Hoare Logic for Parallel Programs

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Abstract

In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [1].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

For excellent descriptions of this work see [2, 4, 1, 3].

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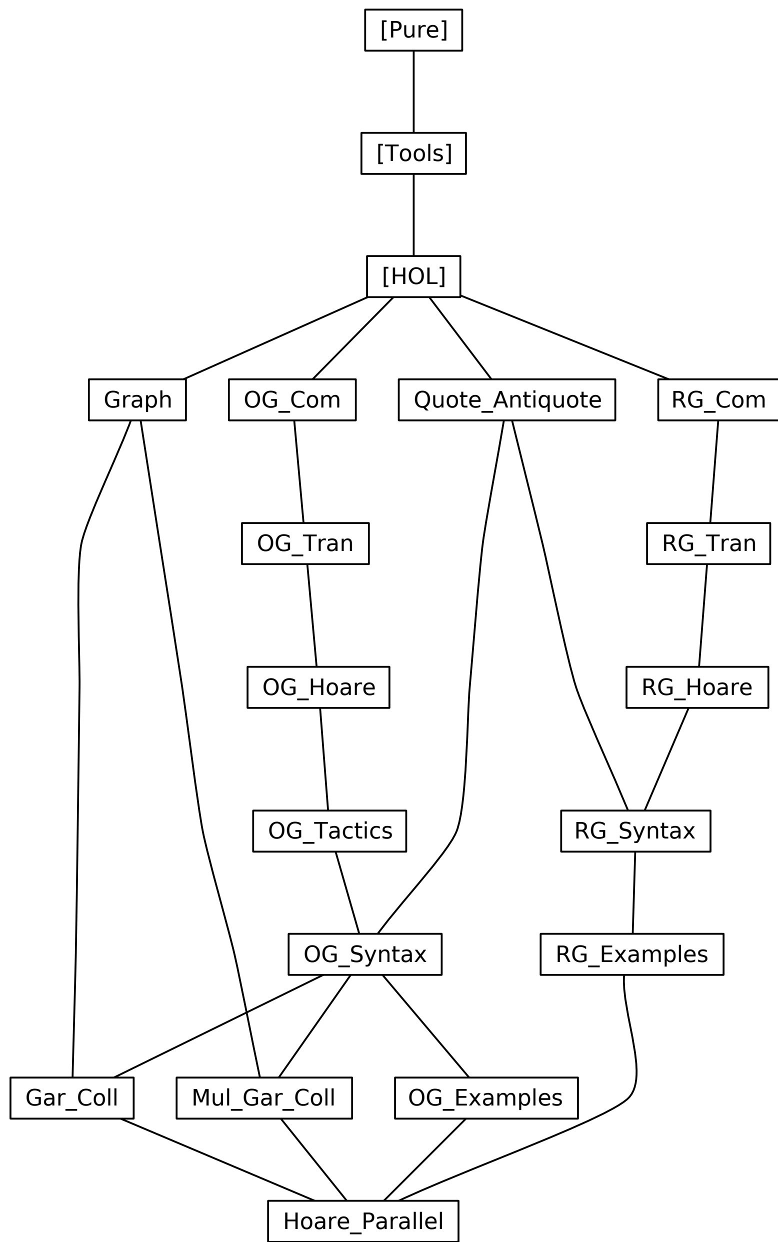
[Pure]
[Tools]
[HOL]
Graph
OG_Com
Quote_Antiquote
RG_Com
OG_Tran
RG_Tran
OG_Hoare
RG_Hoare
OG_Tactics
RG_Syntax
OG_Syntax
RG_Examples
Gar_Coll
Mul_Gar_Coll
OG_Examples
Hoare_Parallel

Chapter 1

The Owicki-Gries Method

1.1 Abstract Syntax

theory *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

type-synonym $'a\ bexp = 'a\ set$
type-synonym $'a\ assn = 'a\ set$

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

datatype *'a ann-com* =
  AnnBasic (*'a assn*) (*'a* ⇒ *'a*)
| *AnnSeq* (*'a ann-com*) (*'a ann-com*)
| *AnnCond1* (*'a assn*) (*'a bexp*) (*'a ann-com*) (*'a ann-com*)
| *AnnCond2* (*'a assn*) (*'a bexp*) (*'a ann-com*)
| *AnnWhile* (*'a assn*) (*'a bexp*) (*'a assn*) (*'a ann-com*)
| *AnnAwait* (*'a assn*) (*'a bexp*) (*'a com*)
and *'a com* =
  Parallel (*'a ann-com option* × *'a assn*) *list*
| *Basic* (*'a* ⇒ *'a*)
| *Seq* (*'a com*) (*'a com*)
| *Cond* (*'a bexp*) (*'a com*) (*'a com*)
| *While* (*'a bexp*) (*'a assn*) (*'a com*)

