult-const-minus-self-real-le[of u 1] by simp
  finally have L'' t ≤ 1 / 4 .

  note t(2)
  also have L'' t * h² / 2 ≤ (1 / 4) * h² / 2
    using ‹L'' t ≤ 1 / 4› by (intro mult-right-mono divide-right-mono) auto
  finally show L h ≤ h² / 8 by simp
qed (auto simp: L-def)


locale interval-bounded-random-variable = prob-space +
  fixes f :: 'a ⇒ real and a b :: real
  assumes random-variable [measurable]: random-variable borel f
  assumes AE-in-interval: AE x in M. f x ∈ {a..b}
begin

lemma integrable [intro]: integrable M f
proof (rule integrable-const-bound)
  show AE x in M. norm (f x) ≤ max |a| |b|
    by (intro eventually-mono[OF AE-in-interval]) auto
qed (fact random-variable)
```

We first show Hoeffding's lemma for distributions whose expectation is 0. The general case will easily follow from this later.

```
lemma Hoeffdings-lemma-nn-integral-0:
  assumes l > 0 and E0: expectation f = 0
  shows   nn-integral M (λx. exp (l * f x)) ≤ ennreal (exp (l² * (b - a)² / 8))
proof (cases AE x in M. f x = 0)
  case True
  hence nn-integral M (λx. exp (l * f x)) = nn-integral M (λx. ennreal 1)
    by (intro nn-integral-cong-AE) auto
  also have ... = ennreal (expectation (λ-. 1))
    by (intro nn-integral-eq-integral) auto
  finally show ?thesis by (simp add: prob-space)
next
  case False
  have a < 0
```

```
  proof (rule ccontr)
    assume a: ¬(a < 0)
    have AE x in M. f x = 0
    proof (subst integral-nonneg-eq-0-iff-AE [symmetric])
      show AE x in M. f x ≥ 0
        using AE-in-interval by eventually-elim (use a in auto)
    qed (use E0 in ‹auto simp: id-def integrable›)
    with False show False by contradiction
  qed

  have b > 0
  proof (rule ccontr)
    assume b: ¬(b > 0)
    have AE x in M. −f x = 0
    proof (subst integral-nonneg-eq-0-iff-AE [symmetric])
      show AE x in M. −f x ≥ 0
        using AE-in-interval by eventually-elim (use b in auto)
    qed (use E0 in ‹auto simp: id-def integrable›)
    with False show False by simp
  qed

  have a < b
    using ‹a < 0› ‹b > 0› by linarith

  define p where p = −a / (b − a)
  define L where L = (λt. −t* p + ln (1 − p + p * exp t))
  define z where z = l * (b − a)
  have z > 0
    unfolding z-def using ‹a < b› ‹l > 0› by auto
  have p > 0
    using ‹a < 0› ‹a < b› unfolding p-def by (intro divide-pos-pos) auto

  have (∫+x. exp (l * f x) ∂M) ≤
        (∫+x. (b − f x) / (b − a) * exp (l * a) + (f x − a) / (b − a) * exp (l * b)
∂M)
  proof (intro nn-integral-mono-AE eventually-mono[OF AE-in-interval] ennreal-leI)
    fix x assume x: f x ∈ {a..b}
    define y where y = (b − f x) / (b−a)
    have y: y ∈ {0..1}
      using x ‹a < b› by (auto simp: y-def)
    have conv: convex-on UNIV (λx. exp(l*x))
      using ‹l > 0› by (intro convex-on-exp) auto
    have exp (l * ((1 − y) *R b + y *R a)) ≤ (1 − y) * exp (l * b) + y * exp (l
* a)
      using y ‹l > 0› by (intro convex-onD[OF convex-on-exp]) auto
    also have (1 − y) *R b + y *R a = f x
      using ‹a < b› by (simp add: y-def divide-simps) (simp add: algebra-simps)?
    also have 1 − y = (f x − a) / (b − a)
      using ‹a < b› by (simp add: field-simps y-def)
```

finally show *exp* $(l * f\,x) \leq (b - f\,x)\ /\ (b - a) * exp\ (l{*}a) + (f\,x - a)/(b{-}a)$
$*$ *exp* $(l{*}b)$
by (*simp add*: *y-def*)
qed
also have $\ldots = (\int^+ x.\ ennreal\ (b - f\,x) * exp\ (l * a)\ /\ (b - a) +$
ennreal $(f\,x - a) * exp\ (l * b)\ /\ (b - a)\ \partial M)$
using ‹$a < 0$› ‹$b > 0$›
by (*intro nn-integral-cong-AE eventually-mono*[*OF AE-in-interval*])
(*simp add*: *ennreal-plus ennreal-mult flip*: *divide-ennreal*)
also have $\ldots = ((\int^+ x.\ ennreal\ (b - f\,x)\ \partial M) * ennreal\ (exp\ (l * a)) +$
$(\int^+ x.\ ennreal\ (f\,x - a)\ \partial M) * ennreal\ (exp\ (l * b)))\ /\ ennreal\ (b$
$- a)$
by (*simp add*: *nn-integral-add nn-integral-divide nn-integral-multc add-divide-distrib-ennreal*)
also have $(\int^+ x.\ ennreal\ (b - f\,x)\ \partial M) = ennreal\ (expectation\ (\lambda x.\ b - f\,x))$
by (*intro nn-integral-eq-integral Bochner-Integration.integrable-diff*
eventually-mono[*OF AE-in-interval*] *integrable-const integrable*) *auto*
also have *expectation* $(\lambda x.\ b - f\,x) = b$
using *assms* **by** (*subst Bochner-Integration.integral-diff*) (*auto simp*: *prob-space*)
also have $(\int^+ x.\ ennreal\ (f\,x - a)\ \partial M) = ennreal\ (expectation\ (\lambda x.\ f\,x - a))$
by (*intro nn-integral-eq-integral Bochner-Integration.integrable-diff*
eventually-mono[*OF AE-in-interval*] *integrable-const integrable*) *auto*
also have *expectation* $(\lambda x.\ f\,x - a) = (-a)$
using *assms* **by** (*subst Bochner-Integration.integral-diff*) (*auto simp*: *prob-space*)
also have $(ennreal\ b * (exp\ (l * a)) + ennreal\ (-a) * (exp\ (l * b)))\ /\ (b - a) =$
$ennreal\ (b * exp\ (l * a) - a * exp\ (l * b))\ /\ ennreal\ (b - a)$
using ‹$a < 0$› ‹$b > 0$›
by (*simp flip*: *ennreal-mult ennreal-plus add*: *mult-nonpos-nonneg divide-ennreal*
mult-mono)
also have $b * exp\ (l * a) - a * exp\ (l * b) = exp\ (L\ z) * (b - a)$
proof −
have *pos*: $1 - p + p * exp\ z > 0$
proof −
have $exp\ z > 1$ **using** ‹$l > 0$› **and** ‹$a < b$›
by (*subst one-less-exp-iff*) (*auto simp*: *z-def intro*!: *mult-pos-pos*)
hence $(exp\ z - 1) * p \geq 0$
unfolding *p-def* **using** ‹$a < 0$› **and** ‹$a < b$›
by (*intro mult-nonneg-nonneg divide-nonneg-pos*) *auto*
thus *?thesis*
by (*simp add*: *algebra-simps*)
qed

have $exp\ (L\ z) * (b - a) = exp\ (-z * p) * (1 - p + p * exp\ z) * (b - a)$
using *pos* **by** (*simp add*: *exp-add L-def exp-diff exp-minus divide-simps*)
also have $\ldots = b * exp\ (l * a) - a * exp\ (l * b)$ **using** ‹$a < b$›
by (*simp add*: *p-def z-def divide-simps*) (*simp add*: *exp-diff algebra-simps*)*?*
finally show *?thesis* **by** *simp*
qed
also have $ennreal\ (exp\ (L\ z) * (b - a))\ /\ ennreal\ (b - a) = ennreal\ (exp\ (L\ z))$
using ‹$a < b$› **by** (*simp add*: *divide-ennreal*)

also have $L\ z = -z * p + ln\ (1 + p * (exp\ z - 1))$
by (*simp add*: *L-def algebra-simps*)
also have $\ldots \leq z^2\ /\ 8$
unfolding *L-def* **by** (*rule Hoeffdings-lemma-aux*[**where** $p = p$]) (*use* ‹$z > 0$› ‹$p > 0$› **in** *simp-all*)
hence *ennreal* $(exp\ (-z * p + ln\ (1 + p * (exp\ z - 1)))) \leq ennreal\ (exp\ (z^2\ /\ 8))$
by (*intro ennreal-leI*) *auto*
finally show *?thesis*
by (*simp add*: *z-def power-mult-distrib*)
qed

context
begin

interpretation *shift*: *interval-bounded-random-variable M* $\lambda x.\ f\ x - \mu\ a - \mu\ b - \mu$
rewrites $b - \mu - (a - \mu) \equiv b - a$
by *unfold-locales* (*auto intro*!: *eventually-mono*[*OF AE-in-interval*])

lemma *expectation-shift*: *expectation* $(\lambda x.\ f\ x - expectation\ f) = 0$
by (*subst Bochner-Integration.integral-diff*) (*auto simp*: *integrable prob-space*)

lemmas *Hoeffdings-lemma-nn-integral* = *shift.Hoeffdings-lemma-nn-integral-0*[*OF - expectation-shift*]

end

end

27.2 Hoeffding's Inequality

Consider n independent real random variables $X_1, \ldots, X_n$ that each almost surely lie in a compact interval $[a_i, b_i]$. Hoeffding's inequality states that the distribution of the sum of the X_i is tightly concentrated around the sum of the expected values: the probability of it being above or below the sum of the expected values by more than some ε decreases exponentially with ε.

locale *indep-interval-bounded-random-variables* = *prob-space* +
fixes $I :: 'b\ set$ **and** $X :: 'b \Rightarrow 'a \Rightarrow real$
fixes $a\ b :: 'b \Rightarrow real$
assumes *fin*: *finite I*
assumes *indep*: *indep-vars* (λ-. *borel*) *X I*
assumes *AE-in-interval*: $\bigwedge i.\ i \in I \Longrightarrow AE\ x\ in\ M.\ X\ i\ x \in \{a\ i..b\ i\}$
begin

lemma *random-variable* [*measurable*]:
assumes *i*: $i \in I$
shows *random-variable borel* $(X\ i)$

using *i indep* **unfolding** *indep-vars-def* **by** *blast*

lemma *bounded-random-variable* [*intro*]:
assumes *i*: $i \in I$
shows *interval-bounded-random-variable M* (*X i*) (*a i*) (*b i*)
by *unfold-locales* (*use AE-in-interval*[*OF i*] *i* **in** *auto*)

end

locale *Hoeffding-ineq* = *indep-interval-bounded-random-variables* +
fixes μ :: *real*
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))$
begin

theorem *Hoeffding-ineq-ge*:
assumes $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} \leq exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
proof (*cases* $\varepsilon = 0$)
case [*simp*]: *True*
have *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} \leq 1$
by *simp*
thus *?thesis* **by** *simp*
next
case *False*
with ‹$\varepsilon \geq 0$› **have** ε: $\varepsilon > 0$
by *auto*

define *d* **where** $d = (\sum i{\in}I.\ (b\ i - a\ i)^2)$
define *l* :: *real* **where** $l = 4 * \varepsilon\ /\ d$
have *d*: $d > 0$
using *assms* **by** (*simp add*: *d-def*)
have *l*: $l > 0$
using ε *d* **by** (*simp add*: *l-def*)
define μ' **where** $\mu' = (\lambda i.\ expectation\ (X\ i))$

have $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} = \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) - \mu \geq \varepsilon\}$
by (*simp add*: *algebra-simps*)
hence *ennreal* (*prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\}$) = *emeasure M* ...
by (*simp add*: *emeasure-eq-measure*)
also have ... $\leq$ *ennreal* (*exp* $(-l{*}\varepsilon)$) $*$ $(\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x) - \mu))\ \partial M)$
by (*intro Chernoff-ineq-nn-integral-ge l*) *auto*
also have $(\lambda x.\ (\sum i{\in}I.\ X\ i\ x) - \mu) = (\lambda x.\ (\sum i{\in}I.\ X\ i\ x - \mu'\ i))$
by (*simp add*: μ*-def sum-subtractf* μ'*-def*)
also have $(\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x - \mu'\ i)))\ \partial M) =$

$(\int^+ x.\ (\prod i{\in}I.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i))))\ \partial M)$
by (*intro nn-integral-cong*)
(*simp-all add*: *sum-distrib-left ring-distribs exp-diff exp-sum fin prod-ennreal*)
also have $\ldots = (\prod i{\in}I.\ \int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M)$
by (*intro indep-vars-nn-integral fin indep-vars-compose2*[*OF indep*]) *auto*
also have $ennreal\ (exp\ (-l * \varepsilon)) * \ldots \leq$
$ennreal\ (exp\ (-l * \varepsilon)) * (\prod i{\in}I.\ ennreal\ (exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)))$
proof (*intro mult-left-mono prod-mono-ennreal*)
fix i **assume** i: $i \in I$
from i **interpret** *interval-bounded-random-variable M X i a i b i* **..**
show $(\int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M) \leq ennreal\ (exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8))$
unfolding μ'*-def* **by** (*rule Hoeffdings-lemma-nn-integral*) *fact+*
qed *auto*
also have $\ldots = ennreal\ (exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)))$
by (*simp add*: *prod-ennreal prod-nonneg flip*: *ennreal-mult*)
also have $exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)) = exp\ (d * l^2\ /\ 8 - l * \varepsilon)$
by (*simp add*: *exp-diff exp-minus sum-divide-distrib sum-distrib-left*
sum-distrib-right exp-sum fin divide-simps mult-ac d-def)
also have $d * l^2\ /\ 8 - l * \varepsilon = -2 * \varepsilon^2\ /\ d$
using d **by** (*simp add*: *l-def field-simps power2-eq-square*)
finally show *?thesis*
by (*subst* (*asm*) *ennreal-le-iff*) (*simp-all add*: *d-def*)
qed

corollary *Hoeffding-ineq-le*:
assumes ε: $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows $prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu - \varepsilon\} \leq exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
proof −
interpret *flip*: *Hoeffding-ineq M I* $\lambda i\ x.\ -X\ i\ x\ \lambda i.\ -b\ i\ \lambda i.\ -a\ i\ -\mu$
proof *unfold-locales*
fix i **assume** $i \in I$
then interpret *interval-bounded-random-variable M X i a i b i* **..**
show *AE x in M.* $-\ X\ i\ x \in \{-\ b\ i..-\ a\ i\}$
by (*intro eventually-mono*[*OF AE-in-interval*]) *auto*
qed (*auto simp*: *fin* μ*-def sum-negf intro*: *indep-vars-compose2*[*OF indep*])

have $prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu - \varepsilon\} = prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ -X\ i\ x) \geq -\mu + \varepsilon\}$
by (*simp add*: *sum-negf algebra-simps*)
also have $\ldots \leq exp\ (-\ 2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
using *flip.Hoeffding-ineq-ge*[*OF* ε] *assms*(*2*) **by** *simp*
finally show *?thesis* **.**
qed

corollary *Hoeffding-ineq-abs-ge*:

assumes ε: $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows *prob* $\{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x) - \mu| \geq \varepsilon\} \leq 2 * exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
proof −
have $\{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x) - \mu| \geq \varepsilon\} =$
$\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} \cup \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu - \varepsilon\}$
by *auto*
also have *prob* $\ldots \leq$ *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} +$ *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu - \varepsilon\}$
by (*intro measure-Un-le*) *auto*
also have $\ldots \leq$ *exp* $(-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2)) +$ *exp* $(-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
by (*intro add-mono Hoeffding-ineq-ge Hoeffding-ineq-le assms*)
finally show *?thesis* **by** *simp*
qed

end

27.3 Hoeffding's inequality for i.i.d. bounded random variables

If we have n even identically-distributed random variables, the statement of Hoeffding's lemma simplifies a bit more: it shows that the probability that the average of the X_i is more than ε above the expected value is no greater than $e^{\frac{-2ny^2}{(b-a)^2}}$.

This essentially gives us a more concrete version of the weak law of large numbers: the law states that the probability vanishes for $n \to \infty$ for any $\varepsilon > 0$. Unlike Hoeffding's inequality, it does not assume the variables to have bounded support, but it does not provide concrete bounds.

locale *iid-interval-bounded-random-variables* = *prob-space* +
fixes I :: *'b set* **and** X :: *'b* ⇒ *'a* ⇒ *real* **and** Y :: *'a* ⇒ *real*
fixes a b :: *real*
assumes *fin*: *finite I*
assumes *indep*: *indep-vars* (λ-. *borel*) *X I*
assumes *distr-X*: $i \in I \Longrightarrow$ *distr M borel* $(X\ i)$ = *distr M borel Y*
assumes *rv-Y* [*measurable*]: *random-variable borel Y*
assumes *AE-in-interval*: *AE x in M. Y x* ∈ {*a*..*b*}
begin

lemma *random-variable* [*measurable*]:
assumes *i*: $i \in I$
shows *random-variable borel* $(X\ i)$
using *i indep* **unfolding** *indep-vars-def* **by** *blast*

sublocale *X*: *indep-interval-bounded-random-variables M I X* λ-. *a* λ-. *b*

proof
fix i **assume** i: $i \in I$
have $AE\ x\ in\ M.\ Y\ x \in \{a..b\}$
by (*fact AE-in-interval*)
also have *?this* $\longleftrightarrow$ $(AE\ x\ in\ distr\ M\ borel\ Y.\ x \in \{a..b\})$
by (*subst AE-distr-iff*) *auto*
also have $distr\ M\ borel\ Y = distr\ M\ borel\ (X\ i)$
using i **by** (*simp add*: *distr-X*)
also have $(AE\ x\ in\ ....\ x \in \{a..b\}) \longleftrightarrow (AE\ x\ in\ M.\ X\ i\ x \in \{a..b\})$
using i **by** (*subst AE-distr-iff*) *auto*
finally show $AE\ x\ in\ M.\ X\ i\ x \in \{a..b\}$ **.**
qed (*simp-all add*: *fin indep*)

lemma *expectation-X* [*simp*]:
assumes i: $i \in I$
shows $expectation\ (X\ i) = expectation\ Y$
proof −
have $expectation\ (X\ i) = lebesgue\text{-}integral\ (distr\ M\ borel\ (X\ i))\ (\lambda x.\ x)$
using i **by** (*intro integral-distr* [*symmetric*]) *auto*
also have $distr\ M\ borel\ (X\ i) = distr\ M\ borel\ Y$
using i **by** (*rule distr-X*)
also have $lebesgue\text{-}integral\ \ldots\ (\lambda x.\ x) = expectation\ Y$
by (*rule integral-distr*) *auto*
finally show $expectation\ (X\ i) = expectation\ Y$ **.**
qed

end

locale *Hoeffding-ineq-iid* = *iid-interval-bounded-random-variables* +
fixes μ :: *real*
defines $\mu \equiv expectation\ Y$
begin

sublocale *X*: *Hoeffding-ineq M I X* λ-. *a* λ-. *b real* $(card\ I) * \mu$
by *unfold-locales* (*simp-all add*: μ-*def*)

corollary
assumes ε: $\varepsilon \geq 0$
assumes $a < b\ I \neq \{\}$
defines $n \equiv card\ I$
shows *Hoeffding-ineq-ge*:
$prob\ \{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) \geq n * \mu + \varepsilon\} \leq$
$exp\ (-2 * \varepsilon^2\ /\ (n * (b - a)^2))$ (**is** *?le*)
and *Hoeffding-ineq-le*:
$prob\ \{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) \leq n * \mu - \varepsilon\} \leq$
$exp\ (-2 * \varepsilon^2\ /\ (n * (b - a)^2))$ (**is** *?ge*)
and *Hoeffding-ineq-abs-ge*:
$prob\ \{x \in space\ M.\ |(\sum i \in I.\ X\ i\ x) - n * \mu| \geq \varepsilon\} \leq$

$2 * exp\ (-2 * \varepsilon^2\ /\ (n * (b - a)^2))$ (**is** *?abs-ge*)
proof −
have *pos*: $(\sum i{\in}I.\ (b - a)^2) > 0$
using ‹$a < b$› ‹$I \neq \{\}$› *fin* **by** (*intro sum-pos*) *auto*
show *?le*
using *X.Hoeffding-ineq-ge*[*OF* ε *pos*] **by** (*simp add*: *n-def*)
show *?ge*
using *X.Hoeffding-ineq-le*[*OF* ε *pos*] **by** (*simp add*: *n-def*)
show *?abs-ge*
using *X.Hoeffding-ineq-abs-ge*[*OF* ε *pos*] **by** (*simp add*: *n-def*)
qed

lemma
assumes ε: $\varepsilon \geq 0$
assumes $a < b$ $I \neq \{\}$
defines $n \equiv card\ I$
shows *Hoeffding-ineq-ge′*:
$prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \geq \mu + \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?ge*)
and *Hoeffding-ineq-le′*:
$prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \leq \mu - \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?le*)
and *Hoeffding-ineq-abs-ge′*:
$prob\ \{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x)\ /\ n - \mu| \geq \varepsilon\} \leq$
$2 * exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?abs-ge*)
proof −
have $n > 0$
using *assms fin* **by** (*auto simp*: *field-simps*)
have ε': $\varepsilon * n \geq 0$
using ‹$n > 0$› ‹$\varepsilon \geq 0$› **by** *auto*
have *eq*: $-\ (2 * (\varepsilon * real\ n)^2\ /\ (real\ (card\ I) * (b - a)^2)) =$
$-\ (2 * real\ n * \varepsilon^2\ /\ (b - a)^2)$
using ‹$n > 0$› **by** (*simp add*: *power2-eq-square divide-simps n-def*)

have $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \geq \mu + \varepsilon\} =$
$\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu * n + \varepsilon * n\}$
using ‹$n > 0$› **by** (*intro Collect-cong conj-cong refl*) (*auto simp*: *field-simps*)
with *Hoeffding-ineq-ge*[*OF* ε' ‹$a < b$› ‹$I \neq \{\}$›] ‹$n > 0$› *eq* **show** *?ge*
by (*simp add*: *n-def mult-ac*)

have $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \leq \mu - \varepsilon\} =$
$\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu * n - \varepsilon * n\}$
using ‹$n > 0$› **by** (*intro Collect-cong conj-cong refl*) (*auto simp*: *field-simps*)
with *Hoeffding-ineq-le*[*OF* ε' ‹$a < b$› ‹$I \neq \{\}$›] ‹$n > 0$› *eq* **show** *?le*
by (*simp add*: *n-def mult-ac*)

have $\{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x)\ /\ n - \mu| \geq \varepsilon\} =$
$\{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x) - \mu * n| \geq \varepsilon * n\}$
using ‹$n > 0$› **by** (*intro Collect-cong conj-cong refl*) (*auto simp*: *field-simps*)

with *Hoeffding-ineq-abs-ge*[*OF* ε' ‹$a < b$› ‹$I \neq \{\}$›] ‹$n > 0$› *eq* **show** *?abs-ge*
by (*simp add*: *n-def mult-ac*)
qed

end

27.4 Hoeffding's Inequality for the Binomial distribution

We can now apply Hoeffding's inequality to the Binomial distribution, which can be seen as the sum of n i.i.d. coin flips (the support of each of which is contained in $[0, 1]$).

locale *binomial-distribution* =
fixes n :: *nat* **and** p :: *real*
assumes *p*: $p \in \{0..1\}$
begin

context
fixes *coins* :: (*nat* ⇒ *bool*) *pmf* **and** μ
assumes *n*: $n > 0$
defines *coins* ≡ *Pi-pmf* {..<*n*} *False* (λ-. *bernoulli-pmf p*)
begin

lemma *coins-component*:
assumes *i*: $i < n$
shows *distr* (*measure-pmf coins*) *borel* (λ*f*. *if f i then 1 else 0*) =
distr (*measure-pmf* (*bernoulli-pmf p*)) *borel* (λ*b*. *if b then 1 else 0*)
proof −
have *distr* (*measure-pmf coins*) *borel* (λ*f*. *if f i then 1 else 0*) =
distr (*measure-pmf* (*map-pmf* (λ*f*. *f i*) *coins*)) *borel* (λ*b*. *if b then 1 else 0*)
unfolding *map-pmf-rep-eq* **by** (*subst distr-distr*) (*auto simp*: *o-def*)
also have *map-pmf* (λ*f*. *f i*) *coins* = *bernoulli-pmf p*
unfolding *coins-def* **using** *i* **by** (*subst Pi-pmf-component*) *auto*
finally show *?thesis*
unfolding *map-pmf-rep-eq* **.**
qed

lemma *prob-binomial-pmf-conv-coins*:
measure-pmf.prob (*binomial-pmf n p*) {*x*. *P* (*real x*)} =
measure-pmf.prob coins {*x*. *P* (∑ *i*<*n*. *if x i then 1 else 0*)}
proof −
have *eq1*: (∑ *i*<*n*. *if x i then 1 else 0*) = *real* (*card* {*i*∈{..<*n*}. *x i*}) **for** *x*
proof −
have (∑ *i*<*n*. *if x i then 1 else* (*0*::*real*)) = (∑ *i*∈{*i*∈{..<*n*}. *x i*}. *1*)
by (*intro sum.mono-neutral-cong-right*) *auto*
thus *?thesis* **by** *simp*
qed
have *eq2*: *binomial-pmf n p* = *map-pmf* (λ*v*. *card* {*i*∈{..<*n*}. *v i*}) *coins*
unfolding *coins-def* **by** (*rule binomial-pmf-altdef'*) (*use p* **in** *auto*)
show *?thesis*

by (*subst eq2*) (*simp-all add*: *eq1*)
qed

interpretation *Hoeffding-ineq-iid*
coins {..<*n*} λ*i f*. *if f i then 1 else 0* λ*f*. *if f 0 then 1 else 0 0 1 p*
proof *unfold-locales*
show *prob-space.indep-vars* (*measure-pmf coins*) (λ-. *borel*) (λ*i f*. *if f i then 1 else 0*) {..<*n*}
unfolding *coins-def*
by (*intro prob-space.indep-vars-compose2*[*OF - indep-vars-Pi-pmf*])
(*auto simp*: *measure-pmf.prob-space-axioms*)
next
have *measure-pmf.expectation coins* (λ*f*. *if f 0 then 1 else 0* :: *real*) =
measure-pmf.expectation (*map-pmf* (λ*f*. *f 0*) *coins*) (λ*b*. *if b then 1 else 0* :: *real*)
by (*simp add*: *coins-def*)
also have *map-pmf* (λ*f*. *f 0*) *coins* = *bernoulli-pmf p*
using *n* **by** (*simp add*: *coins-def Pi-pmf-component*)
also have *measure-pmf.expectation* ... (λ*b*. *if b then 1 else 0*) = *p*
using *p* **by** *simp*
finally show *p* ≡ *measure-pmf.expectation coins* (λ*f*. *if f 0 then 1 else 0*) **by** *simp*
qed (*auto simp*: *coins-component*)

corollary
fixes ε :: *real*
assumes ε: ε ≥ *0*
shows *prob-ge*: *measure-pmf.prob* (*binomial-pmf n p*) {*x*. *x* ≥ *n* ∗ *p* + ε} ≤ *exp* (−*2* ∗ ε^2 / *n*)
and *prob-le*: *measure-pmf.prob* (*binomial-pmf n p*) {*x*. *x* ≤ *n* ∗ *p* − ε} ≤ *exp* (−*2* ∗ ε^2 / *n*)
and *prob-abs-ge*:
measure-pmf.prob (*binomial-pmf n p*) {*x*. |*x* − *n* ∗ *p*| ≥ ε} ≤ *2* ∗ *exp* (−*2* ∗ ε^2 / *n*)
proof −
have [*simp*]: {..<*n*} ≠ {}
using *n* **by** *auto*
show *measure-pmf.prob* (*binomial-pmf n p*) {*x*. *x* ≥ *n* ∗ *p* + ε} ≤ *exp* (−*2* ∗ ε^2 / *n*)
using *Hoeffding-ineq-ge*[*of* ε] **by** (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)
show *measure-pmf.prob* (*binomial-pmf n p*) {*x*. *x* ≤ *n* ∗ *p* − ε} ≤ *exp* (−*2* ∗ ε^2 / *n*)
using *Hoeffding-ineq-le*[*of* ε] **by** (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)
show *measure-pmf.prob* (*binomial-pmf n p*) {*x*. |*x* − *n* ∗ *p*| ≥ ε} ≤ *2* ∗ *exp* (−*2* ∗ ε^2 / *n*)
using *Hoeffding-ineq-abs-ge*[*of* ε]
by (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)

qed

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x\ /\ n \geq p + \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
and *prob-le'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x\ /\ n \leq p - \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
and *prob-abs-ge'*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x\ /\ n - p| \geq \varepsilon\} \leq 2 * exp\ (-2 * n * \varepsilon^2)$
proof −
have [*simp*]: $\{..<n\} \neq \{\}$
using *n* **by** *auto*
show *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x\ /\ n \geq p + \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
using *Hoeffding-ineq-ge'*[*of* ε] **by** (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)
show *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x\ /\ n \leq p - \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
using *Hoeffding-ineq-le'*[*of* ε] **by** (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)
show *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ |x\ /\ n - p| \geq \varepsilon\} \leq 2 * exp\ (-2 * n * \varepsilon^2)$
using *Hoeffding-ineq-abs-ge'*[*of* ε]
by (*subst prob-binomial-pmf-conv-coins*) (*use assms* **in** *simp-all*)
qed

end

end

27.5 Tail bounds for the negative binomial distribution

Since the tail probabilities of a negative Binomial distribution are equal to the tail probabilities of some Binomial distribution, we can obtain tail bounds for the negative Binomial distribution through the Hoeffding tail bounds for the Binomial distribution.

context
fixes *p q* :: *real*
assumes *p*: $p \in \{0<..<1\}$
defines $q \equiv 1 - p$
begin

lemma *prob-neg-binomial-pmf-ge-bound*:
fixes *n* :: *nat* **and** *k* :: *real*
defines $\mu \equiv real\ n * q\ /\ p$

