Hoare Logic for Parallel Programs

Leonor Prensa Nieto

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Abstract

In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [1].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

For excellent descriptions of this work see [2, 4, 1, 3].

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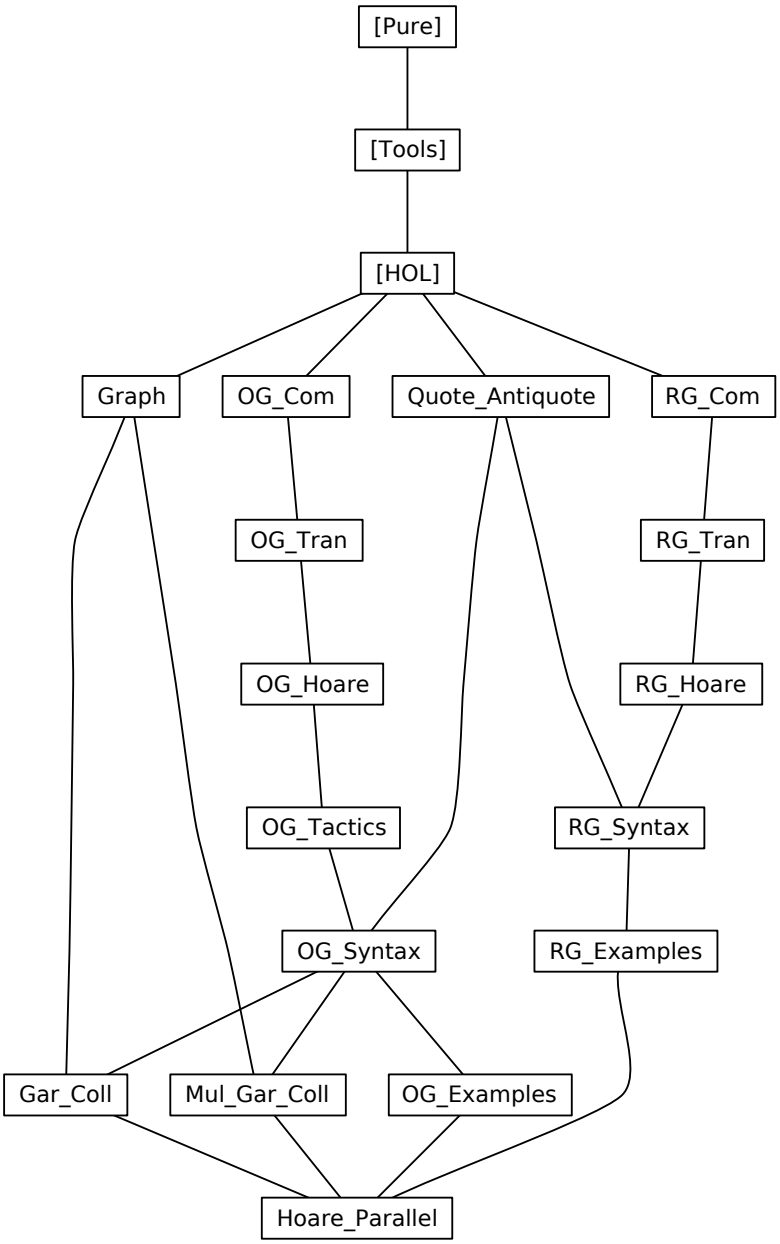
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Chapter 1

The Owicki-Gries Method

1.1 Abstract Syntax

theory *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

type-synonym *'a bexp* = *'a set*
type-synonym *'a assn* = *'a set*

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

datatype *'a ann-com* =
    AnnBasic (*'a assn*) (*'a* ⇒ *'a*)
  | *AnnSeq* (*'a ann-com*) (*'a ann-com*)
  | *AnnCond1* (*'a assn*) (*'a bexp*) (*'a ann-com*) (*'a ann-com*)
  | *AnnCond2* (*'a assn*) (*'a bexp*) (*'a ann-com*)
  | *AnnWhile* (*'a assn*) (*'a bexp*) (*'a assn*) (*'a ann-com*)
  | *AnnAwait* (*'a assn*) (*'a bexp*) (*'a com*)
and *'a com* =
    Parallel (*'a ann-com option* × *'a assn*) *list*
  | *Basic* (*'a* ⇒ *'a*)
  | *Seq* (*'a com*) (*'a com*)
  | *Cond* (*'a bexp*) (*'a com*) (*'a com*)
  | *While* (*'a bexp*) (*'a assn*) (*'a com*)

The function *pre* extracts the precondition of an annotated command:

primrec *pre* ::*'a ann-com* ⇒ *'a assn* **where**
  pre (*AnnBasic r f*) = *r*
| *pre* (*AnnSeq c1 c2*) = *pre c1*
| *pre* (*AnnCond1 r b c1 c2*) = *r*
| *pre* (*AnnCond2 r b c*) = *r*
| *pre* (*AnnWhile r b i c*) = *r*
| *pre* (*AnnAwait r b c*) = *r*

Well-formedness predicate for atomic programs:

primrec *atom-com* :: *'a com* ⇒ *bool* **where**
  atom-com (*Parallel Ts*) = *False*
| *atom-com* (*Basic f*) = *True*
| *atom-com* (*Seq c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
| *atom-com* (*Cond b c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
| *atom-com* (*While b i c*) = *atom-com c*

end

1.2 Operational Semantics

theory *OG-Tran* **imports** *OG-Com* **begin**

type-synonym *'a ann-com-op* = (*'a ann-com*) *option*
type-synonym *'a ann-triple-op* = (*'a ann-com-op* × *'a assn*)

primrec *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op* **where**
  com (*c*, *q*) = *c*

primrec *post* :: *'a ann-triple-op* ⇒ *'a assn* **where**
  post (*c*, *q*) = *q*

definition *All-None* :: *'a ann-triple-op list* ⇒ *bool* **where**
  All-None Ts ≡ ∀(*c*, *q*) ∈ *set Ts*. *c* = *None*

1.2.1 The Transition Relation

inductive-set
  ann-transition :: ((*'a ann-com-op* × *'a*) × (*'a ann-com-op* × *'a*)) *set*
  and *transition* :: ((*'a com* × *'a*) × (*'a com* × *'a*)) *set*
  and *ann-transition'* :: (*'a ann-com-op* × *'a*) ⇒ (*'a ann-com-op* × *'a*) ⇒ *bool*
    (- −*1*→ -[*81*,*81*] *100*)
  and *transition'* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
    (- −*P1*→ -[*81*,*81*] *100*)
  and *transitions* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
    (- −*P*∗→ -[*81*,*81*] *100*)
where
  con-0 −*1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *ann-transition*
| *con-0* −*P1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*
| *con-0* −*P*∗→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*∗

| *AnnBasic*: (*Some* (*AnnBasic r f*), *s*) −*1*→ (*None*, *f s*)

| *AnnSeq1*: (*Some c0*, *s*) −*1*→ (*None*, *t*) ⟹
            (*Some* (*AnnSeq c0 c1*), *s*) −*1*→ (*Some c1*, *t*)
| *AnnSeq2*: (*Some c0*, *s*) −*1*→ (*Some c2*, *t*) ⟹
            (*Some* (*AnnSeq c0 c1*), *s*) −*1*→ (*Some* (*AnnSeq c2 c1*), *t*)

| *AnnCond1T*: *s* ∈ *b* ⟹ (*Some* (*AnnCond1 r b c1 c2*), *s*) −*1*→ (*Some c1*, *s*)

| *AnnCond1F*: $s \notin b \Longrightarrow (Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\rightarrow (Some\ c2,\ s)$

| *AnnCond2T*: $s \in b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (Some\ c,\ s)$
| *AnnCond2F*: $s \notin b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (None,\ s)$

| *AnnWhileF*: $s \notin b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s) -1\rightarrow (None,\ s)$
| *AnnWhileT*: $s \in b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s) -1\rightarrow$
$(Some\ (AnnSeq\ c\ (AnnWhile\ i\ b\ i\ c)),\ s)$

| *AnnAwait*: $[\![\ s \in b;\ atom\text{-}com\ c;\ (c,\ s) -P*\rightarrow (Parallel\ [],\ t)\ ]\!] \Longrightarrow$
$(Some\ (AnnAwait\ r\ b\ c),\ s) -1\rightarrow (None,\ t)$

| *Parallel*: $[\![\ i<length\ Ts;\ Ts!i = (Some\ c,\ q);\ (Some\ c,\ s) -1\rightarrow (r,\ t)\ ]\!]$
$\Longrightarrow (Parallel\ Ts,\ s) -P1\rightarrow (Parallel\ (Ts\ [i:=(r,\ q)]),\ t)$

| *Basic*: $(Basic\ f,\ s) -P1\rightarrow (Parallel\ [],\ f\ s)$

| *Seq1*: $All\text{-}None\ Ts \Longrightarrow (Seq\ (Parallel\ Ts)\ c,\ s) -P1\rightarrow (c,\ s)$
| *Seq2*: $(c0,\ s) -P1\rightarrow (c2,\ t) \Longrightarrow (Seq\ c0\ c1,\ s) -P1\rightarrow (Seq\ c2\ c1,\ t)$

| *CondT*: $s \in b \Longrightarrow (Cond\ b\ c1\ c2,\ s) -P1\rightarrow (c1,\ s)$
| *CondF*: $s \notin b \Longrightarrow (Cond\ b\ c1\ c2,\ s) -P1\rightarrow (c2,\ s)$

| *WhileF*: $s \notin b \Longrightarrow (While\ b\ i\ c,\ s) -P1\rightarrow (Parallel\ [],\ s)$
| *WhileT*: $s \in b \Longrightarrow (While\ b\ i\ c,\ s) -P1\rightarrow (Seq\ c\ (While\ b\ i\ c),\ s)$

monos *rtrancl-mono*

The corresponding abbreviations are:

abbreviation
ann-transition-n :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow nat \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a)$
$\Rightarrow bool$ (- -→ -[81,81] 100) **where**
con-0 $-n\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition\ ^^\ n$

abbreviation
ann-transitions :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow bool$
(- −*→ -[81,81] 100) **where**
con-0 $-*\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition^*$

abbreviation
transition-n :: $('a\ com \times 'a) \Rightarrow nat \Rightarrow ('a\ com \times 'a) \Rightarrow bool$
(- −P-→ -[81,81,81] 100) **where**
con-0 $-Pn\rightarrow$ *con-1* $\equiv (con\text{-}0,\ con\text{-}1) \in transition\ ^^\ n$

1.2.2 Definition of Semantics

definition *ann-sem* :: $'a\ ann\text{-}com \Rightarrow 'a \Rightarrow 'a\ set$ **where**
ann-sem $c \equiv \lambda s.\ \{t.\ (Some\ c,\ s) -*\rightarrow (None,\ t)\}$

definition *ann-SEM* :: *'a ann-com* ⇒ *'a set* ⇒ *'a set* **where**
ann-SEM c S ≡ ⋃(*ann-sem c* ‘ *S*)

definition *sem* :: *'a com* ⇒ *'a* ⇒ *'a set* **where**
sem c ≡ λ*s*. {*t*. ∃ *Ts*. (*c*, *s*) −*P*∗→ (*Parallel Ts*, *t*) ∧ *All-None Ts*}

definition *SEM* :: *'a com* ⇒ *'a set* ⇒ *'a set* **where**
SEM c S ≡ ⋃(*sem c* ‘ *S*)

abbreviation *Omega* :: *'a com* (Ω *63*)
where Ω ≡ *While UNIV UNIV* (*Basic id*)

primrec *fwhile* :: *'a bexp* ⇒ *'a com* ⇒ *nat* ⇒ *'a com* **where**
fwhile b c 0 = Ω
| *fwhile b c* (*Suc n*) = *Cond b* (*Seq c* (*fwhile b c n*)) (*Basic id*)

Proofs

declare *ann-transition-transition.intros* [*intro*]
inductive-cases *transition-cases*:
(*Parallel T*,*s*) −*P1*→ *t*
(*Basic f*, *s*) −*P1*→ *t*
(*Seq c1 c2*, *s*) −*P1*→ *t*
(*Cond b c1 c2*, *s*) −*P1*→ *t*
(*While b i c*, *s*) −*P1*→ *t*

lemma *Parallel-empty-lemma* [*rule-format* (*no-asm*)]:
(*Parallel* []*,s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=[] ∧ *n*=*0* ∧ *s*=*t*
⟨*proof*⟩

lemma *Parallel-AllNone-lemma* [*rule-format* (*no-asm*)]:
All-None Ss ⟶ (*Parallel Ss*,*s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=*Ss* ∧ *n*=*0* ∧ *s*=*t*
⟨*proof*⟩

lemma *Parallel-AllNone*: *All-None Ts* ⟹ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

lemma *Parallel-empty*: *Ts*=[] ⟹ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

lemma *L3-5i*: *X*⊆*Y* ⟹ *SEM c X* ⊆ *SEM c Y*
⟨*proof*⟩

lemma *L3-5ii-lemma1*:
⟦ (*c1*, *s1*) −*P*∗→ (*Parallel Ts*, *s2*); *All-None Ts*;
(*c2*, *s2*) −*P*∗→ (*Parallel Ss*, *s3*); *All-None Ss* ⟧
⟹ (*Seq c1 c2*, *s1*) −*P*∗→ (*Parallel Ss*, *s3*)

⟨*proof*⟩

lemma *L3-5ii-lemma2* [*rule-format* (*no-asm*)]:
$\forall$ *c1 c2 s t*. (*Seq c1 c2*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$
(*All-None Ts*) $\longrightarrow$ ($\exists$ *y m Rs*. (*c1*,*s*) $-P*\rightarrow$ (*Parallel Rs*, *y*) $\wedge$
(*All-None Rs*) $\wedge$ (*c2*, *y*) $-Pm\rightarrow$ (*Parallel Ts*, *t*) $\wedge$ $m \leq n$)
⟨*proof*⟩

lemma *L3-5ii-lemma3*:
$[\![$(*Seq c1 c2*,*s*) $-P*\rightarrow$ (*Parallel Ts*,*t*); *All-None Ts*$]\!] \Longrightarrow$
($\exists$ *y Rs*. (*c1*,*s*) $-P*\rightarrow$ (*Parallel Rs*,*y*) $\wedge$ *All-None Rs*
$\wedge$ (*c2*,*y*) $-P*\rightarrow$ (*Parallel Ts*,*t*))
⟨*proof*⟩

lemma *L3-5ii*: *SEM* (*Seq c1 c2*) *X* = *SEM c2* (*SEM c1 X*)
⟨*proof*⟩

lemma *L3-5iii*: *SEM* (*Seq* (*Seq c1 c2*) *c3*) *X* = *SEM* (*Seq c1* (*Seq c2 c3*)) *X*
⟨*proof*⟩

lemma *L3-5iv*:
SEM (*Cond b c1 c2*) *X* = (*SEM c1* (*X* $\cap$ *b*)) *Un* (*SEM c2* (*X* $\cap$ ($-b$)))
⟨*proof*⟩

lemma *L3-5v-lemma1*[*rule-format*]:
(*S*,*s*) $-Pn\rightarrow$ (*T*,*t*) $\longrightarrow$ *S*=Ω $\longrightarrow$ ($\neg$($\exists$ *Rs*. *T*=(*Parallel Rs*) $\wedge$ *All-None Rs*))
⟨*proof*⟩

lemma *L3-5v-lemma2*: $[\![$(Ω, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*); *All-None Ts* $]\!] \Longrightarrow$ *False*
⟨*proof*⟩

lemma *L3-5v-lemma3*: *SEM* (Ω) *S* = {}
⟨*proof*⟩

lemma *L3-5v-lemma4* [*rule-format*]:
$\forall$ *s*. (*While b i c*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
($\exists$ *k*. (*fwhile b c k*, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*))
⟨*proof*⟩

lemma *L3-5v-lemma5* [*rule-format*]:
$\forall$ *s*. (*fwhile b c k*, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
(*While b i c*, *s*) $-P*\rightarrow$ (*Parallel Ts*,*t*)
⟨*proof*⟩

lemma *L3-5v*: *SEM* (*While b i c*) = (λ*x*. ($\bigcup$ *k*. *SEM* (*fwhile b c k*) *x*))
⟨*proof*⟩

1.3 Validity of Correctness Formulas

definition *com-validity* :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* ((*3*‖= -// -//-) [*90*,*55*,*90*] *50*) **where**
  ‖= *p c q* ≡ *SEM c p* ⊆ *q*

definition *ann-com-validity* :: *'a ann-com* ⇒ *'a assn* ⇒ *bool* (⊨ - - [*60*,*90*] *45*) **where**
  ⊨ *c q* ≡ *ann-SEM c* (*pre c*) ⊆ *q*

end

1.4 The Proof System

theory *OG-Hoare* **imports** *OG-Tran* **begin**

primrec *assertions* :: *'a ann-com* ⇒ (*'a assn*) *set* **where**
  assertions (*AnnBasic r f*) = {*r*}
| *assertions* (*AnnSeq c1 c2*) = *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond1 r b c1 c2*) = {*r*} ∪ *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond2 r b c*) = {*r*} ∪ *assertions c*
| *assertions* (*AnnWhile r b i c*) = {*r*, *i*} ∪ *assertions c*
| *assertions* (*AnnAwait r b c*) = {*r*}

primrec *atomics* :: *'a ann-com* ⇒ (*'a assn* × *'a com*) *set* **where**
  atomics (*AnnBasic r f*) = {(*r*, *Basic f*)}
| *atomics* (*AnnSeq c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond1 r b c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond2 r b c*) = *atomics c*
| *atomics* (*AnnWhile r b i c*) = *atomics c*
| *atomics* (*AnnAwait r b c*) = {(*r* ∩ *b*, *c*)}

primrec *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op* **where**
  com (*c*, *q*) = *c*

primrec *post* :: *'a ann-triple-op* ⇒ *'a assn* **where**
  post (*c*, *q*) = *q*

definition *interfree-aux* :: (*'a ann-com-op* × *'a assn* × *'a ann-com-op*) ⇒ *bool* **where**
  interfree-aux ≡ λ(*co*, *q*, *co'*). *co'*= *None* ∨
                (∀(*r*,*a*) ∈ *atomics* (*the co'*). ‖= (*q* ∩ *r*) *a q* ∧
                (*co* = *None* ∨ (∀ *p* ∈ *assertions* (*the co*). ‖= (*p* ∩ *r*) *a p*)))

definition *interfree* :: ((*'a ann-triple-op*) *list*) ⇒ *bool* **where**
  interfree Ts ≡ ∀ *i j*. *i* < *length Ts* ∧ *j* < *length Ts* ∧ *i* ≠ *j* ⟶
                  interfree-aux (*com* (*Ts*!*i*), *post* (*Ts*!*i*), *com* (*Ts*!*j*))

inductive

oghoare :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* ((*3*⊨/ -//-//-) [*90*,*55*,*90*] *50*)
and *ann-hoare* :: *'a ann-com* ⇒ *'a assn* ⇒ *bool* ((*2*⊢ -// -) [*60*,*90*] *45*)
where
AnnBasic: $r \subseteq \{s.\ f\ s \in q\} \Longrightarrow \vdash (AnnBasic\ r\ f)\ q$

| *AnnSeq*: $[\![\vdash c0\ pre\ c1; \vdash c1\ q]\!] \Longrightarrow \vdash (AnnSeq\ c0\ c1)\ q$

| *AnnCond1*: $[\![r \cap b \subseteq pre\ c1; \vdash c1\ q; r \cap -b \subseteq pre\ c2; \vdash c2\ q]\!]$
$\Longrightarrow \vdash (AnnCond1\ r\ b\ c1\ c2)\ q$
| *AnnCond2*: $[\![r \cap b \subseteq pre\ c; \vdash c\ q; r \cap -b \subseteq q]\!] \Longrightarrow \vdash (AnnCond2\ r\ b\ c)\ q$

| *AnnWhile*: $[\![r \subseteq i; i \cap b \subseteq pre\ c; \vdash c\ i; i \cap -b \subseteq q]\!]$
$\Longrightarrow \vdash (AnnWhile\ r\ b\ i\ c)\ q$

| *AnnAwait*: $[\![atom\text{-}com\ c; \models (r \cap b)\ c\ q]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ c)\ q$

| *AnnConseq*: $[\![\vdash c\ q; q \subseteq q']\!] \Longrightarrow \vdash c\ q'$

| *Parallel*: $[\![\forall i{<}length\ Ts.\ \exists c\ q.\ Ts!i = (Some\ c,\ q) \wedge \vdash c\ q; interfree\ Ts]\!]$
$\Longrightarrow \models (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts!i))))$
$Parallel\ Ts$
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts!i))$

| *Basic*: $\models \{s.\ f\ s \in q\}\ (Basic\ f)\ q$

| *Seq*: $[\![\models p\ c1\ r; \models r\ c2\ q]\!] \Longrightarrow \models p\ (Seq\ c1\ c2)\ q$

| *Cond*: $[\![\models (p \cap b)\ c1\ q; \models (p \cap -b)\ c2\ q]\!] \Longrightarrow \models p\ (Cond\ b\ c1\ c2)\ q$

| *While*: $[\![\models (p \cap b)\ c\ p]\!] \Longrightarrow \models p\ (While\ b\ i\ c)\ (p \cap -b)$

| *Conseq*: $[\![p' \subseteq p; \models p\ c\ q\ ;\ q \subseteq q']\!] \Longrightarrow \models p'\ c\ q'$