The function *pre* extracts the precondition of an annotated command:

primrec *pre* ::*'a ann-com* ⇒ *'a assn* **where**
  pre (*AnnBasic r f*) = *r*
| *pre* (*AnnSeq c1 c2*) = *pre c1*
| *pre* (*AnnCond1 r b c1 c2*) = *r*
| *pre* (*AnnCond2 r b c*) = *r*
| *pre* (*AnnWhile r b i c*) = *r*
| *pre* (*AnnAwait r b c*) = *r*

Well-formedness predicate for atomic programs:

primrec *atom-com* :: 'a com ⇒ bool **where**
atom-com (Parallel Ts) = False
| atom-com (Basic f) = True
| atom-com (Seq c1 c2) = (atom-com c1 ∧ atom-com c2)
| atom-com (Cond b c1 c2) = (atom-com c1 ∧ atom-com c2)
| atom-com (While b i c) = atom-com c

end

1.2 Operational Semantics

theory *OG-Tran* **imports** *OG-Com* **begin**

type-synonym 'a ann-com-op = ('a ann-com) option
type-synonym 'a ann-triple-op = ('a ann-com-op × 'a assn)

primrec *com* :: 'a ann-triple-op ⇒ 'a ann-com-op **where**
com (c, q) = c

primrec *post* :: 'a ann-triple-op ⇒ 'a assn **where**
post (c, q) = q

definition *All-None* :: 'a ann-triple-op list ⇒ bool **where**
All-None Ts ≡ ∀(c, q) ∈ set Ts. c = None

1.2.1 The Transition Relation

inductive-set
ann-transition :: (('a ann-com-op × 'a) × ('a ann-com-op × 'a)) set
and transition :: (('a com × 'a) × ('a com × 'a)) set
and ann-transition' :: ('a ann-com-op × 'a) ⇒ ('a ann-com-op × 'a) ⇒ bool
(- −1→ -[81,81] 100)
and transition' :: ('a com × 'a) ⇒ ('a com × 'a) ⇒ bool
(- −P1→ -[81,81] 100)
and transitions :: ('a com × 'a) ⇒ ('a com × 'a) ⇒ bool
(- −P∗→ -[81,81] 100)
where
con-0 −1→ con-1 ≡ (con-0, con-1) ∈ ann-transition
| con-0 −P1→ con-1 ≡ (con-0, con-1) ∈ transition
| con-0 −P∗→ con-1 ≡ (con-0, con-1) ∈ transition*

| AnnBasic: (Some (AnnBasic r f), s) −1→ (None, f s)

| AnnSeq1: (Some c0, s) −1→ (None, t) ⟹
(Some (AnnSeq c0 c1), s) −1→ (Some c1, t)
| AnnSeq2: (Some c0, s) −1→ (Some c2, t) ⟹
(Some (AnnSeq c0 c1), s) −1→ (Some (AnnSeq c2 c1), t)

| AnnCond1T: s ∈ b ⟹ (Some (AnnCond1 r b c1 c2), s) −1→ (Some c1, s)

| *AnnCond1F*: $s \notin b \Longrightarrow (Some\ (AnnCond1\ r\ b\ c1\ c2),\ s)\ -1\rightarrow\ (Some\ c2,\ s)$

| *AnnCond2T*: $s \in b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s)\ -1\rightarrow\ (Some\ c,\ s)$
| *AnnCond2F*: $s \notin b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s)\ -1\rightarrow\ (None,\ s)$

| *AnnWhileF*: $s \notin b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s)\ -1\rightarrow\ (None,\ s)$
| *AnnWhileT*: $s \in b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s)\ -1\rightarrow$
$(Some\ (AnnSeq\ c\ (AnnWhile\ i\ b\ i\ c)),\ s)$

| *AnnAwait*: $[\![\ s \in b;\ atom\text{-}com\ c;\ (c,\ s)\ -P{*}\rightarrow\ (Parallel\ [],\ t)\ ]\!] \Longrightarrow$
$(Some\ (AnnAwait\ r\ b\ c),\ s)\ -1\rightarrow\ (None,\ t)$