```
  assumes k: k ≥ 0
  shows measure-pmf.prob (neg-binomial-pmf n p) {x. real x ≥ μ + k}
        ≤ exp (− 2 ∗ p ^ 3 ∗ k² / (n + p ∗ k))
proof −
  consider n = 0 | p = 1 | n > 0 p ≠ 1
    by blast
  thus ?thesis
  proof cases
    assume [simp]: n = 0
    show ?thesis using k
      by (simp add: indicator-def μ-def)
  next
    assume [simp]: p = 1
    show ?thesis using k
      by (auto simp add: indicator-def μ-def q-def)
  next
    assume n: n > 0 and p ≠ 1
    from ‹p ≠ 1› and p have p: p ∈ {0<..<1}
      by auto
    from p have q: q ∈ {0<..<1}
      by (auto simp: q-def)

    define k1 where k1 = μ + k
    have k1: k1 ≥ μ
      using k by (simp add: k1-def)
    have k1 > 0
      by (rule less-le-trans[OF - k1]) (use p n in ‹auto simp: q-def μ-def›)

    define k1' where k1' = nat (ceiling k1)
    have μ ≥ 0 using p
      by (auto simp: μ-def q-def)
    have ¬(x < k1') ⟷ real x ≥ k1 for x
      unfolding k1'-def by linarith
    hence eq: UNIV − {..<k1'} = {x. x ≥ k1}
      by auto
    hence measure-pmf.prob (neg-binomial-pmf n p) {n. n ≥ k1} =
          1 − measure-pmf.prob (neg-binomial-pmf n p) {..<k1'}
      using measure-pmf.prob-compl[of {..<k1'} neg-binomial-pmf n p] by simp
    also have measure-pmf.prob (neg-binomial-pmf n p) {..<k1'} =
              measure-pmf.prob (binomial-pmf (n + k1' − 1) q) {..<k1'}
      unfolding q-def using p by (intro prob-neg-binomial-pmf-lessThan) auto
    also have 1 − ... = measure-pmf.prob (binomial-pmf (n + k1' − 1) q) {n.
n ≥ k1}
      using measure-pmf.prob-compl[of {..<k1'} binomial-pmf (n + k1' − 1) q]
eq by simp
    also have {x. real x ≥ k1} = {x. x ≥ real (n + k1' − 1) ∗ q + (k1 − real (n
+ k1' − 1) ∗ q)}
      by simp
    also have measure-pmf.prob (binomial-pmf (n + k1' − 1) q) ... ≤
```

```
            exp (−2 * (k1 − real (n + k1' − 1) * q)^2 / real (n + k1' − 1))
  proof (rule binomial-distribution.prob-ge)
    show binomial-distribution q
      by unfold-locales (use q in auto)
  next
    show n + k1' − 1 > 0
      using ‹k1 > 0› n unfolding k1'-def by linarith
  next
    have real (n + nat ⌈k1⌉ − 1) ≤ real n + k1
      using ‹k1 > 0› by linarith
    hence real (n + k1' − 1) * q ≤ (real n + k1) * q
      unfolding k1'-def by (intro mult-right-mono) (use p in ‹simp-all add:
q-def›)
    also have ... ≤ k1
      using k1 p by (simp add: q-def field-simps μ-def)
    finally show 0 ≤ k1 − real (n + k1' − 1) * q
      by simp
  qed
  also have {x. real (n + k1' − 1) * q + (k1 − real (n + k1' − 1) * q) ≤ real
x} = {x. real x ≥ k1}
    by simp
  also have exp (−2 * (k1 − real (n + k1' − 1) * q)^2 / real (n + k1' − 1)) ≤
          exp (−2 * (k1 − (n + k1) * q)^2 / (n + k1))
  proof −
    have real (n + k1' − Suc 0) ≤ real n + k1
      unfolding k1'-def using ‹k1 > 0› by linarith
    moreover have (real n + k1) * q ≤ k1
      using k1 p by (auto simp: q-def field-simps μ-def)
    moreover have 1 < n + k1'
      using n ‹k1 > 0› unfolding k1'-def by linarith
    ultimately have 2 * (k1 − real (n + k1' − 1) * q)^2 / real (n + k1' − 1) ≥
                    2 * (k1 − (n + k1) * q)^2 / (n + k1)
    by (intro frac-le mult-left-mono power-mono mult-nonneg-nonneg mult-right-mono
diff-mono)
       (use q in simp-all)
    thus ?thesis
      by simp
  qed
  also have ... = exp (−2 * (p * k1 − q * n)^2 / (k1 + n))
    by (simp add: q-def algebra-simps)
  also have −2 * (p * k1 − q * n)^2 = −2 * p^2 * (k1 − μ)^2
    using p by (auto simp: field-simps μ-def)
  also have k1 − μ = k
    by (simp add: k1-def μ-def)
  also note k1-def
  also have μ + k + real n = real n / p + k
    using p by (simp add: μ-def q-def field-simps)
  also have − 2 * p^2 * k^2 / (real n / p + k) = − 2 * p ^ 3 * k^2 / (p * k + n)
    using p by (simp add: field-simps power3-eq-cube power2-eq-square)
```

```
    finally show ?thesis by (simp add: add-ac)
  qed
qed

lemma prob-neg-binomial-pmf-le-bound:
  fixes n :: nat and k :: real
  defines μ ≡ real n * q / p
  assumes k: k ≥ 0
  shows measure-pmf.prob (neg-binomial-pmf n p) {x. real x ≤ μ − k}
        ≤ exp (−2 * p ^ 3 * k² / (n − p * k))
proof −
  consider n = 0 | p = 1 | k > μ | n > 0 p ≠ 1 k ≤ μ
    by force
  thus ?thesis
  proof cases
    assume [simp]: n = 0
    show ?thesis using k
      by (simp add: indicator-def μ-def)
  next
    assume [simp]: p = 1
    show ?thesis using k
      by (auto simp add: indicator-def μ-def q-def)
  next
    assume k > μ
    hence {x. real x ≤ μ − k} = {}
      by auto
    thus ?thesis by simp
  next
    assume n: n > 0 and p ≠ 1 and k ≤ μ
    from ‹p ≠ 1› and p have p: p ∈ {0<..<1}
      by auto
    from p have q: q ∈ {0<..<1}
      by (auto simp: q-def)

    define f :: real ⇒ real where f = (λx. (p * x − q * n)² / (x + n))
    have f-mono: f x ≥ f y if x ≥ 0 y ≤ n * q / p x ≤ y for x y :: real
      using that(3)
    proof (rule DERIV-nonpos-imp-nonincreasing)
      fix t assume t: t ≥ x t ≤ y
      have x > −n
        using n ‹x ≥ 0› by linarith
      hence (f has-field-derivative ((p * t − q * n) * (n * (1 + p) + p * t) / (n +
t) ^ 2)) (at t)
        unfolding f-def using t
      by (auto intro!: derivative-eq-intros simp: algebra-simps q-def power2-eq-square)
      moreover {
        have p * t ≤ p * y
          using p by (intro mult-left-mono t) auto
        also have p * y ≤ q * n
```

```
        using ‹y ≤ n * q / p› p by (simp add: field-simps)
      finally have p * t ≤ q * n .
    }
    hence (p * t − q * n) * (n * (1 + p) + p * t) / (n + t) ^ 2 ≤ 0
      using p ‹x ≥ 0› t
        by (intro mult-nonpos-nonneg divide-nonpos-nonneg add-nonneg-nonneg
mult-nonneg-nonneg) auto
    ultimately show ∃y. (f has-real-derivative y) (at t) ∧ y ≤ 0
      by blast
  qed

  define k1 where k1 = μ − k
  have k1: k1 ≤ real n * q / p
    using assms by (simp add: μ-def k1-def)
  have k1 ≥ 0
    using k ‹k ≤ μ› by (simp add: μ-def k1-def)

  define k1' where k1' = nat (floor k1)
  have μ ≥ 0 using p
    by (auto simp: μ-def q-def)
  have (x ≤ k1') ⟷ real x ≤ k1 for x
    unfolding k1'-def not-less using ‹k1 ≥ 0› by linarith
  hence eq: {n. n ≤ k1}  = {..k1'}
    by auto
  hence measure-pmf.prob (neg-binomial-pmf n p) {n. n ≤ k1} =
        measure-pmf.prob (neg-binomial-pmf n p) {..k1'}
    by simp
  also have measure-pmf.prob (neg-binomial-pmf n p) {..k1'} =
          measure-pmf.prob (binomial-pmf (n + k1') q) {..k1'}
    unfolding q-def using p by (intro prob-neg-binomial-pmf-atMost) auto
  also note eq [symmetric]
  also have {x. real x ≤ k1} = {x. x ≤ real (n + k1') * q − (real (n + k1') *
q − real k1')}
    using eq by auto
  also have measure-pmf.prob (binomial-pmf (n + k1') q) ... ≤
            exp (−2 * (real (n + k1') * q − real k1')^2 / real (n + k1'))
  proof (rule binomial-distribution.prob-le)
    show binomial-distribution q
      by unfold-locales (use q in auto)
  next
    show n + k1' > 0
      using ‹k1 ≥ 0› n unfolding k1'-def by linarith
  next
    have p * k1' ≤ p * k1
      using p ‹k1 ≥ 0› by (intro mult-left-mono) (auto simp: k1'-def)
    also have ... ≤ q * n
      using k1 p by (simp add: field-simps)
    finally show 0 ≤ real (n + k1') * q − real k1'
      by (simp add: algebra-simps q-def)
```

```
    qed
    also have {x. real x ≤ real (n + k1') * q - (real (n + k1') * q - k1')} =
{..k1'}
      by auto
    also have real (n + k1') * q - k1' = -(p * k1' - q * n)
      by (simp add: q-def algebra-simps)
    also have ... ^ 2 = (p * k1' - q * n) ^ 2
      by algebra
    also have - 2 * (p * real k1' - q * real n)^2 / real (n + k1') = -2 * f (real
k1')
      by (simp add: f-def)
    also have f (real k1') ≥ f k1
      by (rule f-mono) (use ‹k1 ≥ 0› k1 in ‹auto simp: k1'-def›)
    hence exp (-2 * f (real k1')) ≤ exp (-2 * f k1)
      by simp
    also have ... = exp (-2 * (p * k1 - q * n)^2 / (k1 + n))
      by (simp add: f-def)

    also have -2 * (p * k1 - q * n)^2 = -2 * p^2 * (k1 - μ)^2
      using p by (auto simp: field-simps μ-def)
    also have (k1 - μ) ^ 2 = k ^ 2
      by (simp add: k1-def μ-def)
    also note k1-def
    also have μ - k + real n = real n / p - k
      using p by (simp add: μ-def q-def field-simps)
    also have - 2 * p^2 * k^2 / (real n / p - k) = - 2 * p ^ 3 * k^2 / (n - p * k)
      using p by (simp add: field-simps power3-eq-cube power2-eq-square)
    also have {..k1'} = {x. real x ≤ μ - k}
      using eq by (simp add: k1-def)
    finally show ?thesis .
  qed
qed
```

Due to the function $exp(-l/x)$ being concave for $x \geq \frac{l}{2}$, the above two bounds can be combined into the following one for moderate values of k. (cf. https://math.stackexchange.com/questions/1565559)

```
lemma prob-neg-binomial-pmf-abs-ge-bound:
  fixes n :: nat and k :: real
  defines μ ≡ real n * q / p
  assumes k ≥ 0 and n-ge: n ≥ p * k * (p^2 * k + 1)
  shows measure-pmf.prob (neg-binomial-pmf n p) {x. |real x - μ| ≥ k} ≤
        2 * exp (-2 * p ^ 3 * k ^ 2 / n)
proof (cases k = 0)
  case False
  with ‹k ≥ 0› have k: k > 0
    by auto
  define l :: real where l = 2 * p ^ 3 * k ^ 2
  have l: l > 0
    using p k by (auto simp: l-def)
```

define $f :: real \Rightarrow real$ **where** $f = (\lambda x.\ exp\ (-l\ /\ x))$
define f' **where** $f' = (\lambda x.\ -l * exp\ (-l\ /\ x)\ /\ x\ \hat{}\ 2)$

have *f'-mono*: $f'\ x \leq f'\ y$ **if** $x \geq l\ /\ 2\ x \leq y$ **for** $x\ y :: real$
using *that*(2)
proof (*rule DERIV-nonneg-imp-nondecreasing*)
fix t **assume** t: $x \leq t\ t \leq y$
have $t > 0$
using *that l t* **by** *auto*
have (f' *has-field-derivative* $(l * (2 * t - l)\ /\ (exp\ (l\ /\ t) * t\ \hat{}\ 4)))\ (at\ t)$
unfolding *f'-def* **using** *t that* ‹$t > 0$›
by (*auto intro*!: *derivative-eq-intros simp*: *field-simps exp-minus simp flip*: *power-Suc*)
moreover have $l * (2 * t - l)\ /\ (exp\ (l\ /\ t) * t\ \hat{}\ 4) \geq 0$
using *that t l* **by** (*intro divide-nonneg-pos mult-nonneg-nonneg*) *auto*
ultimately show $\exists y.\ (f'\ has\text{-}real\text{-}derivative\ y)\ (at\ t) \wedge 0 \leq y$ **by** *blast*
qed

have *convex*: *convex-on* $\{l/2..\}\ (\lambda x.\ -f\ x)$ **unfolding** *f-def*
proof (*intro convex-on-realI*[**where** $f' = f'$])
show $((\lambda x.\ -\ exp\ (-\ l\ /\ x))\ has\text{-}real\text{-}derivative\ f'\ x)\ (at\ x)$ **if** $x \in \{l/2..\}$ **for** x
using *that l*
by (*auto intro*!: *derivative-eq-intros simp*: *f'-def power2-eq-square algebra-simps*)
qed (*use l* **in** ‹*auto intro*!: *f'-mono*›)

have *eq*: $\{x.\ |real\ x - \mu| \geq k\} = \{x.\ real\ x \leq \mu - k\} \cup \{x.\ real\ x \geq \mu + k\}$
by *auto*
have *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ |real\ x - \mu| \geq k\} \leq$
measure-pmf.prob (*neg-binomial-pmf n p*) $\{x.\ real\ x \leq \mu - k\} +$
measure-pmf.prob (*neg-binomial-pmf n p*) $\{x.\ real\ x \geq \mu + k\}$
by (*subst eq*, *rule measure-Un-le*) *auto*
also have $\ldots \leq exp\ (-2 * p\ \hat{}\ 3 * k^2\ /\ (n - p * k)) + exp\ (-2 * p\ \hat{}\ 3 * k^2\ /\ (n + p * k))$
unfolding μ*-def*
by (*intro prob-neg-binomial-pmf-le-bound prob-neg-binomial-pmf-ge-bound add-mono* ‹$k \geq 0$›)
also have $\ldots = 2 * (1/2 * f\ (n - p * k) + 1/2 * f\ (n + p * k))$
by (*simp add*: *f-def l-def*)
also have $1/2 * f\ (n - p * k) + 1/2 * f\ (n + p * k) \leq f\ (1/2 * (n - p * k) + 1/2 * (n + p * k))$
proof −
let $?x = n - p * k$ **and** $?y = n + p * k$
have *le1*: $l\ /\ 2 \leq ?x$ **using** *n-ge*
by (*simp add*: *l-def power2-eq-square power3-eq-cube algebra-simps*)
also have $\ldots \leq ?y$
using *p k* **by** *simp*
finally have *le2*: $l\ /\ 2 \leq ?y$ **.**
have $-f\ ((1 - 1\ /\ 2) *_R ?x + (1\ /\ 2) *_R ?y) \leq (1 - 1\ /\ 2) * - f\ ?x + 1\ /\ 2 * - f\ ?y$

```
    using le1 le2 by (intro convex-onD[OF convex]) auto
  thus ?thesis by simp
 qed
 also have 1/2 * (n - p * k) + 1/2 * (n + p * k) = n
  by (simp add: algebra-simps)
 also have 2 * f n = 2 * exp (-l / n)
  by (simp add: f-def)
 finally show ?thesis
  by (simp add: l-def)
qed auto

end

end


theory Stream-Space
imports
 Infinite-Product-Measure
 HOL−Library.Stream
 HOL−Library.Linear-Temporal-Logic-on-Streams
begin

lemma stream-eq-Stream-iff: s = x ## t ⟷ (shd s = x ∧ stl s = t)
 by (cases s) simp

lemma Stream-snth: (x ## s) !! n = (case n of 0 ⇒ x | Suc n ⇒ s !! n)
 by (cases n) simp-all

definition to-stream :: (nat ⇒ 'a) ⇒ 'a stream where
 to-stream X = smap X nats

lemma to-stream-nat-case: to-stream (case-nat x X) = x ## to-stream X
 unfolding to-stream-def
  by (subst siterate.ctr) (simp add: smap-siterate[symmetric] stream.map-comp
comp-def)

lemma to-stream-in-streams: to-stream X ∈ streams S ⟷ (∀ n. X n ∈ S)
 by (simp add: to-stream-def streams-iff-snth)

definition stream-space :: 'a measure ⇒ 'a stream measure where
 stream-space M =
  distr (Π_M i∈UNIV. M) (vimage-algebra (streams (space M)) snth (Π_M i∈UNIV.
M)) to-stream

lemma space-stream-space: space (stream-space M) = streams (space M)
 by (simp add: stream-space-def)

lemma streams-stream-space[intro]: streams (space M) ∈ sets (stream-space M)
```

using *sets.top*[*of stream-space M*] **by** (*simp add*: *space-stream-space*)

lemma *stream-space-Stream*:
x ## ω ∈ space (*stream-space M*) ⟷ *x ∈ space M ∧ ω ∈ space* (*stream-space M*)
by (*simp add*: *space-stream-space streams-Stream*)

lemma *stream-space-eq-distr*: *stream-space M* = *distr* (Π_M *i*∈*UNIV*. *M*) (*stream-space M*) *to-stream*
unfolding *stream-space-def* **by** (*rule distr-cong*) *auto*

lemma *sets-stream-space-cong*[*measurable-cong*]:
sets M = *sets N* ⟹ *sets* (*stream-space M*) = *sets* (*stream-space N*)
using *sets-eq-imp-space-eq*[*of M N*] **by** (*simp add*: *stream-space-def vimage-algebra-def cong*: *sets-PiM-cong*)

lemma *measurable-snth-PiM*: (λω *n*. ω !! *n*) ∈ *measurable* (*stream-space M*) (Π_M *i*∈*UNIV*. *M*)
by (*auto intro*!: *measurable-vimage-algebra1*
simp: *space-PiM streams-iff-sset sset-range image-subset-iff stream-space-def*)

lemma *measurable-snth*[*measurable*]: (λω. ω !! *n*) ∈ *measurable* (*stream-space M*) *M*
using *measurable-snth-PiM measurable-component-singleton* **by** (*rule measurable-compose*) *simp*

lemma *measurable-shd*[*measurable*]: *shd* ∈ *measurable* (*stream-space M*) *M*
using *measurable-snth*[*of 0*] **by** *simp*

lemma *measurable-stream-space2*:
assumes *f-snth*: ⋀*n*. (λ*x*. *f x* !! *n*) ∈ *measurable N M*
shows *f* ∈ *measurable N* (*stream-space M*)
unfolding *stream-space-def measurable-distr-eq2*
proof (*rule measurable-vimage-algebra2*)
show *f* ∈ *space N* → *streams* (*space M*)
using *f-snth*[*THEN measurable-space*] **by** (*auto simp add*: *streams-iff-sset sset-range*)
show (λ*x*. (!!) (*f x*)) ∈ *measurable N* (*Pi*$_M$ *UNIV* (λ*i*. *M*))
proof (*rule measurable-PiM-single′*)
show (λ*x*. (!!) (*f x*)) ∈ *space N* → *UNIV* →$_E$ *space M*
using *f-snth*[*THEN measurable-space*] **by** *auto*
qed (*rule f-snth*)
qed

lemma *measurable-stream-coinduct*[*consumes 1*, *case-names shd stl*, *coinduct set*: *measurable*]:
assumes *F f*
assumes *h*: ⋀*f*. *F f* ⟹ (λ*x*. *shd* (*f x*)) ∈ *measurable N M*
assumes *t*: ⋀*f*. *F f* ⟹ *F* (λ*x*. *stl* (*f x*))
shows *f* ∈ *measurable N* (*stream-space M*)

proof (*rule measurable-stream-space2*)
fix *n* **show** (λ*x*. *f x* !! *n*) ∈ *measurable N M*
using ‹*F f*› **by** (*induction n arbitrary*: *f*) (*auto intro*: *h t*)
qed

lemma *measurable-sdrop*[*measurable*]: *sdrop n* ∈ *measurable* (*stream-space M*) (*stream-space M*)
by (*rule measurable-stream-space2*) (*simp add*: *sdrop-snth*)

lemma *measurable-stl*[*measurable*]: (λω. *stl* ω) ∈ *measurable* (*stream-space M*) (*stream-space M*)
by (*rule measurable-stream-space2*) (*simp del*: *snth.simps add*: *snth.simps*[*symmetric*])

lemma *measurable-to-stream*[*measurable*]: *to-stream* ∈ *measurable* (Π$_M$ *i*∈*UNIV*. *M*) (*stream-space M*)
by (*rule measurable-stream-space2*) (*simp add*: *to-stream-def*)

lemma *measurable-Stream*[*measurable* (*raw*)]:
assumes *f*[*measurable*]: *f* ∈ *measurable N M*
assumes *g*[*measurable*]: *g* ∈ *measurable N* (*stream-space M*)
shows (λ*x*. *f x* ## *g x*) ∈ *measurable N* (*stream-space M*)
by (*rule measurable-stream-space2*) (*simp add*: *Stream-snth*)

lemma *measurable-smap*[*measurable*]:
assumes *X*[*measurable*]: *X* ∈ *measurable N M*
shows *smap X* ∈ *measurable* (*stream-space N*) (*stream-space M*)
by (*rule measurable-stream-space2*) *simp*

lemma *measurable-stake*[*measurable*]:
stake i ∈ *measurable* (*stream-space* (*count-space UNIV*)) (*count-space* (*UNIV* :: 'a::*countable list set*))
by (*induct i*) *auto*

lemma *measurable-shift*[*measurable*]:
assumes *f*: *f* ∈ *measurable N* (*stream-space M*)
assumes [*measurable*]: *g* ∈ *measurable N* (*stream-space M*)
shows (λ*x*. *stake n* (*f x*) @− *g x*) ∈ *measurable N* (*stream-space M*)
using *f* **by** (*induction n arbitrary*: *f*) *simp-all*

lemma *measurable-case-stream-replace*[*measurable* (*raw*)]:
(λ*x*. *f x* (*shd* (*g x*)) (*stl* (*g x*))) ∈ *measurable M N* ⟹ (λ*x*. *case-stream* (*f x*) (*g x*)) ∈ *measurable M N*
unfolding *stream.case-eq-if* **.**

lemma *measurable-ev-at*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*ev-at P n*)
by (*induction n*) *auto*

lemma *measurable-alw*[*measurable*]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*alw P*)
unfolding *alw-def*
by (*coinduction rule*: *measurable-gfp-coinduct*) (*auto simp*: *inf-continuous-def*)

lemma *measurable-ev*[*measurable*]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*ev P*)
unfolding *ev-def*
by (*coinduction rule*: *measurable-lfp-coinduct*) (*auto simp*: *sup-continuous-def*)

lemma *measurable-until*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *φ Measurable.pred* (*stream-space M*) *ψ*
shows *Measurable.pred* (*stream-space M*) (*φ until ψ*)
unfolding *UNTIL-def*
by (*coinduction rule*: *measurable-gfp-coinduct*) (*simp-all add*: *inf-continuous-def fun-eq-iff*)

lemma *measurable-holds* [*measurable*]: *Measurable.pred M P* ⟹ *Measurable.pred* (*stream-space M*) (*holds P*)
unfolding *holds.simps*[*abs-def*]
by (*rule measurable-compose*[*OF measurable-shd*]) *simp*

lemma *measurable-hld*[*measurable*]: **assumes** [*measurable*]: *t* ∈ *sets M* **shows** *Measurable.pred* (*stream-space M*) (*HLD t*)
unfolding *HLD-def* **by** *measurable*

lemma *measurable-nxt*[*measurable* (*raw*)]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*nxt P*)
unfolding *nxt.simps*[*abs-def*] **by** *simp*

lemma *measurable-suntil*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *Q Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*Q suntil P*)
unfolding *suntil-def* **by** (*coinduction rule*: *measurable-lfp-coinduct*) (*auto simp*: *sup-continuous-def*)

lemma *measurable-szip*:
(λ(*ω1*, *ω2*). *szip ω1 ω2*) ∈ *measurable* (*stream-space M* ⨂$_M$ *stream-space N*) (*stream-space* (*M* ⨂$_M$ *N*))
proof (*rule measurable-stream-space2*)
fix *n*
have (λ*x*. (*case x of* (*ω1*, *ω2*) ⇒ *szip ω1 ω2*) !! *n*) = (λ(*ω1*, *ω2*). (*ω1* !! *n*, *ω2* !! *n*))
by *auto*

also have ... ∈ *measurable* (*stream-space M* ⨂$_M$ *stream-space N*) (*M* ⨂$_M$ *N*)
by *measurable*
finally show (λ*x*. (*case x of* (*ω1*, *ω2*) ⇒ *szip ω1 ω2*) !! *n*) ∈ *measurable* (*stream-space M* ⨂$_M$ *stream-space N*) (*M* ⨂$_M$ *N*)
.
qed

lemma (**in** *prob-space*) *prob-space-stream-space*: *prob-space* (*stream-space M*)
proof −
interpret *product-prob-space* λ-. *M UNIV* **..**
show *?thesis*
by (*subst stream-space-eq-distr*) (*auto intro*!: *P.prob-space-distr*)
qed

lemma (**in** *prob-space*) *nn-integral-stream-space*:
assumes [*measurable*]: *f* ∈ *borel-measurable* (*stream-space M*)
shows (∫$^+$*X*. *f X* ∂*stream-space M*) = (∫$^+$*x*. (∫$^+$*X*. *f* (*x* ## *X*) ∂*stream-space M*) ∂*M*)
proof −
interpret *S*: *sequence-space M* **..**
interpret *P*: *pair-sigma-finite M* Π$_M$ *i*::*nat*∈*UNIV*. *M* **..**

have (∫$^+$*X*. *f X* ∂*stream-space M*) = (∫$^+$*X*. *f* (*to-stream X*) ∂*S.S*)
by (*subst stream-space-eq-distr*) (*simp add*: *nn-integral-distr*)
also have ... = (∫$^+$*X*. *f* (*to-stream* ((λ(*s*, *ω*). *case-nat s ω*) *X*)) ∂(*M* ⨂$_M$ *S.S*))
by (*subst S.PiM-iter*[*symmetric*]) (*simp add*: *nn-integral-distr*)
also have ... = (∫$^+$*x*. ∫$^+$*X*. *f* (*to-stream* ((λ(*s*, *ω*). *case-nat s ω*) (*x*, *X*))) ∂*S.S* ∂*M*)
by (*subst S.nn-integral-fst*) *simp-all*
also have ... = (∫$^+$*x*. ∫$^+$*X*. *f* (*x* ## *to-stream X*) ∂*S.S* ∂*M*)
by (*auto intro*!: *nn-integral-cong simp*: *to-stream-nat-case*)
also have ... = (∫$^+$*x*. ∫$^+$*X*. *f* (*x* ## *X*) ∂*stream-space M* ∂*M*)
by (*subst stream-space-eq-distr*)
(*simp add*: *nn-integral-distr cong*: *nn-integral-cong*)
finally show *?thesis* **.**
qed

lemma (**in** *prob-space*) *emeasure-stream-space*:
assumes *X*[*measurable*]: *X* ∈ *sets* (*stream-space M*)
shows *emeasure* (*stream-space M*) *X* = (∫$^+$*t*. *emeasure* (*stream-space M*) {*x*∈*space* (*stream-space M*). *t* ## *x* ∈ *X* } ∂*M*)
proof −
have *eq*: ⋀*x xs*. *xs* ∈ *space* (*stream-space M*) ⟹ *x* ∈ *space M* ⟹
indicator X (*x* ## *xs*) = *indicator* {*xs*∈*space* (*stream-space M*). *x* ## *xs* ∈ *X* } *xs*
by (*auto split*: *split-indicator*)
show *?thesis*
using *nn-integral-stream-space*[*of indicator X*]

```
    apply (auto intro!: nn-integral-cong)
    apply (subst nn-integral-cong)
    apply (rule eq)
    apply simp-all
    done
qed

lemma (in prob-space) prob-stream-space:
  assumes P[measurable]: {x∈space (stream-space M). P x} ∈ sets (stream-space
M)
  shows 𝒫(x in stream-space M. P x) = (∫⁺t. 𝒫(x in stream-space M. P (t ##
x)) ∂M)
proof −
  interpret S: prob-space stream-space M
    by (rule prob-space-stream-space)
  show ?thesis
    unfolding S.emeasure-eq-measure[symmetric]
   by (subst emeasure-stream-space) (auto simp: stream-space-Stream intro!: nn-integral-cong)
qed

lemma (in prob-space) AE-stream-space:
  assumes [measurable]: Measurable.pred (stream-space M) P
  shows (AE X in stream-space M. P X) = (AE x in M. AE X in stream-space
M. P (x ## X))
proof −
  interpret stream: prob-space stream-space M
    by (rule prob-space-stream-space)

  have eq: ⋀x X. indicator {x. ¬ P x} (x ## X) = indicator {X. ¬ P (x ##
X)} X
    by (auto split: split-indicator)
  show ?thesis
    apply (subst AE-iff-nn-integral, simp)
    apply (subst nn-integral-stream-space, simp)
    apply (subst eq)
    apply (subst nn-integral-0-iff-AE, simp)
    apply (simp add: AE-iff-nn-integral[symmetric])
    done
qed

lemma (in prob-space) AE-stream-all:
  assumes [measurable]: Measurable.pred M P and P: AE x in M. P x
  shows AE x in stream-space M. stream-all P x
proof −
  { fix n have AE x in stream-space M. P (x !! n)
    proof (induct n)
      case 0 with P show ?case
        by (subst AE-stream-space) (auto elim!: eventually-mono)
    next
```