1.5 Soundness

lemmas [*cong del*] = *if-weak-cong*

lemmas *ann-hoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct2*]
lemmas *oghoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct1*]

lemmas *AnnBasic* = *oghoare-ann-hoare.AnnBasic*
lemmas *AnnSeq* = *oghoare-ann-hoare.AnnSeq*
lemmas *AnnCond1* = *oghoare-ann-hoare.AnnCond1*
lemmas *AnnCond2* = *oghoare-ann-hoare.AnnCond2*
lemmas *AnnWhile* = *oghoare-ann-hoare.AnnWhile*
lemmas *AnnAwait* = *oghoare-ann-hoare.AnnAwait*
lemmas *AnnConseq* = *oghoare-ann-hoare.AnnConseq*

lemmas *Parallel* = *oghoare-ann-hoare.Parallel*
lemmas *Basic* = *oghoare-ann-hoare.Basic*
lemmas *Seq* = *oghoare-ann-hoare.Seq*
lemmas *Cond* = *oghoare-ann-hoare.Cond*
lemmas *While* = *oghoare-ann-hoare.While*
lemmas *Conseq* = *oghoare-ann-hoare.Conseq*

1.5.1 Soundness of the System for Atomic Programs

lemma *Basic-ntran* [*rule-format*]:
$(Basic\ f,\ s) -Pn\rightarrow (Parallel\ Ts,\ t) \longrightarrow All\text{-}None\ Ts \longrightarrow t = f\ s$
⟨*proof*⟩

lemma *SEM-fwhile*: $SEM\ S\ (p \cap b) \subseteq p \Longrightarrow SEM\ (fwhile\ b\ S\ k)\ p \subseteq (p \cap -b)$
⟨*proof*⟩

lemma *atom-hoare-sound* [*rule-format*]:
$\|-\ p\ c\ q \longrightarrow atom\text{-}com(c) \longrightarrow \|=\ p\ c\ q$
⟨*proof*⟩

1.5.2 Soundness of the System for Component Programs

inductive-cases *ann-transition-cases*:
$(None,s) -1\rightarrow (c',\ s')$
$(Some\ (AnnBasic\ r\ f),s) -1\rightarrow (c',\ s')$
$(Some\ (AnnSeq\ c1\ c2),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnWhile\ r\ b\ I\ c),\ s) -1\rightarrow (c',\ s')$
$(Some\ (AnnAwait\ r\ b\ c),s) -1\rightarrow (c',\ s')$

Strong Soundness for Component Programs:

lemma *ann-hoare-case-analysis* [*rule-format*]: $\vdash\ C\ q' \longrightarrow$
$((\forall\, r\ f.\ C = AnnBasic\ r\ f \longrightarrow (\exists\, q.\ r \subseteq \{s.\ f\ s \in q\} \wedge q \subseteq q')) \wedge$
$(\forall\, c0\ c1.\ C = AnnSeq\ c0\ c1 \longrightarrow (\exists\, q.\ q \subseteq q' \wedge \vdash c0\ pre\ c1 \wedge \vdash c1\ q)) \wedge$
$(\forall\, r\ b\ c1\ c2.\ C = AnnCond1\ r\ b\ c1\ c2 \longrightarrow (\exists\, q.\ q \subseteq q' \wedge$
$r \cap b \subseteq pre\ c1 \wedge \vdash c1\ q \wedge r \cap -b \subseteq pre\ c2 \wedge \vdash c2\ q)) \wedge$
$(\forall\, r\ b\ c.\ C = AnnCond2\ r\ b\ c \longrightarrow$
$(\exists\, q.\ q \subseteq q' \wedge r \cap b \subseteq pre\ c\ \wedge \vdash c\ q \wedge r \cap -b \subseteq q)) \wedge$
$(\forall\, r\ i\ b\ c.\ C = AnnWhile\ r\ b\ i\ c \longrightarrow$
$(\exists\, q.\ q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre\ c \wedge \vdash c\ i \wedge i \cap -b \subseteq q)) \wedge$
$(\forall\, r\ b\ c.\ C = AnnAwait\ r\ b\ c \longrightarrow (\exists\, q.\ q \subseteq q' \wedge \|-\ (r \cap b)\ c\ q)))$
⟨*proof*⟩

lemma *Help*: $(transition \cap \{(x,y).\ True\}) = (transition)$
⟨*proof*⟩

lemma *Strong-Soundness-aux-aux* [*rule-format*]:
$(co,\ s) -1\rightarrow (co',\ t) \longrightarrow (\forall\, c.\ co = Some\ c \longrightarrow s \in pre\ c \longrightarrow$

$(\forall\, q.\ \vdash\ c\ q \longrightarrow (\mathit{if}\ co' = \mathit{None}\ \mathit{then}\ t{\in}q\ \mathit{else}\ t \in \mathit{pre}(\mathit{the}\ co') \wedge \vdash\ (\mathit{the}\ co')\ q\ )))$
⟨*proof*⟩

lemma *Strong-Soundness-aux*: ⟦ (*Some* c, s) −∗→ (co, t); s ∈ *pre* c; ⊢ c q ⟧
⟹ *if* co = *None then* t ∈ q *else* t ∈ *pre* (*the* co) ∧ ⊢ (*the* co) q
⟨*proof*⟩

lemma *Strong-Soundness*: ⟦ (*Some* c, s)−∗→(co, t); s ∈ *pre* c; ⊢ c q ⟧
⟹ *if* co = *None then* t∈q *else* t ∈ *pre* (*the* co)
⟨*proof*⟩

lemma *ann-hoare-sound*: ⊢ c q ⟹ ⊨ c q
⟨*proof*⟩

1.5.3 Soundness of the System for Parallel Programs

lemma *Parallel-length-post-P1*: (*Parallel Ts*,s) −*P1*→ (*R*′, t) ⟹
(∃ *Rs*. *R*′ = (*Parallel Rs*) ∧ (*length Rs*) = (*length Ts*) ∧
(∀ *i*. *i*<*length Ts* ⟶ *post*(*Rs* ! *i*) = *post*(*Ts* ! *i*)))
⟨*proof*⟩

lemma *Parallel-length-post-PStar*: (*Parallel Ts*,s) −*P*∗→ (*R*′,t) ⟹
(∃ *Rs*. *R*′ = (*Parallel Rs*) ∧ (*length Rs*) = (*length Ts*) ∧
(∀ *i*. *i*<*length Ts* ⟶ *post*(*Ts* ! *i*) = *post*(*Rs* ! *i*)))
⟨*proof*⟩

lemma *assertions-lemma*: *pre* c ∈ *assertions* c
⟨*proof*⟩

lemma *interfree-aux1* [*rule-format*]:
(c,s) −1→ (r,t) ⟶ (*interfree-aux*(*c1*, *q1*, c) ⟶ *interfree-aux*(*c1*, *q1*, r))
⟨*proof*⟩

lemma *interfree-aux2* [*rule-format*]:
(c,s) −1→ (r,t) ⟶ (*interfree-aux*(c, q, a) ⟶ *interfree-aux*(r, q, a))
⟨*proof*⟩

lemma *interfree-lemma*: ⟦ (*Some* c, s) −1→ (r, t);*interfree Ts* ; *i*<*length Ts*;
Ts!*i* = (*Some* c, q) ⟧ ⟹ *interfree* (*Ts*[*i*:= (r, q)])
⟨*proof*⟩

Strong Soundness Theorem for Parallel Programs:

lemma *Parallel-Strong-Soundness-Seq-aux*:
⟦*interfree Ts*; *i*<*length Ts*; *com*(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*) ⟧
⟹ *interfree* (*Ts*[*i*:=(*Some c0*, *pre c1*)])
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-Seq* [*rule-format* (*no-asm*)]:
⟦ ∀ *i*<*length Ts*. (*if com*(*Ts*!*i*) = *None then b* ∈ *post*(*Ts*!*i*)

else $b \in pre(the(com(Ts!i))) \wedge \vdash the(com(Ts!i))\ post(Ts!i)$);
$com(Ts\ !\ i) = Some(AnnSeq\ c0\ c1)$; $i<length\ Ts$; *interfree* Ts $\rrbracket \Longrightarrow$
$(\forall ia<length\ Ts.$ (*if* $com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia) = None$
then $b \in post(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia)$
else $b \in pre(the(com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia))) \wedge$
$\vdash the(com(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia))\ post(Ts[i:=(Some\ c0,\ pre\ c1)]!\ ia)))$
$\wedge$ *interfree* $(Ts[i:=(Some\ c0,\ pre\ c1)])$
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-aux-aux* [*rule-format*]:
$(Some\ c,\ b) -1\rightarrow (co,\ t) \longrightarrow$
$(\forall Ts.\ i<length\ Ts \longrightarrow com(Ts\ !\ i) = Some\ c \longrightarrow$
$(\forall i<length\ Ts.$ (*if* $com(Ts\ !\ i) = None$ *then* $b \in post(Ts!i)$
else $b \in pre(the(com(Ts!i))) \wedge \vdash the(com(Ts!i))\ post(Ts!i))) \longrightarrow$
interfree $Ts \longrightarrow$
$(\forall j.\ j<length\ Ts \wedge i \neq j \longrightarrow$ (*if* $com(Ts!j) = None$ *then* $t \in post(Ts!j)$
else $t \in pre(the(com(Ts!j))) \wedge \vdash the(com(Ts!j))\ post(Ts!j))))$
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-aux* [*rule-format*]:
$\llbracket(Ts',s) -P*\rightarrow (Rs',t);\ \ Ts' = (Parallel\ Ts);$ *interfree* $Ts;$
$\forall i.\ i<length\ Ts \longrightarrow (\exists c\ q.\ (Ts\ !\ i) = (Some\ c,\ q) \wedge s \in (pre\ c) \wedge \vdash c\ q\ ) \rrbracket \Longrightarrow$
$\forall Rs.\ Rs' = (Parallel\ Rs) \longrightarrow (\forall j.\ j<length\ Rs \longrightarrow$
(*if* $com(Rs\ !\ j) = None$ *then* $t \in post(Ts\ !\ j)$
else $t \in pre(the(com(Rs\ !\ j))) \wedge \vdash the(com(Rs\ !\ j))\ post(Ts\ !\ j))) \wedge$ *interfree* Rs
⟨*proof*⟩

lemma *Parallel-Strong-Soundness*:
$\llbracket(Parallel\ Ts,\ s) -P*\rightarrow (Parallel\ Rs,\ t);$ *interfree* $Ts;\ j<length\ Rs;$
$\forall i.\ i<length\ Ts \longrightarrow (\exists c\ q.\ Ts\ !\ i = (Some\ c,\ q) \wedge s \in pre\ c \wedge \vdash c\ q) \rrbracket \Longrightarrow$
if $com(Rs\ !\ j) = None$ *then* $t \in post(Ts\ !\ j)$ *else* $t \in pre\ (the(com(Rs\ !\ j)))$
⟨*proof*⟩

lemma *oghoare-sound* [*rule-format*]: $\|-\ p\ c\ q \longrightarrow \|= p\ c\ q$
⟨*proof*⟩

end

1.6 Generation of Verification Conditions

theory *OG-Tactics*
imports *OG-Hoare*
begin

lemmas *ann-hoare-intros=AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq*
lemmas *oghoare-intros=Parallel Basic Seq Cond While Conseq*

lemma *ParallelConseqRule*:

$[\![\ p \subseteq (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))));$
$\ \|{-}\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))))$
$\quad (Parallel\ Ts)$
$\quad (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i));$
$\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i))\subseteq q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Parallel\ Ts)\ q$
⟨*proof*⟩

lemma *SkipRule*: $p \subseteq q \Longrightarrow \|{-}\ p\ (Basic\ id)\ q$
⟨*proof*⟩

lemma *BasicRule*: $p \subseteq \{s.\ (f\ s){\in}q\} \Longrightarrow \|{-}\ p\ (Basic\ f)\ q$
⟨*proof*⟩

lemma *SeqRule*: $[\![\ \|{-}\ p\ c1\ r;\ \|{-}\ r\ c2\ q\ ]\!] \Longrightarrow \|{-}\ p\ (Seq\ c1\ c2)\ q$
⟨*proof*⟩

lemma *CondRule*:
$[\![\ p \subseteq \{s.\ (s{\in}b \longrightarrow s{\in}w) \wedge (s{\notin}b \longrightarrow s{\in}w')\};\ \|{-}\ w\ c1\ q;\ \|{-}\ w'\ c2\ q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Cond\ b\ c1\ c2)\ q$
⟨*proof*⟩

lemma *WhileRule*: $[\![\ p \subseteq i;\ \|{-}\ (i \cap b)\ c\ i\ ;\ (i \cap (-b)) \subseteq q\ ]\!]$
$\quad \Longrightarrow \|{-}\ p\ (While\ b\ i\ c)\ q$
⟨*proof*⟩

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is $\{s.\ True\}$:

lemma *AnnatomRule*:
$[\![\ atom\text{-}com(c);\ \|{-}\ r\ c\ q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ \{s.\ True\}\ c)\ q$
⟨*proof*⟩

lemma *AnnskipRule*:
$r \subseteq q \Longrightarrow \vdash (AnnBasic\ r\ id)\ q$
⟨*proof*⟩

lemma *AnnwaitRule*:
$[\![\ (r \cap b) \subseteq q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ (Basic\ id))\ q$
⟨*proof*⟩

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

lemma *interfree-aux-rule1*: *interfree-aux*(*co*, *q*, *None*)
⟨*proof*⟩

lemma *interfree-aux-rule2*:
$\forall (R,r){\in}(atomics\ a).\ \|{-}\ (q \cap R)\ r\ q \Longrightarrow interfree\text{-}aux(None,\ q,\ Some\ a)$

⟨*proof*⟩

lemma *interfree-aux-rule3*:
(∀ (*R*, *r*)∈(*atomics a*). ‖− (*q* ∩ *R*) *r q* ∧ (∀ *p*∈(*assertions c*). ‖− (*p* ∩ *R*) *r p*))
⟹ *interfree-aux*(*Some c*, *q*, *Some a*)
⟨*proof*⟩

lemma *AnnBasic-assertions*:
⟦*interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧ ⟹
interfree-aux(*Some* (*AnnBasic r f*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnSeq-assertions*:
⟦ *interfree-aux*(*Some c1*, *q*, *Some a*); *interfree-aux*(*Some c2*, *q*, *Some a*)⟧⟹
interfree-aux(*Some* (*AnnSeq c1 c2*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnCond1-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c1*, *q*, *Some a*);
interfree-aux(*Some c2*, *q*, *Some a*)⟧⟹
interfree-aux(*Some*(*AnnCond1 r b c1 c2*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnCond2-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c*, *q*, *Some a*)⟧⟹
interfree-aux(*Some* (*AnnCond2 r b c*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnWhile-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *i*, *Some a*);
interfree-aux(*Some c*, *q*, *Some a*)⟧⟹
interfree-aux(*Some* (*AnnWhile r b i c*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnAwait-assertions*:
⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧⟹
interfree-aux(*Some* (*AnnAwait r b c*), *q*, *Some a*)
⟨*proof*⟩

lemma *AnnBasic-atomics*:
‖− (*q* ∩ *r*) (*Basic f*) *q* ⟹ *interfree-aux*(*None*, *q*, *Some* (*AnnBasic r f*))
⟨*proof*⟩

lemma *AnnSeq-atomics*:
⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
interfree-aux(*Any*, *q*, *Some* (*AnnSeq a1 a2*))
⟨*proof*⟩

lemma *AnnCond1-atomics*:

⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
interfree-aux(*Any*, *q*, *Some* (*AnnCond1 r b a1 a2*))
⟨*proof*⟩

lemma *AnnCond2-atomics*:
interfree-aux (*Any*, *q*, *Some a*)⟹ *interfree-aux*(*Any*, *q*, *Some* (*AnnCond2 r b a*))
⟨*proof*⟩

lemma *AnnWhile-atomics*: *interfree-aux* (*Any*, *q*, *Some a*)
⟹ *interfree-aux*(*Any*, *q*, *Some* (*AnnWhile r b i a*))
⟨*proof*⟩

lemma *Annatom-atomics*:
‖− (*q* ∩ *r*) *a q* ⟹ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r* {*x*. *True*} *a*))
⟨*proof*⟩

lemma *AnnAwait-atomics*:
‖− (*q* ∩ (*r* ∩ *b*)) *a q* ⟹ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r b a*))
⟨*proof*⟩

definition *interfree-swap* :: ('*a ann-triple-op* ∗ ('*a ann-triple-op*) *list*) ⇒ *bool* **where**
interfree-swap == λ(*x*, *xs*). ∀ *y*∈*set xs*. *interfree-aux* (*com x*, *post x*, *com y*)
∧ *interfree-aux*(*com y*, *post y*, *com x*)

lemma *interfree-swap-Empty*: *interfree-swap* (*x*, [])
⟨*proof*⟩

lemma *interfree-swap-List*:
⟦ *interfree-aux* (*com x*, *post x*, *com y*);
interfree-aux (*com y*, *post y* ,*com x*); *interfree-swap* (*x*, *xs*) ⟧
⟹ *interfree-swap* (*x*, *y*#*xs*)
⟨*proof*⟩

lemma *interfree-swap-Map*: ∀ *k*. *i*≤*k* ∧ *k*<*j* ⟶ *interfree-aux* (*com x*, *post x*, *c k*)
∧ *interfree-aux* (*c k*, *Q k*, *com x*)
⟹ *interfree-swap* (*x*, *map* (λ*k*. (*c k*, *Q k*)) [*i*..<*j*])
⟨*proof*⟩

lemma *interfree-Empty*: *interfree* []
⟨*proof*⟩

lemma *interfree-List*:
⟦ *interfree-swap*(*x*, *xs*); *interfree xs* ⟧ ⟹ *interfree* (*x*#*xs*)
⟨*proof*⟩

lemma *interfree-Map*:
(∀ *i j*. *a*≤*i* ∧ *i*<*b* ∧ *a*≤*j* ∧ *j*<*b* ∧ *i*≠*j* ⟶ *interfree-aux* (*c i*, *Q i*, *c j*))
⟹ *interfree* (*map* (λ*k*. (*c k*, *Q k*)) [*a*..<*b*])