| *Parallel*: $[\![\ i{<}length\ Ts;\ Ts!i = (Some\ c,\ q);\ (Some\ c,\ s)\ -1\rightarrow\ (r,\ t)\ ]\!]$
$\Longrightarrow (Parallel\ Ts,\ s)\ -P1\rightarrow\ (Parallel\ (Ts\ [i{:=}(r,\ q)]),\ t)$

| *Basic*: $(Basic\ f,\ s)\ -P1\rightarrow\ (Parallel\ [],\ f\ s)$

| *Seq1*: $All\text{-}None\ Ts \Longrightarrow (Seq\ (Parallel\ Ts)\ c,\ s)\ -P1\rightarrow\ (c,\ s)$
| *Seq2*: $(c0,\ s)\ -P1\rightarrow\ (c2,\ t) \Longrightarrow (Seq\ c0\ c1,\ s)\ -P1\rightarrow\ (Seq\ c2\ c1,\ t)$

| *CondT*: $s \in b \Longrightarrow (Cond\ b\ c1\ c2,\ s)\ -P1\rightarrow\ (c1,\ s)$
| *CondF*: $s \notin b \Longrightarrow (Cond\ b\ c1\ c2,\ s)\ -P1\rightarrow\ (c2,\ s)$

| *WhileF*: $s \notin b \Longrightarrow (While\ b\ i\ c,\ s)\ -P1\rightarrow\ (Parallel\ [],\ s)$
| *WhileT*: $s \in b \Longrightarrow (While\ b\ i\ c,\ s)\ -P1\rightarrow\ (Seq\ c\ (While\ b\ i\ c),\ s)$

monos *rtrancl-mono*

The corresponding abbreviations are:

abbreviation
ann-transition-n :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow nat \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a)$
$\Rightarrow bool$ (- −-→ -[81,81] 100) **where**
$con\text{-}0\ -n\rightarrow\ con\text{-}1 \equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition\ \widehat{}\widehat{}\ n$

abbreviation
ann-transitions :: $('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow bool$
(- −∗→ -[81,81] 100) **where**
$con\text{-}0\ -{*}\rightarrow\ con\text{-}1 \equiv (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition^*$

abbreviation
transition-n :: $('a\ com \times 'a) \Rightarrow nat \Rightarrow ('a\ com \times 'a) \Rightarrow bool$
(- −P-→ -[81,81,81] 100) **where**
$con\text{-}0\ -Pn\rightarrow\ con\text{-}1 \equiv (con\text{-}0,\ con\text{-}1) \in transition\ \widehat{}\widehat{}\ n$

1.2.2 Definition of Semantics

definition *ann-sem* :: $'a\ ann\text{-}com \Rightarrow 'a \Rightarrow 'a\ set$ **where**
$ann\text{-}sem\ c \equiv \lambda s.\ \{t.\ (Some\ c,\ s)\ -{*}\rightarrow\ (None,\ t)\}$

definition *ann-SEM* :: *'a ann-com* ⇒ *'a set* ⇒ *'a set* **where**
ann-SEM c S ≡ ⋃(*ann-sem c ‘ S*)

definition *sem* :: *'a com* ⇒ *'a* ⇒ *'a set* **where**
sem c ≡ λ*s*. {*t*. ∃ *Ts*. (*c*, *s*) −*P*∗→ (*Parallel Ts*, *t*) ∧ *All-None Ts*}

definition *SEM* :: *'a com* ⇒ *'a set* ⇒ *'a set* **where**
SEM c S ≡ ⋃(*sem c ‘ S*)

abbreviation *Omega* :: *'a com* (Ω *63*)
where Ω ≡ *While UNIV UNIV* (*Basic id*)

primrec *fwhile* :: *'a bexp* ⇒ *'a com* ⇒ *nat* ⇒ *'a com* **where**
fwhile b c 0 = Ω
| *fwhile b c* (*Suc n*) = *Cond b* (*Seq c* (*fwhile b c n*)) (*Basic id*)