```
    case (Suc n) then show ?case
      by (subst AE-stream-space) auto
   qed }
  then show ?thesis
   unfolding stream-all-def by (simp add: AE-all-countable)
qed

lemma streams-sets:
  assumes X[measurable]: X ∈ sets M shows streams X ∈ sets (stream-space M)
proof −
  have streams X = {x∈space (stream-space M). x ∈ streams X}
   using streams-mono[OF - sets.sets-into-space[OF X]] by (auto simp: space-stream-space)
  also have ... = {x∈space (stream-space M). gfp (λp x. shd x ∈ X ∧ p (stl x))
x}
    apply (simp add: set-eq-iff streams-def streamsp-def)
    apply (intro allI conj-cong refl arg-cong2[where f=gfp] ext)
    apply (case-tac xa)
    apply auto
    done
  also have ... ∈ sets (stream-space M)
    apply (intro predE)
    apply (coinduction rule: measurable-gfp-coinduct)
    apply (auto simp: inf-continuous-def)
    done
  finally show ?thesis .
qed

lemma sets-stream-space-in-sets:
  assumes space: space N = streams (space M)
  assumes sets: ⋀i. (λx. x !! i) ∈ measurable N M
  shows sets (stream-space M) ⊆ sets N
  unfolding stream-space-def sets-distr
  by (auto intro!: sets-image-in-sets measurable-Sup2 measurable-vimage-algebra2
del: subsetI equalityI
          simp add: sets-PiM-eq-proj snth-in space sets cong: measurable-cong-sets)

lemma sets-stream-space-eq: sets (stream-space M) =
    sets (SUP i∈UNIV. vimage-algebra (streams (space M)) (λs. s !! i) M)
  by (auto intro!: sets-stream-space-in-sets sets-Sup-in-sets sets-image-in-sets
                measurable-Sup1 snth-in measurable-vimage-algebra1 del: subsetI
          simp: space-Sup-eq-UN space-stream-space)

lemma sets-restrict-stream-space:
  assumes S[measurable]: S ∈ sets M
  shows sets (restrict-space (stream-space M) (streams S)) = sets (stream-space
(restrict-space M S))
  using S[THEN sets.sets-into-space]
  apply (subst restrict-space-eq-vimage-algebra)
  apply (simp add: space-stream-space streams-mono2)
```

apply (*subst vimage-algebra-cong*[*OF refl refl sets-stream-space-eq*])
apply (*subst sets-stream-space-eq*)
apply (*subst sets-vimage-Sup-eq*[**where** *Y*=*streams* (*space M*)])
apply *simp*
apply *auto* []
apply (*auto intro*: *streams-mono*) []
apply *auto* []
apply (*simp add*: *image-image space-restrict-space*)
apply (*simp add*: *vimage-algebra-cong*[*OF refl refl restrict-space-eq-vimage-algebra*])
apply (*subst* (*1 2*) *vimage-algebra-vimage-algebra-eq*)
apply (*auto simp*: *streams-mono snth-in*)
done

primrec *sstart* :: *'a set* ⇒ *'a list* ⇒ *'a stream set* **where**
sstart S [] = *streams S*
| [*simp del*]: *sstart S* (*x* # *xs*) = (##) *x* ‘ *sstart S xs*

lemma *in-sstart*[*simp*]: *s* ∈ *sstart S* (*x* # *xs*) ⟷ *shd s* = *x* ∧ *stl s* ∈ *sstart S xs*
by (*cases s*) (*auto simp*: *sstart.simps*(*2*))

lemma *sstart-in-streams*: *xs* ∈ *lists S* ⟹ *sstart S xs* ⊆ *streams S*
by (*induction xs*) (*auto simp*: *sstart.simps*(*2*))

lemma *sstart-eq*: *x* ∈ *streams S* ⟹ *x* ∈ *sstart S xs* = (∀ *i*<*length xs*. *x* !! *i* = *xs* ! *i*)
by (*induction xs arbitrary*: *x*) (*auto simp*: *nth-Cons streams-stl split*: *nat.splits*)

lemma *sstart-sets*: *sstart S xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
proof (*induction xs*)
case (*Cons x xs*)
note *Cons*[*measurable*]
have *sstart S* (*x* # *xs*) =
{*s*∈*space* (*stream-space* (*count-space UNIV*)). *shd s* = *x* ∧ *stl s* ∈ *sstart S xs*}
by (*simp add*: *set-eq-iff space-stream-space*)
also have ... ∈ *sets* (*stream-space* (*count-space UNIV*))
by *measurable*
finally show *?case* **.**
qed (*simp add*: *streams-sets*)

lemma *sigma-sets-singletons*:
assumes *countable S*
shows *sigma-sets S* ((λ*s*. {*s*}) ‘*S*) = *Pow S*
proof *safe*
interpret *sigma-algebra S sigma-sets S* ((λ*s*. {*s*}) ‘*S*)
by (*rule sigma-algebra-sigma-sets*) *auto*
fix *A* **assume** *A* ⊆ *S*
with *assms* **have** (⋃ *a*∈*A*. {*a*}) ∈ *sigma-sets S* ((λ*s*. {*s*}) ‘*S*)
by (*intro countable-UN'*) (*auto dest*: *countable-subset*)
then show *A* ∈ *sigma-sets S* ((λ*s*. {*s*}) ‘*S*)

by *simp*
qed (*auto dest*: *sigma-sets-into-sp*[*rotated*])

lemma *sets-count-space-eq-sigma*:
countable S ⟹ *sets* (*count-space S*) = *sets* (*sigma S* ((λ*s*. {*s*})‘*S*))
by (*subst sets-measure-of*) (*auto simp*: *sigma-sets-singletons*)

lemma *sets-stream-space-sstart*:
assumes *S*[*simp*]: *countable S*
shows *sets* (*stream-space* (*count-space S*)) = *sets* (*sigma* (*streams S*) (*sstart S‘lists S* ∪ {{}}))
proof
have [*simp*]: *sstart S ‘ lists S* ⊆ *Pow* (*streams S*)
by (*simp add*: *image-subset-iff sstart-in-streams*)

let *?S* = *sigma* (*streams S*) (*sstart S ‘ lists S* ∪ {{}})
{ fix *i a* **assume** *a* ∈ *S*
{ fix *x* **have** (*x* !! *i* = *a* ∧ *x* ∈ *streams S*) ⟷ (∃ *xs*∈*lists S*. *length xs* = *i* ∧ *x* ∈ *sstart S* (*xs* @ [*a*]))
proof (*induction i arbitrary*: *x*)
case (*Suc i*) **from** *this*[*of stl x*] **show** *?case*
by (*simp add*: *length-Suc-conv Bex-def ex-simps*[*symmetric*] *del*: *ex-simps*)
(*metis stream.collapse streams-Stream*)
qed (*insert* ‹*a* ∈ *S*›, *auto intro*: *streams-stl in-streams*) **}**
then have (λ*x*. *x* !! *i*) −‘ {*a*} ∩ *streams S* = (⋃ *xs*∈{*xs*∈*lists S*. *length xs* = *i*}. *sstart S* (*xs* @ [*a*]))
by (*auto simp add*: *set-eq-iff*)
also have ... ∈ *sets ?S*
using ‹*a*∈*S*› **by** (*intro sets.countable-UN′*) (*auto intro*!: *sigma-sets.Basic image-eqI*)
finally have (λ*x*. *x* !! *i*) −‘ {*a*} ∩ *streams S* ∈ *sets ?S* **. }**
then show *sets* (*stream-space* (*count-space S*)) ⊆ *sets* (*sigma* (*streams S*) (*sstart S‘lists S* ∪ {{}}))
by (*intro sets-stream-space-in-sets*) (*auto simp*: *measurable-count-space-eq-countable snth-in*)

have *sigma-sets* (*space* (*stream-space* (*count-space S*))) (*sstart S‘lists S* ∪ {{}}) ⊆ *sets* (*stream-space* (*count-space S*))
proof (*safe intro*!: *sets.sigma-sets-subset*)
fix *xs* **assume** ∀ *x*∈*set xs*. *x* ∈ *S*
then have *sstart S xs* = {*x*∈*space* (*stream-space* (*count-space S*)). ∀ *i*<*length xs*. *x* !! *i* = *xs* ! *i*}
by (*induction xs*)
(*auto simp*: *space-stream-space nth-Cons split*: *nat.split intro*: *in-streams streams-stl*)
also have ... ∈ *sets* (*stream-space* (*count-space S*))
by *measurable*
finally show *sstart S xs* ∈ *sets* (*stream-space* (*count-space S*)) **.**
qed

then show *sets* (*sigma* (*streams S*) (*sstart S*‘*lists S* ∪ {{}})) ⊆ *sets* (*stream-space* (*count-space S*))
by (*simp add*: *space-stream-space*)
qed

lemma *Int-stable-sstart*: *Int-stable* (*sstart S*‘*lists S* ∪ {{}})
proof −
{ fix *xs ys* **assume** *xs* ∈ *lists S ys* ∈ *lists S*
then have *sstart S xs* ∩ *sstart S ys* ∈ *sstart S* ‘ *lists S* ∪ {{}}
proof (*induction xs ys rule*: *list-induct2*′)
case (*4 x xs y ys*)
show *?case*
proof *cases*
assume *x* = *y*
then have *sstart S* (*x* # *xs*) ∩ *sstart S* (*y* # *ys*) = (##) *x* ‘ (*sstart S xs* ∩ *sstart S ys*)
by (*auto simp*: *image-iff intro*!: *stream.collapse*[*symmetric*])
also have . . . ∈ *sstart S* ‘ *lists S* ∪ {{}}
using *4* **by** (*auto simp*: *sstart.simps*(*2*)[*symmetric*] *del*: *in-listsD*)
finally show *?case* **.**
qed *auto*
qed (*simp-all add*: *sstart-in-streams inf.absorb1 inf.absorb2 image-eqI*[**where** *x*=[]]) **}**
then show *?thesis*
by (*auto simp*: *Int-stable-def*)
qed

lemma *stream-space-eq-sstart*:
assumes *S*[*simp*]: *countable S*
assumes *P*: *prob-space M prob-space N*
assumes *ae*: *AE x in M*. *x* ∈ *streams S AE x in N*. *x* ∈ *streams S*
assumes *sets-M*: *sets M* = *sets* (*stream-space* (*count-space UNIV*))
assumes *sets-N*: *sets N* = *sets* (*stream-space* (*count-space UNIV*))
assumes ∗: ⋀*xs*. *xs* ≠ [] ⟹ *xs* ∈ *lists S* ⟹ *emeasure M* (*sstart S xs*) = *emeasure N* (*sstart S xs*)
shows *M* = *N*
proof (*rule measure-eqI-restrict-generator*[*OF Int-stable-sstart*])
have [*simp*]: *sstart S* ‘ *lists S* ⊆ *Pow* (*streams S*)
by (*simp add*: *image-subset-iff sstart-in-streams*)

interpret *M*: *prob-space M* **by** *fact*

show *sstart S* ‘ *lists S* ∪ {{}} ⊆ *Pow* (*streams S*)
by (*auto dest*: *sstart-in-streams del*: *in-listsD*)

{ fix *M* :: ′*a stream measure* **assume** *M*: *sets M* = *sets* (*stream-space* (*count-space UNIV*))
have *sets* (*restrict-space M* (*streams S*)) = *sigma-sets* (*streams S*) (*sstart S* ‘ *lists S* ∪ {{}})

```
    by (subst sets-restrict-space-cong[OF M])
     (simp add: sets-restrict-stream-space restrict-count-space sets-stream-space-sstart)
}
  from this[OF sets-M] this[OF sets-N]
  show sets (restrict-space M (streams S)) = sigma-sets (streams S) (sstart S ‘
lists S ∪ {{}})
       sets (restrict-space N (streams S)) = sigma-sets (streams S) (sstart S ‘ lists
S ∪ {{}})
    by auto
  show {streams S} ⊆ sstart S ‘ lists S ∪ {{}}
    ⋃{streams S} = streams S ⋀s. s ∈ {streams S} ⟹ emeasure M s ≠ ∞
   using M.emeasure-space-1 space-stream-space[of count-space S] sets-eq-imp-space-eq[OF
sets-M]
    by (auto simp add: image-eqI[where x=[]])
  show sets M = sets N
    by (simp add: sets-M sets-N)
next
  fix X assume X ∈ sstart S ‘ lists S ∪ {{}}
  then obtain xs where X = {} ∨ (xs ∈ lists S ∧ X = sstart S xs)
    by auto
  moreover have emeasure M (streams S) = 1
    using ae by (intro prob-space.emeasure-eq-1-AE[OF P(1)]) (auto simp: sets-M
streams-sets)
  moreover have emeasure N (streams S) = 1
    using ae by (intro prob-space.emeasure-eq-1-AE[OF P(2)]) (auto simp: sets-N
streams-sets)
  ultimately show emeasure M X = emeasure N X
    using P[THEN prob-space.emeasure-space-1]
    by (cases xs = []) (auto simp: * space-stream-space del: in-listsD)
qed (auto simp: * ae sets-M del: in-listsD intro!: streams-sets)

lemma sets-sstart[measurable]: sstart Ω xs ∈ sets (stream-space (count-space UNIV))
proof (induction xs)
  case (Cons x xs)
  note this[measurable]
   have sstart Ω (x # xs) = {ω∈space (stream-space (count-space UNIV)). ω ∈
sstart Ω (x # xs)}
    by (auto simp: space-stream-space)
  also have ... ∈ sets (stream-space (count-space UNIV))
    unfolding in-sstart by measurable
  finally show ?case .
qed (auto intro!: streams-sets)

primrec scylinder :: 'a set ⇒ 'a set list ⇒ 'a stream set
where
  scylinder S [] = streams S
| scylinder S (A # As) = {ω∈streams S. shd ω ∈ A ∧ stl ω ∈ scylinder S As}

lemma scylinder-streams: scylinder S xs ⊆ streams S
```

by (*induction xs*) *auto*

lemma *sets-scylinder*: ($\forall x{\in}$*set xs*. *x* $\in$ *sets S*) $\Longrightarrow$ *scylinder* (*space S*) *xs* $\in$ *sets* (*stream-space S*)
by (*induction xs*) (*auto simp*: *space-stream-space*[*symmetric*])

lemma *stream-space-eq-scylinder*:
assumes *P*: *prob-space M prob-space N*
assumes *Int-stable G* **and** *S*: *sets S* = *sets* (*sigma* (*space S*) *G*)
and *C*: *countable C C* $\subseteq$ *G* $\bigcup$ *C* = *space S* **and** *G*: *G* $\subseteq$ *Pow* (*space S*)
assumes *sets-M*: *sets M* = *sets* (*stream-space S*)
assumes *sets-N*: *sets N* = *sets* (*stream-space S*)
assumes $*$: $\bigwedge$*xs*. *xs* $\neq$ [] $\Longrightarrow$ *xs* $\in$ *lists G* $\Longrightarrow$ *emeasure M* (*scylinder* (*space S*) *xs*) = *emeasure N* (*scylinder* (*space S*) *xs*)
shows *M* = *N*
proof (*rule measure-eqI-generator-eq*)
interpret *M*: *prob-space M* **by** *fact*
interpret *N*: *prob-space N* **by** *fact*

let *?G* = *scylinder* (*space S*) ‘ *lists G*
show *sc-Pow*: *?G* $\subseteq$ *Pow* (*streams* (*space S*))
using *scylinder-streams* **by** *auto*

have *sets* (*stream-space S*) = *sets* (*sigma* (*streams* (*space S*)) *?G*)
(**is** *?S* = *sets ?R*)
proof (*rule antisym*)
let *?V* = λ*i*. *vimage-algebra* (*streams* (*space S*)) (λ*s*. *s* !! *i*) *S*
show *?S* $\subseteq$ *sets ?R*
unfolding *sets-stream-space-eq*
proof (*safe intro*!: *sets-Sup-in-sets del*: *subsetI equalityI*)
fix *i* :: *nat*
show *space* (*?V i*) = *space ?R*
using *scylinder-streams* **by** (*subst space-measure-of*) *auto*
{ fix *A* **assume** *A* $\in$ *G*
then have *scylinder* (*space S*) (*replicate i* (*space S*) @ [*A*]) = (λ*s*. *s* !! *i*) −‘ *A* $\cap$ *streams* (*space S*)
by (*induction i*) (*auto simp add*: *streams-shd streams-stl cong*: *conj-cong*)
also have *scylinder* (*space S*) (*replicate i* (*space S*) @ [*A*]) = ($\bigcup$ *xs*$\in${*xs*$\in$*lists C*. *length xs* = *i*}. *scylinder* (*space S*) (*xs* @ [*A*]))
apply (*induction i*)
apply *auto* []
apply (*simp add*: *length-Suc-conv set-eq-iff ex-simps*(*1*,*2*)[*symmetric*] *cong*: *conj-cong del*: *ex-simps*(*1*,*2*))
apply *rule*
subgoal for *i x*
apply (*cases x*)
apply (*subst* (*2*) *C*(*3*)[*symmetric*])
apply (*simp del*: *ex-simps*(*1*,*2*) *add*: *ex-simps*(*1*,*2*)[*symmetric*] *ac-simps Bex-def*)

```
          apply auto
          done
        done
       finally have (λs. s !! i) −‘ A ∩ streams (space S) = (⋃xs∈{xs∈lists C.
length xs = i}. scylinder (space S) (xs @ [A]))
        ..
      also have ... ∈ ?R
        using C(2) ‹A∈G›
        by (intro sets.countable-UN′ countable-Collect countable-lists C)
           (auto intro!: in-measure-of[OF sc-Pow] imageI)
      finally have (λs. s !! i) −‘ A ∩ streams (space S) ∈ ?R . }
    then show sets (?V i) ⊆ ?R
      apply (subst vimage-algebra-cong[OF refl refl S])
      apply (subst vimage-algebra-sigma[OF G])
      apply (simp add: streams-iff-snth) []
      apply (subst sigma-le-sets)
      apply auto
      done
  qed
  have G ⊆ sets S
    unfolding S using G by auto
  with C(2) show sets ?R ⊆ ?S
    unfolding sigma-le-sets[OF sc-Pow] by (auto intro!: sets-scylinder)
 qed
 then show sets M = sigma-sets (streams (space S)) (scylinder (space S) ‘ lists
G)
  sets N = sigma-sets (streams (space S)) (scylinder (space S) ‘ lists G)
  unfolding sets-M sets-N by (simp-all add: sc-Pow)

 show Int-stable ?G
 proof (rule Int-stableI-image)
  fix xs ys assume xs ∈ lists G ys ∈ lists G
   then show ∃zs∈lists G. scylinder (space S) xs ∩ scylinder (space S) ys =
scylinder (space S) zs
  proof (induction xs arbitrary: ys)
    case Nil then show ?case
      by (auto simp add: Int-absorb1 scylinder-streams)
  next
    case xs: (Cons x xs)
    show ?case
    proof (cases ys)
      case Nil with xs.hyps show ?thesis
        by (auto simp add: Int-absorb2 scylinder-streams intro!: bexI[of - x#xs])
    next
      case ys: (Cons y ys′)
      with xs.IH[of ys′] xs.prems obtain zs where
        zs ∈ lists G and eq: scylinder (space S) xs ∩ scylinder (space S) ys′ =
scylinder (space S) zs
        by auto
```

```
      show ?thesis
      proof (intro bexI[of - (x ∩ y)#zs])
        show x ∩ y # zs ∈ lists G
        using ‹zs∈lists G› ‹x∈G› ‹ys∈lists G› ys ‹Int-stable G›[THEN Int-stableD,
of x y] by auto
         show scylinder (space S) (x # xs) ∩ scylinder (space S) ys = scylinder
(space S) (x ∩ y # zs)
          by (auto simp add: eq[symmetric] ys)
      qed
    qed
  qed

  show range (λ-::nat. streams (space S)) ⊆ scylinder (space S) ‘ lists G
    (⋃i. streams (space S)) = streams (space S)
    emeasure M (streams (space S)) ≠ ∞
    by (auto intro!: image-eqI[of - - []])

  fix X assume X ∈ scylinder (space S) ‘ lists G
  then obtain xs where xs: xs ∈ lists G and eq: X = scylinder (space S) xs
    by auto
  then show emeasure M X = emeasure N X
  proof (cases xs = [])
    assume xs = [] then show ?thesis
      unfolding eq
     using sets-M[THEN sets-eq-imp-space-eq] sets-N[THEN sets-eq-imp-space-eq]
        M.emeasure-space-1 N.emeasure-space-1
      by (simp add: space-stream-space[symmetric])
  next
    assume xs ≠ [] with xs show ?thesis
      unfolding eq by (intro ∗)
  qed
qed

lemma stream-space-coinduct:
  fixes R :: 'a stream measure ⇒ 'a stream measure ⇒ bool
  assumes R A B
  assumes R: ⋀A B. R A B ⟹ ∃K∈space (prob-algebra M).
   ∃A'∈M →M prob-algebra (stream-space M). ∃B'∈M →M prob-algebra (stream-space
M).
    (AE y in K. R (A' y) (B' y) ∨ A' y = B' y) ∧
    A = do { y ← K; ω ← A' y; return (stream-space M) (y ## ω) } ∧
    B = do { y ← K; ω ← B' y; return (stream-space M) (y ## ω) }
  shows A = B
proof (rule stream-space-eq-scylinder)
  let ?step = λK L. do { y ← K; ω ← L y; return (stream-space M) (y ## ω) }
  { fix K A A' assume K: K ∈ space (prob-algebra M)
      and A'[measurable]: A' ∈ M →M prob-algebra (stream-space M) and A-eq:
A = ?step K A'
```

have *ps*: *prob-space A*
unfolding *A-eq* **by** (*rule prob-space-bind'*[*OF K*]) *measurable*
have *sets A* = *sets* (*stream-space M*)
unfolding *A-eq* **by** (*rule sets-bind'*[*OF K*]) *measurable*
note *ps this* **}**
note ∗∗ = *this*

{ fix *A B* **assume** *R A B*
obtain *K A' B'* **where** *K*: *K* ∈ *space* (*prob-algebra M*)
and *A'*: *A'* ∈ *M* →$_M$ *prob-algebra* (*stream-space M*) *A* = *?step K A'*
and *B'*: *B'* ∈ *M* →$_M$ *prob-algebra* (*stream-space M*) *B* = *?step K B'*
using *R*[*OF* ‹*R A B*›] **by** *blast*
have *prob-space A prob-space B sets A* = *sets* (*stream-space M*) *sets B* = *sets* (*stream-space M*)
using ∗∗[*OF K A'*] ∗∗[*OF K B'*] **by** *auto* **}**
note *R-D* = *this*

show *prob-space A prob-space B sets A* = *sets* (*stream-space M*) *sets B* = *sets* (*stream-space M*)
using *R-D*[*OF* ‹*R A B*›] **by** *auto*

show *Int-stable* (*sets M*) *sets M* = *sets* (*sigma* (*space M*) (*sets M*)) *countable* {*space M*}
{*space M*} ⊆ *sets M* ⋃{*space M*} = *space M sets M* ⊆ *Pow* (*space M*)
using *sets.space-closed*[*of M*] **by** (*auto simp*: *Int-stable-def*)

{ fix *A As L K* **assume** *K*[*measurable*]: *K* ∈ *space* (*prob-algebra M*) **and** *A*: *A* ∈ *sets M As* ∈ *lists* (*sets M*)
and *L*[*measurable*]: *L* ∈ *M* →$_M$ *prob-algebra* (*stream-space M*)
from *A* **have** [*measurable*]: ∀ *x*∈*set* (*A* # *As*). *x* ∈ *sets M* ∀ *x*∈*set As*. *x* ∈ *sets M*
by *auto*
have [*simp*]: *space K* = *space M sets K* = *sets M*
using *K* **by** (*auto simp*: *space-prob-algebra intro*!: *sets-eq-imp-space-eq*)
have [*simp*]: *x* ∈ *space M* ⟹ *sets* (*L x*) = *sets* (*stream-space M*) **for** *x*
using *measurable-space*[*OF L*] **by** (*auto simp*: *space-prob-algebra*)
note *sets-scylinder*[*measurable*]
have ∗: *indicator* (*scylinder* (*space M*) (*A* # *As*)) (*x* ## ω) =
(*indicator A x* ∗ *indicator* (*scylinder* (*space M*) *As*) ω :: *ennreal*) **for** ω *x*
using *scylinder-streams*[*of space M As*] ‹*A* ∈ *sets M*›[*THEN sets.sets-into-space*]
by (*auto split*: *split-indicator*)
have *emeasure* (*?step K L*) (*scylinder* (*space M*) (*A* # *As*)) = (∫$^+$*y*. *L y* (*scylinder* (*space M*) *As*) ∗ *indicator A y* ∂*K*)
apply (*subst emeasure-bind-prob-algebra*[*OF K*])
apply *measurable*
apply (*rule nn-integral-cong*)
apply (*subst emeasure-bind-prob-algebra*[*OF L*[*THEN measurable-space*]])
apply (*simp-all add*: *ac-simps* ∗ *nn-integral-cmult-indicator del*: *scylinder.simps*)

```
    apply measurable
    done }
  note emeasure-step = this

  fix Xs assume Xs ∈ lists (sets M)
  from this ‹R A B› show emeasure A (scylinder (space M) Xs) = emeasure B
(scylinder (space M) Xs)
  proof (induction Xs arbitrary: A B)
    case (Cons X Xs)
    obtain K A' B' where K: K ∈ space (prob-algebra M)
      and A'[measurable]: A' ∈ M →M prob-algebra (stream-space M) and A: A =
?step K A'
      and B'[measurable]: B' ∈ M →M prob-algebra (stream-space M) and B: B =
?step K B'
      and AE-R: AE x in K. R (A' x) (B' x) ∨ A' x = B' x
      using R[OF ‹R A B›] by blast

    show ?case
      unfolding A B emeasure-step[OF K Cons.hyps A'] emeasure-step[OF K
Cons.hyps B']
      apply (rule nn-integral-cong-AE)
      using AE-R by eventually-elim (auto simp add: Cons.IH)
  next
    case Nil
    note R-D[OF this]
   from this(1,2)[THEN prob-space.emeasure-space-1] this(3,4)[THEN sets-eq-imp-space-eq]
    show ?case
      by (simp add: space-stream-space)
  qed
qed

end


theory Tree-Space
  imports HOL−Analysis.Analysis HOL−Library.Tree
begin

lemma countable-lfp:
  assumes step: ⋀Y. countable Y ⟹ countable (F Y)
  and cont: Order-Continuity.sup-continuous F
  shows countable (lfp F)
by(subst sup-continuous-lfp[OF cont])(simp add: countable-funpow[OF step])

lemma countable-lfp-apply:
  assumes step: ⋀Y x. (⋀x. countable (Y x)) ⟹ countable (F Y x)
  and cont: Order-Continuity.sup-continuous F
  shows countable (lfp F x)
proof −
```

```
  { fix n
    have ⋀x. countable ((F ^^ n) bot x)
      by(induct n)(auto intro: step) }
  thus ?thesis using cont by(simp add: sup-continuous-lfp)
qed

inductive-set trees :: 'a set ⇒ 'a tree set for S :: 'a set where
  [intro!]: Leaf ∈ trees S
| l ∈ trees S ⟹ r ∈ trees S ⟹ v ∈ S ⟹ Node l v r ∈ trees S

lemma Node-in-trees-iff[simp]: Node l v r ∈ trees S ⟷ (l ∈ trees S ∧ v ∈ S ∧
r ∈ trees S)
  by (subst trees.simps) auto

lemma trees-sub-lfp: trees S ⊆ lfp (λT. T ∪ {Leaf} ∪ (⋃l∈T. (⋃v∈S. (⋃r∈T.
{Node l v r}))))
proof
  have mono: mono (λT. T ∪ {Leaf} ∪ (⋃l∈T. (⋃v∈S. (⋃r∈T. {Node l v r}))))
    by (auto simp: mono-def)
  fix t assume t ∈ trees S then show t ∈ lfp (λT. T ∪ {Leaf} ∪ (⋃l∈T. (⋃v∈S.
(⋃r∈T. {Node l v r}))))
  proof induction
    case 1 then show ?case
      by (subst lfp-unfold[OF mono]) auto
  next
    case 2 then show ?case
      by (subst lfp-unfold[OF mono]) auto
  qed
qed

lemma countable-trees: countable A ⟹ countable (trees A)
proof (intro countable-subset[OF trees-sub-lfp] countable-lfp
        sup-continuous-sup sup-continuous-const sup-continuous-id)
  show sup-continuous (λT. (⋃l∈T. ⋃v∈A. ⋃r∈T. {⟨l, v, r⟩}))
    unfolding sup-continuous-def
  proof (intro allI impI equalityI subsetI, goal-cases)
    case (1 M t)
    then obtain i j :: nat and l x r where t = Node l x r x ∈ A l ∈ M i r ∈ M j
      by auto
    hence l ∈ M (max i j) r ∈ M (max i j)
      using incseqD[OF ‹incseq M›, of i max i j] incseqD[OF ‹incseq M›, of j max
i j] by auto
    with ‹t = Node l x r› and ‹x ∈ A› show ?case by auto
  qed auto
qed auto