⟨*proof*⟩

definition *map-ann-hoare* :: (('*a ann-com-op* ∗ '*a assn*) *list*) ⇒ *bool* ([⊢] - [*0*] *45*)
where
[⊢] *Ts* == (∀ *i*<*length Ts*. ∃ *c q*. *Ts*!*i*=(*Some c*, *q*) ∧ ⊢ *c q*)

lemma *MapAnnEmpty*: [⊢] []
⟨*proof*⟩

lemma *MapAnnList*: ⟦ ⊢ *c q* ; [⊢] *xs* ⟧ ⟹ [⊢] (*Some c*,*q*)#*xs*
⟨*proof*⟩

lemma *MapAnnMap*:
∀ *k*. *i*≤*k* ∧ *k*<*j* ⟶ ⊢ (*c k*) (*Q k*) ⟹ [⊢] *map* (λ*k*. (*Some* (*c k*), *Q k*)) [*i*..<*j*]
⟨*proof*⟩

lemma *ParallelRule*:⟦ [⊢] *Ts* ; *interfree Ts* ⟧
⟹ ||− (⋂ *i*∈{*i*. *i*<*length Ts*}. *pre*(*the*(*com*(*Ts*!*i*))))
Parallel Ts
(⋂ *i*∈{*i*. *i*<*length Ts*}. *post*(*Ts*!*i*))
⟨*proof*⟩

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

lemma *Compl-Collect*: −(*Collect b*) = {*x*. ¬(*b x*)}
⟨*proof*⟩

lemma *list-length*: *length* []=*0 length* (*x*#*xs*) = *Suc*(*length xs*)
⟨*proof*⟩
lemma *list-lemmas*: *length* []=*0 length* (*x*#*xs*) = *Suc*(*length xs*)
(*x*#*xs*) ! *0* = *x* (*x*#*xs*) ! *Suc n* = *xs* ! *n*
⟨*proof*⟩
lemma *le-Suc-eq-insert*: {*i*. *i* <*Suc n*} = *insert n* {*i*. *i*< *n*}
⟨*proof*⟩
lemmas *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*
lemmas *my-simp-list* = *list-lemmas fst-conv snd-conv*
not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc nat.inject
Collect-mem-eq ball-simps option.simps primrecdef-list
lemmas *ParallelConseq-list* = *INTER-eq Collect-conj-eq length-map length-upt length-append*

⟨*ML*⟩

The following tactic applies *tac* to each conjunct in a subgoal of the form *A* ⟹ *a1* ∧ *a2* ∧ .. ∧ *an* returning *n* subgoals, one for each conjunct:

⟨*ML*⟩

Tactic for the generation of the verification conditions

The tactic basically uses two subtactics:

HoareRuleTac is called at the level of parallel programs, it uses the ParallelTac to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. $\langle\ \rangle$ and *AWAIT b THEN - END*, and at each interference freedom test.

AnnHoareRuleTac is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter precond, see NOTE).

NOTE: precond(::bool) informs if the subgoal has the form $\|-$ *?p c q*, in this case we have precond=False and the generated verification condition would have the form *?p* $\subseteq \ldots$ which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

⟨*ML*⟩

The final tactic is given the name *oghoare*:

⟨*ML*⟩

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

⟨*ML*⟩

The so defined ML tactics are then "exported" to be used in Isabelle proofs.

⟨*ML*⟩

Tactics useful for dealing with the generated verification conditions:

⟨*ML*⟩

end

1.7 Concrete Syntax

theory *Quote-Antiquote* **imports** *Main* **begin**

syntax

-quote	:: $'b \Rightarrow ('a \Rightarrow 'b)$	$((\langle\!\langle\text{-}\rangle\!\rangle)\ [0]\ 1000)$
-antiquote	:: $('a \Rightarrow 'b) \Rightarrow 'b$	$(\text{´-}\ [1000]\ 1000)$

-Assert :: *'a ⇒ 'a set* (({|-|}) [0] 1000)

translations
{|b|} ⇀ *CONST Collect «b»*

⟨*ML*⟩

end
theory *OG-Syntax*
imports *OG-Tactics Quote-Antiquote*
begin

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

abbreviation *Skip* :: *'a com* (*SKIP 63*)
where *SKIP* ≡ *Basic id*

abbreviation *AnnSkip* :: *'a assn ⇒ 'a ann-com* (*-//SKIP* [*90*] *63*)
where *r SKIP* ≡ *AnnBasic r id*

notation
Seq (-,,/ - [*55*, *56*] *55*) **and**
AnnSeq (-;;/ - [*60*,*61*] *60*)

syntax
-Assign :: *idt ⇒ 'b ⇒ 'a com* ((´- :=/ -) [*70*, *65*] *61*)
-AnnAssign :: *'a assn ⇒ idt ⇒ 'b ⇒ 'a com* ((- ´- :=/ -) [*90*,*70*,*65*] *61*)

translations
´x := a ⇀ *CONST Basic «´(-update-name x (λ-. a))»*
r ´x := a ⇀ *CONST AnnBasic r «´(-update-name x (λ-. a))»*

syntax
-AnnCond1 :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com ⇒ 'a ann-com*
(- *//IF - /THEN - /ELSE - /FI* [*90*,*0*,*0*,*0*] *61*)
-AnnCond2 :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com*
(- *//IF - /THEN - /FI* [*90*,*0*,*0*] *61*)
-AnnWhile :: *'a assn ⇒ 'a bexp ⇒ 'a assn ⇒ 'a ann-com ⇒ 'a ann-com*
(- *//WHILE - /INV - //DO -//OD* [*90*,*0*,*0*,*0*] *61*)
-AnnAwait :: *'a assn ⇒ 'a bexp ⇒ 'a com ⇒ 'a ann-com*
(- *//AWAIT - /THEN /- /END* [*90*,*0*,*0*] *61*)
-AnnAtom :: *'a assn ⇒ 'a com ⇒ 'a ann-com* (-//⟨-⟩ [*90*,*0*] *61*)
-AnnWait :: *'a assn ⇒ 'a bexp ⇒ 'a ann-com* (-//*WAIT - END* [*90*,*0*] *61*)

-Cond :: *'a bexp ⇒ 'a com ⇒ 'a com ⇒ 'a com*
((*0IF -/ THEN -/ ELSE -/ FI*) [*0*, *0*, *0*] *61*)
-Cond2 :: *'a bexp ⇒ 'a com ⇒ 'a com* (*IF - THEN - FI* [*0*,*0*] *56*)
-While-inv :: *'a bexp ⇒ 'a assn ⇒ 'a com ⇒ 'a com*
((*0WHILE -/ INV - //DO - /OD*) [*0*, *0*, *0*] *61*)

-While :: *'a bexp ⇒ 'a com ⇒ 'a com*
((0WHILE - //DO - /OD) [0, 0] 61)

translations
IF b THEN c1 ELSE c2 FI ⇀ CONST Cond {|b|} c1 c2
IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI
WHILE b INV i DO c OD ⇀ CONST While {|b|} i c
WHILE b DO c OD ⇌ WHILE b INV CONST undefined DO c OD

r IF b THEN c1 ELSE c2 FI ⇀ CONST AnnCond1 r {|b|} c1 c2
r IF b THEN c FI ⇀ CONST AnnCond2 r {|b|} c
r WHILE b INV i DO c OD ⇀ CONST AnnWhile r {|b|} i c
r AWAIT b THEN c END ⇀ CONST AnnAwait r {|b|} c
r ⟨c⟩ ⇌ r AWAIT CONST True THEN c END
r WAIT b END ⇌ r AWAIT b THEN SKIP END

nonterminal *prgs*

syntax
-PAR :: *prgs ⇒ 'a* (*COBEGIN//-//COEND [57] 56*)
-prg :: *['a, 'a] ⇒ prgs* (*-//- [60, 90] 57*)
-prgs :: *['a, 'a, prgs] ⇒ prgs* (*-//-//‖//- [60,90,57] 57*)

-prg-scheme :: *['a, 'a, 'a, 'a, 'a] ⇒ prgs*
(*SCHEME [- ≤ - < -] -// - [0,0,0,60, 90] 57*)

translations
-prg c q ⇌ [(CONST Some c, q)]
-prgs c q ps ⇌ (CONST Some c, q) # ps
-PAR ps ⇌ CONST Parallel ps

-prg-scheme j i k c q ⇌ CONST map (λi. (CONST Some c, q)) [j..<k]

⟨ML⟩

end

1.8 Examples

theory *OG-Examples* **imports** *OG-Syntax* **begin**

1.8.1 Mutual Exclusion

Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

record *Petersons-mutex-1* =
pr1 :: *nat*
pr2 :: *nat*

```
 in1 :: bool
 in2 :: bool
 hold :: nat

lemma Petersons-mutex-1:
  ||- {| ´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 |}
  COBEGIN {| ´pr1=0 ∧ ¬´in1 |}
  WHILE True INV {| ´pr1=0 ∧ ¬´in1 |}
  DO
  {| ´pr1=0 ∧ ¬´in1 |} ⟨ ´in1:=True,,´pr1:=1 ⟩;;
  {| ´pr1=1 ∧ ´in1 |} ⟨ ´hold:=1,,´pr1:=2 ⟩;;
  {| ´pr1=2 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2) |}
  AWAIT (¬´in2 ∨ ¬(´hold=1)) THEN ´pr1:=3 END;;
  {| ´pr1=3 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2) |}
   ⟨´in1:=False,,´pr1:=0⟩
  OD {| ´pr1=0 ∧ ¬´in1 |}
  ||
  {| ´pr2=0 ∧ ¬´in2 |}
  WHILE True INV {| ´pr2=0 ∧ ¬´in2 |}
  DO
  {| ´pr2=0 ∧ ¬´in2 |} ⟨ ´in2:=True,,´pr2:=1 ⟩;;
  {| ´pr2=1 ∧ ´in2 |} ⟨ ´hold:=2,,´pr2:=2 ⟩;;
  {| ´pr2=2 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2)) |}
  AWAIT (¬´in1 ∨ ¬(´hold=2)) THEN ´pr2:=3 END;;
  {| ´pr2=3 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2)) |}
    ⟨´in2:=False,,´pr2:=0⟩
  OD {| ´pr2=0 ∧ ¬´in2 |}
  COEND
  {| ´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 |}
⟨proof⟩
```

Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

```
record Busy-wait-mutex =
 flag1 :: bool
 flag2 :: bool
 turn  :: nat
 after1 :: bool
 after2 :: bool

lemma Busy-wait-mutex:
 ||-  {| True |}
  ´flag1:=False,, ´flag2:=False,,
  COBEGIN {| ¬´flag1 |}
        WHILE True
        INV {| ¬´flag1 |}
        DO {| ¬´flag1 |} ⟨ ´flag1:=True,,´after1:=False ⟩;;
```

```
          {´flag1 ∧ ¬´after1} ⟨ ´turn:=1,,´after1:=True ⟩;;
          {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}
           WHILE ¬(´flag2 ⟶ ´turn=2)
           INV {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}
           DO {´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)} SKIP OD;;
          {´flag1 ∧ ´after1 ∧ (´flag2 ∧ ´after2 ⟶ ´turn=2)}
           ´flag1:=False
       OD
      {False}
  ||
    {¬´flag2}
       WHILE True
       INV {¬´flag2}
       DO {¬´flag2} ⟨ ´flag2:=True,,´after2:=False ⟩;;
          {´flag2 ∧ ¬´after2} ⟨ ´turn:=2,,´after2:=True ⟩;;
          {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}
           WHILE ¬(´flag1 ⟶ ´turn=1)
           INV {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}
           DO {´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)} SKIP OD;;
          {´flag2 ∧ ´after2 ∧ (´flag1 ∧ ´after1 ⟶ ´turn=1)}
           ´flag2:=False
       OD
      {False}
  COEND
  {False}
⟨proof⟩
```

Peterson's Algorithm III: A Solution using Semaphores

```
record Semaphores-mutex =
 out :: bool
 who :: nat

lemma Semaphores-mutex:
 ||− {i≠j}
  ´out:=True ,,
  COBEGIN {i≠j}
       WHILE True INV {i≠j}
       DO {i≠j} AWAIT ´out THEN ´out:=False,, ´who:=i END;;
          {¬´out ∧ ´who=i ∧ i≠j} ´out:=True OD
       {False}
  ||
       {i≠j}
       WHILE True INV {i≠j}
       DO {i≠j} AWAIT ´out THEN ´out:=False,,´who:=j END;;
          {¬´out ∧ ´who=j ∧ i≠j} ´out:=True OD
       {False}
  COEND
  {False}
```

⟨*proof*⟩

Peterson's Algorithm III: Parameterized version:

lemma *Semaphores-parameterized-mutex*:
0<n ⟹ ||− {|*True*|}
´*out*:=*True* ,,
COBEGIN
SCHEME [*0*≤ *i*< *n*]
{|*True*|}
WHILE True INV {|*True*|}
DO {|*True*|} *AWAIT* ´*out THEN* ´*out*:=*False*,, ´*who*:=*i END*;;
{|¬´*out* ∧ ´*who*=*i*|} ´*out*:=*True OD*
{|*False*|}
COEND
{|*False*|}
⟨*proof*⟩

The Ticket Algorithm

record *Ticket-mutex* =
num :: *nat*
nextv :: *nat*
turn :: *nat list*
index :: *nat*

lemma *Ticket-mutex*:
⟦ *0<n*; *I*=«*n*=*length* ´*turn* ∧ *0*<´*nextv* ∧ (∀ *k l*. *k*<*n* ∧ *l*<*n* ∧ *k*≠*l*
⟶ ´*turn*!*k* < ´*num* ∧ (´*turn*!*k* =*0* ∨ ´*turn*!*k*≠´*turn*!*l*))» ⟧
⟹ ||− {|*n*=*length* ´*turn*|}
´*index*:= *0*,,
WHILE ´*index* < *n INV* {|*n*=*length* ´*turn* ∧ (∀ *i*<´*index*. ´*turn*!*i*=*0*)|}
DO ´*turn*:= ´*turn*[´*index*:=*0*],, ´*index*:=´*index* +*1 OD*,,
´*num*:=*1* ,, ´*nextv*:=*1* ,,
COBEGIN
SCHEME [*0*≤ *i*< *n*]
{|´*I*|}
WHILE True INV {|´*I*|}
DO {|´*I*|} ⟨ ´*turn* :=´*turn*[*i*:=´*num*],, ´*num*:=´*num*+*1* ⟩;;
{|´*I*|} *WAIT* ´*turn*!*i*=´*nextv END*;;
{|´*I* ∧ ´*turn*!*i*=´*nextv*|} ´*nextv*:=´*nextv*+*1*
OD
{|*False*|}
COEND
{|*False*|}
⟨*proof*⟩

1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

Apt and Olderog. "Verification of sequential and concurrent Programs" page 294:

record *Zero-search* =
  turn :: *nat*
  found :: *bool*
  x :: *nat*
  y :: *nat*

lemma *Zero-search*:
 ⟦*I1*= « *a*≤´*x* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
    ∧ (¬´*found* ∧ *a*<´ *x* ⟶ *f*(´*x*)≠*0*) » ;
   I2= «´*y*≤*a*+*1* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
    ∧ (¬´*found* ∧ ´*y*≤*a* ⟶ *f*(´*y*)≠*0*) » ⟧ ⟹
 ‖− {|∃ *u*. *f*(*u*)=*0*|}
 ´*turn*:=*1*,, ´*found*:= *False*,,
 ´*x*:=*a*,, ´*y*:=*a*+*1* ,,
 COBEGIN {|´*I1*|}
     WHILE ¬´*found*
     INV {|´*I1*|}
     DO {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        WAIT ´*turn*=*1* *END*;;
        {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        ´*turn*:=*2*;;
        {|*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)|}
        ⟨ ´*x*:=´*x*+*1*,,
          IF *f*(´*x*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
     OD;;
     {|´*I1* ∧ ´*found*|}
     ´*turn*:=*2*
     {|´*I1* ∧ ´*found*|}
 ‖
    {|´*I2*|}
     WHILE ¬´*found*
     INV {|´*I2*|}
     DO {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        WAIT ´*turn*=*2* *END*;;
        {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        ´*turn*:=*1*;;
        {|´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)|}
        ⟨ ´*y*:=(´*y* − *1*),,
          IF *f*(´*y*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
     OD;;
     {|´*I2* ∧ ´*found*|}
     ´*turn*:=*1*
     {|´*I2* ∧ ´*found*|}
 COEND
 {|*f*(´*x*)=*0* ∨ *f*(´*y*)=*0*|}
⟨*proof*⟩

Easier Version: without AWAIT. Apt and Olderog. page 256:

lemma *Zero-Search-2*:
$[\![I1 = \ll a \leq \acute{} x \wedge (\acute{} found \longrightarrow (a < \acute{} x \wedge f(\acute{} x) = 0) \vee (\acute{} y \leq a \wedge f(\acute{} y) = 0))$
$\wedge (\neg \acute{} found \wedge a < \acute{} x \longrightarrow f(\acute{} x) \neq 0) \gg;$
$I2 = \ll \acute{} y \leq a+1 \wedge (\acute{} found \longrightarrow (a < \acute{} x \wedge f(\acute{} x) = 0) \vee (\acute{} y \leq a \wedge f(\acute{} y) = 0))$
$\wedge (\neg \acute{} found \wedge \acute{} y \leq a \longrightarrow f(\acute{} y) \neq 0) \gg]\!] \Longrightarrow$
$\| - \{\!| \exists u.\ f(u) = 0 |\!\}$
$\acute{} found := False,,$
$\acute{} x := a,,\ \acute{} y := a+1,,$
$COBEGIN\ \{\!| \acute{} I1 |\!\}$
$WHILE\ \neg \acute{} found$
$INV\ \{\!| \acute{} I1 |\!\}$
$DO\ \{\!| a \leq \acute{} x \wedge (\acute{} found \longrightarrow \acute{} y \leq a \wedge f(\acute{} y) = 0) \wedge (a < \acute{} x \longrightarrow f(\acute{} x) \neq 0) |\!\}$
$\langle\ \acute{} x := \acute{} x + 1,, IF\ f(\acute{} x) = 0\ THEN\ \acute{} found := True\ ELSE\ SKIP\ FI \rangle$
OD
$\{\!| \acute{} I1 \wedge \acute{} found |\!\}$
$\|$
$\{\!| \acute{} I2 |\!\}$
$WHILE\ \neg \acute{} found$
$INV\ \{\!| \acute{} I2 |\!\}$
$DO\ \{\!| \acute{} y \leq a+1 \wedge (\acute{} found \longrightarrow a < \acute{} x \wedge f(\acute{} x) = 0) \wedge (\acute{} y \leq a \longrightarrow f(\acute{} y) \neq 0) |\!\}$
$\langle\ \acute{} y := (\acute{} y - 1),, IF\ f(\acute{} y) = 0\ THEN\ \acute{} found := True\ ELSE\ SKIP\ FI \rangle$
OD
$\{\!| \acute{} I2 \wedge \acute{} found |\!\}$
$COEND$
$\{\!| f(\acute{} x) = 0 \vee f(\acute{} y) = 0 |\!\}$
$\langle proof \rangle$