Proofs

declare *ann-transition-transition.intros* [*intro*]
inductive-cases *transition-cases*:
(*Parallel T*,*s*) −*P1*→ *t*
(*Basic f*, *s*) −*P1*→ *t*
(*Seq c1 c2*, *s*) −*P1*→ *t*
(*Cond b c1 c2*, *s*) −*P1*→ *t*
(*While b i c*, *s*) −*P1*→ *t*

lemma *Parallel-empty-lemma* [*rule-format* (*no-asm*)]:
(*Parallel* [],*s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=[] ∧ *n*=*0* ∧ *s*=*t*
apply(*induct n*)
apply(*simp* (*no-asm*))
apply *clarify*
apply(*drule relpow-Suc-D2*)
apply(*force elim*:*transition-cases*)
done

lemma *Parallel-AllNone-lemma* [*rule-format* (*no-asm*)]:
All-None Ss ⟶ (*Parallel Ss*,*s*) −*Pn*→ (*Parallel Ts*,*t*) ⟶ *Ts*=*Ss* ∧ *n*=*0* ∧ *s*=*t*
apply(*induct n*)
apply(*simp* (*no-asm*))
apply *clarify*
apply(*drule relpow-Suc-D2*)
apply *clarify*
apply(*erule transition-cases*,*simp-all*)
apply(*force dest*:*nth-mem simp add*:*All-None-def*)
done

lemma *Parallel-AllNone*: *All-None Ts* ⟹ (*SEM* (*Parallel Ts*) *X*) = *X*
apply (*unfold SEM-def sem-def*)
apply *auto*

apply(*drule rtrancl-imp-UN-relpow*)
apply *clarify*
apply(*drule Parallel-AllNone-lemma*)
apply *auto*
done

lemma *Parallel-empty*: $Ts=[] \Longrightarrow (SEM\ (Parallel\ Ts)\ X) = X$
apply(*rule Parallel-AllNone*)
apply(*simp add*:*All-None-def*)
done

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

lemma *L3-5i*: $X \subseteq Y \Longrightarrow SEM\ c\ X \subseteq SEM\ c\ Y$
apply (*unfold SEM-def*)
apply *force*
done

lemma *L3-5ii-lemma1*:
$[\![$ (*c1*, *s1*) $-P{*}\rightarrow$ (*Parallel Ts*, *s2*); *All-None Ts*;
(*c2*, *s2*) $-P{*}\rightarrow$ (*Parallel Ss*, *s3*); *All-None Ss* $]\!]$
$\Longrightarrow$ (*Seq c1 c2*, *s1*) $-P{*}\rightarrow$ (*Parallel Ss*, *s3*)
apply(*erule converse-rtrancl-induct2*)
apply(*force intro*:*converse-rtrancl-into-rtrancl*)+
done

lemma *L3-5ii-lemma2* [*rule-format* (*no-asm*)]:
$\forall$ *c1 c2 s t*. (*Seq c1 c2*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$
(*All-None Ts*) $\longrightarrow$ ($\exists$ *y m Rs*. (*c1*,*s*) $-P{*}\rightarrow$ (*Parallel Rs*, *y*) $\wedge$
(*All-None Rs*) $\wedge$ (*c2*, *y*) $-Pm\rightarrow$ (*Parallel Ts*, *t*) $\wedge$ $m \leq n$)
apply(*induct n*)
apply(*force*)
apply(*safe dest*!: *relpow-Suc-D2*)
apply(*erule transition-cases*,*simp-all*)
apply (*fast intro*!: *le-SucI*)
apply (*fast intro*!: *le-SucI elim*!: *relpow-imp-rtrancl converse-rtrancl-into-rtrancl*)
done

lemma *L3-5ii-lemma3*:
$[\![$(*Seq c1 c2*,*s*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*); *All-None Ts*$]\!] \Longrightarrow$
($\exists$ *y Rs*. (*c1*,*s*) $-P{*}\rightarrow$ (*Parallel Rs*,*y*) $\wedge$ *All-None Rs*
$\wedge$ (*c2*,*y*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*))
apply(*drule rtrancl-imp-UN-relpow*)
apply(*fast dest*: *L3-5ii-lemma2 relpow-imp-rtrancl*)
done