lemma trees-UNIV[simp]: trees UNIV = UNIV
proof −
  have t ∈ trees UNIV for t :: 'a tree
```

by (*induction t*) (*auto intro*: *trees.intros*(*2*))
then show *?thesis* **by** *auto*
qed

instance *tree* :: (*countable*) *countable*
proof
have *countable* (*UNIV* :: *'a tree set*)
by (*subst trees-UNIV*[*symmetric*]) (*intro countable-trees*[*OF countableI-type*])
then show ∃ *to-nat*::*'a tree* ⇒ *nat*. *inj to-nat*
by (*auto simp*: *countable-def*)
qed

lemma *map-in-trees*[*intro*]: (⋀*x*. *x* ∈ *set-tree t* ⟹ *f x* ∈ *S*) ⟹ *map-tree f t* ∈ *trees S*
by (*induction t*) (*auto intro*: *trees.intros*(*2*))

primrec *trees-cyl* :: *'a set tree* ⇒ *'a tree set* **where**
trees-cyl Leaf = {*Leaf*}
| *trees-cyl* (*Node l v r*) = (⋃ *l'*∈*trees-cyl l*. (⋃ *v'*∈*v*. (⋃ *r'*∈*trees-cyl r*. {*Node l' v' r'*})))

definition *tree-sigma* :: *'a measure* ⇒ *'a tree measure*
where
tree-sigma M = *sigma* (*trees* (*space M*)) (*trees-cyl* ‘ *trees* (*sets M*))

lemma *Node-in-trees-cyl*: *Node l' v' r'* ∈ *trees-cyl t* ⟷
(∃ *l v r*. *t* = *Node l v r* ∧ *l'* ∈ *trees-cyl l* ∧ *r'* ∈ *trees-cyl r* ∧ *v'* ∈ *v*)
by (*cases t*) *auto*

lemma *trees-cyl-sub-trees*:
assumes *t* ∈ *trees A A* ⊆ *Pow B* **shows** *trees-cyl t* ⊆ *trees B*
using *assms*(*1*)
proof *induction*
case (*2 l v r*) **with** ‹*A* ⊆ *Pow B*› **show** *?case*
by (*auto intro*!: *trees.intros*(*2*))
qed *auto*

lemma *trees-cyl-sets-in-space*: *trees-cyl* ‘ *trees* (*sets M*) ⊆ *Pow* (*trees* (*space M*))
using *trees-cyl-sub-trees*[*OF - sets.space-closed*, *of - M*] **by** *auto*

lemma *space-tree-sigma*: *space* (*tree-sigma M*) = *trees* (*space M*)
unfolding *tree-sigma-def* **by** (*rule space-measure-of-conv*)

lemma *sets-tree-sigma-eq*: *sets* (*tree-sigma M*) = *sigma-sets* (*trees* (*space M*)) (*trees-cyl* ‘ *trees* (*sets M*))
unfolding *tree-sigma-def* **by** (*rule sets-measure-of*) (*rule trees-cyl-sets-in-space*)

lemma *Leaf-in-space-tree-sigma* [*measurable*, *simp*, *intro*]: *Leaf* ∈ *space* (*tree-sigma M*)

by (*auto simp*: *space-tree-sigma*)

lemma *Leaf-in-tree-sigma* [*measurable*, *simp*, *intro*]: {*Leaf*} ∈ *sets* (*tree-sigma M*)
unfolding *sets-tree-sigma-eq*
by (*rule sigma-sets.Basic*) (*auto intro*: *trees.intros*(*2*) *image-eqI*[**where** *x*=*Leaf*])

lemma *trees-cyl-map-treeI*: *t* ∈ *trees-cyl* (*map-tree* (λ*x*. *A*) *t*) **if** ∗: *t* ∈ *trees A*
using ∗ **by** *induction auto*

lemma *trees-cyl-map-in-sets*:
(⋀*x*. *x* ∈ *set-tree t* ⟹ *f x* ∈ *sets M*) ⟹ *trees-cyl* (*map-tree f t*) ∈ *sets* (*tree-sigma M*)
by (*subst sets-tree-sigma-eq*) *auto*

lemma *Node-in-tree-sigma*:
assumes *L*: *X* ∈ *sets* (*M* ⨂$_M$ (*tree-sigma M* ⨂$_M$ *tree-sigma M*))
shows {*Node l v r* | *l v r*. (*v*, *l*, *r*) ∈ *X*} ∈ *sets* (*tree-sigma M*)
proof −
let *?E* = λ*s*::*unit tree*. *trees-cyl* (*map-tree* (λ-. *space M*) *s*)
have *1*: *countable* (*range ?E*)
by (*intro countable-image countableI-type*)
have *2*: *trees-cyl* ‘ *trees* (*sets M*) ⊆ *Pow* (*space* (*tree-sigma M*))
using *trees-cyl-sets-in-space*[*of M*] **by** (*simp add*: *space-tree-sigma*)
have *3*: *sets* (*tree-sigma M*) = *sigma-sets* (*space* (*tree-sigma M*)) (*trees-cyl* ‘ *trees* (*sets M*))
unfolding *sets-tree-sigma-eq* **by** (*simp add*: *space-tree-sigma*)
have *4*: (⋃*s*. *?E s*) = *space* (*tree-sigma M*)
proof (*safe*; *clarsimp simp*: *space-tree-sigma*)
fix *t s* **assume** *t* ∈ *trees-cyl* (*map-tree* (λ-::*unit*. *space M*) *s*)
then show *t* ∈ *trees* (*space M*)
by (*induction s arbitrary*: *t*) *auto*
next
fix *t* **assume** *t* ∈ *trees* (*space M*)
then show ∃ *t*′. *t* ∈ *?E t*′
by (*intro exI*[*of* - *map-tree* (λ-. ()) *t*])
(*auto simp*: *tree.map-comp comp-def intro*: *trees-cyl-map-treeI*)
qed
have *5*: *range ?E* ⊆ *trees-cyl* ‘ *trees* (*sets M*) **by** *auto*
let *?P* = {*A* × *B* | *A B*. *A* ∈ *trees-cyl* ‘ *trees* (*sets M*) ∧ *B* ∈ *trees-cyl* ‘ *trees* (*sets M*)}
have *P*: *sets* (*tree-sigma M* ⨂$_M$ *tree-sigma M*) = *sets* (*sigma* (*space* (*tree-sigma M*) × *space* (*tree-sigma M*)) *?P*)
by (*rule sets-pair-eq*[*OF 2 3 1 5 4 2 3 1 5 4*])

have *sets* (*M* ⨂$_M$ (*tree-sigma M* ⨂$_M$ *tree-sigma M*)) =
sets (*sigma* (*space M* × *space* (*tree-sigma M* ⨂$_M$ *tree-sigma M*)) {*A* × *BC* | *A BC*. *A* ∈ *sets M* ∧ *BC* ∈ *?P*})
proof (*rule sets-pair-eq*)
show *sets M* ⊆ *Pow* (*space M*) *sets M* = *sigma-sets* (*space M*) (*sets M*)

by (*auto simp*: *sets.sigma-sets-eq sets.space-closed*)
show *countable* {*space M*} {*space M*} ⊆ *sets M* ⋃{*space M*} = *space M*
by *auto*
show *?P* ⊆ *Pow* (*space* (*tree-sigma M* ⨂$_M$ *tree-sigma M*))
using *trees-cyl-sets-in-space*[*of M*]
by (*auto simp*: *space-pair-measure space-tree-sigma subset-eq*)
then show *sets* (*tree-sigma M* ⨂$_M$ *tree-sigma M*) =
sigma-sets (*space* (*tree-sigma M* ⨂$_M$ *tree-sigma M*)) *?P*
by (*subst P*, *subst sets-measure-of*) (*auto simp*: *space-tree-sigma space-pair-measure*)
show *countable* ((λ(*a*, *b*). *a* × *b*) ' (*range ?E* × *range ?E*))
by (*intro countable-image countable-SIGMA countableI-type*)
show (λ(*a*, *b*). *a* × *b*) ' (*range ?E* × *range ?E*) ⊆ *?P*
by *auto*
qed (*insert 4*, *auto simp*: *space-pair-measure space-tree-sigma set-eq-iff*)
also have ... = *sigma-sets* (*space M* × *trees* (*space M*) × *trees* (*space M*))
{*A* × *BC* |*A BC*. *A* ∈ *sets M* ∧ *BC* ∈ {*A* × *B* |*A B*.
A ∈ *trees-cyl* ' *trees* (*sets M*) ∧ *B* ∈ *trees-cyl* ' *trees* (*sets M*)}}
(**is** - = *sigma-sets ?X ?Y*) **using** *sets.space-closed*[*of M*] *trees-cyl-sub-trees*[*of* - *sets M space M*]
by (*subst sets-measure-of*)
(*auto simp*: *space-pair-measure space-tree-sigma*)
also have *?Y* = {*A* × *trees-cyl B* × *trees-cyl C* | *A B C*. *A* ∈ *sets M* ∧
B ∈ *trees* (*sets M*) ∧ *C* ∈ *trees* (*sets M*)} **by** *blast*
finally have *X* ∈ *sigma-sets* (*space M* × *trees* (*space M*) × *trees* (*space M*))
{*A* × *trees-cyl B* × *trees-cyl C* | *A B C*. *A* ∈ *sets M* ∧ *B* ∈ *trees* (*sets M*) ∧ *C* ∈ *trees* (*sets M*) }
using *assms* **by** *blast*
then show *?thesis*
proof *induction*
case (*Basic A'*)
then obtain *A B C* **where** *A'* = *A* × *trees-cyl B* × *trees-cyl C*
and ∗: *A* ∈ *sets M B* ∈ *trees* (*sets M*) *C* ∈ *trees* (*sets M*)
by *auto*
then have {*Node l v r* |*l v r*. (*v*, *l*, *r*) ∈ *A'*} = *trees-cyl* (*Node B A C*)
by *auto*
then show *?case*
by (*auto simp del*: *trees-cyl.simps simp*: *sets-tree-sigma-eq intro*!: *sigma-sets.Basic* ∗)
next
case *Empty* **show** *?case* **by** *auto*
next
case (*Compl A*)
have {*Node l v r* |*l v r*. (*v*, *l*, *r*) ∈ *space M* × *trees* (*space M*) × *trees* (*space M*) − *A*} =
(*space* (*tree-sigma M*) − {*Node l v r* |*l v r*. (*v*, *l*, *r*) ∈ *A*}) − {*Leaf*}
by (*auto simp*: *space-tree-sigma elim*: *trees.cases*)
also have ... ∈ *sets* (*tree-sigma M*)
by (*intro sets.Diff Compl*) *auto*
finally show *?case* **.**

next
case (*Union I*)
have ∗: {*Node l v r* |*l v r*. (*v*, *l*, *r*) ∈ ⋃(*I* ‘ *UNIV*)} =
(⋃ *i*. {*Node l v r* |*l v r*. (*v*, *l*, *r*) ∈ *I i*}) **by** *auto*
show *?case* **unfolding** ∗ **using** *Union*(*2*) **by** (*intro sets.countable-UN*) *auto*
qed
qed

lemma *measurable-left*[*measurable*]: *left* ∈ *tree-sigma M* →$_M$ *tree-sigma M*
proof (*rule measurableI*)
show *t* ∈ *space* (*tree-sigma M*) ⟹ *left t* ∈ *space* (*tree-sigma M*) **for** *t*
by (*cases t*) (*auto simp*: *space-tree-sigma*)
fix *A* **assume** *A*: *A* ∈ *sets* (*tree-sigma M*)
from *sets.sets-into-space*[*OF this*]
have ∗: *left* −‘ *A* ∩ *space* (*tree-sigma M*) =
(*if Leaf* ∈ *A then* {*Leaf*} *else* {}) ∪
{*Node a v r* | *a v r*. (*v*, *a*, *r*) ∈ *space M* × *A* × *space* (*tree-sigma M*)}
by (*auto simp*: *space-tree-sigma elim*: *trees.cases*)
show *left* −‘ *A* ∩ *space* (*tree-sigma M*) ∈ *sets* (*tree-sigma M*)
unfolding ∗ **using** *A* **by** (*intro sets.Un Node-in-tree-sigma pair-measureI*) *auto*
qed

lemma *measurable-right*[*measurable*]: *right* ∈ *tree-sigma M* →$_M$ *tree-sigma M*
proof (*rule measurableI*)
show *t* ∈ *space* (*tree-sigma M*) ⟹ *right t* ∈ *space* (*tree-sigma M*) **for** *t*
by (*cases t*) (*auto simp*: *space-tree-sigma*)
fix *A* **assume** *A*: *A* ∈ *sets* (*tree-sigma M*)
from *sets.sets-into-space*[*OF this*]
have ∗: *right* −‘ *A* ∩ *space* (*tree-sigma M*) =
(*if Leaf* ∈ *A then* {*Leaf*} *else* {}) ∪
{*Node l v a* | *l v a*. (*v*, *l*, *a*) ∈ *space M* × *space* (*tree-sigma M*) × *A*}
by (*auto simp*: *space-tree-sigma elim*: *trees.cases*)
show *right* −‘ *A* ∩ *space* (*tree-sigma M*) ∈ *sets* (*tree-sigma M*)
unfolding ∗ **using** *A* **by** (*intro sets.Un Node-in-tree-sigma pair-measureI*) *auto*
qed

lemma *measurable-value′*: *value* ∈ *restrict-space* (*tree-sigma M*) (−{*Leaf*}) →$_M$ *M*
proof (*rule measurableI*)
show *t* ∈ *space* (*restrict-space* (*tree-sigma M*) (− {*Leaf*})) ⟹ *value t* ∈ *space M* **for** *t*
by (*cases t*) (*auto simp*: *space-restrict-space space-tree-sigma*)
fix *A* **assume** *A*: *A* ∈ *sets M*
from *sets.sets-into-space*[*OF this*]
have *value* −‘ *A* ∩ *space* (*restrict-space* (*tree-sigma M*) (− {*Leaf*})) =
{*Node l a r* | *l a r*. (*a*, *l*, *r*) ∈ *A* × *space* (*tree-sigma M*) × *space* (*tree-sigma M*)}
by (*auto simp*: *space-tree-sigma space-restrict-space elim*: *trees.cases*)
also have ... ∈ *sets* (*tree-sigma M*)

using *A* **by** (*intro sets.Un Node-in-tree-sigma pair-measureI*) *auto*
finally show *value* −‘ *A* ∩ *space* (*restrict-space* (*tree-sigma M*) (− {*Leaf*})) ∈
sets (*restrict-space* (*tree-sigma M*) (− {*Leaf*}))
by (*auto simp*: *sets-restrict-space-iff space-restrict-space*)
qed

lemma *measurable-value*[*measurable* (*raw*)]:
assumes $f \in X \rightarrow_M$ *tree-sigma M*
and $\bigwedge x.\ x \in$ *space X* $\Longrightarrow f\ x \neq$ *Leaf*
shows $(\lambda\omega.$ *value* $(f\ \omega)) \in X \rightarrow_M M$
proof −
from *assms* **have** $f \in X \rightarrow_M$ *restrict-space* (*tree-sigma M*) (− {*Leaf*})
by (*intro measurable-restrict-space2*) *auto*
from *this* **and** *measurable-value′* **show** *?thesis* **by** (*rule measurable-compose*)
qed

lemma *measurable-Node* [*measurable*]:
$(\lambda(l,x,r).$ *Node l x r*$) \in$ *tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M* $\rightarrow_M$ *tree-sigma M*
proof (*rule measurable-sigma-sets*)
show *sets* (*tree-sigma M*) = *sigma-sets* (*trees* (*space M*)) (*trees-cyl* ‘ *trees* (*sets M*))
by (*simp add*: *sets-tree-sigma-eq*)
show *trees-cyl* ‘ *trees* (*sets M*) ⊆ *Pow* (*trees* (*space M*))
by (*rule trees-cyl-sets-in-space*)
show $(\lambda(l,\ x,\ r).\ \langle l,\ x,\ r\rangle) \in$ *space* (*tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M*) → *trees* (*space M*)
by (*auto simp*: *space-pair-measure space-tree-sigma*)
fix *A* **assume** *t*: *A* ∈ *trees-cyl* ‘ *trees* (*sets M*)
then obtain *t* **where** *t*: *t* ∈ *trees* (*sets M*) *A* = *trees-cyl t* **by** *auto*
show $(\lambda(l,\ x,\ r).\ \langle l,\ x,\ r\rangle)$ −‘ *A* ∩
space (*tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M*)
∈ *sets* (*tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M*)
proof (*cases t*)
case *Leaf*
have $(\lambda(l,\ x,\ r).\ \langle l,\ x,\ r\rangle)$ −‘ {*Leaf* :: *′a tree*} = {} **by** *auto*
with *Leaf* **show** *?thesis* **using** *t* **by** *simp*
next
case (*Node l B r*)
hence $(\lambda(l,\ x,\ r).\ \langle l,\ x,\ r\rangle)$ −‘ *A* ∩ *space* (*tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M*) =
trees-cyl l × *B* × *trees-cyl r*
using *t* **and** *Node* **and** *trees-cyl-sub-trees*[*of* - *sets M space M*]
by (*auto simp*: *space-pair-measure space-tree-sigma*
dest: *sets.sets-into-space*[*of* - *M*])
thus *?thesis* **using** *t* **and** *Node*
by (*auto intro*!: *pair-measureI simp*: *sets-tree-sigma-eq*)
qed

qed

lemma *measurable-Node'* [*measurable* (*raw*)]:
assumes [*measurable*]: $l \in B \to_M$ *tree-sigma* A
assumes [*measurable*]: $x \in B \to_M A$
assumes [*measurable*]: $r \in B \to_M$ *tree-sigma* A
shows $(\lambda y.\ Node\ (l\ y)\ (x\ y)\ (r\ y)) \in B \to_M$ *tree-sigma* A
proof −
have $(\lambda y.\ Node\ (l\ y)\ (x\ y)\ (r\ y)) = (\lambda(a,b,c).\ Node\ a\ b\ c) \circ (\lambda y.\ (l\ y,\ x\ y,\ r\ y))$
by (*simp add*: *o-def*)
also have $\ldots \in B \to_M$ *tree-sigma* A
by (*intro measurable-comp*[*OF - measurable-Node*]) *simp-all*
finally show *?thesis* **.**
qed

lemma *measurable-rec-tree*[*measurable* (*raw*)]:
assumes *t*: $t \in B \to_M$ *tree-sigma* M
assumes *l*: $l \in B \to_M A$
assumes *n*: $(\lambda(x,\ l,\ v,\ r,\ al,\ ar).\ n\ x\ l\ v\ r\ al\ ar) \in$
$(B \bigotimes_M$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \bigotimes_M A \bigotimes_M A) \to_M A$ (**is** *?N* $\in$ *?M* $\to_M A$)
shows $(\lambda x.\ rec\text{-}tree\ (l\ x)\ (n\ x)\ (t\ x)) \in B \to_M A$
proof (*rule measurable-piecewise-restrict*)
let *?C* $= \lambda t.\ \lambda s$::*unit tree*. t −‘ *trees-cyl* (*map-tree* (λ-. *space* M) s)
show *countable* (*range* (*?C t*)) **by** (*intro countable-image countableI-type*)
show *space* $B \subseteq (\bigcup s.$ *?C t s*)
proof (*safe*; *clarsimp*)
fix x **assume** *x*: $x \in$ *space* B **have** $t\ x \in$ *trees* (*space* M)
using *t*[*THEN measurable-space*, *OF x*] **by** (*simp add*: *space-tree-sigma*)
then show $\exists$ *xa*::*unit tree*. $t\ x \in$ *trees-cyl* (*map-tree* (λ-. *space* M) *xa*)
by (*intro exI*[*of* - *map-tree* (λ-. ()) (*t x*)])
(*simp add*: *tree.map-comp comp-def trees-cyl-map-treeI*)
qed
fix Ω **assume** Ω ∈ *range* (*?C t*)
then obtain *s* :: *unit tree* **where** Ω: Ω = *?C t s* **by** *auto*
then show Ω ∩ *space* $B \in$ *sets* B
by (*safe intro*!: *measurable-sets*[*OF t*] *trees-cyl-map-in-sets*)
show $(\lambda x.\ rec\text{-}tree\ (l\ x)\ (n\ x)\ (t\ x)) \in$ *restrict-space* $B\ \Omega \to_M A$
unfolding Ω **using** *t*
proof (*induction s arbitrary*: *t*)
case *Leaf*
show *?case*
proof (*rule measurable-cong*[*THEN iffD2*])
fix ω **assume** ω ∈ *space* (*restrict-space* B (*?C t Leaf*))
then show *rec-tree* (*l* ω) (*n* ω) (*t* ω) = *l* ω
by (*auto simp*: *space-restrict-space*)
next
show $l \in$ *restrict-space* B (*?C t Leaf*) $\to_M A$
using *l* **by** (*rule measurable-restrict-space1*)

```
    qed
  next
    case (Node ls u rs)
    let ?F = λω. ?N (ω, left (t ω), value (t ω), right (t ω),
        rec-tree (l ω) (n ω) (left (t ω)), rec-tree (l ω) (n ω) (right (t ω)))
    show ?case
    proof (rule measurable-cong[THEN iffD2])
      fix ω assume ω ∈ space (restrict-space B (?C t (Node ls u rs)))
      then show rec-tree (l ω) (n ω) (t ω) = ?F ω
        by (auto simp: space-restrict-space)
    next
      show ?F ∈ (restrict-space B (?C t (Node ls u rs))) →M A
        apply (intro measurable-compose[OF - n] measurable-Pair[rotated])
        subgoal
          apply (rule measurable-restrict-mono[OF Node(2)])
          apply (rule measurable-compose[OF Node(3) measurable-right])
          by auto
        subgoal
          apply (rule measurable-restrict-mono[OF Node(1)])
          apply (rule measurable-compose[OF Node(3) measurable-left])
          by auto
        subgoal
          by (rule measurable-restrict-space1)
             (rule measurable-compose[OF Node(3) measurable-right])
        subgoal
          apply (rule measurable-compose[OF - measurable-value'])
          apply (rule measurable-restrict-space3[OF Node(3)])
          by auto
        subgoal
          by (rule measurable-restrict-space1)
             (rule measurable-compose[OF Node(3) measurable-left])
        by (rule measurable-restrict-space1) auto
    qed
  qed
qed