1.8.3 Producer/Consumer

Previous lemmas

lemma *nat-lemma2*: $[\![\ b = m*(n::nat) + t;\ a = s*n + u;\ t = u;\ b - a < n\ ]\!] \Longrightarrow m \leq s$
$\langle proof \rangle$

lemma *mod-lemma*: $[\![\ (c::nat) \leq a;\ a < b;\ b - c < n\ ]\!] \Longrightarrow b\ mod\ n \neq a\ mod\ n$
$\langle proof \rangle$

Producer/Consumer Algorithm

record *Producer-consumer* =
ins :: *nat*
outs :: *nat*
li :: *nat*
lj :: *nat*
vx :: *nat*
vy :: *nat*
buffer :: *nat list*
b :: *nat list*

The whole proof takes aprox. 4 minutes.

lemma *Producer-consumer*:

```
⟦INIT= «0<length a ∧ 0<length ´buffer ∧ length ´b=length a» ;
  I= «(∀k<´ins. ´outs≤k ⟶ (a ! k) = ´buffer ! (k mod (length ´buffer))) ∧
        ´outs≤´ins ∧ ´ins−´outs≤length ´buffer» ;
  I1= «´I ∧ ´li≤length a» ;
  p1= «´I1 ∧ ´li=´ins» ;
  I2 = «´I ∧ (∀k<´lj. (a ! k)=(´b ! k)) ∧ ´lj≤length a» ;
  p2 = «´I2 ∧ ´lj=´outs» ⟧ ⟹
 ||− {|´INIT|}
´ins:=0,, ´outs:=0,, ´li:=0,, ´lj:=0,,
 COBEGIN {|´p1 ∧ ´INIT|}
  WHILE ´li <length a
   INV {|´p1 ∧ ´INIT|}
  DO {|´p1 ∧ ´INIT ∧ ´li<length a|}
     ´vx:= (a ! ´li);;
    {|´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)|}
     WAIT ´ins−´outs < length ´buffer END;;
    {|´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)
      ∧ ´ins−´outs < length ´buffer|}
     ´buffer:=(list-update ´buffer (´ins mod (length ´buffer)) ´vx);;
    {|´p1 ∧ ´INIT ∧ ´li<length a
      ∧ (a ! ´li)=(´buffer ! (´ins mod (length ´buffer)))
      ∧ ´ins−´outs <length ´buffer|}
     ´ins:=´ins+1;;
    {|´I1 ∧ ´INIT ∧ (´li+1)=´ins ∧ ´li<length a|}
     ´li:=´li+1
  OD
 {|´p1 ∧ ´INIT ∧ ´li=length a|}
 ||
 {|´p2 ∧ ´INIT|}
  WHILE ´lj < length a
   INV {|´p2 ∧ ´INIT|}
  DO {|´p2 ∧ ´lj<length a ∧ ´INIT|}
     WAIT ´outs<´ins END;;
    {|´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´INIT|}
     ´vy:=(´buffer ! (´outs mod (length ´buffer)));;
    {|´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´vy=(a ! ´lj) ∧ ´INIT|}
     ´outs:=´outs+1;;
    {|´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ ´vy=(a ! ´lj) ∧ ´INIT|}
     ´b:=(list-update ´b ´lj ´vy);;
    {|´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ (a ! ´lj)=(´b ! ´lj) ∧ ´INIT|}
     ´lj:=´lj+1
  OD
 {|´p2 ∧ ´lj=length a ∧ ´INIT|}
 COEND
 {| ∀k<length a. (a ! k)=(´b ! k)|}
⟨proof⟩
```

1.8.4 Parameterized Examples

Set Elements of an Array to Zero

record *Example1* =
  $a :: nat \Rightarrow nat$

lemma *Example1*:
 $\|- \{\!|True|\!\}$
   COBEGIN SCHEME $[0 \leq i < n]$ $\{\!|True|\!\}$ $\acute{}a := \acute{}a\ (i := 0)$ $\{\!|\acute{}a\ i = 0|\!\}$ *COEND*
  $\{\!|\forall i < n.\ \acute{}a\ i = 0|\!\}$
⟨*proof*⟩

Same example with lists as auxiliary variables.

record *Example1-list* =
  $A :: nat\ list$
lemma *Example1-list*:
 $\|- \{\!|n < length\ \acute{}A|\!\}$
   COBEGIN
     SCHEME $[0 \leq i < n]$ $\{\!|n < length\ \acute{}A|\!\}$ $\acute{}A := \acute{}A[i := 0]$ $\{\!|\acute{}A!i = 0|\!\}$
   COEND
    $\{\!|\forall i < n.\ \acute{}A!i = 0|\!\}$
⟨*proof*⟩

Increment a Variable in Parallel

First some lemmas about summation properties.

lemma *Example2-lemma2-aux*: $!!b.\ j < n \Longrightarrow$
 $(\sum i = 0..<n.\ (b\ i :: nat)) =$
 $(\sum i = 0..<j.\ b\ i) + b\ j + (\sum i = 0..<n - (Suc\ j)\ .\ b\ (Suc\ j + i))$
⟨*proof*⟩

lemma *Example2-lemma2-aux2*:
  $!!b.\ j \leq s \Longrightarrow (\sum i :: nat = 0..<j.\ (b\ (s := t))\ i) = (\sum i = 0..<j.\ b\ i)$
⟨*proof*⟩

lemma *Example2-lemma2*:
 $!!b.\ [\![j < n;\ b\ j = 0]\!] \Longrightarrow Suc\ (\sum i :: nat = 0..<n.\ b\ i) = (\sum i = 0..<n.\ (b\ (j := Suc\ 0))$
$i)$
⟨*proof*⟩

record *Example2* =
 $c :: nat \Rightarrow nat$
 $x :: nat$

lemma *Example-2*: $0 < n \Longrightarrow$
 $\|- \{\!|\acute{}x = 0 \wedge (\sum i = 0..<n.\ \acute{}c\ i) = 0|\!\}$
 COBEGIN

$SCHEME\ [0{\leq}i{<}n]$
$\{\!|\ {}'x{=}(\sum i{=}0..{<}n.\ {}'c\ i) \wedge {}'c\ i{=}0\ |\!\}$
$\langle\ {}'x{:=}{}'x{+}(Suc\ 0),,\ {}'c{:=}{}'c\ (i{:=}(Suc\ 0))\ \rangle$
$\{\!|\ {}'x{=}(\sum i{=}0..{<}n.\ {}'c\ i) \wedge {}'c\ i{=}(Suc\ 0)\ |\!\}$
$COEND$
$\{\!|\ {}'x{=}n\ |\!\}$
$\langle proof \rangle$

end

Chapter 2

Case Study: Single and Multi-Mutator Garbage Collection Algorithms

2.1 Formalization of the Memory

theory *Graph* **imports** *Main* **begin**

datatype *node* = *Black* | *White*

type-synonym *nodes* = *node list*
type-synonym *edge* = *nat* × *nat*
type-synonym *edges* = *edge list*

consts *Roots* :: *nat set*

definition *Proper-Roots* :: *nodes* ⇒ *bool* **where**
Proper-Roots M ≡ *Roots*≠{} ∧ *Roots* ⊆ {*i*. *i*<*length M*}

definition *Proper-Edges* :: (*nodes* × *edges*) ⇒ *bool* **where**
Proper-Edges ≡ (λ(*M*,*E*). ∀ *i*<*length E*. *fst*(*E*!*i*)<*length M* ∧ *snd*(*E*!*i*)<*length M*)

definition *BtoW* :: (*edge* × *nodes*) ⇒ *bool* **where**
BtoW ≡ (λ(*e*,*M*). (*M*!*fst e*)=*Black* ∧ (*M*!*snd e*)≠*Black*)

definition *Blacks* :: *nodes* ⇒ *nat set* **where**
Blacks M ≡ {*i*. *i*<*length M* ∧ *M*!*i*=*Black*}

definition *Reach* :: *edges* ⇒ *nat set* **where**
Reach E ≡ {*x*. (∃ *path*. 1<*length path* ∧ *path*!(*length path* − 1)∈*Roots* ∧ *x*=*path*!0
∧ (∀ *i*<*length path* − 1. (∃ *j*<*length E*. *E*!*j*=(*path*!(*i*+1), *path*!*i*))))
∨ *x*∈*Roots*}

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

2.1.1 Proofs about Graphs

lemmas *Graph-defs= Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def*
declare *Graph-defs* [*simp*]

Graph 1

lemma *Graph1-aux* [*rule-format*]:
$[\![\ Roots{\subseteq}Blacks\ M;\ \forall\, i{<}length\ E.\ \neg BtoW(E!i,M)]\!]$
$\Longrightarrow 1{<}\ length\ path \longrightarrow (path!(length\ path\ -\ 1)){\in}Roots \longrightarrow$
$(\forall\, i{<}length\ path\ -\ 1.\ (\exists\, j.\ j\ <\ length\ E \wedge E!j{=}(path!(Suc\ i),\ path!i)))$
$\longrightarrow M!(path!0) = Black$
⟨*proof*⟩

lemma *Graph1*:
$[\![Roots{\subseteq}Blacks\ M;\ Proper\text{-}Edges(M,\ E);\ \forall\, i{<}length\ E.\ \neg BtoW(E!i,M)\ ]\!]$
$\Longrightarrow Reach\ E{\subseteq}Blacks\ M$
⟨*proof*⟩

Graph 2

lemma *Ex-first-occurrence* [*rule-format*]:
$P\ (n{::}nat) \longrightarrow (\exists\, m.\ P\ m \wedge (\forall\, i.\ i{<}m \longrightarrow \neg\ P\ i))$
⟨*proof*⟩

lemma *Compl-lemma*: $(n{::}nat){\leq}l \Longrightarrow (\exists\, m.\ m{\leq}l \wedge n{=}l\ -\ m)$
⟨*proof*⟩

lemma *Ex-last-occurrence*:
$[\![P\ (n{::}nat);\ n{\leq}l]\!] \Longrightarrow (\exists\, m.\ P\ (l\ -\ m) \wedge (\forall\, i.\ i{<}m \longrightarrow \neg P\ (l\ -\ i)))$
⟨*proof*⟩

lemma *Graph2*:
$[\![T \in Reach\ E;\ R{<}length\ E]\!] \Longrightarrow T \in Reach\ (E[R{:=}(fst(E!R),\ T)])$
⟨*proof*⟩

Graph 3

declare *min.absorb1* [*simp*] *min.absorb2* [*simp*]

lemma *Graph3*:
$[\![\ T{\in}Reach\ E;\ R{<}length\ E\ ]\!] \Longrightarrow Reach(E[R{:=}(fst(E!R),T)]) \subseteq Reach\ E$
⟨*proof*⟩

Graph 4

lemma *Graph4*:
$[\![T \in Reach\ E;\ Roots{\subseteq}Blacks\ M;\ I{\leq}length\ E;\ T{<}length\ M;\ R{<}length\ E;$
$\forall i{<}I.\ \neg BtoW(E!i,M);\ R{<}I;\ M!fst(E!R){=}Black;\ M!T{\neq}Black]\!] \Longrightarrow$
$(\exists r.\ I{\leq}r \wedge r{<}length\ E \wedge BtoW(E[R{:=}(fst(E!R),T)]!r,M))$
⟨*proof*⟩

declare *min.absorb1* [*simp del*] *min.absorb2* [*simp del*]

Graph 5

lemma *Graph5*:
$[\![\ T \in Reach\ E\ ;\ Roots \subseteq Blacks\ M;\ \forall i{<}R.\ \neg BtoW(E!i,M);\ T{<}length\ M;$
$R{<}length\ E;\ M!fst(E!R){=}Black;\ M!snd(E!R){=}Black;\ M!T \neq Black]\!]$
$\Longrightarrow (\exists r.\ R{<}r \wedge r{<}length\ E \wedge BtoW(E[R{:=}(fst(E!R),T)]!r,M))$
⟨*proof*⟩

Other lemmas about graphs

lemma *Graph6*:
$[\![Proper\text{-}Edges(M,E);\ R{<}length\ E\ ;\ T{<}length\ M]\!] \Longrightarrow Proper\text{-}Edges(M,E[R{:=}(fst(E!R),T)])$
⟨*proof*⟩

lemma *Graph7*:
$[\![Proper\text{-}Edges(M,E)]\!] \Longrightarrow Proper\text{-}Edges(M[T{:=}a],E)$
⟨*proof*⟩

lemma *Graph8*:
$[\![Proper\text{-}Roots(M)]\!] \Longrightarrow Proper\text{-}Roots(M[T{:=}a])$
⟨*proof*⟩

Some specific lemmata for the verification of garbage collection algorithms.

lemma *Graph9*: $j{<}length\ M \Longrightarrow Blacks\ M{\subseteq}Blacks\ (M[j := Black])$
⟨*proof*⟩

lemma *Graph10* [*rule-format* (*no-asm*)]: $\forall i.\ M!i{=}a \longrightarrow M[i{:=}a]{=}M$
⟨*proof*⟩

lemma *Graph11* [*rule-format* (*no-asm*)]:
$[\![\ M!j{\neq}Black;j{<}length\ M]\!] \Longrightarrow Blacks\ M \subset Blacks\ (M[j := Black])$
⟨*proof*⟩

lemma *Graph12*: $[\![a{\subseteq}Blacks\ M;j{<}length\ M]\!] \Longrightarrow a{\subseteq}Blacks\ (M[j := Black])$
⟨*proof*⟩

lemma *Graph13*: $[\![a{\subset}\ Blacks\ M;j{<}length\ M]\!] \Longrightarrow a \subset Blacks\ (M[j := Black])$
⟨*proof*⟩

declare *Graph-defs* [*simp del*]

end

2.2 The Single Mutator Case

theory *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

declare *psubsetE* [*rule del*]

Declaration of variables:

record *gar-coll-state* =
  M :: *nodes*
  E :: *edges*
  bc :: *nat set*
  obc :: *nat set*
  Ma :: *nodes*
  ind :: *nat*
  k :: *nat*
  z :: *bool*

2.2.1 The Mutator

The mutator first redirects an arbitrary edge R from an arbitrary accessible node towards an arbitrary accessible node T. It then colors the new target T black.

We declare the arbitrarily selected node and edge as constants:

consts *R* :: *nat* *T* :: *nat*

The following predicate states, given a list of nodes m and a list of edges e, the conditions under which the selected edge R and node T are valid:

definition *Mut-init* :: *gar-coll-state* ⇒ *bool* **where**
  Mut-init ≡ « *T* ∈ *Reach* ´*E* ∧ *R* < *length* ´*E* ∧ *T* < *length* ´*M* »

For the mutator we consider two modules, one for each action. An auxiliary variable ´*z* is set to false if the mutator has already redirected an edge but has not yet colored the new target.

definition *Redirect-Edge* :: *gar-coll-state ann-com* **where**
  Redirect-Edge ≡ {|´*Mut-init* ∧ ´*z*|} ⟨´*E*:=´*E*[*R*:=(*fst*(´*E*!*R*), *T*)],, ´*z*:= (¬´*z*)⟩

definition *Color-Target* :: *gar-coll-state ann-com* **where**
  Color-Target ≡ {|´*Mut-init* ∧ ¬´*z*|} ⟨´*M*:=´*M*[*T*:=*Black*],, ´*z*:= (¬´*z*)⟩

definition *Mutator* :: *gar-coll-state ann-com* **where**
  Mutator ≡
  {|´*Mut-init* ∧ ´*z*|}
  WHILE True INV {|´*Mut-init* ∧ ´*z*|}
  DO Redirect-Edge ;; *Color-Target OD*

Correctness of the mutator

lemmas *mutator-defs = Mut-init-def Redirect-Edge-def Color-Target-def*

lemma *Redirect-Edge*:
⊢ *Redirect-Edge pre*(*Color-Target*)
⟨*proof*⟩

lemma *Color-Target*:
⊢ *Color-Target* {| ´*Mut-init* ∧ ´*z* |}
⟨*proof*⟩

lemma *Mutator*:
⊢ *Mutator* {|*False*|}
⟨*proof*⟩

2.2.2 The Collector

A constant *M-init* is used to give ´*Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *gar-coll-state* ⇒ *bool* **where**
Proper-M-init ≡ « *Blacks M-init=Roots* ∧ *length M-init=length* ´*M* »

definition *Proper* :: *gar-coll-state* ⇒ *bool* **where**
Proper ≡ « *Proper-Roots* ´*M* ∧ *Proper-Edges*(´*M*, ´*E*) ∧ ´*Proper-M-init* »

definition *Safe* :: *gar-coll-state* ⇒ *bool* **where**
Safe ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

lemmas *collector-defs = Proper-M-init-def Proper-def Safe-def*

Blackening the roots

definition *Blacken-Roots* :: *gar-coll-state ann-com* **where**
Blacken-Roots ≡
{| ´*Proper* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ ´*ind=0* |}
WHILE ´*ind*<*length* ´*M*
INV {| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M* |}
DO {| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}
IF ´*ind*∈*Roots THEN*
{| ´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots* |}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{| ´*Proper* ∧ (∀ *i*<´*ind*+*1*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}
´*ind*:=´*ind*+*1*

OD

lemma *Blacken-Roots*:
$\vdash$ *Blacken-Roots* {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* |}
⟨*proof*⟩

Propagating black

definition *PBInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
PBInv ≡ « λ*ind*. ´*obc* < *Blacks* ´*M* ∨ (∀ *i* <*ind*. ¬*BtoW* (´*E*!*i*, ´*M*) ∨
(¬´*z* ∧ *i*=*R* ∧ (*snd*(´*E*!*R*)) = *T* ∧ (∃ *r*. *ind* ≤ *r* ∧ *r* < *length* ´*E* ∧ *BtoW*(´*E*!*r*,´*M*))))»

definition *Propagate-Black-aux* :: *gar-coll-state ann-com* **where**
Propagate-Black-aux ≡
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* ∧ ´*ind*=*0* |}
WHILE ´*ind*<*length* ´*E*
INV {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*≤*length* ´*E* |}
DO {| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* |}
IF ´*M*!(*fst* (´*E*!´*ind*)) = *Black THEN*
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black* |}
´*M*:=´*M*[*snd*(´*E*!´*ind*):=*Black*];;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ ´*PBInv* (´*ind* + *1*) ∧ ´*ind*<*length* ´*E* |}
´*ind*:=´*ind*+*1*
FI
OD

lemma *Propagate-Black-aux*:
$\vdash$ *Propagate-Black-aux*
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M*
∧ (´*obc* < *Blacks* ´*M* ∨ ´*Safe*) |}
⟨*proof*⟩

Refining propagating black

definition *Auxk* :: *gar-coll-state* ⇒ *bool* **where**
Auxk ≡ «´*k*<*length* ´*M* ∧ (´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨
´*obc*<*Blacks* ´*M* ∨ (¬´*z* ∧ ´*ind*=*R* ∧ *snd*(´*E*!*R*)=*T*
∧ (∃ *r*. ´*ind*<*r* ∧ *r*<*length* ´*E* ∧ *BtoW*(´*E*!*r*, ´*M*))))»

definition *Propagate-Black* :: *gar-coll-state ann-com* **where**
Propagate-Black ≡
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* |}
´*ind*:=*0*;;
{| ´*Proper* ∧ *Roots⊆Blacks* ´*M* ∧ ´*obc⊆Blacks* ´*M* ∧ ´*bc⊆Blacks* ´*M* ∧ ´*ind*=*0* |}