lemma *L3-5ii*: $SEM\ (Seq\ c1\ c2)\ X = SEM\ c2\ (SEM\ c1\ X)$
apply (*unfold SEM-def sem-def*)
apply *auto*

```
 apply(fast dest: L3-5ii-lemma3)
apply(fast elim: L3-5ii-lemma1)
done

lemma L3-5iii: SEM (Seq (Seq c1 c2) c3) X = SEM (Seq c1 (Seq c2 c3)) X
apply (simp (no-asm) add: L3-5ii)
done

lemma L3-5iv:
 SEM (Cond b c1 c2) X = (SEM c1 (X ∩ b)) Un (SEM c2 (X ∩ (−b)))
apply (unfold SEM-def sem-def)
apply auto
apply(erule converse-rtranclE)
 prefer 2
 apply (erule transition-cases,simp-all)
  apply(fast intro: converse-rtrancl-into-rtrancl elim: transition-cases)+
done


lemma  L3-5v-lemma1[rule-format]:
 (S,s) −Pn→ (T,t) ⟶ S=Ω ⟶ (¬(∃Rs. T=(Parallel Rs) ∧ All-None Rs))
apply (unfold UNIV-def)
apply(rule nat-less-induct)
apply safe
apply(erule relpow-E2)
 apply simp-all
apply(erule transition-cases)
 apply simp-all
apply(erule relpow-E2)
 apply(simp add: Id-def)
apply(erule transition-cases,simp-all)
apply clarify
apply(erule transition-cases,simp-all)
apply(erule relpow-E2,simp)
apply clarify
apply(erule transition-cases)
 apply simp+
    apply clarify
    apply(erule transition-cases)
apply simp-all
done

lemma L3-5v-lemma2: ⟦(Ω, s) −P∗→ (Parallel Ts, t); All-None Ts ⟧ ⟹ False
apply(fast dest: rtrancl-imp-UN-relpow L3-5v-lemma1)
done

lemma L3-5v-lemma3: SEM (Ω) S = {}
apply (unfold SEM-def sem-def)
apply(fast dest: L3-5v-lemma2)
```

done

lemma *L3-5v-lemma4* [*rule-format*]:
$\forall s.$ (*While b i c*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
($\exists k.$ (*fwhile b c k*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*, *t*))
apply(*rule nat-less-induct*)
apply *safe*
apply(*erule relpow-E2*)
apply *safe*
apply(*erule transition-cases*,*simp-all*)
apply (*rule-tac x* = *1* **in** *exI*)
apply(*force dest*: *Parallel-empty-lemma intro*: *converse-rtrancl-into-rtrancl simp add*: *Id-def*)
apply *safe*
apply(*drule L3-5ii-lemma2*)
apply *safe*
apply(*drule le-imp-less-Suc*)
apply (*erule allE* , *erule impE*,*assumption*)
apply (*erule allE* , *erule impE*, *assumption*)
apply *safe*
apply (*rule-tac x* = *k+1* **in** *exI*)
apply(*simp* (*no-asm*))
apply(*rule converse-rtrancl-into-rtrancl*)
apply *fast*
apply(*fast elim*: *L3-5ii-lemma1*)
done

lemma *L3-5v-lemma5* [*rule-format*]:
$\forall s.$ (*fwhile b c k*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
(*While b i c*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*)
apply(*induct k*)
apply(*force dest*: *L3-5v-lemma2*)
apply *safe*
apply(*erule converse-rtranclE*)
apply *simp-all*
apply(*erule transition-cases*,*simp-all*)
apply(*rule converse-rtrancl-into-rtrancl*)
apply(*fast*)
apply(*fast elim*!: *L3-5ii-lemma1 dest*: *L3-5ii-lemma3*)
apply(*drule rtrancl-imp-UN-relpow*)
apply *clarify*
apply(*erule relpow-E2*)
apply *simp-all*
apply(*erule transition-cases*,*simp-all*)
apply(*fast dest*: *Parallel-empty-lemma*)
done

lemma *L3-5v*: *SEM* (*While b i c*) = ($\lambda x.$ ($\bigcup k.$ *SEM* (*fwhile b c k*) *x*))
apply(*rule ext*)

apply (*simp add*: *SEM-def sem-def*)
apply *safe*
 apply(*drule rtrancl-imp-UN-relpow*,*simp*)
 apply *clarify*
 apply(*fast dest*:*L3-5v-lemma4*)
apply(*fast intro*: *L3-5v-lemma5*)
done