lemma measurable-case-tree [measurable (raw)]:
  assumes t ∈ B →M tree-sigma M
  assumes l ∈ B →M A
  assumes (λ(x, l, v, r). n x l v r)
          ∈ B ⨂M tree-sigma M ⨂M M ⨂M tree-sigma M →M A
  shows   (λx. case-tree (l x) (n x) (t x)) ∈ B →M (A :: 'a measure)
proof −
  define n' where n' = (λx l v r (-::'a) (-::'a). n x l v r)
  have (λx. case-tree (l x) (n x) (t x)) = (λx. rec-tree (l x) (n' x) (t x))
    (is - = (λx. rec-tree - (?n' x) -)) by (rule ext) (auto split: tree.splits simp:
n'-def)
  also have ... ∈ B →M A
  proof (rule measurable-rec-tree)
```

have ($\lambda(x,\ l,\ v,\ r,\ al,\ ar).\ n'\ x\ l\ v\ r\ al\ ar$) =
($\lambda(x,l,v,r).\ n\ x\ l\ v\ r$) $\circ$ ($\lambda(x,l,v,r,al,ar).\ (x,l,v,r)$)
by (*simp add*: *n'-def o-def case-prod-unfold*)
also have ... $\in B \bigotimes_M$ *tree-sigma M* $\bigotimes_M$ *M* $\bigotimes_M$ *tree-sigma M* $\bigotimes_M$ *A* $\bigotimes_M$ *A* $\to_M$ *A*
using *assms*(*3*) **by** *measurable*
finally show ($\lambda(x,\ l,\ v,\ r,\ al,\ ar).\ n'\ x\ l\ v\ r\ al\ ar$) $\in$ ... .
qed (*insert assms*, *simp-all*)
finally show *?thesis* .
qed

hide-const (**open**) *left*
hide-const (**open**) *right*

end

28 Conditional Expectation

theory *Conditional-Expectation*
imports *Probability-Measure*
begin

28.1 Restricting a measure to a sub-sigma-algebra

definition *subalgebra*::*'a measure* ⇒ *'a measure* ⇒ *bool* **where**
subalgebra M F = ((*space F* = *space M*) ∧ (*sets F* ⊆ *sets M*))

lemma *sub-measure-space*:
assumes *i*: *subalgebra M F*
shows *measure-space* (*space M*) (*sets F*) (*emeasure M*)
proof −
have *sigma-algebra* (*space M*) (*sets F*)
by (*metis i measure-space measure-space-def subalgebra-def*)
moreover have *positive* (*sets F*) (*emeasure M*)
using *Sigma-Algebra.positive-def* **by** *auto*
moreover have *countably-additive* (*sets F*) (*emeasure M*)
by (*meson countably-additive-def emeasure-countably-additive i order-trans subalgebra-def subsetCE*)
ultimately show *?thesis* **unfolding** *measure-space-def* **by** *simp*
qed

definition *restr-to-subalg*::*'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
restr-to-subalg M F = *measure-of* (*space M*) (*sets F*) (*emeasure M*)

lemma *space-restr-to-subalg*:
space (*restr-to-subalg M F*) = *space M*
unfolding *restr-to-subalg-def* **by** (*simp add*: *space-measure-of-conv*)

lemma *sets-restr-to-subalg* [*measurable-cong*]:

assumes *subalgebra M F*
shows *sets* (*restr-to-subalg M F*) = *sets F*
unfolding *restr-to-subalg-def* **by** (*metis sets.sets-measure-of-eq assms subalgebra-def*)

lemma *emeasure-restr-to-subalg*:
assumes *subalgebra M F*
A ∈ *sets F*
shows *emeasure* (*restr-to-subalg M F*) *A* = *emeasure M A*
unfolding *restr-to-subalg-def*
by (*metis assms subalgebra-def emeasure-measure-of-conv sub-measure-space sets.sigma-sets-eq*)

lemma *null-sets-restr-to-subalg*:
assumes *subalgebra M F*
shows *null-sets* (*restr-to-subalg M F*) = *null-sets M* ∩ *sets F*
proof
have *x* ∈ *null-sets M* ∩ *sets F* **if** *x* ∈ *null-sets* (*restr-to-subalg M F*) **for** *x*
by (*metis that Int-iff assms emeasure-restr-to-subalg null-setsD1 null-setsD2 null-setsI*
sets-restr-to-subalg subalgebra-def subsetD)
then show *null-sets* (*restr-to-subalg M F*) ⊆ *null-sets M* ∩ *sets F* **by** *auto*
next
have *x* ∈ *null-sets* (*restr-to-subalg M F*) **if** *x* ∈ *null-sets M* ∩ *sets F* **for** *x*
by (*metis that Int-iff assms null-setsD1 null-setsI sets-restr-to-subalg emeasure-restr-to-subalg*[*OF assms*])
then show *null-sets M* ∩ *sets F* ⊆ *null-sets* (*restr-to-subalg M F*) **by** *auto*
qed

lemma *AE-restr-to-subalg*:
assumes *subalgebra M F*
AE x in (*restr-to-subalg M F*). *P x*
shows *AE x in M*. *P x*
proof −
obtain *A* **where** *A*: ⋀*x*. *x* ∈ *space* (*restr-to-subalg M F*) − *A* ⟹ *P x A* ∈ *null-sets* (*restr-to-subalg M F*)
using *AE-E3*[*OF assms*(*2*)] **by** *auto*
then have *A* ∈ *null-sets M* **using** *null-sets-restr-to-subalg*[*OF assms*(*1*)] **by** *auto*
moreover have ⋀*x*. *x* ∈ *space M* − *A* ⟹ *P x*
using *space-restr-to-subalg A*(*1*) **by** *fastforce*
ultimately show *?thesis*
unfolding *eventually-ae-filter* **by** *auto*
qed

lemma *AE-restr-to-subalg2*:
assumes *subalgebra M F*
AE x in M. *P x* **and** [*measurable*]: *P* ∈ *measurable F* (*count-space UNIV*)
shows *AE x in* (*restr-to-subalg M F*). *P x*
proof −
define *U* **where** *U* = {*x* ∈ *space M*. ¬(*P x*)}
then have ∗: *U* = {*x* ∈ *space F*. ¬(*P x*)} **using** *assms*(*1*) **by** (*simp add*:

subalgebra-def)
then have $U \in$ *sets F* **by** *simp*
then have $U \in$ *sets M* **using** *assms*(*1*) **by** (*meson subalgebra-def subsetD*)
then have $U \in$ *null-sets M* **unfolding** *U-def* **using** *assms*(*2*) **using** *AE-iff-measurable*
by *blast*
then have $U \in$ *null-sets* (*restr-to-subalg M F*) **using** *null-sets-restr-to-subalg*[*OF assms*(*1*)] ‹$U \in$ *sets F*› **by** *auto*
then show *?thesis* **using** $*$ **by** (*metis* (*no-types*, *lifting*) *Collect-mono U-def eventually-ae-filter space-restr-to-subalg*)
qed

lemma *prob-space-restr-to-subalg*:
assumes *subalgebra M F*
prob-space M
shows *prob-space* (*restr-to-subalg M F*)
by (*metis* (*no-types*, *lifting*) *assms*(*1*) *assms*(*2*) *emeasure-restr-to-subalg prob-space.emeasure-space-1 prob-spaceI*
sets.top space-restr-to-subalg subalgebra-def)

lemma *finite-measure-restr-to-subalg*:
assumes *subalgebra M F*
finite-measure M
shows *finite-measure* (*restr-to-subalg M F*)
by (*metis* (*no-types*, *lifting*) *assms emeasure-restr-to-subalg finite-measure.finite-emeasure-space finite-measureI sets.top space-restr-to-subalg subalgebra-def infinity-ennreal-def*)

lemma *measurable-in-subalg*:
assumes *subalgebra M F*
$f \in$ *measurable F N*
shows $f \in$ *measurable* (*restr-to-subalg M F*) *N*
by (*metis measurable-cong-sets assms*(*2*) *sets-restr-to-subalg*[*OF assms*(*1*)])

lemma *measurable-in-subalg′*:
assumes *subalgebra M F*
$f \in$ *measurable* (*restr-to-subalg M F*) *N*
shows $f \in$ *measurable F N*
by (*metis measurable-cong-sets assms*(*2*) *sets-restr-to-subalg*[*OF assms*(*1*)])

lemma *measurable-from-subalg*:
assumes *subalgebra M F*
$f \in$ *measurable F N*
shows $f \in$ *measurable M N*
using *assms* **unfolding** *measurable-def subalgebra-def* **by** *auto*

The following is the direct transposition of `nn_integral_subalgebra` (from `Nonnegative_Lebesgue_Integration`) in the current notations, with the removal of the useless assumption $f \geq 0$.

lemma *nn-integral-subalgebra2*:
assumes *subalgebra M F* **and** [*measurable*]: $f \in$ *borel-measurable F*

shows $(\int^+ x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int^+ x.\ f\ x\ \partial M)$
proof (*rule nn-integral-subalgebra*)
show $f \in$ *borel-measurable* (*restr-to-subalg M F*)
by (*rule measurable-in-subalg*[*OF assms*(*1*)]) *simp*
show *sets* (*restr-to-subalg M F*) $\subseteq$ *sets M* **by** (*metis sets-restr-to-subalg*[*OF assms*(*1*)] *assms*(*1*) *subalgebra-def*)
fix *A* **assume** $A \in$ *sets* (*restr-to-subalg M F*)
then show *emeasure* (*restr-to-subalg M F*) *A* = *emeasure M A*
by (*metis sets-restr-to-subalg*[*OF assms*(*1*)] *emeasure-restr-to-subalg*[*OF assms*(*1*)])
qed (*auto simp add*: *assms space-restr-to-subalg sets-restr-to-subalg*[*OF assms*(*1*)])

The following is the direct transposition of `integral_subalgebra` (from `Bochner_Integration`) in the current notations.

lemma *integral-subalgebra2*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F* **and**
[*measurable*]: $f \in$ *borel-measurable F*
shows $(\int x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int x.\ f\ x\ \partial M)$
by (*rule integral-subalgebra*,
metis measurable-in-subalg[*OF assms*(*1*)] *assms*(*2*),
auto simp add: *assms space-restr-to-subalg sets-restr-to-subalg emeasure-restr-to-subalg*,
meson assms(*1*) *subalgebra-def subset-eq*)

lemma *integrable-from-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F*
integrable (*restr-to-subalg M F*) *f*
shows *integrable M f*
proof (*rule integrableI-bounded*)
have [*measurable*]: $f \in$ *borel-measurable F* **using** *assms* **by** *auto*
then show $f \in$ *borel-measurable M* **using** *assms*(*1*) *measurable-from-subalg* **by** *blast*

have $(\int^+ x.\ \textit{ennreal}\ (\textit{norm}\ (f\,x))\ \partial M) = (\int^+ x.\ \textit{ennreal}\ (\textit{norm}\ (f\,x))\ \partial(\textit{restr-to-subalg}\ M\ F))$
by (*rule nn-integral-subalgebra2*[*symmetric*], *auto simp add*: *assms*)
also have ... $< \infty$ **using** *integrable-iff-bounded assms* **by** *auto*
finally show $(\int^+ x.\ \textit{ennreal}\ (\textit{norm}\ (f\,x))\ \partial M) < \infty$ **by** *simp*
qed

lemma *integrable-in-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: *subalgebra M F*
$f \in$ *borel-measurable F*
integrable M f
shows *integrable* (*restr-to-subalg M F*) *f*
proof (*rule integrableI-bounded*)
show $f \in$ *borel-measurable* (*restr-to-subalg M F*) **using** *assms*(*2*) *assms*(*1*) **by** *auto*

have $(\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial(restr\text{-}to\text{-}subalg\ M\ F)) = (\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial M)$
by (*rule nn-integral-subalgebra2*, *auto simp add*: *assms*)
also have $... < \infty$ **using** *integrable-iff-bounded assms* **by** *auto*
finally show $(\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial(restr\text{-}to\text{-}subalg\ M\ F)) < \infty$ **by** *simp*
qed

28.2 Nonnegative conditional expectation

The conditional expectation of a function f, on a measure space M, with respect to a sub sigma algebra F, should be a function g which is F-measurable whose integral on any F-set coincides with the integral of f. Such a function is uniquely defined almost everywhere. The most direct construction is to use the measure fdM, restrict it to the sigma-algebra F, and apply the Radon-Nikodym theorem to write it as $gdM_{|F}$ for some F-measurable function g. Another classical construction for L^2 functions is done by orthogonal projection on F-measurable functions, and then extending by density to L^1. The Radon-Nikodym point of view avoids the L^2 machinery, and works for all positive functions.

In this paragraph, we develop the definition and basic properties for nonnegative functions, as the basics of the general case. As in the definition of integrals, the nonnegative case is done with ennreal-valued functions, without any integrability assumption.

definition *nn-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *ennreal*) ⇒ (*'a* ⇒ *ennreal*)
where
nn-cond-exp M F f =
(*if f* ∈ *borel-measurable M* ∧ *subalgebra M F*
then RN-deriv (*restr-to-subalg M F*) (*restr-to-subalg* (*density M f*) *F*)
else (λ-. *0*))

lemma
shows *borel-measurable-nn-cond-exp* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-nn-cond-exp2* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable M*
by (*simp-all add*: *nn-cond-exp-def*)
(*metis borel-measurable-RN-deriv borel-measurable-subalgebra sets-restr-to-subalg space-restr-to-subalg subalgebra-def*)

The good setting for conditional expectations is the situation where the subalgebra F gives rise to a sigma-finite measure space. To see what goes wrong if it is not sigma-finite, think of $\mathbb{R}$ with the trivial sigma-algebra $\{\emptyset, \mathbb{R}\}$. In this case, conditional expectations have to be constant functions, so they have integral 0 or ∞. This means that a positive integrable function can have no meaningful conditional expectation.

locale *sigma-finite-subalgebra* =
fixes *M F*::*'a measure*
assumes *subalg*: *subalgebra M F*
and *sigma-fin-subalg*: *sigma-finite-measure* (*restr-to-subalg M F*)

lemma *sigma-finite-subalgebra-is-sigma-finite*:
assumes *sigma-finite-subalgebra M F*
shows *sigma-finite-measure M*
proof
have *subalg*: *subalgebra M F*
and *sigma-fin-subalg*: *sigma-finite-measure* (*restr-to-subalg M F*)
using *assms* **unfolding** *sigma-finite-subalgebra-def* **by** *auto*
obtain *A* **where** *Ap*: *countable A* ∧ *A* ⊆ *sets* (*restr-to-subalg M F*) ∧ ⋃ *A* = *space* (*restr-to-subalg M F*) ∧ (∀ *a*∈*A*. *emeasure* (*restr-to-subalg M F*) *a* ≠ ∞)
using *sigma-finite-measure.sigma-finite-countable*[*OF sigma-fin-subalg*] **by** *fastforce*
have *A* ⊆ *sets F* **using** *Ap sets-restr-to-subalg*[*OF subalg*] **by** *fastforce*
then have *A* ⊆ *sets M* **using** *subalg subalgebra-def* **by** *force*
moreover have ⋃ *A* = *space M* **using** *Ap space-restr-to-subalg* **by** *simp*
moreover have ∀ *a*∈*A*. *emeasure M a* ≠ ∞ **by** (*metis subsetD emeasure-restr-to-subalg*[*OF subalg*] ‹*A* ⊆ *sets F*› *Ap*)
ultimately show ∃ *A*. *countable A* ∧ *A* ⊆ *sets M* ∧ ⋃ *A* = *space M* ∧ (∀ *a*∈*A*. *emeasure M a* ≠ ∞) **using** *Ap* **by** *auto*
qed

sublocale *sigma-finite-subalgebra* ⊆ *sigma-finite-measure*
using *sigma-finite-subalgebra-is-sigma-finite sigma-finite-subalgebra-axioms* **by** *blast*

Conditional expectations are very often used in probability spaces. This is a special case of the previous one, as we prove now.

locale *finite-measure-subalgebra* = *finite-measure* +
fixes *F*::*'a measure*
assumes *subalg*: *subalgebra M F*

lemma *finite-measure-subalgebra-is-sigma-finite*:
assumes *finite-measure-subalgebra M F*
shows *sigma-finite-subalgebra M F*
proof −
interpret *finite-measure-subalgebra M F* **using** *assms* **by** *simp*
have *finite-measure* (*restr-to-subalg M F*)
using *finite-measure-restr-to-subalg subalg finite-emeasure-space finite-measureI*
unfolding *infinity-ennreal-def* **by** *blast*
then have *sigma-finite-measure* (*restr-to-subalg M F*)
unfolding *finite-measure-def* **by** *simp*
then show *sigma-finite-subalgebra M F* **unfolding** *sigma-finite-subalgebra-def*
using *subalg* **by** *simp*
qed

sublocale *finite-measure-subalgebra* ⊆ *sigma-finite-subalgebra*

proof −
have *finite-measure* (*restr-to-subalg M F*)
using *finite-measure-restr-to-subalg subalg finite-emeasure-space finite-measureI*
unfolding *infinity-ennreal-def* **by** *blast*
then have *sigma-finite-measure* (*restr-to-subalg M F*)
unfolding *finite-measure-def* **by** *simp*
then show *sigma-finite-subalgebra M F* **unfolding** *sigma-finite-subalgebra-def*
using *subalg* **by** *simp*
qed

context *sigma-finite-subalgebra*
begin

The next lemma is arguably the most fundamental property of conditional expectation: when computing an expectation against an F-measurable function, it is equivalent to work with a function or with its F-conditional expectation.

This property (even for bounded test functions) characterizes conditional expectations, as the second lemma below shows. From this point on, we will only work with it, and forget completely about the definition using Radon-Nikodym derivatives.

lemma *nn-cond-exp-intg*:
assumes [*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
shows (∫⁺ *x*. *f x* ∗ *nn-cond-exp M F g x* ∂*M*) = (∫⁺ *x*. *f x* ∗ *g x* ∂*M*)
proof −
have [*measurable*]: *f* ∈ *borel-measurable M*
by (*meson assms subalg borel-measurable-subalgebra subalgebra-def*)
have *ac*: *absolutely-continuous* (*restr-to-subalg M F*) (*restr-to-subalg* (*density M g*) *F*)
unfolding *absolutely-continuous-def*
proof −
have *null-sets* (*restr-to-subalg M F*) = *null-sets M* ∩ *sets F* **by** (*rule null-sets-restr-to-subalg*[*OF subalg*])
moreover have *null-sets M* ⊆ *null-sets* (*density M g*)
by (*rule absolutely-continuousI-density*[*unfolded absolutely-continuous-def*]) *auto*
ultimately have *null-sets* (*restr-to-subalg M F*) ⊆ *null-sets* (*density M g*) ∩ *sets F* **by** *auto*
moreover have *null-sets* (*density M g*) ∩ *sets F* = *null-sets* (*restr-to-subalg* (*density M g*) *F*)
by (*rule null-sets-restr-to-subalg*[*symmetric*]) (*metis subalg sets-density space-density subalgebra-def*)
ultimately show *null-sets* (*restr-to-subalg M F*) ⊆ *null-sets* (*restr-to-subalg* (*density M g*) *F*) **by** *auto*
qed

have (∫⁺ *x*. *f x* ∗ *nn-cond-exp M F g x* ∂*M*) = (∫⁺ *x*. *f x* ∗ *nn-cond-exp M F g x* ∂(*restr-to-subalg M F*))

by (*rule nn-integral-subalgebra2*[*symmetric*]) (*simp-all add*: *assms subalg*)
also have ... = ($\int^+ x.\ f\ x * RN\text{-}deriv\ (restr\text{-}to\text{-}subalg\ M\ F)\ (restr\text{-}to\text{-}subalg\ (density\ M\ g)\ F)\ x\ \partial(restr\text{-}to\text{-}subalg\ M\ F)$)
unfolding *nn-cond-exp-def* **using** *assms subalg* **by** *simp*
also have ... = ($\int^+ x.\ RN\text{-}deriv\ (restr\text{-}to\text{-}subalg\ M\ F)\ (restr\text{-}to\text{-}subalg\ (density\ M\ g)\ F)\ x * f\ x\ \partial(restr\text{-}to\text{-}subalg\ M\ F)$)
by (*simp add*: *mult.commute*)
also have ... = ($\int^+ x.\ f\ x\ \partial(restr\text{-}to\text{-}subalg\ (density\ M\ g)\ F)$)
proof (*rule sigma-finite-measure.RN-deriv-nn-integral*[*symmetric*])
show *sets* (*restr-to-subalg* (*density M g*) *F*) = *sets* (*restr-to-subalg M F*)
by (*metis subalg restr-to-subalg-def sets.sets-measure-of-eq space-density subalgebra-def*)
qed (*auto simp add*: *assms measurable-restrict ac measurable-in-subalg subalg sigma-fin-subalg*)
also have ... = ($\int^+ x.\ f\ x\ \partial(density\ M\ g)$)
by (*metis nn-integral-subalgebra2 subalg assms*(*1*) *sets-density space-density subalgebra-def*)
also have ... = ($\int^+ x.\ g\ x * f\ x\ \partial M$)
by (*rule nn-integral-density*) (*simp-all add*: *assms*)
also have ... = ($\int^+ x.\ f\ x * g\ x\ \partial M$)
by (*simp add*: *mult.commute*)
finally show *?thesis* **by** *simp*
qed

lemma *nn-cond-exp-charact*:
assumes $\bigwedge A.\ A \in sets\ F \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$ **and**
[*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ F$
shows $AE\ x\ in\ M.\ g\ x = nn\text{-}cond\text{-}exp\ M\ F\ f\ x$
proof −
let *?MF* = *restr-to-subalg M F*
{
fix *A* **assume** $A \in sets\ ?MF$
then have [*measurable*]: $A \in sets\ F$ **using** *sets-restr-to-subalg*[*OF subalg*] **by** *simp*
have ($\int^+ x \in A.\ g\ x\ \partial\ ?MF$) = ($\int^+ x \in A.\ g\ x\ \partial M$)
by (*simp add*: *nn-integral-subalgebra2 subalg*)
also have ... = ($\int^+ x \in A.\ f\ x\ \partial M$) **using** *assms*(*1*) **by** *simp*
also have ... = ($\int^+ x.\ indicator\ A\ x * f\ x\ \partial M$) **by** (*simp add*: *mult.commute*)
also have ... = ($\int^+ x.\ indicator\ A\ x * nn\text{-}cond\text{-}exp\ M\ F\ f\ x\ \partial M$)
by (*rule nn-cond-exp-intg*[*symmetric*]) (*auto simp add*: *assms*)
also have ... = ($\int^+ x \in A.\ nn\text{-}cond\text{-}exp\ M\ F\ f\ x\ \partial M$) **by** (*simp add*: *mult.commute*)
also have ... = ($\int^+ x \in A.\ nn\text{-}cond\text{-}exp\ M\ F\ f\ x\ \partial\ ?MF$)
by (*simp add*: *nn-integral-subalgebra2 subalg*)
finally have ($\int^+ x \in A.\ g\ x\ \partial\ ?MF$) = ($\int^+ x \in A.\ nn\text{-}cond\text{-}exp\ M\ F\ f\ x\ \partial\ ?MF$) **by** *simp*
} **note** $* = this$
have $AE\ x\ in\ ?MF.\ g\ x = nn\text{-}cond\text{-}exp\ M\ F\ f\ x$
by (*rule sigma-finite-measure.density-unique2*)

(*auto simp add*: *assms subalg sigma-fin-subalg AE-restr-to-subalg2* ∗)
then show *?thesis* **using** *AE-restr-to-subalg*[*OF subalg*] **by** *simp*
qed

lemma *nn-cond-exp-F-meas*:
assumes $f \in$ *borel-measurable F*
shows *AE x in M*. $f\ x =$ *nn-cond-exp M F f x*
by (*rule nn-cond-exp-charact*) (*auto simp add*: *assms measurable-from-subalg*[*OF subalg*])

lemma *nn-cond-exp-prod*:
assumes [*measurable*]: $f \in$ *borel-measurable F* $g \in$ *borel-measurable M*
shows *AE x in M*. $f\ x *$ *nn-cond-exp M F g x* = *nn-cond-exp M F* ($\lambda x.\ f\ x * g\ x$) *x*
proof (*rule nn-cond-exp-charact*)
have [*measurable*]: $f \in$ *borel-measurable M* **by** (*rule measurable-from-subalg*[*OF subalg assms*(*1*)])
show ($\lambda x.\ f\ x * g\ x$) $\in$ *borel-measurable M* **by** *measurable*

fix *A* **assume** *A* $\in$ *sets F*
then have [*measurable*]: ($\lambda x.\ f\ x *$ *indicator A x*) $\in$ *borel-measurable F* **by** *measurable*
have ($\int^+ x{\in}A.\ (f\ x * g\ x)\ \partial M$) = $\int^+ x.\ (f\ x *$ *indicator A x*) $* g\ x\ \partial M$
by (*simp add*: *mult.commute mult.left-commute*)
also have ... = $\int^+ x.\ (f\ x *$ *indicator A x*) $*$ *nn-cond-exp M F g x* ∂M
by (*rule nn-cond-exp-intg*[*symmetric*]) (*auto simp add*: *assms*)
also have ... = ($\int^+ x{\in}A.\ (f\ x *$ *nn-cond-exp M F g x*)∂M)
by (*simp add*: *mult.commute mult.left-commute*)
finally show ($\int^+ x{\in}A.\ (f\ x * g\ x)\ \partial M$) = ($\int^+ x{\in}A.\ (f\ x *$ *nn-cond-exp M F g x*)∂M) **by** *simp*
qed (*auto simp add*: *assms*)

lemma *nn-cond-exp-sum*:
assumes [*measurable*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
shows *AE x in M*. *nn-cond-exp M F f x* + *nn-cond-exp M F g x* = *nn-cond-exp M F* ($\lambda x.\ f\ x + g\ x$) *x*
proof (*rule nn-cond-exp-charact*)
fix *A* **assume** [*measurable*]: *A* $\in$ *sets F*
then have *A* $\in$ *sets M* **by** (*meson subalg subalgebra-def subsetD*)
have ($\int^+ x{\in}A.$ (*nn-cond-exp M F f x* + *nn-cond-exp M F g x*)∂M) = ($\int^+ x{\in}A.$ *nn-cond-exp M F f x* ∂M) + ($\int^+ x{\in}A.$ *nn-cond-exp M F g x* ∂M)
by (*rule nn-set-integral-add*) (*auto simp add*: *assms* ‹*A* $\in$ *sets M*›)
also have ... = ($\int^+ x.$ *indicator A x* $*$ *nn-cond-exp M F f x* ∂M) + ($\int^+ x.$ *indicator A x* $*$ *nn-cond-exp M F g x* ∂M)
by (*metis* (*no-types*, *lifting*) *mult.commute nn-integral-cong*)
also have ... = ($\int^+ x.$ *indicator A x* $* f\ x\ \partial M$) + ($\int^+ x.$ *indicator A x* $* g\ x\ \partial M$)
by (*simp add*: *nn-cond-exp-intg*)
also have ... = ($\int^+ x{\in}A.\ f\ x\ \partial M$) + ($\int^+ x{\in}A.\ g\ x\ \partial M$)
by (*metis* (*no-types*, *lifting*) *mult.commute nn-integral-cong*)

also have ... = $(\int^+ x{\in}A.\ (f\ x + g\ x)\partial M)$
by (*rule nn-set-integral-add*[*symmetric*]) (*auto simp add*: *assms* ‹$A \in$ *sets* M›)
finally show $(\int^+ x{\in}A.\ (f\ x + g\ x)\partial M) = (\int^+ x{\in}A.$ (*nn-cond-exp M F f x + nn-cond-exp M F g x*)$\partial M)$
by *simp*
qed (*auto simp add*: *assms*)

lemma *nn-cond-exp-cong*:
assumes *AE x in M*. $f\ x = g\ x$
and [*measurable*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
shows *AE x in M*. *nn-cond-exp M F f x* = *nn-cond-exp M F g x*
proof (*rule nn-cond-exp-charact*)
fix A **assume** [*measurable*]: $A \in$ *sets F*
have $(\int^+ x{\in}A.$ *nn-cond-exp M F f x* $\partial M) = \int^+ x.$ *indicator A x* $*$ *nn-cond-exp M F f x* ∂M
by (*simp add*: *mult.commute*)
also have ... = $\int^+ x.$ *indicator A x* $*$ *f x* ∂M **by** (*simp add*: *nn-cond-exp-intg assms*)
also have ... = $(\int^+ x{\in}A.$ *f x* $\partial M)$ **by** (*simp add*: *mult.commute*)
also have ... = $(\int^+ x{\in}A.$ *g x* $\partial M)$ **by** (*rule nn-set-integral-cong*[*OF assms*(*1*)])
finally show $(\int^+ x{\in}A.$ *g x* $\partial M) = (\int^+ x{\in}A.$ *nn-cond-exp M F f x* $\partial M)$ **by** *simp*
qed (*auto simp add*: *assms*)

lemma *nn-cond-exp-mono*:
assumes *AE x in M*. $f\ x \le g\ x$
and [*measurable*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
shows *AE x in M*. *nn-cond-exp M F f x* $\le$ *nn-cond-exp M F g x*
proof −
define h **where** $h = (\lambda x.\ g\ x - f\ x)$
have [*measurable*]: $h \in$ *borel-measurable M* **unfolding** *h-def* **by** *simp*
have $*$: *AE x in M*. $g\ x = f\ x + h\ x$ **unfolding** *h-def* **using** *assms*(*1*) **by** (*auto simp*: *ennreal-ineq-diff-add*)
have *AE x in M*. *nn-cond-exp M F g x* = *nn-cond-exp M F* $(\lambda x.\ f\ x + h\ x)\ x$
by (*rule nn-cond-exp-cong*) (*auto simp add*: $*$ *assms*)
moreover have *AE x in M*. *nn-cond-exp M F f x* + *nn-cond-exp M F h x* = *nn-cond-exp M F* $(\lambda x.\ f\ x + h\ x)\ x$
by (*rule nn-cond-exp-sum*) (*auto simp add*: *assms*)
ultimately have *AE x in M*. *nn-cond-exp M F g x* = *nn-cond-exp M F f x* + *nn-cond-exp M F h x* **by** *auto*
then show *?thesis* **by** *force*
qed

lemma *nested-subalg-is-sigma-finite*:
assumes *subalgebra M G subalgebra G F*
shows *sigma-finite-subalgebra M G*
unfolding *sigma-finite-subalgebra-def*
proof (*auto simp add*: *assms*)
have $\exists A.$ *countable* $A \wedge A \subseteq$ *sets* (*restr-to-subalg M F*) $\wedge \bigcup A =$ *space* (*restr-to-subalg M F*) $\wedge$ $(\forall a{\in}A.$ *emeasure* (*restr-to-subalg M F*) $a \neq \infty)$

using *sigma-fin-subalg sigma-finite-measure-def* **by** *auto*
then obtain *A* **where** *A*:*countable A* ∧ *A* ⊆ *sets* (*restr-to-subalg M F*) ∧ ⋃ *A* = *space* (*restr-to-subalg M F*) ∧ (∀ *a*∈*A*. *emeasure* (*restr-to-subalg M F*) *a* ≠ ∞)
by *auto*
have *sets F* ⊆ *sets M*
by (*meson assms order-trans subalgebra-def*)
then have *countable A* ∧ *A* ⊆ *sets* (*restr-to-subalg M G*) ∧ ⋃ *A* = *space* (*restr-to-subalg M F*) ∧ (∀ *a*∈*A*. *emeasure* (*restr-to-subalg M G*) *a* ≠ ∞)
by (*metis* (*no-types*) *A assms basic-trans-rules*(*31*) *emeasure-restr-to-subalg order-trans sets-restr-to-subalg subalgebra-def*)
then show *sigma-finite-measure* (*restr-to-subalg M G*)
by (*metis sigma-finite-measure.intro space-restr-to-subalg*)
qed

lemma *nn-cond-exp-nested-subalg*:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *nn-cond-exp M F f x* = *nn-cond-exp M F* (*nn-cond-exp M G f*) *x*
proof (*rule nn-cond-exp-charact*, *auto*)
interpret *G*: *sigma-finite-subalgebra M G* **by** (*rule nested-subalg-is-sigma-finite*[*OF assms*(*1*) *assms*(*2*)])
fix *A* **assume** [*measurable*]: *A* ∈ *sets F*
then have [*measurable*]: *A* ∈ *sets G* **using** *assms*(*2*) **by** (*meson subsetD subalgebra-def*)

have *set-nn-integral M A* (*nn-cond-exp M G f*) = (∫⁺ *x*. *indicator A x* ∗ *nn-cond-exp M G f x*∂*M*)
by (*metis* (*no-types*) *mult.commute*)
also have ... = (∫⁺ *x*. *indicator A x* ∗ *f x* ∂*M*) **by** (*rule G.nn-cond-exp-intg*, *auto simp add*: *assms*)
also have ... = (∫⁺ *x*. *indicator A x* ∗ *nn-cond-exp M F f x* ∂*M*) **by** (*rule nn-cond-exp-intg*[*symmetric*], *auto simp add*: *assms*)
also have ... = *set-nn-integral M A* (*nn-cond-exp M F f*) **by** (*metis* (*no-types*) *mult.commute*)
finally show *set-nn-integral M A* (*nn-cond-exp M G f*) = *set-nn-integral M A* (*nn-cond-exp M F f*)**.**
qed

end

28.3 Real conditional expectation

Once conditional expectations of positive functions are defined, the definition for real-valued functions follows readily, by taking the difference of positive and negative parts. One could also define a conditional expectation of vector-space valued functions, as in `Bochner_Integral`, but since the real-valued case is the most important, and quicker to formalize, I con-

centrate on it. (It is also essential for the case of the most general Pettis integral.)

definition *real-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *real*) ⇒ (*'a* ⇒ *real*)
where
real-cond-exp M F f =
(λ*x*. *enn2real*(*nn-cond-exp M F* (λ*x*. *ennreal* (*f x*)) *x*) − *enn2real*(*nn-cond-exp M F* (λ*x*. *ennreal* (−*f x*)) *x*))

lemma
shows *borel-measurable-cond-exp* [*measurable*]: *real-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-cond-exp2* [*measurable*]: *real-cond-exp M F f* ∈ *borel-measurable M*
unfolding *real-cond-exp-def* **by** *auto*

context *sigma-finite-subalgebra*
begin

lemma *real-cond-exp-abs*:
assumes [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *abs*(*real-cond-exp M F f x*) ≤ *nn-cond-exp M F* (λ*x*. *ennreal* (*abs*(*f x*))) *x*
proof −
define *fp* **where** *fp* = (λ*x*. *ennreal* (*f x*))
define *fm* **where** *fm* = (λ*x*. *ennreal* (− *f x*))
have [*measurable*]: *fp* ∈ *borel-measurable M fm* ∈ *borel-measurable M* **unfolding** *fp-def fm-def* **by** *auto*
have *eq*: ⋀*x*. *ennreal* |*f x*| = *fp x* + *fm x* **unfolding** *fp-def fm-def* **by** (*simp add*: *abs-real-def ennreal-neg*)

{
fix *x* **assume** *H*: *nn-cond-exp M F fp x* + *nn-cond-exp M F fm x* = *nn-cond-exp M F* (λ*x*. *fp x* + *fm x*) *x*
have |*real-cond-exp M F f x*| ≤ |*enn2real*(*nn-cond-exp M F fp x*)| + |*enn2real*(*nn-cond-exp M F fm x*)|
unfolding *real-cond-exp-def fp-def fm-def* **by** (*auto intro*: *abs-triangle-ineq4 simp del*: *enn2real-nonneg*)
from *ennreal-leI*[*OF this*]
have *abs*(*real-cond-exp M F f x*) ≤ *nn-cond-exp M F fp x* + *nn-cond-exp M F fm x*
by *simp* (*metis add.commute ennreal-enn2real le-iff-add not-le top-unique*)
also have ... = *nn-cond-exp M F* (λ*x*. *fp x* + *fm x*) *x* **using** *H* **by** *simp*
finally have *abs*(*real-cond-exp M F f x*) ≤ *nn-cond-exp M F* (λ*x*. *fp x* + *fm x*) *x* **by** *simp*
}
moreover have *AE x in M*. *nn-cond-exp M F fp x* + *nn-cond-exp M F fm x* = *nn-cond-exp M F* (λ*x*. *fp x* + *fm x*) *x*
by (*rule nn-cond-exp-sum*) (*auto simp add*: *fp-def fm-def*)
ultimately have *AE x in M*. *abs*(*real-cond-exp M F f x*) ≤ *nn-cond-exp M F*

$(\lambda x.\ fp\ x + fm\ x)\ x$
by *auto*
then show *?thesis* **using** *eq* **by** *simp*
qed

The next lemma shows that the conditional expectation is an F-measurable function whose average against an F-measurable function f coincides with the average of the original function against f. It is obtained as a consequence of the same property for the positive conditional expectation, taking the difference of the positive and the negative part. The proof is given first assuming $f \geq 0$ for simplicity, and then extended to the general case in the subsequent lemma. The idea of the proof is essentially trivial, but the implementation is slightly tedious as one should check all the integrability properties of the different parts, and go back and forth between positive integral and signed integrals, and between real-valued functions and ennreal-valued functions.

Once this lemma is available, we will use it to characterize the conditional expectation, and never come back to the original technical definition, as we did in the case of the nonnegative conditional expectation.

lemma *real-cond-exp-intg-fpos*:
assumes *integrable M* $(\lambda x.\ f\ x * g\ x)$ **and** *f-pos*[*simp*]: $\bigwedge x.\ f\ x \geq 0$ **and**
[*measurable*]: $f \in$ *borel-measurable F* $g \in$ *borel-measurable M*
shows *integrable M* $(\lambda x.\ f\ x *$ *real-cond-exp M F g* $x)$
$(\int\ x.\ f\ x *$ *real-cond-exp M F g x* $\partial M) = (\int\ x.\ f\ x * g\ x\ \partial M)$
proof −
have [*measurable*]: $f \in$ *borel-measurable M* **by** (*rule measurable-from-subalg*[*OF subalg assms*(*3*)])
define *gp* **where** $gp = (\lambda x.\ ennreal\ (g\ x))$
define *gm* **where** $gm = (\lambda x.\ ennreal\ (-\ g\ x))$
have [*measurable*]: *gp* $\in$ *borel-measurable M gm* $\in$ *borel-measurable M* **unfolding** *gp-def gm-def* **by** *auto*
define *h* **where** $h = (\lambda x.\ ennreal(abs(g\ x)))$
have *hgpgm*: $\bigwedge x.\ h\ x = gp\ x + gm\ x$ **unfolding** *gp-def gm-def h-def* **by** (*simp add*: *abs-real-def ennreal-neg*)
have [*measurable*]: $h \in$ *borel-measurable M* **unfolding** *h-def* **by** *simp*
have *pos*[*simp*]: $\bigwedge x.\ h\ x \geq 0\ \bigwedge x.\ gp\ x \geq 0\ \bigwedge x.\ gm\ x \geq 0$ **unfolding** *h-def gp-def gm-def* **by** *simp-all*
have *gp-real*: $\bigwedge x.\ enn2real(gp\ x) = max\ (g\ x)\ 0$
unfolding *gp-def* **by** (*simp add*: *max-def ennreal-neg*)
have *gm-real*: $\bigwedge x.\ enn2real(gm\ x) = max\ (-g\ x)\ 0$
unfolding *gm-def* **by** (*simp add*: *max-def ennreal-neg*)

have $(\int^+ x.\ norm(f\ x * max\ (g\ x)\ 0)\ \partial M) \leq (\int^+ x.\ norm(f\ x * g\ x)\ \partial M)$
by (*simp add*: *nn-integral-mono*)
also have ... $< \infty$ **using** *assms*(*1*) **by** (*simp add*: *integrable-iff-bounded*)
finally have $(\int^+ x.\ norm(f\ x * max\ (g\ x)\ 0)\ \partial M) < \infty$ **by** *simp*
then have *int1*: *integrable M* $(\lambda x.\ f\ x * max\ (g\ x)\ 0)$ **by** (*simp add*: *inte-*

grableI-bounded)

have ($\int^+ x.\ norm(f\ x * max\ (-g\ x)\ 0)\ \partial M) \leq (\int^+ x.\ norm(f\ x * g\ x)\ \partial M)$
by (*simp add*: *nn-integral-mono*)
also have ... $< \infty$ **using** *assms*(*1*) **by** (*simp add*: *integrable-iff-bounded*)
finally have ($\int^+ x.\ norm(f\ x * max\ (-g\ x)\ 0)\ \partial M) < \infty$ **by** *simp*
then have *int2*: *integrable M* ($\lambda x.\ f\ x * max\ (-g\ x)\ 0$) **by** (*simp add*: *integrableI-bounded*)

have ($\int^+ x.\ f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x\ \partial M) = (\int^+ x.\ f\ x * h\ x\ \partial M)$
by (*rule nn-cond-exp-intg*) *auto*
also have ... $= \int^+ x.\ ennreal\ (f\ x * max\ (g\ x)\ 0 + f\ x * max\ (-\ g\ x)\ 0)\ \partial M$
unfolding *h-def*
by (*intro nn-integral-cong*)(*auto simp*: *ennreal-mult*[*symmetric*] *abs-mult split*: *split-max*)
also have ... $< \infty$
using *Bochner-Integration.integrable-add*[*OF int1 int2*, *THEN integrableD*(*2*)] **by** (*auto simp add*: *less-top*)
finally have $*$: ($\int^+ x.\ f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x\ \partial M) < \infty$ **by** *simp*

have ($\int^+ x.\ norm(f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x)\ \partial M) = (\int^+ x.\ f\ x * abs(real\text{-}cond\text{-}exp\ M\ F\ g\ x)\ \partial M)$
by (*simp add*: *abs-mult*)
also have ... $\leq (\int^+ x.\ f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x\ \partial M)$
proof (*rule nn-integral-mono-AE*)
{
fix *x* **assume** $*$: $abs(real\text{-}cond\text{-}exp\ M\ F\ g\ x) \leq nn\text{-}cond\text{-}exp\ M\ F\ h\ x$
have $ennreal\ (f\ x * |real\text{-}cond\text{-}exp\ M\ F\ g\ x|) = f\ x * ennreal(|real\text{-}cond\text{-}exp\ M\ F\ g\ x|)$
by (*simp add*: *ennreal-mult*)
also have ... $\leq f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x$
using $*$ **by** (*auto intro*!: *mult-left-mono*)
finally have $ennreal\ (f\ x * |real\text{-}cond\text{-}exp\ M\ F\ g\ x|) \leq f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x$
by *simp*
}
then show *AE x in M*. $ennreal\ (f\ x * |real\text{-}cond\text{-}exp\ M\ F\ g\ x|) \leq f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x$
using *real-cond-exp-abs*[*OF assms*(*4*)] *h-def* **by** *auto*
qed
finally have $**$: ($\int^+ x.\ norm(f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x)\ \partial M) < \infty$ **using** $*$ **by** *auto*
show *integrable M* ($\lambda x.\ f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x$)
using $**$ **by** (*intro integrableI-bounded*) *auto*

have ($\int^+ x.\ f\ x * nn\text{-}cond\text{-}exp\ M\ F\ gp\ x\ \partial M) \leq (\int^+ x.\ f\ x * nn\text{-}cond\text{-}exp\ M\ F\ h\ x\ \partial M)$
proof (*rule nn-integral-mono-AE*)

have *AE x in M. nn-cond-exp M F gp x ≤ nn-cond-exp M F h x*
by (*rule nn-cond-exp-mono*) (*auto simp add*: *hgpgm*)
then show *AE x in M. f x * nn-cond-exp M F gp x ≤ f x * nn-cond-exp M F h x*
by (*auto simp*: *mult-left-mono*)
qed
then have *a*: (∫⁺*x*. *f x * nn-cond-exp M F gp x ∂M) < ∞
using * **by** *auto*
have *ennreal*(*norm*(*f x * enn2real*(*nn-cond-exp M F gp x*))) ≤ *f x * nn-cond-exp M F gp x* **for** *x*
by (*auto simp add*: *ennreal-mult intro*!: *mult-left-mono*)
(*metis enn2real-ennreal enn2real-nonneg le-cases le-ennreal-iff*)
then have (∫⁺*x*. *norm*(*f x * enn2real*(*nn-cond-exp M F gp x*)) *∂M*) ≤ (∫⁺*x*. *f x * nn-cond-exp M F gp x ∂M*)
by (*simp add*: *nn-integral-mono*)
then have (∫⁺*x*. *norm*(*f x * enn2real*(*nn-cond-exp M F gp x*)) *∂M*) < ∞ **using** *a* **by** *auto*
then have *gp-int*: *integrable M* (λ*x*. *f x * enn2real*(*nn-cond-exp M F gp x*)) **by** (*simp add*: *integrableI-bounded*)
have *gp-fin*: *AE x in M. f x * nn-cond-exp M F gp x ≠ ∞*
apply (*rule nn-integral-PInf-AE*) **using** *a* **by** *auto*

have (∫ *x*. *f x * enn2real*(*nn-cond-exp M F gp x*) *∂M*) = *enn2real* (∫⁺ *x*. *f x * enn2real*(*nn-cond-exp M F gp x*) *∂M*)
by (*rule integral-eq-nn-integral*) *auto*
also have ... = *enn2real*(∫⁺ *x*. *ennreal*(*f x * enn2real*(*gp x*)) *∂M*)
proof −
{
fix *x* **assume** *f x * nn-cond-exp M F gp x ≠ ∞*
then have *ennreal* (*f x * enn2real* (*nn-cond-exp M F gp x*)) = *ennreal* (*f x*) * *nn-cond-exp M F gp x*
by (*auto simp add*: *ennreal-mult ennreal-mult-eq-top-iff less-top intro*!: *ennreal-mult-left-cong*)
}
then have *AE x in M. ennreal* (*f x * enn2real* (*nn-cond-exp M F gp x*)) = *ennreal* (*f x*) * *nn-cond-exp M F gp x*
using *gp-fin* **by** *auto*
then have (∫⁺ *x*. *f x * enn2real*(*nn-cond-exp M F gp x*) *∂M*) = (∫⁺ *x*. *f x * nn-cond-exp M F gp x ∂M*)
by (*rule nn-integral-cong-AE*)
also have ... = (∫⁺ *x*. *f x * gp x ∂M*)
by (*rule nn-cond-exp-intg*) (*auto simp add*: *gp-def*)
also have ... = (∫⁺ *x*. *ennreal*(*f x * enn2real*(*gp x*)) *∂M*)
by (*rule nn-integral-cong-AE*) (*auto simp*: *ennreal-mult gp-def*)
finally have (∫⁺ *x*. *f x * enn2real*(*nn-cond-exp M F gp x*) *∂M*) = (∫⁺ *x*. *ennreal*(*f x * enn2real*(*gp x*)) *∂M*) **by** *simp*
then show *?thesis* **by** *simp*
qed
also have ... = (∫ *x*. *f x * enn2real*(*gp x*) *∂M*)

by (*rule integral-eq-nn-integral*[*symmetric*]) (*auto simp add*: *gp-def*)
finally have *gp-expr*: $(\int\ x.\ f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gp\ x)\ \partial M) = (\int\ x.\ f\ x * \mathit{enn2real}(gp\ x)\ \partial M)$ **by** *simp*

have $(\int^+ x.\ f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x\ \partial M) \leq (\int^+ x.\ f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ h\ x\ \partial M)$
proof (*rule nn-integral-mono-AE*)
have *AE x in M*. $\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x \leq \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ h\ x$
by (*rule nn-cond-exp-mono*) (*auto simp add*: *hgpgm*)
then show *AE x in M*. $f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x \leq f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ h\ x$
by (*auto simp*: *mult-left-mono*)
qed
then have *a*: $(\int^+ x.\ f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x\ \partial M) < \infty$
using $*$ **by** *auto*
have $\bigwedge x.\ \mathit{ennreal}(\mathit{norm}(f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x))) \leq f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x$
by (*auto simp add*: *ennreal-mult intro*!: *mult-left-mono*)
(*metis enn2real-ennreal enn2real-nonneg le-cases le-ennreal-iff*)
then have $(\int^+ x.\ \mathit{norm}(f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x))\ \partial M) \leq (\int^+ x.\ f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x\ \partial M)$
by (*simp add*: *nn-integral-mono*)
then have $(\int^+ x.\ \mathit{norm}(f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x))\ \partial M) < \infty$ **using** *a* **by** *auto*
then have *gm-int*: *integrable M* $(\lambda x.\ f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x))$ **by** (*simp add*: *integrableI-bounded*)
have *gm-fin*: *AE x in M*. $f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x \neq \infty$
apply (*rule nn-integral-PInf-AE*) **using** *a* **by** *auto*

have $(\int\ x.\ f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x)\ \partial M) = \mathit{enn2real}\ (\int^+\ x.\ f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x)\ \partial M)$
by (*rule integral-eq-nn-integral*) *auto*
also have ... $= \mathit{enn2real}(\int^+\ x.\ \mathit{ennreal}(f\ x * \mathit{enn2real}(gm\ x))\ \partial M)$
proof −
{
fix *x* **assume** $f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x \neq \infty$
then have $\mathit{ennreal}\ (f\ x * \mathit{enn2real}\ (\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x)) = \mathit{ennreal}\ (f\ x) * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x$
by (*auto simp add*: *ennreal-mult ennreal-mult-eq-top-iff less-top intro*!: *ennreal-mult-left-cong*)
}
then have *AE x in M*. $\mathit{ennreal}\ (f\ x * \mathit{enn2real}\ (\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x)) = \mathit{ennreal}\ (f\ x) * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x$
using *gm-fin* **by** *auto*
then have $(\int^+\ x.\ f\ x * \mathit{enn2real}(\mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x)\ \partial M) = (\int^+\ x.\ f\ x * \mathit{nn\text{-}cond\text{-}exp}\ M\ F\ gm\ x\ \partial M)$
by (*rule nn-integral-cong-AE*)
also have ... $= (\int^+\ x.\ f\ x * gm\ x\ \partial M)$

by (*rule nn-cond-exp-intg*) (*auto simp add*: *gm-def*)
also have ... = $(\int^+ x.\ ennreal(f\ x * enn2real(gm\ x))\ \partial M)$
by (*rule nn-integral-cong-AE*) (*auto simp*: *ennreal-mult gm-def*)
finally have $(\int^+ x.\ f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gm\ x)\ \partial M) = (\int^+ x.\ ennreal(f\ x * enn2real(gm\ x))\ \partial M)$ **by** *simp*
then show *?thesis* **by** *simp*
qed
also have ... = $(\int x.\ f\ x * enn2real(gm\ x)\ \partial M)$
by (*rule integral-eq-nn-integral*[*symmetric*]) (*auto simp add*: *gm-def*)
finally have *gm-expr*: $(\int x.\ f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gm\ x)\ \partial M) = (\int x.\ f\ x * enn2real(gm\ x)\ \partial M)$ **by** *simp*

have $(\int x.\ f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x\ \partial M) = (\int x.\ f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gp\ x) - f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gm\ x)\ \partial M)$
unfolding *real-cond-exp-def gp-def gm-def* **by** (*simp add*: *right-diff-distrib*)
also have ... = $(\int x.\ f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gp\ x)\ \partial M) - (\int x.\ f\ x * enn2real(nn\text{-}cond\text{-}exp\ M\ F\ gm\ x)\ \partial M)$
by (*rule Bochner-Integration.integral-diff*) (*simp-all add*: *gp-int gm-int*)
also have ... = $(\int x.\ f\ x * enn2real(gp\ x)\ \partial M) - (\int x.\ f\ x * enn2real(gm\ x)\ \partial M)$
using *gp-expr gm-expr* **by** *simp*
also have ... = $(\int x.\ f\ x * max\ (g\ x)\ 0\ \partial M) - (\int x.\ f\ x * max\ (-g\ x)\ 0\ \partial M)$
using *gp-real gm-real* **by** *simp*
also have ... = $(\int x.\ f\ x * max\ (g\ x)\ 0 - f\ x * max\ (-g\ x)\ 0\ \partial M)$
by (*rule Bochner-Integration.integral-diff*[*symmetric*]) (*simp-all add*: *int1 int2*)
also have ... = $(\int x.\ f\ x * g\ x\ \partial M)$
by (*metis* (*mono-tags*, *opaque-lifting*) *diff-0 diff-zero eq-iff max.cobounded2 max-def minus-minus neg-le-0-iff-le right-diff-distrib*)
finally show $(\int x.\ f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x\ \partial M) = (\int x.\ f\ x * g\ x\ \partial M)$
by *simp*
qed

lemma *real-cond-exp-intg*:
assumes *integrable M* $(\lambda x.\ f\ x * g\ x)$ **and**
[*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
shows *integrable M* $(\lambda x.\ f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x)$
$(\int x.\ f\ x * real\text{-}cond\text{-}exp\ M\ F\ g\ x\ \partial M) = (\int x.\ f\ x * g\ x\ \partial M)$
proof −
have [*measurable*]: *f* ∈ *borel-measurable M* **by** (*rule measurable-from-subalg*[*OF subalg assms*(*2*)])
define *fp* **where** $fp = (\lambda x.\ max\ (f\ x)\ 0)$
define *fm* **where** $fm = (\lambda x.\ max\ (-f\ x)\ 0)$
have [*measurable*]: *fp* ∈ *borel-measurable M fm* ∈ *borel-measurable M*
unfolding *fp-def fm-def* **by** *simp-all*
have [*measurable*]: *fp* ∈ *borel-measurable F fm* ∈ *borel-measurable F*
unfolding *fp-def fm-def* **by** *simp-all*

have $(\int^+ x.\ norm(fp\ x * g\ x)\ \partial M) \leq (\int^+ x.\ norm(f\ x * g\ x)\ \partial M)$

by (*simp add*: *fp-def nn-integral-mono*)
also have ... $< \infty$ **using** *assms*(*1*) **by** (*simp add*: *integrable-iff-bounded*)
finally have $(\int^+ x.\ norm(fp\ x * g\ x)\ \partial M) < \infty$ **by** *simp*
then have *intp*: *integrable M* $(\lambda x.\ fp\ x * g\ x)$ **by** (*simp add*: *integrableI-bounded*)
moreover have $\bigwedge x.\ fp\ x \geq 0$ **unfolding** *fp-def* **by** *simp*
ultimately have *Rp*: *integrable M* $(\lambda x.\ fp\ x *$ *real-cond-exp M F g x*$)$
$(\int\ x.\ fp\ x *$ *real-cond-exp M F g x* $\partial M) = (\int\ x.\ fp\ x * g\ x\ \partial M)$
using *real-cond-exp-intg-fpos* **by** *auto*

have $(\int^+ x.\ norm(fm\ x * g\ x)\ \partial M) \leq (\int^+ x.\ norm(f\ x * g\ x)\ \partial M)$
by (*simp add*: *fm-def nn-integral-mono*)
also have ... $< \infty$ **using** *assms*(*1*) **by** (*simp add*: *integrable-iff-bounded*)
finally have $(\int^+ x.\ norm(fm\ x * g\ x)\ \partial M) < \infty$ **by** *simp*
then have *intm*: *integrable M* $(\lambda x.\ fm\ x * g\ x)$ **by** (*simp add*: *integrableI-bounded*)
moreover have $\bigwedge x.\ fm\ x \geq 0$ **unfolding** *fm-def* **by** *simp*
ultimately have *Rm*: *integrable M* $(\lambda x.\ fm\ x *$ *real-cond-exp M F g x*$)$
$(\int\ x.\ fm\ x *$ *real-cond-exp M F g x* $\partial M) = (\int\ x.\ fm\ x * g\ x\ \partial M)$
using *real-cond-exp-intg-fpos* **by** *auto*

have *integrable M* $(\lambda x.\ fp\ x *$ *real-cond-exp M F g x* $-\ fm\ x *$ *real-cond-exp M F g x*$)$
using *Rp*(*1*) *Rm*(*1*) *integrable-diff* **by** *simp*
moreover have $*$: $\bigwedge x.\ f\ x *$ *real-cond-exp M F g x* $= fp\ x *$ *real-cond-exp M F g x* $-\ fm\ x *$ *real-cond-exp M F g x*
unfolding *fp-def fm-def* **by** (*simp add*: *max-def*)
ultimately show *integrable M* $(\lambda x.\ f\ x *$ *real-cond-exp M F g x*$)$
by *simp*

have $(\int\ x.\ f\ x *$ *real-cond-exp M F g x* $\partial M) = (\int\ x.\ fp\ x *$ *real-cond-exp M F g x* $-\ fm\ x *$ *real-cond-exp M F g x* $\partial M)$
using $*$ **by** *simp*
also have ... $= (\int\ x.\ fp\ x *$ *real-cond-exp M F g x* $\partial M) - (\int\ x.\ fm\ x *$ *real-cond-exp M F g x* $\partial M)$
using *Rp*(*1*) *Rm*(*1*) **by** *simp*
also have ... $= (\int\ x.\ fp\ x * g\ x\ \partial M) - (\int\ x.\ fm\ x * g\ x\ \partial M)$
using *Rp*(*2*) *Rm*(*2*) **by** *simp*
also have ... $= (\int\ x.\ fp\ x * g\ x - fm\ x * g\ x\ \partial M)$
using *intm intp* **by** *simp*
also have ... $= (\int\ x.\ f\ x * g\ x\ \partial M)$
unfolding *fp-def fm-def* **by** (*metis* (*no-types*, *opaque-lifting*) *diff-0 diff-zero max.commute*
max-def minus-minus mult.commute neg-le-iff-le right-diff-distrib)
finally show $(\int\ x.\ f\ x *$ *real-cond-exp M F g x* $\partial M) = (\int\ x.\ f\ x * g\ x\ \partial M)$ **by** *simp*
qed

lemma *real-cond-exp-intA*:
assumes [*measurable*]: *integrable M f A* $\in$ *sets F*
shows $(\int\ x \in A.\ f\ x\ \partial M) = (\int x \in A.$ *real-cond-exp M F f x* $\partial M)$

proof −
have $A \in$ *sets* M **by** (*meson assms*(2) *subalg subalgebra-def subsetD*)
have *integrable* M (λx. *indicator* A $x * f\ x$) **using** *integrable-mult-indicator*[*OF* ‹$A \in$ *sets* M› *assms*(1)] **by** *auto*
then show *?thesis* **using** *real-cond-exp-intg*(2)[**where** *?f* = *indicator* A **and** *?g* = f, *symmetric*]
unfolding *set-lebesgue-integral-def* **by** *auto*
qed

lemma *real-cond-exp-int* [*intro*]:
assumes *integrable* M f
shows *integrable* M (*real-cond-exp* M F f) ($\int x$. *real-cond-exp* M F f x ∂M) = ($\int x$. f x ∂M)
using *real-cond-exp-intg*[**where** *?f* = λx. *1* **and** *?g* = f] *assms* **by** *auto*

lemma *real-cond-exp-charact*:
assumes $\bigwedge A$. $A \in$ *sets* $F \Longrightarrow$ ($\int$ $x \in A$. f x ∂M) = ($\int$ $x \in A$. g x ∂M)
and [*measurable*]: *integrable* M f *integrable* M g
$g \in$ *borel-measurable* F
shows *AE* x *in* M. *real-cond-exp* M F f x = g x
proof −
let *?MF* = *restr-to-subalg* M F
have *AE* x *in* *?MF*. *real-cond-exp* M F f x = g x
proof (*rule AE-symmetric*[*OF density-unique-real*])
fix A **assume** $A \in$ *sets* *?MF*
then have [*measurable*]: $A \in$ *sets* F **using** *sets-restr-to-subalg*[*OF subalg*] **by** *simp*
then have *a* [*measurable*]: $A \in$ *sets* M **by** (*meson subalg subalgebra-def subsetD*)
have ($\int x \in A$. g x ∂ *?MF*) = ($\int x \in A$. g x ∂M)
unfolding *set-lebesgue-integral-def* **by** (*simp add*: *integral-subalgebra2 subalg*)
also have ... = ($\int x \in A$. f x ∂M) **using** *assms*(1) **by** *simp*
also have ... = ($\int x$. *indicator* A $x * f$ x ∂M) **by** (*simp add*: *mult.commute set-lebesgue-integral-def*)
also have ... = ($\int x$. *indicator* A x * *real-cond-exp* M F f x ∂M)
apply (*rule real-cond-exp-intg*(2)[*symmetric*]) **using** *integrable-mult-indicator*[*OF a assms*(2)] **by** (*auto simp add*: *assms*)
also have ... = ($\int x \in A$. *real-cond-exp* M F f x ∂M) **by** (*simp add*: *mult.commute set-lebesgue-integral-def*)
also have ... = ($\int x \in A$. *real-cond-exp* M F f x ∂ *?MF*)
by (*simp add*: *integral-subalgebra2 subalg set-lebesgue-integral-def*)
finally show ($\int x \in A$. g x ∂ *?MF*) = ($\int x \in A$. *real-cond-exp* M F f x ∂ *?MF*) **by** *simp*
next
have *integrable* M (*real-cond-exp* M F f) **by** (*rule real-cond-exp-int*(1)[*OF assms*(2)])
then show *integrable* *?MF* (*real-cond-exp* M F f) **by** (*metis borel-measurable-cond-exp integrable-in-subalg*[*OF subalg*])
show *integrable* (*restr-to-subalg* M F) g **by** (*simp add*: *assms*(3) *integrable-in-subalg*[*OF subalg*])

qed
then show *?thesis* **using** *AE-restr-to-subalg*[*OF subalg*] **by** *auto*
qed

lemma *real-cond-exp-F-meas* [*intro*, *simp*]:
assumes *integrable M f*
f ∈ *borel-measurable F*
shows *AE x in M*. *real-cond-exp M F f x* = *f x*
by (*rule real-cond-exp-charact*, *auto simp add*: *assms measurable-from-subalg*[*OF subalg*])

lemma *real-cond-exp-mult*:
assumes [*measurable*]:*f* ∈ *borel-measurable F g* ∈ *borel-measurable M integrable M* (λ*x*. *f x* ∗ *g x*)
shows *AE x in M*. *real-cond-exp M F* (λ*x*. *f x* ∗ *g x*) *x* = *f x* ∗ *real-cond-exp M F g x*
proof (*rule real-cond-exp-charact*)
fix *A* **assume** *A* ∈ *sets F*
then have [*measurable*]: (λ*x*. *f x* ∗ *indicator A x*) ∈ *borel-measurable F* **by** *measurable*
have [*measurable*]: *A* ∈ *sets M* **using** *subalg* **by** (*meson* ‹*A* ∈ *sets F*› *subalgebra-def subsetD*)
have (∫ *x*∈*A*. (*f x* ∗ *g x*) ∂*M*) = ∫ *x*. (*f x* ∗ *indicator A x*) ∗ *g x* ∂*M*
by (*simp add*: *mult.commute mult.left-commute set-lebesgue-integral-def*)
also have ... = ∫ *x*. (*f x* ∗ *indicator A x*) ∗ *real-cond-exp M F g x* ∂*M*
apply (*rule real-cond-exp-intg*(*2*)[*symmetric*], *auto simp add*: *assms*)
using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *assms*(*3*)] **by** (*simp add*: *mult.commute mult.left-commute*)
also have ... = (∫ *x*∈*A*. (*f x* ∗ *real-cond-exp M F g x*)∂*M*)
by (*simp add*: *mult.commute mult.left-commute set-lebesgue-integral-def*)
finally show (∫ *x*∈*A*. (*f x* ∗ *g x*) ∂*M*) = (∫ *x*∈*A*. (*f x* ∗ *real-cond-exp M F g x*)∂*M*) **by** *simp*
qed (*auto simp add*: *real-cond-exp-intg*(*1*) *assms*)

lemma *real-cond-exp-add* [*intro*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-cond-exp M F* (λ*x*. *f x* + *g x*) *x* = *real-cond-exp M F f x* + *real-cond-exp M F g x*
proof (*rule real-cond-exp-charact*)
have *integrable M* (*real-cond-exp M F f*) *integrable M* (*real-cond-exp M F g*)
using *real-cond-exp-int*(*1*) *assms* **by** *auto*
then show *integrable M* (λ*x*. *real-cond-exp M F f x* + *real-cond-exp M F g x*)
by *auto*

fix *A* **assume** [*measurable*]: *A* ∈ *sets F*
then have *A* ∈ *sets M* **by** (*meson subalg subalgebra-def subsetD*)
have *intAf*: *integrable M* (λ*x*. *indicator A x* ∗ *f x*)
using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *assms*(*1*)] **by** *auto*
have *intAg*: *integrable M* (λ*x*. *indicator A x* ∗ *g x*)

using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *assms*(*2*)] **by** *auto*

have (∫ *x*∈*A*. (*real-cond-exp M F f x* + *real-cond-exp M F g x*)∂*M*) = (∫ *x*∈*A*. *real-cond-exp M F f x* ∂*M*) + (∫ *x*∈*A*. *real-cond-exp M F g x* ∂*M*)
apply (*rule set-integral-add*, *auto simp add*: *assms set-integrable-def*)
using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *real-cond-exp-int*(*1*)[*OF assms*(*1*)]]
integrable-mult-indicator[*OF* ‹*A* ∈ *sets M*› *real-cond-exp-int*(*1*)[*OF assms*(*2*)]] **by** *simp-all*
also have ... = (∫ *x*. *indicator A x* ∗ *real-cond-exp M F f x* ∂*M*) + (∫ *x*. *indicator A x* ∗ *real-cond-exp M F g x* ∂*M*)
unfolding *set-lebesgue-integral-def* **by** *auto*
also have ... = (∫ *x*. *indicator A x* ∗ *f x* ∂*M*) + (∫ *x*. *indicator A x* ∗ *g x* ∂*M*)
using *real-cond-exp-intg*(*2*) *assms* ‹*A* ∈ *sets F*› *intAf intAg* **by** *auto*
also have ... = (∫ *x*∈*A*. *f x* ∂*M*) + (∫ *x*∈*A*. *g x* ∂*M*)
unfolding *set-lebesgue-integral-def* **by** *auto*
also have ... = (∫ *x*∈*A*. (*f x* + *g x*)∂*M*)
by (*rule set-integral-add*(*2*)[*symmetric*]) (*auto simp add*: *assms set-integrable-def* ‹*A* ∈ *sets M*› *intAf intAg*)
finally show (∫ *x*∈*A*. (*f x* + *g x*)∂*M*) = (∫ *x*∈*A*. (*real-cond-exp M F f x* + *real-cond-exp M F g x*)∂*M*)
by *simp*
qed (*auto simp add*: *assms*)

lemma *real-cond-exp-cong*:
assumes *ae*: *AE x in M*. *f x* = *g x* **and** [*measurable*]: *f* ∈ *borel-measurable M g* ∈ *borel-measurable M*
shows *AE x in M*. *real-cond-exp M F f x* = *real-cond-exp M F g x*
proof −
have *AE x in M*. *nn-cond-exp M F* (λ*x*. *ennreal* (*f x*)) *x* = *nn-cond-exp M F* (λ*x*. *ennreal* (*g x*)) *x*
apply (*rule nn-cond-exp-cong*) **using** *assms* **by** *auto*
moreover have *AE x in M*. *nn-cond-exp M F* (λ*x*. *ennreal* (−*f x*)) *x* = *nn-cond-exp M F* (λ*x*. *ennreal*(−*g x*)) *x*
apply (*rule nn-cond-exp-cong*) **using** *assms* **by** *auto*
ultimately show *AE x in M*. *real-cond-exp M F f x* = *real-cond-exp M F g x*
unfolding *real-cond-exp-def* **by** *auto*
qed

lemma *real-cond-exp-cmult* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M*. *real-cond-exp M F* (λ*x*. *c* ∗ *f x*) *x* = *c* ∗ *real-cond-exp M F f x*
by (*rule real-cond-exp-mult*[**where** *?f* = λ*x*. *c* **and** *?g* = *f*], *auto simp add*: *assms borel-measurable-integrable*)

lemma *real-cond-exp-cdiv* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*

shows *AE x in M. real-cond-exp M F* (λx. *f x / c*) *x = real-cond-exp M F f x / c*
using *real-cond-exp-cmult*[*of - 1 / c*, *OF assms*] **by** (*auto simp add*: *field-split-simps*)

lemma *real-cond-exp-diff* [*intro*, *simp*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F* (λx. *f x − g x*) *x = real-cond-exp M F f x − real-cond-exp M F g x*
proof −
have *AE x in M. real-cond-exp M F* (λx. *f x + (− g x)*) *x = real-cond-exp M F f x + real-cond-exp M F* (λx. *−g x*) *x*
using *real-cond-exp-add*[**where** *?f = f* **and** *?g = λx. − g x*] *assms* **by** *auto*
moreover have *AE x in M. real-cond-exp M F* (λx. *−g x*) *x = − real-cond-exp M F g x*
using *real-cond-exp-cmult*[**where** *?f = g* **and** *?c = −1*] *assms*(*2*) **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *real-cond-exp-pos* [*intro*]:
assumes *AE x in M. f x ≥ 0* **and** [*measurable*]: *f ∈ borel-measurable M*
shows *AE x in M. real-cond-exp M F f x ≥ 0*
proof −
define *g* **where** *g* = (λx. *max* (*f x*) *0*)
have *AE x in M. f x = g x* **using** *assms g-def* **by** *auto*
then have ∗: *AE x in M. real-cond-exp M F f x = real-cond-exp M F g x* **using** *real-cond-exp-cong g-def* **by** *auto*

have ⋀x. *g x ≥ 0* **unfolding** *g-def* **by** *simp*
then have (λx. *ennreal*(*−g x*)) = (λx. *0*)
by (*simp add*: *ennreal-neg*)
moreover have *AE x in M. 0 = nn-cond-exp M F* (λx. *0*) *x*
by (*rule nn-cond-exp-F-meas*, *auto*)
ultimately have *AE x in M. nn-cond-exp M F* (λx. *ennreal*(*−g x*)) *x = 0*
by *simp*
then have *AE x in M. real-cond-exp M F g x = enn2real*(*nn-cond-exp M F* (λx. *ennreal* (*g x*)) *x*)
unfolding *real-cond-exp-def* **by** *auto*
then have *AE x in M. real-cond-exp M F g x ≥ 0* **by** *auto*
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *real-cond-exp-mono*:
assumes *AE x in M. f x ≤ g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F f x ≤ real-cond-exp M F g x*
proof −
have *AE x in M. real-cond-exp M F g x − real-cond-exp M F f x = real-cond-exp M F* (λx. *g x − f x*) *x*
by (*rule AE-symmetric*[*OF real-cond-exp-diff*], *auto simp add*: *assms*)
moreover have *AE x in M. real-cond-exp M F* (λx. *g x − f x*) *x ≥ 0*