WHILE ´ind<length ´E
INV {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´PBInv ´ind ∧ ´ind≤length ´E|}
DO {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´PBInv ´ind ∧ ´ind<length ´E|}
IF (´M!(fst (´E!´ind)))=Black THEN
{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black|}
´k:=(snd(´E!´ind));;
{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black
∧ ´Auxk|}
⟨´M:=´M[´k:=Black],, ´ind:=´ind+1⟩
ELSE {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´PBInv ´ind ∧ ´ind<length ´E|}
⟨IF (´M!(fst (´E!´ind)))≠Black THEN ´ind:=´ind+1 FI⟩
FI
OD

lemma *Propagate-Black*:
⊢ Propagate-Black
{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ (´obc < Blacks ´M ∨ ´Safe)|}
⟨*proof*⟩

Counting black nodes

definition *CountInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
CountInv ≡ « λ*ind*. {*i*. *i*<*ind* ∧ ´*Ma*!*i*=*Black*}⊆´*bc* »

definition *Count* :: *gar-coll-state ann-com* **where**
Count ≡
{|´Proper ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}|}
´ind:=0;;
{|´Proper ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}
∧ ´ind=0|}
WHILE ´ind<length ´M
INV {|´Proper ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ length ´Ma=length ´M ∧ ´CountInv ´ind
∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind≤length ´M|}
DO {|´Proper ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ length ´Ma=length ´M ∧ ´CountInv ´ind
∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}

```
    IF ´M!´ind=Black
      THEN {| ´Proper ∧ Roots⊆Blacks ´M
            ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
            ∧ length ´Ma=length ´M ∧ ´CountInv ´ind
          ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
      ´bc:=insert ´ind ´bc
    FI;;
    {| ´Proper ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ length ´Ma=length ´M ∧ ´CountInv (´ind+1)
     ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}
    ´ind:=´ind+1
  OD
```

lemma *Count*:
```
 ⊢ Count
 {| ´Proper ∧ Roots⊆Blacks ´M
  ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆´bc ∧ ´bc⊆Blacks ´M ∧ length ´Ma=length
´M
  ∧ (´obc < Blacks ´Ma ∨ ´Safe)|}
⟨proof⟩
```

Appending garbage nodes to the free list

axiomatization *Append-to-free* :: *nat* × *edges* ⇒ *edges*
where
```
 Append-to-free0: length (Append-to-free (i, e)) = length e and
 Append-to-free1: Proper-Edges (m, e)
                ⟹ Proper-Edges (m, Append-to-free(i, e)) and
 Append-to-free2: i ∉ Reach e
   ⟹ n ∈ Reach (Append-to-free(i, e)) = ( n = i ∨ n ∈ Reach e)
```

definition *AppendInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
```
 AppendInv ≡ «λind. ∀ i<length ´M. ind≤i ⟶ i∈Reach ´E ⟶ ´M!i=Black»
```

definition *Append* :: *gar-coll-state ann-com* **where**
```
  Append ≡
 {| ´Proper ∧ Roots⊆Blacks ´M ∧ ´Safe|}
 ´ind:=0;;
 {| ´Proper ∧ Roots⊆Blacks ´M ∧ ´Safe ∧ ´ind=0|}
   WHILE ´ind<length ´M
    INV {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind≤length ´M|}
   DO {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M|}
     IF ´M!´ind=Black THEN
       {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
       ´M:=´M[´ind:=White]
     ELSE {| ´Proper ∧ ´AppendInv ´ind ∧ ´ind<length ´M ∧ ´ind∉Reach ´E|}
          ´E:=Append-to-free(´ind,´E)
     FI;;
```

$\{\!|\ \acute{}Proper \wedge \acute{}AppendInv\ (\acute{}ind{+}1) \wedge \acute{}ind{<}length\ \acute{}M\ |\!\}$
$\acute{}ind{:=}\acute{}ind{+}1$
OD

lemma *Append*:
$\vdash$ *Append* $\{\!|\ \acute{}Proper |\!\}$
⟨*proof*⟩

Correctness of the Collector

definition *Collector* :: *gar-coll-state ann-com* **where**
Collector ≡
$\{\!|\ \acute{}Proper |\!\}$
WHILE True INV $\{\!|\ \acute{}Proper |\!\}$
DO
Blacken-Roots;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M |\!\}$
$\acute{}obc{:=}\{\}$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} |\!\}$
$\acute{}bc{:=}Roots$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} \wedge \acute{}bc{=}Roots |\!\}$
$\acute{}Ma{:=}M\text{-}init$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{=}\{\} \wedge \acute{}bc{=}Roots \wedge \acute{}Ma{=}M\text{-}init |\!\}$
WHILE $\acute{}obc{\neq}\acute{}bc$
INV $\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M$
$\wedge\ \acute{}obc{\subseteq}Blacks\ \acute{}Ma \wedge Blacks\ \acute{}Ma{\subseteq}\acute{}bc \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M$
$\wedge\ length\ \acute{}Ma{=}length\ \acute{}M \wedge (\acute{}obc < Blacks\ \acute{}Ma \vee \acute{}Safe) |\!\}$
DO $\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M |\!\}$
$\acute{}obc{:=}\acute{}bc$;;
Propagate-Black;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M$
$\wedge\ (\acute{}obc < Blacks\ \acute{}M \vee \acute{}Safe) |\!\}$
$\acute{}Ma{:=}\acute{}M$;;
$\{\!|\ \acute{}Proper \wedge Roots{\subseteq}Blacks\ \acute{}M \wedge \acute{}obc{\subseteq}Blacks\ \acute{}Ma$
$\wedge\ Blacks\ \acute{}Ma{\subseteq}Blacks\ \acute{}M \wedge \acute{}bc{\subseteq}Blacks\ \acute{}M \wedge length\ \acute{}Ma{=}length\ \acute{}M$
$\wedge\ (\acute{}obc < Blacks\ \acute{}Ma \vee \acute{}Safe) |\!\}$
$\acute{}bc{:=}\{\}$;;
Count
OD;;
Append
OD

lemma *Collector*:
$\vdash$ *Collector* $\{\!|False|\!\}$
⟨*proof*⟩

2.2.3 Interference Freedom

lemmas *modules* = *Redirect-Edge-def Color-Target-def Blacken-Roots-def*
Propagate-Black-def Count-def Append-def

lemmas *Invariants = PBInv-def Auxk-def CountInv-def AppendInv-def*
lemmas *abbrev = collector-defs mutator-defs Invariants*

lemma *interfree-Blacken-Roots-Redirect-Edge*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Blacken-Roots*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

lemma *interfree-Blacken-Roots-Color-Target*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Blacken-Roots*:
interfree-aux (*Some Color-Target*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

lemma *interfree-Propagate-Black-Redirect-Edge*:
interfree-aux (*Some Propagate-Black*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Propagate-Black*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Propagate-Black*)
⟨*proof*⟩

lemma *interfree-Propagate-Black-Color-Target*:
interfree-aux (*Some Propagate-Black*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Propagate-Black*:
interfree-aux (*Some Color-Target*, {}, *Some Propagate-Black*)
⟨*proof*⟩

lemma *interfree-Count-Redirect-Edge*:
interfree-aux (*Some Count*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Count*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Count*)
⟨*proof*⟩

lemma *interfree-Count-Color-Target*:
interfree-aux (*Some Count*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Count*:
interfree-aux (*Some Color-Target*, {}, *Some Count*)

⟨*proof*⟩

lemma *interfree-Append-Redirect-Edge*:
interfree-aux (*Some Append*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Append*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Append*)
⟨*proof*⟩

lemma *interfree-Append-Color-Target*:
interfree-aux (*Some Append*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Append*:
interfree-aux (*Some Color-Target*, {}, *Some Append*)
⟨*proof*⟩

lemmas *collector-mutator-interfree* =
interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target
interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target
interfree-Count-Redirect-Edge interfree-Count-Color-Target
interfree-Append-Redirect-Edge interfree-Append-Color-Target
interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots
interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black
interfree-Redirect-Edge-Count interfree-Color-Target-Count
interfree-Redirect-Edge-Append interfree-Color-Target-Append

Interference freedom Collector-Mutator

lemma *interfree-Collector-Mutator*:
interfree-aux (*Some Collector*, {}, *Some Mutator*)
⟨*proof*⟩

Interference freedom Mutator-Collector

lemma *interfree-Mutator-Collector*:
interfree-aux (*Some Mutator*, {}, *Some Collector*)
⟨*proof*⟩

The Garbage Collection algorithm

In total there are 289 verification conditions.

lemma *Gar-Coll*:
∥− {| ´*Proper* ∧ ´*Mut-init* ∧ ´*z* |}
COBEGIN
Collector
{| *False* |}
∥

Mutator
{|*False*|}
COEND
{|*False*|}
⟨*proof*⟩

end

2.3 The Multi-Mutator Case

theory *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

record *mut* =
Z :: *bool*
R :: *nat*
T :: *nat*

Declaration of variables:

record *mul-gar-coll-state* =
M :: *nodes*
E :: *edges*
bc :: *nat set*
obc :: *nat set*
Ma :: *nodes*
ind :: *nat*
k :: *nat*
q :: *nat*
l :: *nat*
Muts :: *mut list*

2.3.1 The Mutators

definition *Mul-mut-init* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
Mul-mut-init ≡ « λ*n*. *n*=*length* ´*Muts* ∧ (∀ *i*<*n*. *R* (´*Muts*!*i*)<*length* ´*E*
∧ *T* (´*Muts*!*i*)<*length* ´*M*) »

definition *Mul-Redirect-Edge* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Redirect-Edge j n ≡
{|´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)|}
⟨*IF T*(´*Muts*!*j*) ∈ *Reach* ´*E THEN*
´*E*:= ´*E*[*R* (´*Muts*!*j*):= (*fst* (´*E*!*R*(´*Muts*!*j*)), *T* (´*Muts*!*j*))] *FI*,,
´*Muts*:= ´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*False*|)]⟩

definition *Mul-Color-Target* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Color-Target j n ≡
{|´*Mul-mut-init n* ∧ ¬ *Z* (´*Muts*!*j*)|}
⟨´*M*:=´*M*[*T* (´*Muts*!*j*):=*Black*],, ´*Muts*:=´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*True*|)]⟩

definition *Mul-Mutator* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Mutator j n ≡
{| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
WHILE True
INV {| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
DO Mul-Redirect-Edge j n ;;
Mul-Color-Target j n
OD

lemmas *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

Correctness of the proof outline of one mutator

lemma *Mul-Redirect-Edge*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Redirect-Edge j n*
pre(*Mul-Color-Target j n*)
⟨*proof*⟩

lemma *Mul-Color-Target*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Color-Target j n*
{| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*) |}
⟨*proof*⟩

lemma *Mul-Mutator*: *0*≤*j* ∧ *j*<*n* ⟹
⊢ *Mul-Mutator j n* {|*False*|}
⟨*proof*⟩

Interference freedom between mutators

lemma *Mul-interfree-Redirect-Edge-Redirect-Edge*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
interfree-aux (*Some* (*Mul-Redirect-Edge i n*),{}, *Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Color-Target*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
interfree-aux (*Some*(*Mul-Redirect-Edge i n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Redirect-Edge*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Color-Target*:
⟦*0*≤*i*; *i*<*n*; *0*≤*j*; *j*<*n*; *i*≠*j*⟧ ⟹
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemmas *mul-mutator-interfree* =
Mul-interfree-Redirect-Edge-Redirect-Edge Mul-interfree-Redirect-Edge-Color-Target
Mul-interfree-Color-Target-Redirect-Edge Mul-interfree-Color-Target-Color-Target

lemma *Mul-interfree-Mutator-Mutator*: $[\![i < n; j < n; i \neq j]\!] \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator i n*), {}, *Some* (*Mul-Mutator j n*))
⟨*proof*⟩

Modular Parameterized Mutators

lemma *Mul-Parameterized-Mutators*: $0<n \Longrightarrow$
$\|-$ {| ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*)) |}
COBEGIN
SCHEME [0≤ *j*< *n*]
Mul-Mutator j n
{|*False*|}
COEND
{|*False*|}
⟨*proof*⟩

2.3.2 The Collector

definition *Queue* :: *mul-gar-coll-state* ⇒ *nat* **where**
Queue ≡ « *length* (*filter* (λ*i*. ¬ *Z i* ∧ ´*M*!(*T i*) ≠ *Black*) ´*Muts*) »

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *mul-gar-coll-state* ⇒ *bool* **where**
Proper-M-init ≡ « *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* »

definition *Mul-Proper* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
Mul-Proper ≡ « λ*n*. *Proper-Roots* ´*M* ∧ *Proper-Edges* (´*M*, ´*E*) ∧ ´*Proper-M-init*
∧ *n*=*length* ´*Muts* »

definition *Safe* :: *mul-gar-coll-state* ⇒ *bool* **where**
Safe ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

lemmas *mul-collector-defs* = *Proper-M-init-def Mul-Proper-def Safe-def*

Blackening Roots

definition *Mul-Blacken-Roots* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Blacken-Roots n ≡
{| ´*Mul-Proper n* |}
´*ind*:=*0*;;
{| ´*Mul-Proper n* ∧ ´*ind*=*0* |}
WHILE ´*ind*<*length* ´*M*
INV {| ´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length*
´*M* |}
DO {| ´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* |}

IF ´ind∈Roots THEN
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*|}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*+*1*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
´*ind*:=´*ind*+*1*
OD

lemma *Mul-Blacken-Roots*:
⊢ *Mul-Blacken-Roots n*
{|´*Mul-Proper n* ∧ *Roots* ⊆ *Blacks* ´*M*|}
⟨*proof*⟩

Propagating Black

definition *Mul-PBInv* :: *mul-gar-coll-state* ⇒ *bool* **where**
Mul-PBInv ≡ «´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue*
∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*)) ∧ ´*l*≤´*Queue*»

definition *Mul-Auxk* :: *mul-gar-coll-state* ⇒ *bool* **where**
Mul-Auxk ≡ «´*l*<´*Queue* ∨ ´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨ ´*obc*⊂*Blacks* ´*M*»

definition *Mul-Propagate-Black* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Propagate-Black n ≡
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*)|}
´*ind*:=*0*;;
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ *Blacks* ´*M*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*) ∧ ´*ind*=*0*|}
WHILE ´ind<*length ´E*
INV {|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ ´*ind*≤*length* ´*E*|}
DO {|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ ´*ind*<*length* ´*E*|}
IF ´M!(*fst* (´*E*!´*ind*))=*Black THEN*
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*Mul-PBInv* ∧ (´*M*!*fst*(´*E*!´*ind*))=*Black* ∧ ´*ind*<*length* ´*E*|}
´*k*:=*snd*(´*E*!´*ind*);;
{|´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue* ∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*))
∧ ´*l*≤´*Queue* ∧ ´*Mul-Auxk*) ∧ ´*k*<*length* ´*M* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black*
∧ ´*ind*<*length* ´*E*|}

```
  ⟨´M:=´M[´k:=Black],,´ind:=´ind+1⟩
  ELSE {|´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ ´Mul-PBInv ∧ ´ind<length ´E|}
       ⟨IF ´M!(fst (´E!´ind))≠Black THEN ´ind:=´ind+1 FI⟩ FI
OD
```

lemma *Mul-Propagate-Black*:

```
⊢ Mul-Propagate-Black n
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue ∧ (´l≤´Queue ∨ ´obc⊂Blacks
´M))|}
⟨proof⟩
```

Counting Black Nodes

definition *Mul-CountInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**

```
Mul-CountInv ≡ « λind. {i. i<ind ∧ ´Ma!i=Black}⊆´bc »
```

definition *Mul-Count* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**

```
Mul-Count n ≡
{|´Mul-Proper n ∧ Roots⊆Blacks ´M
 ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
 ∧ length ´Ma=length ´M
 ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
 ∧ ´q<n+1 ∧ ´bc={}|}
´ind:=0;;
{|´Mul-Proper n ∧ Roots⊆Blacks ´M
 ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
 ∧ length ´Ma=length ´M
 ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
 ∧ ´q<n+1 ∧ ´bc={} ∧ ´ind=0|}
WHILE ´ind<length ´M
  INV {|´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
       ∧ ´q<n+1 ∧ ´ind≤length ´M|}
DO {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M|}
  IF ´M!´ind=Black
  THEN {|´Mul-Proper n ∧ Roots⊆Blacks ´M
        ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
         ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
```

```
          ∧ ´q<n+1 ∧ ´ind<length ´M ∧ ´M!´ind=Black⦘
        ´bc:=insert ´ind ´bc
    FI;;
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv (´ind+1)
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M⦘
  ´ind:=´ind+1
  OD
```

lemma *Mul-Count*:
```
  ⊢ Mul-Count n
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1⦘
⟨proof⟩
```

Appending garbage nodes to the free list

axiomatization *Append-to-free* :: *nat* × *edges* ⇒ *edges*
where
```
  Append-to-free0: length (Append-to-free (i, e)) = length e and
  Append-to-free1: Proper-Edges (m, e)
                   ⟹ Proper-Edges (m, Append-to-free(i, e)) and
  Append-to-free2: i ∉ Reach e
          ⟹ n ∈ Reach (Append-to-free(i, e)) = ( n = i ∨ n ∈ Reach e)
```

definition *Mul-AppendInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
```
  Mul-AppendInv ≡ « λind. (∀ i. ind≤i ⟶ i<length ´M ⟶ i∈Reach ´E ⟶
´M!i=Black)»
```

definition *Mul-Append* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
```
  Mul-Append n ≡
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe⦘
  ´ind:=0;;
  ⦇´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe ∧ ´ind=0⦘
  WHILE ´ind<length ´M
    INV ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind≤length ´M⦘
  DO ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M⦘
      IF ´M!´ind=Black THEN
    ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´M!´ind=Black⦘
      ´M:=´M[´ind:=White]
      ELSE
      ⦇´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´ind∉Reach
´E⦘
      ´E:=Append-to-free(´ind,´E)
```

```
    FI;;
  {´Mul-Proper n ∧ ´Mul-AppendInv (´ind+1) ∧ ´ind<length ´M}
   ´ind:=´ind+1
  OD

lemma Mul-Append:
  ⊢ Mul-Append n
     {´Mul-Proper n}
⟨proof⟩
```

Collector

```
definition Mul-Collector :: nat ⇒  mul-gar-coll-state ann-com where
  Mul-Collector n ≡
{´Mul-Proper n}
WHILE True INV {´Mul-Proper n}
DO
Mul-Blacken-Roots n ;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M}
 ´obc:={};;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={}}
 ´bc:=Roots;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots}
 ´l:=0;;
{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´l=0}
 WHILE ´l<n+1
   INV {´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧
        (´Safe ∨ (´l≤´Queue ∨ ´bc⊂Blacks ´M) ∧ ´l<n+1)}
 DO {´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´l≤´Queue ∨ ´bc⊂Blacks ´M)}
    ´obc:=´bc;;
    Mul-Propagate-Black n;;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
     ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M))}
    ´bc:={};;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
     ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M)) ∧ ´bc={}}
      ⟨ ´Ma:=´M,, ´q:=´Queue ⟩;;
    Mul-Count n;;
    {´Mul-Proper n ∧ Roots⊆Blacks ´M
     ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
     ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
     ∧ ´q<n+1}
    IF ´obc=´bc THEN
```