```
    by (rule real-cond-exp-pos, auto simp add: assms(1))
  ultimately have AE x in M. real-cond-exp M F g x − real-cond-exp M F f x ≥
0 by auto
  then show ?thesis by auto
qed

lemma (in −) measurable-P-restriction [measurable (raw)]:
  assumes [measurable]: Measurable.pred M P A ∈ sets M
  shows {x ∈ A. P x} ∈ sets M
proof −
  have A ⊆ space M using sets.sets-into-space[OF assms(2)].
  then have {x ∈ A. P x} = A ∩ {x ∈ space M. P x} by blast
  then show ?thesis by auto
qed

lemma real-cond-exp-gr-c:
  assumes [measurable]: integrable M f
      and AE: AE x in M. f x > c
  shows AE x in M. real-cond-exp M F f x > c
proof −
  define X where X = {x ∈ space M. real-cond-exp M F f x ≤ c}
  have [measurable]: X ∈ sets F
    unfolding X-def apply measurable by (metis sets.top subalg subalgebra-def)
  then have [measurable]: X ∈ sets M using sets-restr-to-subalg subalg subalge-
bra-def by blast
  have emeasure M X = 0
  proof (rule ccontr)
    assume ¬(emeasure M X) = 0
    have emeasure (restr-to-subalg M F) X = emeasure M X
      by (simp add: emeasure-restr-to-subalg subalg)
    then have emeasure (restr-to-subalg M F) X > 0
      using ‹¬(emeasure M X) = 0› gr-zeroI by auto
      then obtain A where A ∈ sets (restr-to-subalg M F) A ⊆ X emeasure
(restr-to-subalg M F) A > 0 emeasure (restr-to-subalg M F) A < ∞
    using sigma-fin-subalg by (metis emeasure-notin-sets ennreal-0 infinity-ennreal-def
le-less-linear neq-top-trans
      not-gr-zero order-refl sigma-finite-measure.approx-PInf-emeasure-with-finite)
    then have [measurable]: A ∈ sets F using subalg sets-restr-to-subalg by blast
    then have [measurable]: A ∈ sets M using sets-restr-to-subalg subalg subalge-
bra-def by blast
    have Ic: set-integrable M A (λx. c)
      unfolding set-integrable-def
     using ‹emeasure (restr-to-subalg M F) A < ∞› emeasure-restr-to-subalg subalg
by fastforce
    have If: set-integrable M A f
      unfolding set-integrable-def
      by (rule integrable-mult-indicator, auto simp add: ‹integrable M f›)
    have AE x in M. indicator A x ∗ c = indicator A x ∗ f x
    proof (rule integral-ineq-eq-0-then-AE)
```