$\{\!|\ ´Mul\text{-}Proper\ n \wedge Roots{\subseteq}Blacks\ ´M$
$\wedge\ ´obc{\subseteq}Blacks\ ´Ma \wedge Blacks\ ´Ma{\subseteq}Blacks\ ´M \wedge ´bc{\subseteq}Blacks\ ´M$
$\wedge\ length\ ´Ma{=}length\ ´M \wedge Blacks\ ´Ma{\subseteq}´bc$
$\wedge\ (´Safe \vee ´obc{\subset}Blacks\ ´Ma \vee ´l{<}´q \wedge (´q{\leq}´Queue \vee ´obc{\subset}Blacks\ ´M))$
$\wedge\ ´q{<}n{+}1 \wedge ´obc{=}´bc\ |\!\}$
$´l{:=}´l{+}1$
$ELSE\ \{\!|\ ´Mul\text{-}Proper\ n \wedge Roots{\subseteq}Blacks\ ´M$
$\wedge\ ´obc{\subseteq}Blacks\ ´Ma \wedge Blacks\ ´Ma{\subseteq}Blacks\ ´M \wedge ´bc{\subseteq}Blacks\ ´M$
$\wedge\ length\ ´Ma{=}length\ ´M \wedge Blacks\ ´Ma{\subseteq}´bc$
$\wedge\ (´Safe \vee ´obc{\subset}Blacks\ ´Ma \vee ´l{<}´q \wedge (´q{\leq}´Queue \vee ´obc{\subset}Blacks\ ´M))$
$\wedge\ ´q{<}n{+}1 \wedge ´obc{\neq}´bc\ |\!\}$
$´l{:=}0\ FI$
OD;;
$Mul\text{-}Append\ n$
OD

lemmas *mul-modules = Mul-Redirect-Edge-def Mul-Color-Target-def Mul-Blacken-Roots-def Mul-Propagate-Black-def Mul-Count-def Mul-Append-def*

lemma *Mul-Collector*:
$\vdash Mul\text{-}Collector\ n$
$\{\!|False|\!\}$
⟨*proof*⟩

2.3.3 Interference Freedom

lemma *le-length-filter-update*[*rule-format*]:
$\forall i.\ (\neg P\ (list!i) \vee P\ j) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) \leq length(filter\ P\ (list[i{:=}j]))$
⟨*proof*⟩

lemma *less-length-filter-update* [*rule-format*]:
$\forall i.\ P\ j \wedge \neg(P\ (list!i)) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) < length(filter\ P\ (list[i{:=}j]))$
⟨*proof*⟩

lemma *Mul-interfree-Blacken-Roots-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Blacken-Roots*: $[\![0{\leq}j;\ j{<}n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Blacken-Roots n*))
⟨*proof*⟩

lemma *Mul-interfree-Blacken-Roots-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some* (*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Blacken-Roots*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some* (*Mul-Blacken-Roots n*))
⟨*proof*⟩

lemma *Mul-interfree-Propagate-Black-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Propagate-Black*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Propagate-Black n*))
⟨*proof*⟩

lemma *Mul-interfree-Propagate-Black-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Propagate-Black*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some*(*Mul-Propagate-Black n*))
⟨*proof*⟩

lemma *Mul-interfree-Count-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Count*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Count n*))
⟨*proof*⟩

lemma *Mul-interfree-Count-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Count*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Count n*))
⟨*proof*⟩

lemma *Mul-interfree-Append-Redirect-Edge*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Append*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Append n*))
⟨*proof*⟩

lemma *Mul-interfree-Append-Color-Target*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Append*: $[\![0{\leq}j;\ j{<}n]\!]{\Longrightarrow}$

interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Append n*))
⟨*proof*⟩

Interference freedom Collector-Mutator

lemmas *mul-collector-mutator-interfree* =
Mul-interfree-Blacken-Roots-Redirect-Edge Mul-interfree-Blacken-Roots-Color-Target
Mul-interfree-Propagate-Black-Redirect-Edge Mul-interfree-Propagate-Black-Color-Target
Mul-interfree-Count-Redirect-Edge Mul-interfree-Count-Color-Target
Mul-interfree-Append-Redirect-Edge Mul-interfree-Append-Color-Target
Mul-interfree-Redirect-Edge-Blacken-Roots Mul-interfree-Color-Target-Blacken-Roots
Mul-interfree-Redirect-Edge-Propagate-Black Mul-interfree-Color-Target-Propagate-Black
Mul-interfree-Redirect-Edge-Count Mul-interfree-Color-Target-Count
Mul-interfree-Redirect-Edge-Append Mul-interfree-Color-Target-Append

lemma *Mul-interfree-Collector-Mutator*: *j*<*n* ⟹
interfree-aux (*Some* (*Mul-Collector n*), {}, *Some* (*Mul-Mutator j n*))
⟨*proof*⟩

Interference freedom Mutator-Collector

lemma *Mul-interfree-Mutator-Collector*: *j* < *n* ⟹
interfree-aux (*Some* (*Mul-Mutator j n*), {}, *Some* (*Mul-Collector n*))
⟨*proof*⟩

The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

lemma *Mul-Gar-Coll*:
⊨− {| ´*Mul-Proper n* ∧ ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))|}
COBEGIN
Mul-Collector n
{|*False*|}
‖
SCHEME [0≤ *j*< *n*]
Mul-Mutator j n
{|*False*|}
COEND
{|*False*|}
⟨*proof*⟩

end

Chapter 3

The Rely-Guarantee Method

3.1 Abstract Syntax

theory *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (bexp) as sets of states. Syntax of commands *com* and parallel commands *par-com*.

type-synonym *'a bexp* = *'a set*

datatype *'a com* =
  Basic 'a ⇒'a
| *Seq 'a com 'a com*
| *Cond 'a bexp 'a com 'a com*
| *While 'a bexp 'a com*
| *Await 'a bexp 'a com*

type-synonym *'a par-com* = *'a com option list*

end

3.2 Operational Semantics

theory *RG-Tran*
imports *RG-Com*
begin

3.2.1 Semantics of Component Programs

Environment transitions

type-synonym *'a conf* = (*('a com) option*) × *'a*

inductive-set
  etran :: (*'a conf* × *'a conf*) *set*
  and *etran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*e*→ - [*81*,*81*] *80*)

where
  $P -e\rightarrow Q \equiv (P,Q) \in etran$
| *Env*: $(P, s) -e\rightarrow (P, t)$

lemma *etranE*: $c -e\rightarrow c' \Longrightarrow (\bigwedge P\ s\ t.\ c = (P, s) \Longrightarrow c' = (P, t) \Longrightarrow Q) \Longrightarrow Q$
  ⟨*proof*⟩

Component transitions

inductive-set
  ctran :: (*'a conf* × *'a conf*) *set*
  and *ctran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*c*→ - [*81*,*81*] *80*)
  and *ctrans* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*c**→ - [*81*,*81*] *80*)
where
  $P -c\rightarrow Q \equiv (P,Q) \in ctran$
| $P -c*\rightarrow Q \equiv (P,Q) \in ctran^*$

| *Basic*: $(Some(Basic\ f), s) -c\rightarrow (None, f\ s)$

| *Seq1*: $(Some\ P0, s) -c\rightarrow (None, t) \Longrightarrow (Some(Seq\ P0\ P1), s) -c\rightarrow (Some\ P1, t)$

| *Seq2*: $(Some\ P0, s) -c\rightarrow (Some\ P2, t) \Longrightarrow (Some(Seq\ P0\ P1), s) -c\rightarrow (Some(Seq\ P2\ P1), t)$

| *CondT*: $s\in b \Longrightarrow (Some(Cond\ b\ P1\ P2), s) -c\rightarrow (Some\ P1, s)$
| *CondF*: $s\notin b \Longrightarrow (Some(Cond\ b\ P1\ P2), s) -c\rightarrow (Some\ P2, s)$

| *WhileF*: $s\notin b \Longrightarrow (Some(While\ b\ P), s) -c\rightarrow (None, s)$
| *WhileT*: $s\in b \Longrightarrow (Some(While\ b\ P), s) -c\rightarrow (Some(Seq\ P\ (While\ b\ P)), s)$

| *Await*: $[\![s\in b; (Some\ P, s) -c*\rightarrow (None, t)]\!] \Longrightarrow (Some(Await\ b\ P), s) -c\rightarrow (None, t)$

monos *rtrancl-mono*

3.2.2 Semantics of Parallel Programs

type-synonym *'a par-conf* = (*'a par-com*) × *'a*

inductive-set
  par-etran :: (*'a par-conf* × *'a par-conf*) *set*
  and *par-etran'* :: [*'a par-conf*,*'a par-conf*] ⇒ *bool* (- −*pe*→ - [*81*,*81*] *80*)
where
  $P -pe\rightarrow Q \equiv (P,Q) \in par\text{-}etran$
| *ParEnv*: $(Ps, s) -pe\rightarrow (Ps, t)$

inductive-set
  par-ctran :: (*'a par-conf* × *'a par-conf*) *set*
  and *par-ctran'* :: [*'a par-conf*,*'a par-conf*] ⇒ *bool* (- −*pc*→ - [*81*,*81*] *80*)

where
  $P\ -pc\rightarrow\ Q \equiv (P,Q) \in$ *par-ctran*
| *ParComp*: $[\![i<$*length Ps*; $(Ps!i,\ s)\ -c\rightarrow\ (r,\ t)]\!] \Longrightarrow (Ps,\ s)\ -pc\rightarrow\ (Ps[i:=r],\ t)$

lemma *par-ctranE*: $c\ -pc\rightarrow\ c' \Longrightarrow$
  $(\bigwedge i\ Ps\ s\ r\ t.\ c = (Ps,\ s) \Longrightarrow c' = (Ps[i := r],\ t) \Longrightarrow i < length\ Ps \Longrightarrow$
    $(Ps\ !\ i,\ s)\ -c\rightarrow\ (r,\ t) \Longrightarrow P) \Longrightarrow P$
  ⟨*proof*⟩

3.2.3 Computations

Sequential computations

type-synonym *'a confs* = *'a conf list*

inductive-set *cptn* :: *'a confs set*
where
  CptnOne: $[(P,s)] \in$ *cptn*
| *CptnEnv*: $(P,\ t)\#xs \in$ *cptn* $\Longrightarrow (P,s)\#(P,t)\#xs \in$ *cptn*
| *CptnComp*: $[\![(P,s)\ -c\rightarrow\ (Q,t);\ (Q,\ t)\#xs \in$ *cptn* $]\!] \Longrightarrow (P,s)\#(Q,t)\#xs \in$ *cptn*

definition *cp* :: *('a com) option* ⇒ *'a* ⇒ *('a confs) set* **where**
  cp P s ≡ $\{l.\ l!0=(P,s) \wedge l \in$ *cptn*$\}$

Parallel computations

type-synonym *'a par-confs* = *'a par-conf list*

inductive-set *par-cptn* :: *'a par-confs set*
where
  ParCptnOne: $[(P,s)] \in$ *par-cptn*
| *ParCptnEnv*: $(P,t)\#xs \in$ *par-cptn* $\Longrightarrow (P,s)\#(P,t)\#xs \in$ *par-cptn*
| *ParCptnComp*: $[\![\ (P,s)\ -pc\rightarrow\ (Q,t);\ (Q,t)\#xs \in$ *par-cptn* $]\!] \Longrightarrow (P,s)\#(Q,t)\#xs$
$\in$ *par-cptn*

definition *par-cp* :: *'a par-com* ⇒ *'a* ⇒ *('a par-confs) set* **where**
  par-cp P s ≡ $\{l.\ l!0=(P,s) \wedge l \in$ *par-cptn*$\}$

3.2.4 Modular Definition of Computation

definition *lift* :: *'a com* ⇒ *'a conf* ⇒ *'a conf* **where**
  lift Q ≡ $\lambda(P,\ s).$ (*if P=None then* (*Some Q*,*s*) *else* (*Some*(*Seq* (*the P*) *Q*), *s*))

inductive-set *cptn-mod* :: *('a confs) set*
where
  CptnModOne: $[(P,\ s)] \in$ *cptn-mod*
| *CptnModEnv*: $(P,\ t)\#xs \in$ *cptn-mod* $\Longrightarrow (P,\ s)\#(P,\ t)\#xs \in$ *cptn-mod*
| *CptnModNone*: $[\![(Some\ P,\ s)\ -c\rightarrow\ (None,\ t);\ (None,\ t)\#xs \in$ *cptn-mod* $]\!] \Longrightarrow$
$(Some\ P,s)\#(None,\ t)\#xs \in$*cptn-mod*

| *CptnModCondT*: $[\![(Some\ P0,\ s)\#ys \in cptn\text{-}mod;\ s \in b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P0,\ s)\#ys \in cptn\text{-}mod$
| *CptnModCondF*: $[\![(Some\ P1,\ s)\#ys \in cptn\text{-}mod;\ s \notin b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P1,\ s)\#ys \in cptn\text{-}mod$
| *CptnModSeq1*: $[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ zs=map\ (lift\ P1)\ xs\ ]\!]$
$\Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$
| *CptnModSeq2*:
$[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P0,\ s)\#xs)) = None;$
$(Some\ P1,\ snd(last\ ((Some\ P0,\ s)\#xs)))\#ys \in cptn\text{-}mod;$
$zs=(map\ (lift\ P1)\ xs)@ys\ ]\!] \Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$

| *CptnModWhile1*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ s \in b;\ zs=map\ (lift\ (While\ b\ P))\ xs\ ]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$
| *CptnModWhile2*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P,\ s)\#xs))=None;\ s \in b;$
$zs=(map\ (lift\ (While\ b\ P))\ xs)@ys;$
$(Some(While\ b\ P),\ snd(last\ ((Some\ P,\ s)\#xs)))\#ys \in cptn\text{-}mod]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$

3.2.5 Equivalence of Both Definitions.

lemma *last-length*: $((a\#xs)!(length\ xs))=last\ (a\#xs)$
⟨*proof*⟩

lemma *div-seq* [*rule-format*]: $list \in cptn\text{-}mod \Longrightarrow$
$(\forall\ s\ P\ Q\ zs.\ list=(Some\ (Seq\ P\ Q),\ s)\#zs \longrightarrow$
$(\exists\ xs.\ (Some\ P,\ s)\#xs \in cptn\text{-}mod\ \wedge\ (zs=(map\ (lift\ Q)\ xs)\ \vee$
$(\ fst(((Some\ P,\ s)\#xs)!length\ xs)=None\ \wedge$
$(\exists\ ys.\ (Some\ Q,\ snd(((Some\ P,\ s)\#xs)!length\ xs))\#ys \in cptn\text{-}mod$
$\wedge\ zs=(map\ (lift\ (Q))\ xs)@ys)))))$
⟨*proof*⟩

lemma *cptn-onlyif-cptn-mod-aux* [*rule-format*]:
$\forall\ s\ Q\ t\ xs.((Some\ a,\ s),\ Q,\ t) \in ctran \longrightarrow (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
$\longrightarrow (Some\ a,\ s)\ \#\ (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
⟨*proof*⟩

lemma *cptn-onlyif-cptn-mod* [*rule-format*]: $c \in cptn \Longrightarrow c \in cptn\text{-}mod$
⟨*proof*⟩

lemma *lift-is-cptn*: $c{\in}cptn \Longrightarrow map\ (lift\ P)\ c \in cptn$
⟨*proof*⟩

lemma *cptn-append-is-cptn* [*rule-format*]:
$\forall\ b\ a.\ b\#c1{\in}cptn \longrightarrow\ a\#c2{\in}cptn \longrightarrow (b\#c1)!length\ c1=a \longrightarrow b\#c1@c2{\in}cptn$
⟨*proof*⟩

lemma *last-lift*: $[\![xs{\neq}[];\ fst(xs!(length\ xs - (Suc\ 0)))=None]\!]$

$\Longrightarrow$ *fst*((*map* (*lift P*) *xs*)!(*length* (*map* (*lift P*) *xs*)− (*Suc 0*)))=(*Some P*)
⟨*proof*⟩

lemma *last-fst* [*rule-format*]: *P*((*a*#*x*)!*length x*) ⟶ ¬*P a* ⟶ *P* (*x*!(*length x* − (*Suc 0*)))
⟨*proof*⟩

lemma *last-fst-esp*:
fst(((*Some a*,*s*)#*xs*)!(*length xs*))=*None* $\Longrightarrow$ *fst*(*xs*!(*length xs* − (*Suc 0*)))=*None*
⟨*proof*⟩

lemma *last-snd*: *xs*≠[] $\Longrightarrow$
snd(((*map* (*lift P*) *xs*))!(*length* (*map* (*lift P*) *xs*) − (*Suc 0*)))=*snd*(*xs*!(*length xs* − (*Suc 0*)))
⟨*proof*⟩

lemma *Cons-lift*: (*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)
⟨*proof*⟩

lemma *Cons-lift-append*:
(*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) @ *ys* = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)@ *ys*
⟨*proof*⟩

lemma *lift-nth*: *i*<*length xs* $\Longrightarrow$ *map* (*lift Q*) *xs* ! *i* = *lift Q* (*xs*! *i*)
⟨*proof*⟩

lemma *snd-lift*: *i*< *length xs* $\Longrightarrow$ *snd*(*lift Q* (*xs* ! *i*))= *snd* (*xs* ! *i*)
⟨*proof*⟩

lemma *cptn-if-cptn-mod*: *c* ∈ *cptn-mod* $\Longrightarrow$ *c* ∈ *cptn*
⟨*proof*⟩

theorem *cptn-iff-cptn-mod*: (*c* ∈ *cptn*) = (*c* ∈ *cptn-mod*)
⟨*proof*⟩

3.3 Validity of Correctness Formulas

3.3.1 Validity for Component Programs.

type-synonym *'a rgformula* =
'a com × *'a set* × (*'a* × *'a*) *set* × (*'a* × *'a*) *set* × *'a set*

definition *assum* :: (*'a set* × (*'a* × *'a*) *set*) ⇒ (*'a confs*) *set* **where**
assum ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*e*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

definition *comm* :: ((*'a* × *'a*) *set* × *'a set*) ⇒ (*'a confs*) *set* **where**

comm ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*c*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
(*fst* (*last c*) = *None* ⟶ *snd* (*last c*) ∈ *post*)}

definition *com-validity* :: *'a com* ⇒ *'a set* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set* ⇒ *'a set* ⇒ *bool*
(⊨ - *sat* [-, -, -, -] [60,0,0,0,0] 45) **where**
⊨ *P sat* [*pre*, *rely*, *guar*, *post*] ≡
∀ *s*. *cp* (*Some P*) *s* ∩ *assum*(*pre*, *rely*) ⊆ *comm*(*guar*, *post*)

3.3.2 Validity for Parallel Programs.

definition *All-None* :: ((*'a com*) *option*) *list* ⇒ *bool* **where**
All-None xs ≡ ∀ *c*∈*set xs*. *c*=*None*

definition *par-assum* :: (*'a set* × (*'a* × *'a*) *set*) ⇒ (*'a par-confs*) *set* **where**
par-assum ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*pe*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

definition *par-comm* :: ((*'a* × *'a*) *set* × *'a set*) ⇒ (*'a par-confs*) *set* **where**
par-comm ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*pc*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
(*All-None* (*fst* (*last c*)) ⟶ *snd*(*last c*) ∈ *post*)}

definition *par-com-validity* :: *'a par-com* ⇒ *'a set* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set*
⇒ *'a set* ⇒ *bool* (⊨ - *SAT* [-, -, -, -] [60,0,0,0,0] 45) **where**
⊨ *Ps SAT* [*pre*, *rely*, *guar*, *post*] ≡
∀ *s*. *par-cp Ps s* ∩ *par-assum*(*pre*, *rely*) ⊆ *par-comm*(*guar*, *post*)

3.3.3 Compositionality of the Semantics

Definition of the conjoin operator

definition *same-length* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-length c clist ≡ (∀ *i*<*length clist*. *length*(*clist*!*i*)=*length c*)

definition *same-state* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-state c clist ≡ (∀ *i* <*length clist*. ∀ *j*<*length c*. *snd*(*c*!*j*) = *snd*((*clist*!*i*)!*j*))

definition *same-program* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-program c clist ≡ (∀ *j*<*length c*. *fst*(*c*!*j*) = *map* (λ*x*. *fst*(*nth x j*)) *clist*)

definition *compat-label* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
compat-label c clist ≡ (∀ *j*. *Suc j*<*length c* ⟶
(*c*!*j* −*pc*→ *c*!*Suc j* ∧ (∃ *i*<*length clist*. (*clist*!*i*)!*j* −*c*→ (*clist*!*i*)! *Suc j* ∧
(∀ *l*<*length clist*. *l*≠*i* ⟶ (*clist*!*l*)!*j* −*e*→ (*clist*!*l*)! *Suc j*))) ∨
(*c*!*j* −*pe*→ *c*!*Suc j* ∧ (∀ *i*<*length clist*. (*clist*!*i*)!*j* −*e*→ (*clist*!*i*)! *Suc j*)))

definition *conjoin* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* (- ∝ - [*65*,*65*] *64*)
where
c ∝ *clist* ≡ (*same-length c clist*) ∧ (*same-state c clist*) ∧ (*same-program c clist*) ∧ (*compat-label c clist*)

Some previous lemmas

lemma *list-eq-if* [*rule-format*]:
∀ *ys*. *xs*=*ys* ⟶ (*length xs* = *length ys*) ⟶ (∀ *i*<*length xs*. *xs*!*i*=*ys*!*i*)
⟨*proof*⟩

lemma *list-eq*: (*length xs* = *length ys* ∧ (∀ *i*<*length xs*. *xs*!*i*=*ys*!*i*)) = (*xs*=*ys*)
⟨*proof*⟩

lemma *nth-tl*: ⟦ *ys*!*0*=*a*; *ys*≠[] ⟧ ⟹ *ys*=(*a*#(*tl ys*))
⟨*proof*⟩

lemma *nth-tl-if* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P ys* ⟶ *P* (*a*#(*tl ys*))
⟨*proof*⟩

lemma *nth-tl-onlyif* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P* (*a*#(*tl ys*)) ⟶ *P ys*
⟨*proof*⟩

lemma *seq-not-eq1*: *Seq c1 c2*≠*c1*
⟨*proof*⟩

lemma *seq-not-eq2*: *Seq c1 c2*≠*c2*
⟨*proof*⟩

lemma *if-not-eq1*: *Cond b c1 c2* ≠*c1*
⟨*proof*⟩

lemma *if-not-eq2*: *Cond b c1 c2*≠*c2*
⟨*proof*⟩

lemmas *seq-and-if-not-eq* [*simp*] = *seq-not-eq1 seq-not-eq2*
seq-not-eq1 [*THEN not-sym*] *seq-not-eq2* [*THEN not-sym*]
if-not-eq1 if-not-eq2 if-not-eq1 [*THEN not-sym*] *if-not-eq2* [*THEN not-sym*]

lemma *prog-not-eq-in-ctran-aux*:
assumes *c*: (*P*,*s*) −*c*→ (*Q*,*t*)
shows *P*≠*Q* ⟨*proof*⟩

lemma *prog-not-eq-in-ctran* [*simp*]: ¬ (*P*,*s*) −*c*→ (*P*,*t*)
⟨*proof*⟩

lemma *prog-not-eq-in-par-ctran-aux* [*rule-format*]: (*P*,*s*) −*pc*→ (*Q*,*t*) ⟹ (*P*≠*Q*)
⟨*proof*⟩

lemma *prog-not-eq-in-par-ctran* [*simp*]: ¬ (*P*,*s*) −*pc*→ (*P*,*t*)
⟨*proof*⟩

lemma *tl-in-cptn*: ⟦ *a*#*xs* ∈*cptn*; *xs*≠[] ⟧ ⟹ *xs*∈*cptn*
⟨*proof*⟩

lemma *tl-zero*[*rule-format*]:
P (*ys*!*Suc j*) ⟶ *Suc j*<*length ys* ⟶ *ys*≠[] ⟶ *P* (*tl*(*ys*)!*j*)
⟨*proof*⟩

3.3.4 The Semantics is Compositional

lemma *aux-if* [*rule-format*]:
∀ *xs s clist*. (*length clist* = *length xs* ∧ (∀ *i*<*length xs*. (*xs*!*i*,*s*)#*clist*!*i* ∈ *cptn*)
∧ ((*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#*snd i*) (*zip xs clist*))
⟶ (*xs*, *s*)#*ys* ∈ *par-cptn*)
⟨*proof*⟩

lemma *aux-onlyif* [*rule-format*]: ∀ *xs s*. (*xs*, *s*)#*ys* ∈ *par-cptn* ⟶
(∃ *clist*. (*length clist* = *length xs*) ∧
(*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#(*snd i*)) (*zip xs clist*) ∧
(∀ *i*<*length xs*. (*xs*!*i*,*s*)#(*clist*!*i*) ∈ *cptn*))
⟨*proof*⟩

lemma *one-iff-aux*: *xs*≠[] ⟹ (∀ *ys*. ((*xs*, *s*)#*ys* ∈ *par-cptn*) =
(∃ *clist*. *length clist*= *length xs* ∧
((*xs*, *s*)#*ys* ∝ *map* (λ*i*. (*fst i*,*s*)#(*snd i*)) (*zip xs clist*)) ∧
(∀ *i*<*length xs*. (*xs*!*i*,*s*)#(*clist*!*i*) ∈ *cptn*))) =
(*par-cp* (*xs*) *s* = {*c*. ∃ *clist*. (*length clist*)=(*length xs*) ∧
(∀ *i*<*length clist*. (*clist*!*i*) ∈ *cp*(*xs*!*i*) *s*) ∧ *c* ∝ *clist*})
⟨*proof*⟩

theorem *one*: *xs*≠[] ⟹
par-cp xs s = {*c*. ∃ *clist*. (*length clist*)=(*length xs*) ∧
(∀ *i*<*length clist*. (*clist*!*i*) ∈ *cp*(*xs*!*i*) *s*) ∧ *c* ∝ *clist*}
⟨*proof*⟩

end

3.4 The Proof System

theory *RG-Hoare* **imports** *RG-Tran* **begin**

3.4.1 Proof System for Component Programs

declare *Un-subset-iff* [*simp del*] *sup.bounded-iff* [*simp del*]

definition *stable* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *bool* **where**
stable ≡ λ*f g*. (∀ *x y*. *x* ∈ *f* ⟶ (*x*, *y*) ∈ *g* ⟶ *y* ∈ *f*)

inductive
rghoare :: [*'a com*, *'a set*, (*'a* × *'a*) *set*, (*'a* × *'a*) *set*, *'a set*] ⇒ *bool*
(⊢ - *sat* [-, -, -, -] [*60*,*0*,*0*,*0*,*0*] *45*)
where
Basic: ⟦ *pre* ⊆ {*s*. *f s* ∈ *post*}; {(*s*,*t*). *s* ∈ *pre* ∧ (*t*=*f s* ∨ *t*=*s*)} ⊆ *guar*;
stable pre rely; *stable post rely* ⟧
⟹ ⊢ *Basic f sat* [*pre*, *rely*, *guar*, *post*]

| *Seq*: ⟦ ⊢ *P sat* [*pre*, *rely*, *guar*, *mid*]; ⊢ *Q sat* [*mid*, *rely*, *guar*, *post*] ⟧
⟹ ⊢ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]

| *Cond*: ⟦ *stable pre rely*; ⊢ *P1 sat* [*pre* ∩ *b*, *rely*, *guar*, *post*];
⊢ *P2 sat* [*pre* ∩ −*b*, *rely*, *guar*, *post*]; ∀ *s*. (*s*,*s*)∈*guar* ⟧
⟹ ⊢ *Cond b P1 P2 sat* [*pre*, *rely*, *guar*, *post*]

| *While*: ⟦ *stable pre rely*; (*pre* ∩ −*b*) ⊆ *post*; *stable post rely*;
⊢ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*,*s*)∈*guar* ⟧
⟹ ⊢ *While b P sat* [*pre*, *rely*, *guar*, *post*]

| *Await*: ⟦ *stable pre rely*; *stable post rely*;
∀ *V*. ⊢ *P sat* [*pre* ∩ *b* ∩ {*V*}, {(*s*, *t*). *s* = *t*},
UNIV, {*s*. (*V*, *s*) ∈ *guar*} ∩ *post*] ⟧
⟹ ⊢ *Await b P sat* [*pre*, *rely*, *guar*, *post*]

| *Conseq*: ⟦ *pre* ⊆ *pre'*; *rely* ⊆ *rely'*; *guar'* ⊆ *guar*; *post'* ⊆ *post*;
⊢ *P sat* [*pre'*, *rely'*, *guar'*, *post'*] ⟧
⟹ ⊢ *P sat* [*pre*, *rely*, *guar*, *post*]

definition *Pre* :: *'a rgformula* ⇒ *'a set* **where**
Pre x ≡ *fst*(*snd x*)

definition *Post* :: *'a rgformula* ⇒ *'a set* **where**
Post x ≡ *snd*(*snd*(*snd*(*snd x*)))

definition *Rely* :: *'a rgformula* ⇒ (*'a* × *'a*) *set* **where**
Rely x ≡ *fst*(*snd*(*snd x*))

definition *Guar* :: *'a rgformula* ⇒ (*'a* × *'a*) *set* **where**
Guar x ≡ *fst*(*snd*(*snd*(*snd x*)))

definition *Com* :: *'a rgformula* ⇒ *'a com* **where**
Com x ≡ *fst x*

3.4.2 Proof System for Parallel Programs

type-synonym *'a par-rgformula* =
(*'a rgformula*) *list* × *'a set* × (*'a* × *'a*) *set* × (*'a* × *'a*) *set* × *'a set*

inductive
par-rghoare :: ('*a rgformula*) *list* ⇒ '*a set* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* ⇒ '*a set* ⇒ *bool*
(⊢ - *SAT* [-, -, -, -] [*60*,*0*,*0*,*0*,*0*] *45*)
where
Parallel:
⟦ ∀ *i*<*length xs*. *rely* ∪ (⋃ *j*∈{*j*. *j*<*length xs* ∧ *j*≠*i*}. *Guar*(*xs*!*j*)) ⊆ *Rely*(*xs*!*i*);
(⋃ *j*∈{*j*. *j*<*length xs*}. *Guar*(*xs*!*j*)) ⊆ *guar*;
pre ⊆ (⋂ *i*∈{*i*. *i*<*length xs*}. *Pre*(*xs*!*i*));
(⋂ *i*∈{*i*. *i*<*length xs*}. *Post*(*xs*!*i*)) ⊆ *post*;
∀ *i*<*length xs*. ⊢ *Com*(*xs*!*i*) *sat* [*Pre*(*xs*!*i*),*Rely*(*xs*!*i*),*Guar*(*xs*!*i*),*Post*(*xs*!*i*)] ⟧
⟹ ⊢ *xs SAT* [*pre*, *rely*, *guar*, *post*]

3.5 Soundness

Some previous lemmas

lemma *tl-of-assum-in-assum*:
(*P*, *s*) # (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*) ⟹ *stable pre rely*
⟹ (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*)
⟨*proof*⟩

lemma *etran-in-comm*:
(*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*) ⟹ (*P*, *s*) # (*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*)
⟨*proof*⟩

lemma *ctran-in-comm*:
⟦(*s*, *s*) ∈ *guar*; (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)⟧
⟹ (*P*, *s*) # (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)
⟨*proof*⟩

lemma *takecptn-is-cptn* [*rule-format*, *elim*!]:
∀ *j*. *c* ∈ *cptn* ⟶ *take* (*Suc j*) *c* ∈ *cptn*
⟨*proof*⟩

lemma *dropcptn-is-cptn* [*rule-format*,*elim*!]:
∀ *j*<*length c*. *c* ∈ *cptn* ⟶ *drop j c* ∈ *cptn*
⟨*proof*⟩

lemma *takepar-cptn-is-par-cptn* [*rule-format*,*elim*]:
∀ *j*. *c* ∈ *par-cptn* ⟶ *take* (*Suc j*) *c* ∈ *par-cptn*
⟨*proof*⟩

lemma *droppar-cptn-is-par-cptn* [*rule-format*]:
∀ *j*<*length c*. *c* ∈ *par-cptn* ⟶ *drop j c* ∈ *par-cptn*
⟨*proof*⟩

lemma *tl-of-cptn-is-cptn*: ⟦*x* # *xs* ∈ *cptn*; *xs* ≠ []⟧ ⟹ *xs* ∈ *cptn*
⟨*proof*⟩

lemma *not-ctran-None* [*rule-format*]:
$\forall s.\ (None,\ s)\#xs \in cptn \longrightarrow (\forall i{<}length\ xs.\ ((None,\ s)\#xs)!i\ -e\rightarrow\ xs!i)$
⟨*proof*⟩

lemma *cptn-not-empty* [*simp*]:$[] \notin cptn$
⟨*proof*⟩

lemma *etran-or-ctran* [*rule-format*]:
$\forall m\ i.\ x{\in}cptn \longrightarrow m \leq length\ x$
$\longrightarrow (\forall i.\ Suc\ i < m \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i) \longrightarrow Suc\ i < m$
$\longrightarrow x!i\ -e\rightarrow\ x!Suc\ i$
⟨*proof*⟩

lemma *etran-or-ctran2* [*rule-format*]:
$\forall i.\ Suc\ i{<}length\ x \longrightarrow x{\in}cptn \longrightarrow (x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i)$
$\vee\ (x!i\ -e\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i)$
⟨*proof*⟩

lemma *etran-or-ctran2-disjI1*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -c\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i$
⟨*proof*⟩

lemma *etran-or-ctran2-disjI2*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -e\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i$
⟨*proof*⟩

lemma *not-ctran-None2* [*rule-format*]:
$[\![\ (None,\ s)\ \#\ xs\ {\in}cptn;\ i{<}length\ xs]\!] \Longrightarrow \neg\ ((None,\ s)\ \#\ xs)\ !\ i\ -c\rightarrow\ xs\ !\ i$
⟨*proof*⟩

lemma *Ex-first-occurrence* [*rule-format*]: $P\ (n{::}nat) \longrightarrow (\exists m.\ P\ m \wedge (\forall i{<}m.\ \neg\ P\ i))$
⟨*proof*⟩

lemma *stability* [*rule-format*]:
$\forall j\ k.\ x \in cptn \longrightarrow stable\ p\ rely \longrightarrow j{\leq}k \longrightarrow k{<}length\ x \longrightarrow snd(x!j){\in}p\ \longrightarrow$
$(\forall i.\ (Suc\ i){<}length\ x \longrightarrow$
$(x!i\ -e\rightarrow\ x!(Suc\ i)) \longrightarrow (snd(x!i),\ snd(x!(Suc\ i))) \in rely) \longrightarrow$
$(\forall i.\ j{\leq}i \wedge i{<}k \longrightarrow x!i\ -e\rightarrow\ x!Suc\ i) \longrightarrow snd(x!k){\in}p \wedge fst(x!j){=}fst(x!k)$
⟨*proof*⟩

3.5.1 Soundness of the System for Component Programs

Soundness of the Basic rule

lemma *unique-ctran-Basic* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$

⟨*proof*⟩

lemma *exists-ctran-Basic-None* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *Basic-sound*:
$[\![pre \subseteq \{s.\ f\ s \in post\};\ \{(s,\ t).\ s \in pre \wedge t = f\ s\} \subseteq guar;$
stable pre rely; *stable post rely*$]\!]$
$\Longrightarrow \models Basic\ f\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

Soundness of the Await rule

lemma *unique-ctran-Await* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *exists-ctran-Await-None* [*rule-format*]:
$\forall s\ i.\ \ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *Star-imp-cptn*:
$(P,\ s)\ -c{*}\rightarrow\ (R,\ t) \Longrightarrow \exists l \in cp\ P\ s.\ (last\ l){=}(R,\ t)$
$\wedge\ (\forall i.\ Suc\ i{<}length\ l \longrightarrow l!i\ -c\rightarrow\ l!Suc\ i)$
⟨*proof*⟩

lemma *Await-sound*:
$[\![stable\ pre\ rely;\ stable\ post\ rely;$
$\forall V.\ \vdash P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post] \wedge$
$\models P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post]\ ]\!]$
$\Longrightarrow \models Await\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

Soundness of the Conditional rule

lemma *Cond-sound*:
$[\![\ stable\ pre\ rely;\ \models P1\ sat\ [pre \cap b,\ rely,\ guar,\ post];$
$\models P2\ sat\ [pre \cap -\ b,\ rely,\ guar,\ post];\ \forall s.\ (s,s){\in}guar]\!]$
$\Longrightarrow \models (Cond\ b\ P1\ P2)\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

Soundness of the Sequential rule

inductive-cases *Seq-cases* [*elim*!]: (*Some* (*Seq P Q*), *s*) $-c\rightarrow$ *t*

lemma *last-lift-not-None*: *fst* ((*lift Q*) ((*x*#*xs*)!(*length xs*))) $\neq$ *None*
⟨*proof*⟩

lemma *Seq-sound1* [*rule-format*]:
x∈ *cptn-mod* $\Longrightarrow$ ∀ *s P*. *x* !*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$
(∀ *i*<*length x*. *fst*(*x*!*i*)≠*Some Q*) $\longrightarrow$
(∃ *xs*∈ *cp* (*Some P*) *s*. *x*=*map* (*lift Q*) *xs*)
⟨*proof*⟩

lemma *Seq-sound2* [*rule-format*]:
x ∈ *cptn* $\Longrightarrow$ ∀ *s P i*. *x*!*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$ *i*<*length x*
$\longrightarrow$ *fst*(*x*!*i*)=*Some Q* $\longrightarrow$
(∀ *j*<*i*. *fst*(*x*!*j*)≠(*Some Q*)) $\longrightarrow$
(∃ *xs ys*. *xs* ∈ *cp* (*Some P*) *s* ∧ *length xs*=*Suc i*
∧ *ys* ∈ *cp* (*Some Q*) (*snd*(*xs* !*i*)) ∧ *x*=(*map* (*lift Q*) *xs*)@*tl ys*)
⟨*proof*⟩

lemma *last-lift-not-None2*: *fst* ((*lift Q*) (*last* (*x*#*xs*))) $\neq$ *None*
⟨*proof*⟩

lemma *Seq-sound*:
⟦⊨ *P sat* [*pre*, *rely*, *guar*, *mid*]; ⊨ *Q sat* [*mid*, *rely*, *guar*, *post*]⟧
$\Longrightarrow$ ⊨ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

Soundness of the While rule

lemma *last-append*[*rule-format*]:
∀ *xs*. *ys*≠[] $\longrightarrow$ ((*xs*@*ys*)!(*length* (*xs*@*ys*) − (*Suc 0*)))=(*ys*!(*length ys* − (*Suc 0*)))
⟨*proof*⟩

lemma *assum-after-body*:
⟦ ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*];
(*Some P*, *s*) # *xs* ∈ *cptn-mod*; *fst* (*last* ((*Some P*, *s*) # *xs*)) = *None*; *s* ∈ *b*;
(*Some* (*While b P*), *s*) # (*Some* (*Seq P* (*While b P*)), *s*) #
map (*lift* (*While b P*)) *xs* @ *ys* ∈ *assum* (*pre*, *rely*)⟧
$\Longrightarrow$ (*Some* (*While b P*), *snd* (*last* ((*Some P*, *s*) # *xs*))) # *ys* ∈ *assum* (*pre*, *rely*)
⟨*proof*⟩

lemma *While-sound-aux* [*rule-format*]:
⟦ *pre* ∩ − *b* ⊆ *post*; ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*, *s*) ∈ *guar*;
stable pre rely; *stable post rely*; *x* ∈ *cptn-mod* ⟧
$\Longrightarrow$ ∀ *s xs*. *x*=(*Some*(*While b P*),*s*)#*xs* $\longrightarrow$ *x*∈*assum*(*pre*, *rely*) $\longrightarrow$ *x* ∈ *comm*
(*guar*, *post*)
⟨*proof*⟩

lemma *While-sound*:
⟦*stable pre rely*; *pre* ∩ − *b* ⊆ *post*; *stable post rely*;
⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*,*s*)∈*guar*⟧
⟹ ⊨ *While b P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