```
    have (∫ x∈A. c ∂M) = (∫ x∈A. f x ∂M)
    proof (rule antisym)
      show (∫ x∈A. c ∂M) ≤ (∫ x∈A. f x ∂M)
        apply (rule set-integral-mono-AE) using Ic If assms(2) by auto
      have (∫ x∈A. f x ∂M) = (∫ x∈A. real-cond-exp M F f x ∂M)
        by (rule real-cond-exp-intA, auto simp add: ‹integrable M f›)
      also have ... ≤ (∫ x∈A. c ∂M)
        apply (rule set-integral-mono)
        unfolding set-integrable-def
       apply (rule integrable-mult-indicator, simp, simp add: real-cond-exp-int(1)[OF
‹integrable M f›])
        using Ic X-def ‹A ⊆ X› by (auto simp: set-integrable-def)
      finally show (∫ x∈A. f x ∂M) ≤ (∫ x∈A. c ∂M) by simp
    qed
    then have measure M A * c = LINT x|M. indicat-real A x * f x
      by (auto simp: set-lebesgue-integral-def)
    then show LINT x|M. indicat-real A x * c = LINT x|M. indicat-real A x *
f x
      by auto
    show AE x in M. indicat-real A x * c ≤ indicat-real A x * f x
    using AE unfolding indicator-def by auto
  qed (use Ic If in ‹auto simp: set-integrable-def›)
  then have AE x∈A in M. c = f x by auto
  then have AE x∈A in M. False using assms(2) by auto
  have A ∈ null-sets M unfolding ae-filter-def by (meson AE-iff-null-sets ‹A
∈ sets M› ‹AE x∈A in M. False›)
  then show False using ‹emeasure (restr-to-subalg M F) A > 0›
    by (simp add: emeasure-restr-to-subalg null-setsD1 subalg)
 qed
 then show ?thesis using AE-iff-null-sets[OF ‹X ∈ sets M›] unfolding X-def
by auto
qed

lemma real-cond-exp-less-c:
 assumes [measurable]: integrable M f
    and AE x in M. f x < c
 shows AE x in M. real-cond-exp M F f x < c
proof −
 have AE x in M. real-cond-exp M F f x = −real-cond-exp M F (λx. −f x) x
  using real-cond-exp-cmult[OF ‹integrable M f›, of −1] by auto
 moreover have AE x in M. real-cond-exp M F (λx. −f x) x > −c
  apply (rule real-cond-exp-gr-c) using assms by auto
 ultimately show ?thesis by auto
qed

lemma real-cond-exp-ge-c:
 assumes [measurable]: integrable M f
    and AE x in M. f x ≥ c
 shows AE x in M. real-cond-exp M F f x ≥ c
```

proof −
obtain *u*::*nat* ⇒ *real* **where** *u*: ⋀*n*. *u n* < *c u* ⟶ *c*
using *approx-from-below-dense-linorder*[*of c*−*1 c*] **by** *auto*
have ∗: *AE x in M*. *real-cond-exp M F f x* > *u n* **for** *n*::*nat*
apply (*rule real-cond-exp-gr-c*) **using** *assms* ‹*u n* < *c*› **by** *auto*
have *AE x in M*. ∀ *n*. *real-cond-exp M F f x* > *u n*
by (*subst AE-all-countable*, *auto simp add*: ∗)
moreover have *real-cond-exp M F f x* ≥ *c* **if** ∀ *n*. *real-cond-exp M F f x* > *u n*
for *x*
proof −
have *real-cond-exp M F f x* ≥ *u n* **for** *n* **using** *that less-imp-le* **by** *auto*
then show *?thesis* **using** *u*(*2*) *LIMSEQ-le-const2* **by** *metis*
qed
ultimately show *?thesis* **by** *auto*
qed

lemma *real-cond-exp-le-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M*. *f x* ≤ *c*
shows *AE x in M*. *real-cond-exp M F f x* ≤ *c*
proof −
have *AE x in M*. *real-cond-exp M F f x* = −*real-cond-exp M F* (λ*x*. −*f x*) *x*
using *real-cond-exp-cmult*[*OF* ‹*integrable M f*›, *of* −*1*] **by** *auto*
moreover have *AE x in M*. *real-cond-exp M F* (λ*x*. −*f x*) *x* ≥ −*c*
apply (*rule real-cond-exp-ge-c*) **using** *assms* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *real-cond-exp-mono-strict*:
assumes *AE x in M*. *f x* < *g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-cond-exp M F f x* < *real-cond-exp M F g x*
proof −
have *AE x in M*. *real-cond-exp M F g x* − *real-cond-exp M F f x* = *real-cond-exp M F* (λ*x*. *g x* − *f x*) *x*
by (*rule AE-symmetric*[*OF real-cond-exp-diff*], *auto simp add*: *assms*)
moreover have *AE x in M*. *real-cond-exp M F* (λ*x*. *g x* − *f x*) *x* > *0*
by (*rule real-cond-exp-gr-c*, *auto simp add*: *assms*)
ultimately have *AE x in M*. *real-cond-exp M F g x* − *real-cond-exp M F f x* > *0* **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *real-cond-exp-nested-subalg* [*intro*, *simp*]:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *integrable M f*
shows *AE x in M*. *real-cond-exp M F* (*real-cond-exp M G f*) *x* = *real-cond-exp M F f x*
proof (*rule real-cond-exp-charact*)
interpret *G*: *sigma-finite-subalgebra M G* **by** (*rule nested-subalg-is-sigma-finite*[*OF*

assms(1) assms(2)])
show *integrable M* (*real-cond-exp M G f*) **by** (*auto simp add*: *assms G.real-cond-exp-int*(*1*))

fix *A* **assume** [*measurable*]: *A* ∈ *sets F*
then have [*measurable*]: *A* ∈ *sets G* **using** *assms*(*2*) **by** (*meson subsetD subalgebra-def*)
have *set-lebesgue-integral M A* (*real-cond-exp M G f*) = *set-lebesgue-integral M A f*
by (*rule G.real-cond-exp-intA*[*symmetric*], *auto simp add*: *assms*(*3*))
also have ... = *set-lebesgue-integral M A* (*real-cond-exp M F f*)
by (*rule real-cond-exp-intA*, *auto simp add*: *assms*(*3*))
finally show *set-lebesgue-integral M A* (*real-cond-exp M G f*) = *set-lebesgue-integral M A* (*real-cond-exp M F f*) **by** *auto*
qed (*auto simp add*: *assms real-cond-exp-int*(*1*))

lemma *real-cond-exp-sum* [*intro*, *simp*]:
fixes *f*::*'b* ⇒ *'a* ⇒ *real*
assumes [*measurable*]: ⋀*i*. *integrable M* (*f i*)
shows *AE x in M*. *real-cond-exp M F* (λ*x*. ∑ *i*∈*I*. *f i x*) *x* = (∑ *i*∈*I*. *real-cond-exp M F* (*f i*) *x*)
proof (*rule real-cond-exp-charact*)
fix *A* **assume** [*measurable*]: *A* ∈ *sets F*
then have *A-meas* [*measurable*]: *A* ∈ *sets M* **by** (*meson subsetD subalg subalgebra-def*)
have ∗: *integrable M* (λ*x*. *indicator A x* ∗ *f i x*) **for** *i*
using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *assms*(*1*)] **by** *auto*
have ∗∗: *integrable M* (λ*x*. *indicator A x* ∗ *real-cond-exp M F* (*f i*) *x*) **for** *i*
using *integrable-mult-indicator*[*OF* ‹*A* ∈ *sets M*› *real-cond-exp-int*(*1*)[*OF assms*(*1*)]]
by *auto*
have *inti*: (∫ *x*. *indicator A x* ∗ *f i x* ∂*M*) = (∫ *x*. *indicator A x* ∗ *real-cond-exp M F* (*f i*) *x* ∂*M*) **for** *i*
by (*rule real-cond-exp-intg*(*2*)[*symmetric*], *auto simp add*: ∗)

have (∫ *x*∈*A*. (∑ *i*∈*I*. *f i x*)∂*M*) = (∫ *x*. (∑ *i*∈*I*. *indicator A x* ∗ *f i x*)∂*M*)
by (*simp add*: *sum-distrib-left set-lebesgue-integral-def*)
also have ... = (∑ *i*∈*I*. (∫ *x*. *indicator A x* ∗ *f i x* ∂*M*))
by (*rule Bochner-Integration.integral-sum*, *simp add*: ∗)
also have ... = (∑ *i*∈*I*. (∫ *x*. *indicator A x* ∗ *real-cond-exp M F* (*f i*) *x* ∂*M*))
using *inti* **by** *auto*
also have ... = (∫ *x*. (∑ *i*∈*I*. *indicator A x* ∗ *real-cond-exp M F* (*f i*) *x*)∂*M*)
by (*rule Bochner-Integration.integral-sum*[*symmetric*], *simp add*: ∗∗)
also have ... = (∫ *x*∈*A*. (∑ *i*∈*I*. *real-cond-exp M F* (*f i*) *x*)∂*M*)
by (*simp add*: *sum-distrib-left set-lebesgue-integral-def*)
finally show (∫ *x*∈*A*. (∑ *i*∈*I*. *f i x*)∂*M*) = (∫ *x*∈*A*. (∑ *i*∈*I*. *real-cond-exp M F* (*f i*) *x*)∂*M*) **by** *auto*
qed (*auto simp add*: *assms real-cond-exp-int*(*1*)[*OF assms*(*1*)])

Jensen's inequality, describing the behavior of the integral under a convex function, admits a version for the conditional expectation, as follows.

theorem *real-cond-exp-jensens-inequality*:
fixes $q :: real \Rightarrow real$
assumes X: *integrable M X AE x in M. X x ∈ I*
assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I = UNIV$
assumes q: *integrable M (λx. q (X x)) convex-on I q q ∈ borel-measurable borel*
shows *AE x in M. real-cond-exp M F X x ∈ I*
AE x in M. q (real-cond-exp M F X x) ≤ real-cond-exp M F (λx. q (X x)) x
proof −
have *open I* **using** *I* **by** *auto*
then have *interior I = I* **by** (*simp add*: *interior-eq*)
have [*measurable*]: *I ∈ sets borel* **using** *I* **by** *auto*
define *phi* **where** *phi = (λx. Inf ((λt. (q x − q t) / (x − t)) ' ({x<..} ∩ I)))*
have ∗∗: *q (X x) ≥ q (real-cond-exp M F X x) + phi (real-cond-exp M F X x) ∗ (X x − real-cond-exp M F X x)*
if *X x ∈ I real-cond-exp M F X x ∈ I* **for** *x*
unfolding *phi-def* **apply** (*rule convex-le-Inf-differential*)
using ‹*convex-on I q*› *that* ‹*interior I = I*› **by** *auto*

It is not clear that the function ϕ is measurable. We replace it by a version which is better behaved.

define *psi* **where** *psi = (λx. phi x ∗ indicator I x)*
have *A*: *psi y = phi y* **if** *y ∈ I* **for** *y* **unfolding** *psi-def indicator-def* **using** *that* **by** *auto*
have ∗: *q (X x) ≥ q (real-cond-exp M F X x) + psi (real-cond-exp M F X x) ∗ (X x − real-cond-exp M F X x)*
if *X x ∈ I real-cond-exp M F X x ∈ I* **for** *x*
unfolding *A*[*OF* ‹*real-cond-exp M F X x ∈ I*›] **using** ∗∗ *that* **by** *auto*

note *I*
moreover have *AE x in M. real-cond-exp M F X x > a* **if** *I ⊆ {a <..}* **for** *a*
apply (*rule real-cond-exp-gr-c*) **using** *X that* **by** *auto*
moreover have *AE x in M. real-cond-exp M F X x < b* **if** *I ⊆ {..<b}* **for** *b*
apply (*rule real-cond-exp-less-c*) **using** *X that* **by** *auto*
ultimately show *AE x in M. real-cond-exp M F X x ∈ I*
by (*elim disjE*) (*auto simp*: *subset-eq*)
then have *main-ineq*: *AE x in M. q (X x) ≥ q (real-cond-exp M F X x) + psi (real-cond-exp M F X x) ∗ (X x − real-cond-exp M F X x)*
using ∗ *X*(*2*) **by** *auto*

Then, one wants to take the conditional expectation of this inequality. On the left, one gets the conditional expectation of $q \circ X$. On the right, the last term vanishes, and one is left with q of the conditional expectation, as desired. Unfortunately, this argument only works if $\psi \cdot X$ and $q(E(X|F))$ are integrable, and there is no reason why this should be true. The trick is to multiply by a F-measurable function which is small enough to make everything integrable.

obtain *f*::*'a ⇒ real* **where** [*measurable*]: *f ∈ borel-measurable* (*restr-to-subalg M F*)

integrable (*restr-to-subalg M F*) *f*
and *f*: $\bigwedge x.\ f\ x > 0\ \bigwedge x.\ f\ x \leq 1$
using *sigma-finite-measure.obtain-positive-integrable-function*[*OF sigma-fin-subalg*]
by *metis*
then have [*measurable*]: $f \in$ *borel-measurable F* **by** (*simp add*: *subalg*)
then have [*measurable*]: $f \in$ *borel-measurable M* **using** *measurable-from-subalg*[*OF subalg*] **by** *blast*
define *g* **where** $g = (\lambda x.\ f\ x/(1+ |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)| + |q\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)|))$
define *G* **where** $G = (\lambda x.\ g\ x * psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x))$
have *g*: $g\ x > 0\ g\ x \leq 1$ **for** *x* **unfolding** *G-def g-def* **using** *f*[*of x*] **by** (*auto simp add*: *abs-mult*)
have *G*: $|G\ x| \leq 1$ **for** *x* **unfolding** *G-def g-def* **using** *f*[*of x*]
proof (*auto simp add*: *abs-mult*)
have $f\ x * |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)| \leq 1 * |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)|$
apply (*rule mult-mono*) **using** *f*[*of x*] **by** *auto*
also have $... \leq 1 + |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)| + |q\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)|$ **by** *auto*
finally show $f\ x * |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)| \leq 1 + |psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)| + |q\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x)|$
by *simp*
qed
have *AE x in M*. $g\ x * q\ (X\ x) \geq g\ x * (q\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x) + psi\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x) * (X\ x - real\text{-}cond\text{-}exp\ M\ F\ X\ x))$
using *main-ineq g* **by** (*auto simp add*: *divide-simps*)
then have *main-G*: *AE x in M*. $g\ x * q\ (X\ x) \geq g\ x * q\ (real\text{-}cond\text{-}exp\ M\ F\ X\ x) + G\ x * (X\ x - real\text{-}cond\text{-}exp\ M\ F\ X\ x)$
unfolding *G-def* **by** (*auto simp add*: *algebra-simps*)

To proceed, we need to know that ψ is measurable.

have *phi-mono*: $phi\ x \leq phi\ y$ **if** $x \leq y\ x \in I\ y \in I$ **for** *x y*
proof (*cases* $x < y$)
case *True*
have $q\ x + phi\ x * (y-x) \leq q\ y$
unfolding *phi-def* **apply** (*rule convex-le-Inf-differential*) **using** ‹*convex-on I q*› *that* ‹*interior I* = *I*› **by** *auto*
then have $phi\ x \leq (q\ x - q\ y)\ /\ (x - y)$
using *that* ‹$x < y$› **by** (*auto simp add*: *field-split-simps algebra-simps*)
moreover have $(q\ x - q\ y)/(x - y) \leq phi\ y$
unfolding *phi-def* **proof** (*rule cInf-greatest*, *auto*)
fix *t* **assume** $t \in I\ y < t$
have $(q\ x - q\ y)\ /\ (x - y) \leq (q\ x - q\ t)\ /\ (x - t)$
apply (*rule convex-on-slope-le*[*OF q*(*2*)]) **using** ‹$y < t$› ‹$x < y$› ‹$t \in I$› ‹$x \in I$› **by** *auto*
also have $... \leq (q\ y - q\ t)\ /\ (y - t)$
apply (*rule convex-on-slope-le*[*OF q*(*2*)]) **using** ‹$y < t$› ‹$x < y$› ‹$t \in I$› ‹$x \in I$› **by** *auto*
finally show $(q\ x - q\ y)\ /\ (x - y) \leq (q\ y - q\ t)\ /\ (y - t)$ **by** *simp*
next

```
    obtain e where 0 < e ball y e ⊆ I using ‹open I› ‹y ∈ I› openE by blast
    then have y + e/2 ∈ {y<..} ∩ I by (auto simp: dist-real-def)
    then show {y<..} ∩ I = {} ⟹ False by auto
  qed
  ultimately show phi x ≤ phi y by auto
 next
  case False
  then have x = y using ‹x ≤ y› by auto
  then show ?thesis by auto
 qed
 have [measurable]: psi ∈ borel-measurable borel
  by (rule borel-measurable-piecewise-mono[of {I, -I}])
     (auto simp add: psi-def indicator-def phi-mono intro: mono-onI)
 have [measurable]: q ∈ borel-measurable borel using q by simp

 have [measurable]: X ∈ borel-measurable M
                    real-cond-exp M F X ∈ borel-measurable F
                    g ∈ borel-measurable F g ∈ borel-measurable M
                    G ∈ borel-measurable F G ∈ borel-measurable M
   using X measurable-from-subalg[OF subalg] unfolding G-def g-def by auto
 have int1: integrable (restr-to-subalg M F) (λx. g x * q (real-cond-exp M F X x))
   apply (rule Bochner-Integration.integrable-bound[of - f], auto simp add: subalg
‹integrable (restr-to-subalg M F) f›)
   unfolding g-def by (auto simp add: field-split-simps abs-mult algebra-simps)
 have int2: integrable M (λx. G x * (X x - real-cond-exp M F X x))
  apply (rule Bochner-Integration.integrable-bound[of - λx. |X x| + |real-cond-exp
M F X x|])
  apply (auto intro!: Bochner-Integration.integrable-add integrable-abs real-cond-exp-int
‹integrable M X› AE-I2)
  using G unfolding abs-mult by (meson abs-ge-zero abs-triangle-ineq4 dual-order.trans
mult-left-le-one-le)
 have int3: integrable M (λx. g x * q (X x))
   apply (rule Bochner-Integration.integrable-bound[of - λx. q(X x)], auto simp
add: q(1) abs-mult)
   using g by (simp add: less-imp-le mult-left-le-one-le)
```

Taking the conditional expectation of the main convexity inequality `main_G`, we get the following.