Soundness of the Rule of Consequence

lemma *Conseq-sound*:
⟦*pre* ⊆ *pre*′; *rely* ⊆ *rely*′; *guar*′ ⊆ *guar*; *post*′ ⊆ *post*;
⊨ *P sat* [*pre*′, *rely*′, *guar*′, *post*′]⟧
⟹ ⊨ *P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

Soundness of the system for sequential component programs

theorem *rgsound*:
⊢ *P sat* [*pre*, *rely*, *guar*, *post*] ⟹ ⊨ *P sat* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

3.5.2 Soundness of the System for Parallel Programs

definition *ParallelCom* :: (′*a rgformula*) *list* ⇒ ′*a par-com* **where**
ParallelCom Ps ≡ *map* (*Some* ∘ *fst*) *Ps*

lemma *two*:
⟦ ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
pre ⊆ (⋂ *i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i*<*length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
length xs=*length clist*; *x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x*∈*par-assum*(*pre*, *rely*);
∀ *i*<*length clist*. *clist*!*i*∈*cp* (*Some*(*Com*(*xs*!*i*))) *s*; *x* ∝ *clist* ⟧
⟹ ∀ *j i*. *i*<*length clist* ∧ *Suc j*<*length x* ⟶ (*clist*!*i*!*j*) −*c*→ (*clist*!*i*!*Suc j*)
⟶ (*snd*(*clist*!*i*!*j*), *snd*(*clist*!*i*!*Suc j*)) ∈ *Guar*(*xs*!*i*)
⟨*proof*⟩

lemma *three* [*rule-format*]:
⟦ *xs*≠[]; ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
pre ⊆ (⋂ *i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i*<*length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
length xs=*length clist*; *x* ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum*(*pre*, *rely*);
∀ *i*<*length clist*. *clist*!*i*∈*cp* (*Some*(*Com*(*xs*!*i*))) *s*; *x* ∝ *clist* ⟧
⟹ ∀ *j i*. *i*<*length clist* ∧ *Suc j*<*length x* ⟶ (*clist*!*i*!*j*) −*e*→ (*clist*!*i*!*Suc j*)
⟶ (*snd*(*clist*!*i*!*j*), *snd*(*clist*!*i*!*Suc j*)) ∈ *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}.
Guar (*xs* ! *j*))

⟨*proof*⟩

lemma *four*:
⟦*xs*≠[]; ∀ *i* < *length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
(⋃*j*∈{*j*. *j* < *length xs*}. *Guar* (*xs* ! *j*)) ⊆ *guar*;
pre ⊆ (⋂*i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
∀ *i* < *length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
x ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum* (*pre*, *rely*); *Suc i* < *length x*;
x ! *i* −*pc*→ *x* ! *Suc i*⟧
⟹ (*snd* (*x* ! *i*), *snd* (*x* ! *Suc i*)) ∈ *guar*
⟨*proof*⟩

lemma *parcptn-not-empty* [*simp*]:[] ∉ *par-cptn*
⟨*proof*⟩

lemma *five*:
⟦*xs*≠[]; ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))
⊆ *Rely* (*xs* ! *i*);
pre ⊆ (⋂*i*∈{*i*. *i* < *length xs*}. *Pre* (*xs* ! *i*));
(⋂*i*∈{*i*. *i* < *length xs*}. *Post* (*xs* ! *i*)) ⊆ *post*;
∀ *i* < *length xs*.
⊨ *Com* (*xs* ! *i*) *sat* [*Pre* (*xs* ! *i*), *Rely* (*xs* ! *i*), *Guar* (*xs* ! *i*), *Post* (*xs* ! *i*)];
x ∈ *par-cp* (*ParallelCom xs*) *s*; *x* ∈ *par-assum* (*pre*, *rely*);
All-None (*fst* (*last x*)) ⟧ ⟹ *snd* (*last x*) ∈ *post*
⟨*proof*⟩

lemma *ParallelEmpty* [*rule-format*]:
∀ *i s*. *x* ∈ *par-cp* (*ParallelCom* []) *s* ⟶
Suc i < *length x* ⟶ (*x* ! *i*, *x* ! *Suc i*) ∉ *par-ctran*
⟨*proof*⟩

theorem *par-rgsound*:
⊢ *c SAT* [*pre*, *rely*, *guar*, *post*] ⟹
⊨ (*ParallelCom c*) *SAT* [*pre*, *rely*, *guar*, *post*]
⟨*proof*⟩

end

3.6 Concrete Syntax

theory *RG-Syntax*
imports *RG-Hoare Quote-Antiquote*
begin

abbreviation *Skip* :: *'a com* (*SKIP*)
where *SKIP* ≡ *Basic id*

notation *Seq* ((-;;/ -) [60,61] 60)

syntax
-Assign :: idt ⇒ 'b ⇒ 'a com ((´- :=/ -) [70, 65] 61)
-Cond :: 'a bexp ⇒ 'a com ⇒ 'a com ⇒ 'a com ((0IF -/ THEN -/ ELSE -/FI) [0, 0, 0] 61)
-Cond2 :: 'a bexp ⇒ 'a com ⇒ 'a com ((0IF - THEN - FI) [0,0] 56)
-While :: 'a bexp ⇒ 'a com ⇒ 'a com ((0WHILE - /DO - /OD) [0, 0] 61)
-Await :: 'a bexp ⇒ 'a com ⇒ 'a com ((0AWAIT - /THEN /- /END) [0,0] 61)
-Atom :: 'a com ⇒ 'a com ((⟨-⟩) 61)
-Wait :: 'a bexp ⇒ 'a com ((0WAIT - END) 61)

translations
´x := a ⇀ CONST Basic «´(-update-name x (λ-. a))»
IF b THEN c1 ELSE c2 FI ⇀ CONST Cond {|b|} c1 c2
IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI
WHILE b DO c OD ⇀ CONST While {|b|} c
AWAIT b THEN c END ⇌ CONST Await {|b|} c
⟨c⟩ ⇌ AWAIT CONST True THEN c END
WAIT b END ⇌ AWAIT b THEN SKIP END

nonterminal *prgs*

syntax
-PAR :: prgs ⇒ 'a (COBEGIN//-//COEND 60)
-prg :: 'a ⇒ prgs (- 57)
-prgs :: ['a, prgs] ⇒ prgs (-//‖//- [60,57] 57)

translations
-prg a ⇀ [a]
-prgs a ps ⇀ a # ps
-PAR ps ⇀ ps

syntax
-prg-scheme :: ['a, 'a, 'a, 'a] ⇒ prgs (SCHEME [- ≤ - < -] - [0,0,0,60] 57)

translations
-prg-scheme j i k c ⇌ (CONST map (λi. c) [j..<k])

Translations for variables before and after a transition:

syntax
-before :: id ⇒ 'a (^{o}-)
-after :: id ⇒ 'a (^{a}-)

translations
^{o}x ⇌ x ´CONST fst
^{a}x ⇌ x ´CONST snd

⟨*ML*⟩

end

3.7 Examples

theory *RG-Examples*
imports *RG-Syntax*
begin

lemmas *definitions* [*simp*]= *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

3.7.1 Set Elements of an Array to Zero

lemma *le-less-trans2*: ⟦(*j*::*nat*)<*k*; *i*≤ *j*⟧ ⟹ *i*<*k*
⟨*proof*⟩

lemma *add-le-less-mono*: ⟦ (*a*::*nat*) < *c*; *b*≤*d* ⟧ ⟹ *a* + *b* < *c* + *d*
⟨*proof*⟩

record *Example1* =
A :: *nat list*

lemma *Example1*:
⊢ *COBEGIN*
SCHEME [0 ≤ *i* < *n*]
(´*A* := ´*A* [*i* := 0],
{| *n* < *length* ´*A* |},
{| *length* ᵒ*A* = *length* ᵃ*A* ∧ ᵒ*A* ! *i* = ᵃ*A* ! *i* |},
{| *length* ᵒ*A* = *length* ᵃ*A* ∧ (∀ *j*<*n*. *i* ≠ *j* ⟶ ᵒ*A* ! *j* = ᵃ*A* ! *j*) |},
{| ´*A* ! *i* = 0 |})
COEND
SAT [{| *n* < *length* ´*A* |}, {| ᵒ*A* = ᵃ*A* |}, {| *True* |}, {| ∀ *i* < *n*. ´*A* ! *i* = 0 |}]
⟨*proof*⟩

lemma *Example1-parameterized*:
k < *t* ⟹
⊢ *COBEGIN*
SCHEME [*k*∗*n*≤*i*<(*Suc k*)∗*n*] (´*A*:=´*A*[*i*:=0],
{|*t*∗*n* < *length* ´*A*|},
{|*t*∗*n* < *length* ᵒ*A* ∧ *length* ᵒ*A*=*length* ᵃ*A* ∧ ᵒ*A*!*i* = ᵃ*A*!*i*|},
{|*t*∗*n* < *length* ᵒ*A* ∧ *length* ᵒ*A*=*length* ᵃ*A* ∧ (∀ *j*<*length* ᵒ*A* . *i*≠*j* ⟶ ᵒ*A*!*j* = ᵃ*A*!*j*)|},
{|´*A*!*i*=0|})
COEND
SAT [{|*t*∗*n* < *length* ´*A*|},
{|*t*∗*n* < *length* ᵒ*A* ∧ *length* ᵒ*A*=*length* ᵃ*A* ∧ (∀ *i*<*n*. ᵒ*A*!(*k*∗*n*+*i*)=ᵃ*A*!(*k*∗*n*+*i*))|},
{|*t*∗*n* < *length* ᵒ*A* ∧ *length* ᵒ*A*=*length* ᵃ*A* ∧

$(\forall\, i{<}length\ {}^{o}A\ .\ (i{<}k{*}n \longrightarrow {}^{o}A!i = {}^{a}A!i) \wedge ((Suc\ k){*}n \leq i \longrightarrow {}^{o}A!i = {}^{a}A!i))\}$,
$\{\!|\forall\, i{<}n.\ {}'A!(k{*}n{+}i) = 0|\!\}$]
⟨*proof*⟩

3.7.2 Increment a Variable in Parallel

Two components

record *Example2* =
x :: *nat*
c-0 :: *nat*
c-1 :: *nat*

lemma *Example2*:
⊢ *COBEGIN*
(⟨ ′*x*:=′*x*+1;; ′*c-0*:=′*c-0* + 1 ⟩,
$\{\!|{}'x{=}{}'c\text{-}0 + {}'c\text{-}1 \ \wedge\ {}'c\text{-}0{=}0|\!\}$,
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\}$,
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\}$,
$\{\!|{}'x{=}{}'c\text{-}0 + {}'c\text{-}1 \wedge {}'c\text{-}0{=}1\ |\!\}$)
‖
(⟨ ′*x*:=′*x*+1;; ′*c-1*:=′*c-1*+1 ⟩,
$\{\!|{}'x{=}{}'c\text{-}0 + {}'c\text{-}1 \wedge {}'c\text{-}1{=}0\ |\!\}$,
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\}$,
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\}$,
$\{\!|{}'x{=}{}'c\text{-}0 + {}'c\text{-}1 \wedge {}'c\text{-}1{=}1|\!\}$)
COEND
SAT [$\{\!|{}'x{=}0 \wedge {}'c\text{-}0{=}0 \wedge {}'c\text{-}1{=}0|\!\}$,
$\{\!|{}^{o}x{=}{}^{a}x \wedge\ {}^{o}c\text{-}0{=}\ {}^{a}c\text{-}0 \wedge {}^{o}c\text{-}1{=}{}^{a}c\text{-}1|\!\}$,
$\{\!|\mathit{True}|\!\}$,
$\{\!|{}'x{=}2|\!\}$]
⟨*proof*⟩

Parameterized

lemma *Example2-lemma2-aux*: *j*<*n* ⟹
$(\sum i{=}0..{<}n.\ (b\ i{::}nat)) =$
$(\sum i{=}0..{<}j.\ b\ i) + b\ j + (\sum i{=}0..{<}n{-}(Suc\ j)\ .\ b\ (Suc\ j + i))$
⟨*proof*⟩

lemma *Example2-lemma2-aux2*:
j≤ *s* ⟹ $(\sum i{::}nat{=}0..{<}j.\ (b\ (s{:=}t))\ i) = (\sum i{=}0..{<}j.\ b\ i)$

⟨*proof*⟩

lemma *Example2-lemma2*:
⟦*j*<*n*; *b j*=*0*⟧ ⟹ *Suc* (∑ *i*::*nat*=*0*..<*n*. *b i*)=(∑ *i*=*0*..<*n*. (*b* (*j* := *Suc 0*)) *i*)
⟨*proof*⟩

lemma *Example2-lemma2-Suc0*: ⟦*j*<*n*; *b j*=*0*⟧ ⟹
Suc (∑ *i*::*nat*=*0*..< *n*. *b i*)=(∑ *i*=*0*..< *n*. (*b* (*j*:=*Suc 0*)) *i*)
⟨*proof*⟩

record *Example2-parameterized* =
C :: *nat* ⇒ *nat*
y :: *nat*

lemma *Example2-parameterized*: *0*<*n* ⟹
⊢ *COBEGIN SCHEME* [*0*≤*i*<*n*]
(⟨ ´*y*:=´*y*+*1*;; ´*C*:=´*C* (*i*:=*1*) ⟩,
{|´*y*=(∑ *i*=*0*..<*n*. ´*C i*) ∧ ´*C i*=*0*|},
{|ᵒ*C i* = ᵃ*C i* ∧
(ᵒ*y*=(∑ *i*=*0*..<*n*. ᵒ*C i*) ⟶ ᵃ*y* =(∑ *i*=*0*..<*n*. ᵃ*C i*))|},
{|(∀ *j*<*n*. *i*≠*j* ⟶ ᵒ*C j* = ᵃ*C j*) ∧
(ᵒ*y*=(∑ *i*=*0*..<*n*. ᵒ*C i*) ⟶ ᵃ*y* =(∑ *i*=*0*..<*n*. ᵃ*C i*))|},
{|´*y*=(∑ *i*=*0*..<*n*. ´*C i*) ∧ ´*C i*=*1*|})
COEND
SAT [{|´*y*=*0* ∧ (∑ *i*=*0*..<*n*. ´*C i*)=*0* |}, {|ᵒ*C*=ᵃ*C* ∧ ᵒ*y*=ᵃ*y*|}, {| *True*|}, {|´*y*=*n*|}]
⟨*proof*⟩

3.7.3 Find Least Element

A previous lemma:

lemma *mod-aux* :⟦*i* < (*n*::*nat*); *a mod n* = *i*; *j* < *a* + *n*; *j mod n* = *i*; *a* < *j*⟧ ⟹
False
⟨*proof*⟩

record *Example3* =
X :: *nat* ⇒ *nat*
Y :: *nat* ⇒ *nat*

lemma *Example3*: *m mod n*=*0* ⟹
⊢ *COBEGIN*
SCHEME [*0*≤*i*<*n*]
(*WHILE* (∀ *j*<*n*. ´*X i* < ´*Y j*) *DO*
IF P(*B*!(´*X i*)) *THEN* ´*Y*:=´*Y* (*i*:=´*X i*)
ELSE ´*X*:= ´*X* (*i*:=(´*X i*)+ *n*) *FI*
OD,
{|(´*X i*) *mod n*=*i* ∧ (∀ *j*<´*X i*. *j mod n*=*i* ⟶ ¬*P*(*B*!*j*)) ∧ (´*Y i*<*m* ⟶ *P*(*B*!(´*Y i*)) ∧ ´*Y i*≤ *m*+*i*)|},
{|(∀ *j*<*n*. *i*≠*j* ⟶ ᵃ*Y j* ≤ ᵒ*Y j*) ∧ ᵒ*X i* = ᵃ*X i* ∧
ᵒ*Y i* = ᵃ*Y i*|},

$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}X\ j = {}^{a}X\ j \wedge {}^{o}Y\ j = {}^{a}Y\ j) \wedge$
$\quad {}^{a}Y\ i \leq {}^{o}Y\ i|\!\}$,
$\{\!|(\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y\ i{<}m \longrightarrow P(B!(\acute{}Y\ i)) \wedge \acute{}Y\ i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y\ j \leq \acute{}X\ i)\ |\!\}$
COEND
SAT $[\{\!|\ \forall i{<}n.\ \acute{}X\ i{=}i \wedge \acute{}Y\ i{=}m{+}i\ |\!\}, \{\!|{}^{o}X{=}{}^{a}X \wedge {}^{o}Y{=}{}^{a}Y|\!\}, \{\!|True|\!\},$
$\quad \{\!|\forall i{<}n.\ (\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$\quad\quad (\acute{}Y\ i{<}m \longrightarrow P(B!(\acute{}Y\ i)) \wedge \acute{}Y\ i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y\ j \leq \acute{}X\ i)|\!\}]$
⟨*proof*⟩

Same but with a list as auxiliary variable:

record *Example3-list* =
$\quad X$ :: *nat list*
$\quad Y$ :: *nat list*

lemma *Example3-list*: $m\ mod\ n{=}0 \Longrightarrow \vdash$ (*COBEGIN SCHEME* $[0{\leq}i{<}n]$
(*WHILE* $(\forall j{<}n.\ \acute{}X!i < \acute{}Y!j)$ *DO*
$\quad$ *IF* $P(B!(\acute{}X!i))$ *THEN* $\acute{}Y{:=}\acute{}Y[i{:=}\acute{}X!i]$ *ELSE* $\acute{}X{:=}\ \acute{}X[i{:=}(\acute{}X!i){+}\ n]$ *FI*
OD,
$\{\!|n{<}length\ \acute{}X \wedge n{<}length\ \acute{}Y \wedge (\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i)|\!\}$,
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{a}Y!j \leq {}^{o}Y!j) \wedge {}^{o}X!i = {}^{a}X!i \wedge$
$\quad {}^{o}Y!i = {}^{a}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y|\!\}$,
$\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}X!j = {}^{a}X!j \wedge {}^{o}Y!j = {}^{a}Y!j) \wedge$
$\quad {}^{a}Y!i \leq {}^{o}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y|\!\}$,
$\{\!|(\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge (\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y!j \leq \acute{}X!i)\ |\!\})$ *COEND*)
SAT $[\{\!|n{<}length\ \acute{}X \wedge n{<}length\ \acute{}Y \wedge (\forall i{<}n.\ \acute{}X!i{=}i \wedge \acute{}Y!i{=}m{+}i)\ |\!\},$
$\quad \{\!|{}^{o}X{=}{}^{a}X \wedge {}^{o}Y{=}{}^{a}Y|\!\},$
$\quad \{\!|True|\!\},$
$\quad \{\!|\forall i{<}n.\ (\acute{}X!i)\ mod\ n{=}i \wedge (\forall j{<}\acute{}X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$\quad\quad (\acute{}Y!i{<}m \longrightarrow P(B!(\acute{}Y!i)) \wedge \acute{}Y!i{\leq}\ m{+}i) \wedge (\exists j{<}n.\ \acute{}Y!j \leq \acute{}X!i)|\!\}]$
⟨*proof*⟩

end
theory *Hoare-Parallel*
imports *OG-Examples Gar-Coll Mul-Gar-Coll RG-Examples*
begin

end

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