```
 have AE x in M. real-cond-exp M F (λx. g x * q (X x)) x ≥ real-cond-exp M F
(λx. g x * q (real-cond-exp M F X x) + G x * (X x - real-cond-exp M F X x)) x
   apply (rule real-cond-exp-mono[OF main-G])
   apply (rule Bochner-Integration.integrable-add[OF integrable-from-subalg[OF
subalg int1]])
   using int2 int3 by auto
```

This reduces to the desired inequality thanks to the properties of conditional expectation, i.e., the conditional expectation of an F-measurable function is this function, and one can multiply an F-measurable function outside of conditional expectations. Since all these equalities only hold almost ev-

erywhere, we formulate them separately, and then combine all of them to simplify the above equation, again almost everywhere.

moreover have *AE x in M. real-cond-exp M F* (λ*x. g x * q* (*X x*)) *x = g x * real-cond-exp M F* (λ*x. q* (*X x*)) *x*
by (*rule real-cond-exp-mult*, *auto simp add*: *int3*)
moreover have *AE x in M. real-cond-exp M F* (λ*x. g x * q* (*real-cond-exp M F X x*) + *G x * (X x − real-cond-exp M F X x*)) *x*
= real-cond-exp M F (λ*x. g x * q* (*real-cond-exp M F X x*)) *x* + *real-cond-exp M F* (λ*x. G x * (X x − real-cond-exp M F X x*)) *x*
by (*rule real-cond-exp-add*, *auto simp add*: *integrable-from-subalg*[*OF subalg int1*] *int2*)
moreover have *AE x in M. real-cond-exp M F* (λ*x. g x * q* (*real-cond-exp M F X x*)) *x = g x * q* (*real-cond-exp M F X x*)
by (*rule real-cond-exp-F-meas*, *auto simp add*: *integrable-from-subalg*[*OF subalg int1*])
moreover have *AE x in M. real-cond-exp M F* (λ*x. G x * (X x − real-cond-exp M F X x*)) *x = G x * real-cond-exp M F* (λ*x.* (*X x − real-cond-exp M F X x*)) *x*
by (*rule real-cond-exp-mult*, *auto simp add*: *int2*)
moreover have *AE x in M. real-cond-exp M F* (λ*x.* (*X x − real-cond-exp M F X x*)) *x = real-cond-exp M F X x − real-cond-exp M F* (λ*x. real-cond-exp M F X x*) *x*
by (*rule real-cond-exp-diff*, *auto intro*!: *real-cond-exp-int* ‹*integrable M X*›)
moreover have *AE x in M. real-cond-exp M F* (λ*x. real-cond-exp M F X x*) *x = real-cond-exp M F X x*
by (*rule real-cond-exp-F-meas*, *auto intro*!: *real-cond-exp-int* ‹*integrable M X*›)
ultimately have *AE x in M. g x * real-cond-exp M F* (λ*x. q* (*X x*)) *x ≥ g x * q* (*real-cond-exp M F X x*)
by *auto*
then show *AE x in M. real-cond-exp M F* (λ*x. q* (*X x*)) *x ≥ q* (*real-cond-exp M F X x*)
using *g*(*1*) **by** (*auto simp add*: *field-split-simps*)
qed

Jensen's inequality does not imply that $q(E(X|F))$ is integrable, as it only proves an upper bound for it. Indeed, this is not true in general, as the following counterexample shows:

on $[1, \infty)$ with Lebesgue measure, let F be the sigma-algebra generated by the intervals $[n, n+1)$ for integer n. Let $q(x) = -\sqrt{x}$ for $x \geq 0$. Define X which is equal to $1/n$ over $[n, n+1/n)$ and 2^{-n} on $[n+1/n, n+1)$. Then X is integrable as $\sum 1/n^2 < \infty$, and $q(X)$ is integrable as $\sum 1/n^{3/2} < \infty$. On the other hand, $E(X|F)$ is essentially equal to $1/n^2$ on $[n, n+1)$ (we neglect the term 2^{-n}, we only put it there because X should take its values in $I = (0, \infty)$). Hence, $q(E(X|F))$ is equal to $-1/n$ on $[n, n+1)$, hence it is not integrable.

However, this counterexample is essentially the only situation where this function is not integrable, as shown by the next lemma.

lemma *integrable-convex-cond-exp*:

fixes $q :: real \Rightarrow real$
assumes X: *integrable M X AE x in M. X x ∈ I*
assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I = UNIV$
assumes q: *integrable M* $(\lambda x.\ q\ (X\ x))$ *convex-on I q q ∈ borel-measurable borel*
assumes H: *emeasure M (space M) = ∞ ⟹ 0 ∈ I*
shows *integrable M* $(\lambda x.\ q$ *(real-cond-exp M F X x))*
proof −
have [*measurable*]: $(\lambda x.\ q$ *(real-cond-exp M F X x)) ∈ borel-measurable M*
q ∈ borel-measurable borel
X ∈ borel-measurable M
using *X(1) q(3)* **by** *auto*
have *open I* **using** *I* **by** *auto*
then have *interior I = I* **by** (*simp add*: *interior-eq*)

consider *emeasure M (space M) = 0 | emeasure M (space M) > 0 ∧ emeasure M (space M) < ∞ | emeasure M (space M) = ∞*
by (*metis infinity-ennreal-def not-gr-zero top.not-eq-extremum*)
then show *?thesis*
proof (*cases*)
case *1*
show *?thesis* **by** (*subst integrable-cong-AE*[*of - - λx. 0*], *auto intro*: *emeasure-0-AE*[*OF 1*])
next
case *2*
interpret *finite-measure M* **using** *2* **by** (*auto intro*!: *finite-measureI*)

have $I \neq \{\}$
using ‹*AE x in M. X x ∈ I*› *2 eventually-mono integral-less-AE-space* **by** *fastforce*
then obtain z **where** $z \in I$ **by** *auto*

define A **where** $A = Inf\ ((\lambda t.\ (q\ z - q\ t)\ /\ (z - t))\ `\ (\{z<..\} \cap I))$
have $q\ y \geq q\ z + A * (y - z)$ **if** $y \in I$ **for** y **unfolding** *A-def* **apply** (*rule convex-le-Inf-differential*)
using ‹$z \in I$› ‹$y \in I$› ‹*interior I = I*› *q(2)* **by** *auto*
then have *AE x in M. q (real-cond-exp M F X x) ≥ q z + A * (real-cond-exp M F X x − z)*
using *real-cond-exp-jensens-inequality(1)*[*OF X I q*] **by** *auto*
moreover have *AE x in M. q (real-cond-exp M F X x) ≤ real-cond-exp M F* $(\lambda x.\ q\ (X\ x))\ x$
using *real-cond-exp-jensens-inequality(2)*[*OF X I q*] **by** *auto*
moreover have $|a| \leq |b| + |c|$ **if** $b \leq a \wedge a \leq c$ **for** $a\ b\ c$::*real*
using *that* **by** *auto*
ultimately have ∗: *AE x in M.* $|q$ *(real-cond-exp M F X x)|*
$\leq |$*real-cond-exp M F* $(\lambda x.\ q\ (X\ x))\ x| + |q\ z + A * ($*real-cond-exp M F X x − z*$)|$
by *auto*

show *integrable M* $(\lambda x.\ q$ *(real-cond-exp M F X x))*

apply (*rule Bochner-Integration.integrable-bound*[*of* - $\lambda x.$ |*real-cond-exp M F* ($\lambda x.$ *q* (*X x*)) *x*| + |*q z* + *A* ∗ (*real-cond-exp M F X x* − *z*)|])
apply (*auto intro*!: *Bochner-Integration.integrable-add integrable-abs integrable-mult-right Bochner-Integration.integrable-diff real-cond-exp-int*(*1*))
using *X*(*1*) *q*(*1*) ∗ **by** *auto*
next
case *3*
then have *0* ∈ *I* **using** *H finite-measure.finite-emeasure-space* **by** *auto*
have *q*(*0*) = *0*
proof (*rule ccontr*)
assume ∗: ¬(*q*(*0*) = *0*)
define *e* **where** *e* = |*q*(*0*)| / *2*
then have *e* > *0* **using** ∗ **by** *auto*
have *continuous* (*at 0*) *q*
using *q*(*2*) ‹*0* ∈ *I*› ‹*open I*› ‹*interior I* = *I*› *continuous-on-interior convex-on-continuous* **by** *blast*
then obtain *d* **where** *d*: *d* > *0* ⋀*y*. |*y* − *0*| < *d* ⟹ |*q y* − *q 0*| < *e* **using** ‹*e* > *0*›
by (*metis continuous-at-real-range real-norm-def*)
then have ∗: |*q*(*y*)| > *e* **if** |*y*| < *d* **for** *y*
proof −
have |*q 0*| ≤ |*q 0* − *q y*| + |*q y*| **by** *auto*
also have ... < *e* + |*q y*| **using** *d*(*2*) *that* **by** *force*
finally have |*q y*| > |*q 0*| − *e* **by** *auto*
then show *?thesis* **unfolding** *e-def* **by** *simp*
qed
have *emeasure M* {*x* ∈ *space M*. |*X x*| < *d*} ≤ *emeasure M* ({*x* ∈ *space M*. *1* ≤ *ennreal*(*1*/*e*) ∗ |*q*(*X x*)|})
by (*rule emeasure-mono*, *auto simp add*: ∗ ‹*e*>*0*› *less-imp-le ennreal-mult*′′[*symmetric*])
also have ... ≤ (*1*/*e*) ∗ (∫ +*x*. *ennreal*(|*q*(*X x*)|) ∗ *indicator* (*space M*) *x* ∂*M*)
by (*rule nn-integral-Markov-inequality*, *auto*)
also have ... = (*1*/*e*) ∗ (∫ +*x*. *ennreal*(|*q*(*X x*)|) ∂*M*) **by** *auto*
also have ... = (*1*/*e*) ∗ *ennreal*(∫ *x*. |*q*(*X x*)| ∂*M*)
using *nn-integral-eq-integral*[*OF integrable-abs*[*OF q*(*1*)]] **by** *auto*
also have ... < ∞
by (*simp add*: *ennreal-mult-less-top*)
finally have *A*: *emeasure M* {*x* ∈ *space M*. |*X x*| < *d*} < ∞ **by** *simp*

have {*x* ∈ *space M*. |*X x*| ≥ *d*} = {*x* ∈ *space M*. *1* ≤ *ennreal*(*1*/*d*) ∗ |*X x*|} ∩ *space M*
by (*auto simp add*: ‹*d*>*0*› *ennreal-mult*′′[*symmetric*])
then have *emeasure M* {*x* ∈ *space M*. |*X x*| ≥ *d*} = *emeasure M* ({*x* ∈ *space M*. *1* ≤ *ennreal*(*1*/*d*) ∗ |*X x*|})
by *auto*
also have ... ≤ (*1*/*d*) ∗ (∫ +*x*. *ennreal*(|*X x*|) ∗ *indicator* (*space M*) *x* ∂*M*)
by (*rule nn-integral-Markov-inequality*, *auto*)
also have ... = (*1*/*d*) ∗ (∫ +*x*. *ennreal*(|*X x*|) ∂*M*) **by** *auto*
also have ... = (*1*/*d*) ∗ *ennreal*(∫ *x*. |*X x*| ∂*M*)
using *nn-integral-eq-integral*[*OF integrable-abs*[*OF X*(*1*)]] **by** *auto*

also have ... $< \infty$
by (*simp add*: *ennreal-mult-less-top*)
finally have *B*: *emeasure M* $\{x \in$ *space M*. $|X\ x| \geq d\} < \infty$ **by** *simp*

have *space M* $= \{x \in$ *space M*. $|X\ x| < d\} \cup \{x \in$ *space M*. $|X\ x| \geq d\}$ **by** *auto*
then have *emeasure M* (*space M*) $=$ *emeasure M* $(\{x \in$ *space M*. $|X\ x| < d\}$ $\cup \{x \in$ *space M*. $|X\ x| \geq d\})$
by *simp*
also have ... $\leq$ *emeasure M* $\{x \in$ *space M*. $|X\ x| < d\}$ + *emeasure M* $\{x \in$ *space M*. $|X\ x| \geq d\}$
by (*auto intro*!: *emeasure-subadditive*)
also have ... $< \infty$ **using** *A B* **by** *auto*
finally show *False* **using** ‹*emeasure M* (*space M*) $= \infty$› **by** *auto*
qed

define *A* **where** *A* = *Inf* $((\lambda t.\ (q\ 0 - q\ t)\ /\ (0 - t))$ ‘ $(\{0<..\} \cap I))$
have *q y* $\geq$ *q 0* + *A* $*$ $(y - 0)$ **if** $y \in I$ **for** *y* **unfolding** *A-def* **apply** (*rule convex-le-Inf-differential*)
using ‹*0* $\in I$› ‹*y* $\in I$› ‹*interior I* = *I*› *q*(*2*) **by** *auto*
then have *q y* $\geq$ *A* $*$ *y* **if** $y \in I$ **for** *y* **using** ‹*q 0* = *0*› *that* **by** *auto*
then have *AE x in M*. *q* (*real-cond-exp M F X x*) $\geq$ *A* $*$ *real-cond-exp M F X x*
using *real-cond-exp-jensens-inequality*(*1*)[*OF X I q*] **by** *auto*
moreover have *AE x in M*. *q* (*real-cond-exp M F X x*) $\leq$ *real-cond-exp M F* $(\lambda x.\ q\ (X\ x))\ x$
using *real-cond-exp-jensens-inequality*(*2*)[*OF X I q*] **by** *auto*
moreover have $|a| \leq |b| + |c|$ **if** $b \leq a \wedge a \leq c$ **for** *a b c*::*real*
using *that* **by** *auto*
ultimately have $*$: *AE x in M*. $|q$ (*real-cond-exp M F X x*)$|$
$\leq |$*real-cond-exp M F* $(\lambda x.\ q\ (X\ x))\ x| + |A *$ *real-cond-exp M F X x*$|$
by *auto*

show *integrable M* $(\lambda x.\ q$ (*real-cond-exp M F X x*))
apply (*rule Bochner-Integration.integrable-bound*[*of* - $\lambda x.$ $|$*real-cond-exp M F* $(\lambda x.\ q\ (X\ x))\ x| + |A *$ *real-cond-exp M F X x* $|$])
apply (*auto intro*!: *Bochner-Integration.integrable-add integrable-abs integrable-mult-right Bochner-Integration.integrable-diff real-cond-exp-int*(*1*))
using *X*(*1*) *q*(*1*) $*$ **by** *auto*
qed
qed

end

end

theory *Essential-Supremum*
imports *HOL−Analysis.Analysis*

begin

lemma *ae-filter-eq-bot-iff*: *ae-filter M = bot* $\longleftrightarrow$ *emeasure M* (*space M*) = *0*
by (*simp add*: *AE-iff-measurable trivial-limit-def*)

29 The essential supremum

In this paragraph, we define the essential supremum and give its basic properties. The essential supremum of a function is its maximum value if one is allowed to throw away a set of measure 0. It is convenient to define it to be infinity for non-measurable functions, as it allows for neater statements in general. This is a prerequisiste to define the space L^∞.

definition *esssup*::*'a measure* $\Rightarrow$ (*'a* $\Rightarrow$ *'b*::{*second-countable-topology*, *dense-linorder*, *linorder-topology*, *complete-linorder*}) $\Rightarrow$ *'b*
where *esssup M f* = (*if f* $\in$ *borel-measurable M then Limsup* (*ae-filter M*) *f else top*)

lemma *esssup-non-measurable*: *f* $\notin$ *M* $\rightarrow_M$ *borel* $\Longrightarrow$ *esssup M f = top*
by (*simp add*: *esssup-def*)

lemma *esssup-eq-AE*:
assumes *f*: *f* $\in$ *M* $\rightarrow_M$ *borel* **shows** *esssup M f = Inf* {*z*. *AE x in M*. *f x* $\leq$ *z*}
unfolding *esssup-def if-P*[*OF f*] *Limsup-def*
proof (*intro antisym INF-greatest Inf-greatest*; *clarsimp*)
fix *y* **assume** *AE x in M*. *f x* $\leq$ *y*
then have (λ*x*. *f x* $\leq$ *y*) $\in$ {*P*. *AE x in M*. *P x*}
by *simp*
then show (*INF P*$\in${*P*. *AE x in M*. *P x*}. *SUP x*$\in$*Collect P*. *f x*) $\leq$ *y*
by (*rule INF-lower2*) (*auto intro*: *SUP-least*)
next
fix *P* **assume** *P*: *AE x in M*. *P x*
show *Inf* {*z*. *AE x in M*. *f x* $\leq$ *z*} $\leq$ (*SUP x*$\in$*Collect P*. *f x*)
proof (*rule Inf-lower*; *clarsimp*)
show *AE x in M*. *f x* $\leq$ (*SUP x*$\in$*Collect P*. *f x*)
using *P* **by** (*auto elim*: *eventually-mono simp*: *SUP-upper*)
qed
qed

lemma *esssup-eq*: *f* $\in$ *M* $\rightarrow_M$ *borel* $\Longrightarrow$ *esssup M f = Inf* {*z*. *emeasure M* {*x* $\in$ *space M*. *f x* > *z*} = *0*}
by (*auto simp add*: *esssup-eq-AE not-less*[*symmetric*] *AE-iff-measurable*[*OF - refl*] *intro*!: *arg-cong*[**where** *f*=*Inf*])

lemma *esssup-zero-measure*:
emeasure M {*x* $\in$ *space M*. *f x* > *esssup M f*} = *0*
proof (*cases esssup M f = top*)
case *True*
then show *?thesis* **by** *auto*

next
case *False*
then have *f*[*measurable*]: $f \in M \to_M$ *borel* **unfolding** *esssup-def* **by** *meson*
have *esssup M f* < *top* **using** *False* **by** (*auto simp*: *less-top*)
have ∗: $\{x \in$ *space M*. *f x* > *z*$\} \in$ *null-sets M* **if** *z* > *esssup M f* **for** *z*
proof −
have ∃ *w*. *w* < *z* ∧ *emeasure M* $\{x \in$ *space M*. *f x* > *w*$\}$ = *0*
using ‹*z* > *esssup M f*› *f* **by** (*auto simp*: *esssup-eq Inf-less-iff*)
then obtain *w* **where** *w* < *z emeasure M* $\{x \in$ *space M*. *f x* > *w*$\}$ = *0* **by** *auto*
then have *a*: $\{x \in$ *space M*. *f x* > *w*$\} \in$ *null-sets M* **by** *auto*
have *b*: $\{x \in$ *space M*. *f x* > *z*$\} \subseteq \{x \in$ *space M*. *f x* > *w*$\}$ **using** ‹*w* < *z*› **by** *auto*
show *?thesis* **using** *null-sets-subset*[*OF a - b*] **by** *simp*
qed
obtain *u*::*nat* ⇒ *'b* **where** *u*: $\bigwedge n.\ u\ n >$ *esssup M f u* ⟶ *esssup M f*
using *approx-from-above-dense-linorder*[*OF* ‹*esssup M f* < *top*›] **by** *auto*
have $\{x \in$ *space M*. *f x* > *esssup M f*$\} = (\bigcup n.\ \{x \in$ *space M*. *f x* > *u n*$\})$
using *u* **apply** *auto*
apply (*metis* (*mono-tags*, *lifting*) *order-tendsto-iff eventually-mono LIMSEQ-unique*)
using *less-imp-le less-le-trans* **by** *blast*
also have ... ∈ *null-sets M*
using ∗[*OF u*(*1*)] **by** *auto*
finally show *?thesis* **by** *auto*
qed

lemma *esssup-AE*: *AE x in M*. *f x* ≤ *esssup M f*
proof (*cases f* ∈ *M* →$_M$ *borel*)
case *True* **then show** *?thesis*
by (*intro AE-I*[*OF - esssup-zero-measure*[*of - f*]]) *auto*
qed (*simp add*: *esssup-non-measurable*)

lemma *esssup-pos-measure*:
f ∈ *borel-measurable M* ⟹ *z* < *esssup M f* ⟹ *emeasure M* $\{x \in$ *space M*. *f x* > *z*$\}$ > *0*
using *Inf-less-iff mem-Collect-eq not-gr-zero* **by** (*force simp*: *esssup-eq*)

lemma *esssup-I* [*intro*]: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *c* ⟹ *esssup M f* ≤ *c*
unfolding *esssup-def* **by** (*simp add*: *Limsup-bounded*)

lemma *esssup-AE-mono*: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *g x* ⟹ *esssup M f* ≤ *esssup M g*
by (*auto simp*: *esssup-def Limsup-mono*)

lemma *esssup-mono*: *f* ∈ *borel-measurable M* ⟹ (⋀*x*. *f x* ≤ *g x*) ⟹ *esssup M f* ≤ *esssup M g*
by (*rule esssup-AE-mono*) *auto*

lemma *esssup-AE-cong*:
f ∈ *borel-measurable M* ⟹ *g* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* = *g x*
⟹ *esssup M f* = *esssup M g*
by (*auto simp*: *esssup-def intro*!: *Limsup-eq*)

lemma *esssup-const*: *emeasure M* (*space M*) ≠ *0* ⟹ *esssup M* (λ*x*. *c*) = *c*
by (*simp add*: *esssup-def Limsup-const ae-filter-eq-bot-iff*)

lemma *esssup-cmult*: **assumes** *c* > (*0*::*real*) **shows** *esssup M* (λ*x*. *c* ∗ *f x*::*ereal*)
= *c* ∗ *esssup M f*
proof −
have (λ*x*. *ereal c* ∗ *f x*) ∈ *M* →$_M$ *borel* ⟹ *f* ∈ *M* →$_M$ *borel*
proof (*subst measurable-cong*)
fix ω **show** *f* ω = *ereal* (*1*/*c*) ∗ (*ereal c* ∗ *f* ω)
using ‹*0* < *c*› **by** (*cases f* ω) *auto*
qed *auto*
then have (λ*x*. *ereal c* ∗ *f x*) ∈ *M* →$_M$ *borel* ⟷ *f* ∈ *M* →$_M$ *borel*
by(*safe intro*!: *borel-measurable-ereal-times borel-measurable-const*)
with ‹*0*<*c*› **show** *?thesis*
by (*cases ae-filter M* = *bot*)
(*auto simp*: *esssup-def bot-ereal-def top-ereal-def Limsup-ereal-mult-left*)
qed

lemma *esssup-add*:
esssup M (λ*x*. *f x* + *g x*::*ereal*) ≤ *esssup M f* + *esssup M g*
proof (*cases f* ∈ *borel-measurable M* ∧ *g* ∈ *borel-measurable M*)
case *True*
then have [*measurable*]: (λ*x*. *f x* + *g x*) ∈ *borel-measurable M* **by** *auto*
have *f x* + *g x* ≤ *esssup M f* + *esssup M g* **if** *f x* ≤ *esssup M f g x* ≤ *esssup M*
g **for** *x*
using *that add-mono* **by** *auto*
then have *AE x in M*. *f x* + *g x* ≤ *esssup M f* + *esssup M g*
using *esssup-AE*[*of f M*] *esssup-AE*[*of g M*] **by** *auto*
then show *?thesis* **using** *esssup-I* **by** *auto*
next
case *False*
then have *esssup M f* + *esssup M g* = ∞ **unfolding** *esssup-def top-ereal-def*
by *auto*
then show *?thesis* **by** *auto*
qed

lemma *esssup-zero-space*:
emeasure M (*space M*) = *0* ⟹ *f* ∈ *borel-measurable M* ⟹ *esssup M f* = (−
∞::*ereal*)
by (*simp add*: *esssup-def ae-filter-eq-bot-iff*[*symmetric*] *bot-ereal-def*)

end

30 Stopping times

theory *Stopping-Time*
imports *HOL−Analysis.Analysis*
begin

30.1 Stopping Time

This is also called strong stopping time. Then stopping time is T with alternative is $T\ x < t$ measurable.

definition *stopping-time* :: ('*t*::*linorder* ⇒ '*a measure*) ⇒ ('*a* ⇒ '*t*) ⇒ *bool*
where
stopping-time F T = (∀ *t*. *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*))

lemma *stopping-time-cong*: (⋀*t x*. *x* ∈ *space* (*F t*) ⟹ *T x* = *S x*) ⟹ *stopping-time F T* = *stopping-time F S*
unfolding *stopping-time-def* **by** (*intro arg-cong*[**where** *f*=*All*] *ext measurable-cong*) *simp*

lemma *stopping-timeD*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*)
by (*auto simp*: *stopping-time-def*)

lemma *stopping-timeD2*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *t* < *T x*)
unfolding *not-le*[*symmetric*] **by** (*auto intro*: *stopping-timeD Measurable.pred-intros-logic*)

lemma *stopping-timeI*[*intro?*]: (⋀*t*. *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*)) ⟹ *stopping-time F T*
by (*auto simp*: *stopping-time-def*)

lemma *measurable-stopping-time*:
fixes *T* :: '*a* ⇒ '*t*::{*linorder-topology*, *second-countable-topology*}
assumes *T*: *stopping-time F T*
and *M*: ⋀*t*. *sets* (*F t*) ⊆ *sets M* ⋀*t*. *space* (*F t*) = *space M*
shows *T* ∈ *M* →$_M$ *borel*
proof (*rule borel-measurableI-le*)
show {*x* ∈ *space M*. *T x* ≤ *t*} ∈ *sets M* **for** *t*
using *stopping-timeD*[*OF T*] *M* **by** (*auto simp*: *Measurable.pred-def*)
qed

lemma *stopping-time-const*: *stopping-time F* (λ*x*. *c*)
by (*auto simp*: *stopping-time-def*)

lemma *stopping-time-min*:
stopping-time F T ⟹ *stopping-time F S* ⟹ *stopping-time F* (λ*x*. *min* (*T x*) (*S x*))
by (*auto simp*: *stopping-time-def min-le-iff-disj intro*!: *pred-intros-logic*)

lemma *stopping-time-max*:
stopping-time F T $\Longrightarrow$ *stopping-time F S* $\Longrightarrow$ *stopping-time F* ($\lambda x.$ *max* (*T x*) (*S x*))
by (*auto simp*: *stopping-time-def intro*!: *pred-intros-logic*)

31 Filtration

locale *filtration* =
fixes Ω :: *'a set* **and** *F* :: *'t*::{*linorder-topology*, *second-countable-topology*} $\Rightarrow$ *'a measure*
assumes *space-F*: $\bigwedge i.$ *space* (*F i*) = Ω
assumes *sets-F-mono*: $\bigwedge i\ j.$ *i* $\leq$ *j* $\Longrightarrow$ *sets* (*F i*) $\subseteq$ *sets* (*F j*)
begin

31.1 σ-algebra of a Stopping Time

definition *pre-sigma* :: (*'a* $\Rightarrow$ *'t*) $\Rightarrow$ *'a measure*
where
pre-sigma T = *sigma* Ω {*A*. $\forall t.$ {$\omega$$\in$*A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)}

lemma *space-pre-sigma*: *space* (*pre-sigma T*) = Ω
unfolding *pre-sigma-def* **using** *sets.space-closed*[*of F* -]
by (*intro space-measure-of*) (*auto simp*: *space-F*)

lemma *measure-pre-sigma*[*simp*]: *emeasure* (*pre-sigma T*) = (λ-. *0*)
by (*simp add*: *pre-sigma-def emeasure-sigma*)

lemma *sigma-algebra-pre-sigma*:
assumes *T*: *stopping-time F T*
shows *sigma-algebra* Ω {*A*. $\forall t.$ {$\omega$$\in$*A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)}
unfolding *sigma-algebra-iff2*
proof (*intro sigma-algebra-iff2*[*THEN iffD2*] *conjI ballI allI impI CollectI*)
show {*A*. $\forall t.$ {ω $\in$ *A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)} $\subseteq$ *Pow* Ω
using *sets.space-closed*[*of F* -] **by** (*auto simp*: *space-F*)
next
fix *A t* **assume** *A* $\in$ {*A*. $\forall t.$ {ω $\in$ *A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)}
then have {ω $\in$ *space* (*F t*). *T* ω $\leq$ *t*} $-$ {ω $\in$ *A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)
using *T stopping-timeD*[*measurable*] **by** *auto*
also have {ω $\in$ *space* (*F t*). *T* ω $\leq$ *t*} $-$ {ω $\in$ *A*. *T* ω $\leq$ *t*} = {ω $\in$ Ω $-$ *A*. *T* ω $\leq$ *t*}
by (*auto simp*: *space-F*)
finally show {ω $\in$ Ω $-$ *A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*) .
next
fix *AA* :: *nat* $\Rightarrow$ *'a set* **and** *t* **assume** *range AA* $\subseteq$ {*A*. $\forall t.$ {ω $\in$ *A*. *T* ω $\leq$ *t*} $\in$ *sets* (*F t*)}
then have ($\bigcup i.$ {ω $\in$ *AA i*. *T* ω $\leq$ *t*}) $\in$ *sets* (*F t*) **for** *t*
by *auto*
also have ($\bigcup i.$ {ω $\in$ *AA i*. *T* ω $\leq$ *t*}) = {ω $\in$ $\bigcup$(*AA* ' *UNIV*). *T* ω $\leq$ *t*}
by *auto*

finally show $\{\omega \in \bigcup (AA \text{ ` } UNIV).\ T\ \omega \leq t\} \in sets\ (F\ t)$.
qed *auto*

lemma *sets-pre-sigma*: *stopping-time F T* $\Longrightarrow$ *sets* (*pre-sigma T*) = $\{A.\ \forall\, t.\ \{\omega \in A.\ T\ \omega \leq t\} \in sets\ (F\ t)\}$
unfolding *pre-sigma-def* **by** (*rule sigma-algebra.sets-measure-of-eq*[*OF sigma-algebra-pre-sigma*])

lemma *sets-pre-sigmaI*: *stopping-time F T* $\Longrightarrow$ ($\bigwedge t.\ \{\omega \in A.\ T\ \omega \leq t\} \in sets\ (F\ t)$) $\Longrightarrow$ $A \in sets\ (pre\text{-}sigma\ T)$
unfolding *sets-pre-sigma* **by** *auto*

lemma *pred-pre-sigmaI*:
assumes *T*: *stopping-time F T*
shows ($\bigwedge t.$ *Measurable.pred* (*F t*) ($\lambda\omega.\ P\ \omega \wedge T\ \omega \leq t$)) $\Longrightarrow$ *Measurable.pred* (*pre-sigma T*) *P*
unfolding *pred-def space-F space-pre-sigma* **by** (*intro sets-pre-sigmaI*[*OF T*]) *simp*

lemma *sets-pre-sigmaD*: *stopping-time F T* $\Longrightarrow$ $A \in sets\ (pre\text{-}sigma\ T)$ $\Longrightarrow$ $\{\omega \in A.\ T\ \omega \leq t\} \in sets\ (F\ t)$
unfolding *sets-pre-sigma* **by** *auto*

lemma *stopping-time-le-const*: *stopping-time F T* $\Longrightarrow$ $s \leq t$ $\Longrightarrow$ *Measurable.pred* (*F t*) ($\lambda\omega.\ T\ \omega \leq s$)
using *stopping-timeD*[*of F T*] *sets-F-mono*[*of - t*] **by** (*auto simp*: *pred-def space-F*)

lemma *measurable-stopping-time-pre-sigma*:
assumes *T*: *stopping-time F T* **shows** $T \in pre\text{-}sigma\ T \rightarrow_M borel$
proof (*intro borel-measurableI-le sets-pre-sigmaI*[*OF T*])
fix *t t'*
have $\{\omega \in space\ (F\ (min\ t'\ t)).\ T\ \omega \leq min\ t'\ t\} \in sets\ (F\ (min\ t'\ t))$
using *T* **unfolding** *pred-def*[*symmetric*] **by** (*rule stopping-timeD*)
also have $\ldots \subseteq sets\ (F\ t)$
by (*rule sets-F-mono*) *simp*
finally show $\{\omega \in \{x \in space\ (pre\text{-}sigma\ T).\ T\ x \leq t'\}.\ T\ \omega \leq t\} \in sets\ (F\ t)$
by (*simp add*: *space-pre-sigma space-F*)
qed

lemma *mono-pre-sigma*:
assumes *T*: *stopping-time F T* **and** *S*: *stopping-time F S*
and *le*: $\bigwedge\omega.\ \omega \in \Omega \Longrightarrow T\ \omega \leq S\ \omega$
shows *sets* (*pre-sigma T*) $\subseteq$ *sets* (*pre-sigma S*)
unfolding *sets-pre-sigma*[*OF S*] *sets-pre-sigma*[*OF T*]
proof *safe*
interpret *sigma-algebra* Ω $\{A.\ \forall\, t.\ \{\omega \in A.\ T\ \omega \leq t\} \in sets\ (F\ t)\}$
using *T* **by** (*rule sigma-algebra-pre-sigma*)
fix *A t* **assume** *A*: $\forall\, t.\ \{\omega \in A.\ T\ \omega \leq t\} \in sets\ (F\ t)$
then have $A \subseteq \Omega$
using *sets-into-space* **by** *auto*

from A **have** $\{\omega{\in}A.\ T\ \omega \le t\} \cap \{\omega{\in}space\ (F\ t).\ S\ \omega \le t\} \in sets\ (F\ t)$
using *stopping-timeD*[*OF S*] **by** (*auto simp*: *pred-def*)
also have $\{\omega{\in}A.\ T\ \omega \le t\} \cap \{\omega{\in}space\ (F\ t).\ S\ \omega \le t\} = \{\omega{\in}A.\ S\ \omega \le t\}$
using ‹$A \subseteq \Omega$› *sets-into-space*[*of A*] *le* **by** (*auto simp*: *space-F intro*: *order-trans*)
finally show $\{\omega{\in}A.\ S\ \omega \le t\} \in sets\ (F\ t)$
by *auto*
qed

lemma *stopping-time-less-const*:
assumes *T*: *stopping-time F T* **shows** *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega < t)$
proof −
obtain $D :: 't\ set$
where *D*: *countable D* $\bigwedge X.$ *open* $X \Longrightarrow X \neq \{\} \Longrightarrow \exists\, d{\in}D.\ d \in X$
using *countable-dense-setE* **by** *auto*
show *?thesis*
proof *cases*
assume $*$: $\forall\, t'{<}t.\ \exists\, t''.\ t' < t'' \wedge t'' < t$
{ fix t' **assume** $t' < t$
with $*$ **have** $\{t' <..< t\} \neq \{\}$
by *fastforce*
with *D*(*2*)[*OF - this*]
have $\exists\, d{\in}D.\ t' < d \wedge d < t$
by *auto* **}**
note $** =$ *this*

show *?thesis*
proof (*rule measurable-cong*[*THEN iffD2*])
show $T\ \omega < t \longleftrightarrow (\exists\, r{\in}\{r{\in}D.\ r < t\}.\ T\ \omega \le r)$ **for** ω
by (*auto dest*: $**$ *intro*: *less-imp-le*)
show *Measurable.pred* $(F\ t)\ (\lambda w.\ \exists\, r{\in}\{r \in D.\ r < t\}.\ T\ w \le r)$
by (*intro measurable-pred-countable stopping-time-le-const*[*OF T*] *countable-Collect D*) *auto*
qed
next
assume $\neg\ (\forall\, t'{<}t.\ \exists\, t''.\ t' < t'' \wedge t'' < t)$
then obtain t' **where** t': $t' < t \bigwedge t''.\ t'' < t \Longrightarrow t'' \le t'$
by (*auto simp*: *not-less*[*symmetric*])
show *?thesis*
proof (*rule measurable-cong*[*THEN iffD2*])
show $T\ \omega < t \longleftrightarrow T\ \omega \le t'$ **for** ω
using t' **by** *auto*
show *Measurable.pred* $(F\ t)\ (\lambda w.\ T\ w \le t')$
using ‹$t'{<}t$› **by** (*intro stopping-time-le-const*[*OF T*]) *auto*
qed
qed
qed

lemma *stopping-time-eq-const*: *stopping-time F T* $\Longrightarrow$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega = t)$

unfolding *eq-iff* **using** *stopping-time-less-const*[*of T t*]
by (*intro pred-intros-logic stopping-time-le-const*) (*auto simp*: *not-less*[*symmetric*]
)

lemma *stopping-time-less*:
assumes *T*: *stopping-time F T* **and** *S*: *stopping-time F S*
shows *Measurable.pred* (*pre-sigma T*) (λω. *T ω* < *S ω*)
proof (*rule pred-pre-sigmaI*[*OF T*])
fix *t*
obtain *D* :: *'t set*
where [*simp*]: *countable D* **and** *semidense-D*: ⋀*x y*. *x* < *y* ⟹ (∃ *b*∈*D*. *x* ≤ *b*
∧ *b* < *y*)
using *countable-separating-set-linorder2* **by** *auto*
show *Measurable.pred* (*F t*) (λω. *T ω* < *S ω* ∧ *T ω* ≤ *t*)
proof (*rule measurable-cong*[*THEN iffD2*])
let *?f* = λω. *if T ω* = *t then* ¬ *S ω* ≤ *t else* ∃ *s*∈{*s*∈*D*. *s* ≤ *t*}. *T ω* ≤ *s* ∧ ¬
(*S ω* ≤ *s*)
{ fix ω **assume** *T ω* ≤ *t T ω* ≠ *t T ω* < *S ω*
then have *T ω* < *min t* (*S ω*)
by *auto*
then obtain *r* **where** *r* ∈ *D T ω* ≤ *r r* < *min t* (*S ω*)
by (*metis semidense-D*)
then have ∃ *s*∈{*s*∈*D*. *s* ≤ *t*}. *T ω* ≤ *s* ∧ *s* < *S ω*
by *auto* **}**
then show (*T ω* < *S ω* ∧ *T ω* ≤ *t*) = *?f ω* **for** ω
by (*auto simp*: *not-le*)
show *Measurable.pred* (*F t*) *?f*
by (*intro pred-intros-logic measurable-If measurable-pred-countable count-
able-Collect*
stopping-time-le-const predE stopping-time-eq-const T S)
auto
qed
qed

end

lemma *stopping-time-SUP-enat*:
fixes *T* :: *nat* ⇒ (*'a* ⇒ *enat*)
shows (⋀*i*. *stopping-time F* (*T i*)) ⟹ *stopping-time F* (*SUP i*. *T i*)
unfolding *stopping-time-def SUP-apply SUP-le-iff* **by** (*auto intro*!: *pred-intros-countable*)

lemma *less-eSuc-iff*: *a* < *eSuc b* ⟷ (*a* ≤ *b* ∧ *a* ≠ ∞)
by (*cases a*) *auto*

lemma *stopping-time-Inf-enat*:
fixes *F* :: *enat* ⇒ *'a measure*
assumes *F*: *filtration* Ω *F*
assumes *P*: ⋀*i*. *Measurable.pred* (*F i*) (*P i*)
shows *stopping-time F* (λω. *Inf* {*i*. *P i ω*})

proof (*rule stopping-timeI*, *cases*)
fix t :: *enat* **assume** $t = \infty$ **then show** *Measurable.pred* (*F t*) ($\lambda\omega$. *Inf* {*i*. *P i* ω} $\leq$ *t*)
by *auto*
next
fix t :: *enat* **assume** $t \neq \infty$
moreover
{ fix *i* ω **assume** *Inf* {*i*. *P i* ω} $\leq$ *t*
with ‹*t* $\neq \infty$› **have** ($\exists$ *i*$\leq$*t*. *P i* ω)
unfolding *Inf-le-iff* **by** (*cases t*) (*auto elim*!: *allE*[*of* - *eSuc t*] *simp*: *less-eSuc-iff*)
}
ultimately have $*$: $\bigwedge\omega$. *Inf* {*i*. *P i* ω} $\leq$ *t* $\longleftrightarrow$ ($\exists$ *i*$\in${..*t*}. *P i* ω)
by (*auto intro*!: *Inf-lower2*)
show *Measurable.pred* (*F t*) ($\lambda\omega$. *Inf* {*i*. *P i* ω} $\leq$ *t*)
unfolding $*$ **using** *filtration.sets-F-mono*[*OF F*, *of* - *t*] *P*
by (*intro pred-intros-countable-bounded*) (*auto simp*: *pred-def filtration.space-F*[*OF F*])
qed

lemma *stopping-time-Inf-nat*:
fixes *F* :: *nat* $\Rightarrow$ *'a measure*
assumes *F*: *filtration* Ω *F*
assumes *P*: $\bigwedge$*i*. *Measurable.pred* (*F i*) (*P i*) **and** *wf*: $\bigwedge$*i* ω. $\omega \in \Omega \Longrightarrow \exists$ *n*. *P n* ω
shows *stopping-time F* ($\lambda\omega$. *Inf* {*i*. *P i* ω})
unfolding *stopping-time-def*
proof (*intro allI*, *subst measurable-cong*)
fix *t* ω **assume** $\omega \in$ *space* (*F t*)
then have $\omega \in \Omega$
using *filtration.space-F*[*OF F*] **by** *auto*
from *wf*[*OF this*] **have** ((*LEAST n*. *P n* ω) $\leq$ *t*) = ($\exists$ *i*$\leq$*t*. *P i* ω)
by (*rule LeastI2-wellorder-ex*) *auto*
then show (*Inf* {*i*. *P i* ω} $\leq$ *t*) = ($\exists$ *i*$\in${..*t*}. *P i* ω)
by (*simp add*: *Inf-nat-def Bex-def*)
next
fix *t* **from** *P* **show** *Measurable.pred* (*F t*) (λw. $\exists$ *i*$\in${..*t*}. *P i w*)
using *filtration.sets-F-mono*[*OF F*, *of* - *t*]
by (*intro pred-intros-countable-bounded*) (*auto simp*: *pred-def filtration.space-F*[*OF F*])
qed

end

theory *Probability*
imports
Central-Limit-Theorem
Discrete-Topology
PMF-Impl

Projective-Limit
Random-Permutations
SPMF
Product-PMF
Hoeffding
Stream-Space
Tree-Space
Conditional-Expectation
Essential-Supremum
Stopping-Time
begin

end