1.3 Validity of Correctness Formulas

definition *com-validity* :: $'a$ *assn* ⇒ $'a$ *com* ⇒ $'a$ *assn* ⇒ *bool* ((*3*‖= -// -//-) [*90*,*55*,*90*] *50*) **where**
  ‖= *p c q* ≡ *SEM c p* ⊆ *q*

definition *ann-com-validity* :: $'a$ *ann-com* ⇒ $'a$ *assn* ⇒ *bool* (⊨ - - [*60*,*90*] *45*) **where**
  ⊨ *c q* ≡ *ann-SEM c* (*pre c*) ⊆ *q*

end

1.4 The Proof System

theory *OG-Hoare* **imports** *OG-Tran* **begin**

primrec *assertions* :: $'a$ *ann-com* ⇒ ($'a$ *assn*) *set* **where**
  assertions (*AnnBasic r f*) = {*r*}
| *assertions* (*AnnSeq c1 c2*) = *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond1 r b c1 c2*) = {*r*} ∪ *assertions c1* ∪ *assertions c2*
| *assertions* (*AnnCond2 r b c*) = {*r*} ∪ *assertions c*
| *assertions* (*AnnWhile r b i c*) = {*r*, *i*} ∪ *assertions c*
| *assertions* (*AnnAwait r b c*) = {*r*}

primrec *atomics* :: $'a$ *ann-com* ⇒ ($'a$ *assn* × $'a$ *com*) *set* **where**
  atomics (*AnnBasic r f*) = {(*r*, *Basic f*)}
| *atomics* (*AnnSeq c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond1 r b c1 c2*) = *atomics c1* ∪ *atomics c2*
| *atomics* (*AnnCond2 r b c*) = *atomics c*
| *atomics* (*AnnWhile r b i c*) = *atomics c*
| *atomics* (*AnnAwait r b c*) = {(*r* ∩ *b*, *c*)}

primrec *com* :: $'a$ *ann-triple-op* ⇒ $'a$ *ann-com-op* **where**
  com (*c*, *q*) = *c*

primrec *post* :: $'a$ *ann-triple-op* ⇒ $'a$ *assn* **where**
  post (*c*, *q*) = *q*

definition *interfree-aux* :: ($'a$ *ann-com-op* × $'a$ *assn* × $'a$ *ann-com-op*) ⇒ *bool* **where**

interfree-aux ≡ λ(*co*, *q*, *co*′). *co*′= *None* ∨
(∀ (*r*,*a*) ∈ *atomics* (*the co*′). ‖= (*q* ∩ *r*) *a q* ∧
(*co* = *None* ∨ (∀ *p* ∈ *assertions* (*the co*). ‖= (*p* ∩ *r*) *a p*)))

definition *interfree* :: ((′*a ann-triple-op*) *list*) ⇒ *bool* **where**
interfree Ts ≡ ∀ *i j*. *i* < *length Ts* ∧ *j* < *length Ts* ∧ *i* ≠ *j* ⟶
interfree-aux (*com* (*Ts*!*i*), *post* (*Ts*!*i*), *com* (*Ts*!*j*))

inductive
oghoare :: ′*a assn* ⇒ ′*a com* ⇒ ′*a assn* ⇒ *bool* ((*3*‖− -//-//-) [*90*,*55*,*90*] *50*)
and *ann-hoare* :: ′*a ann-com* ⇒ ′*a assn* ⇒ *bool* ((*2*⊢ -// -) [*60*,*90*] *45*)
where
AnnBasic: *r* ⊆ {*s*. *f s* ∈ *q*} ⟹ ⊢ (*AnnBasic r f*) *q*

| *AnnSeq*: ⟦ ⊢ *c0 pre c1*; ⊢ *c1 q* ⟧ ⟹ ⊢ (*AnnSeq c0 c1*) *q*

| *AnnCond1*: ⟦ *r* ∩ *b* ⊆ *pre c1*; ⊢ *c1 q*; *r* ∩ −*b* ⊆ *pre c2*; ⊢ *c2 q*⟧
⟹ ⊢ (*AnnCond1 r b c1 c2*) *q*
| *AnnCond2*: ⟦ *r* ∩ *b* ⊆ *pre c*; ⊢ *c q*; *r* ∩ −*b* ⊆ *q* ⟧ ⟹ ⊢ (*AnnCond2 r b c*) *q*

| *AnnWhile*: ⟦ *r* ⊆ *i*; *i* ∩ *b* ⊆ *pre c*; ⊢ *c i*; *i* ∩ −*b* ⊆ *q* ⟧
⟹ ⊢ (*AnnWhile r b i c*) *q*

| *AnnAwait*: ⟦ *atom-com c*; ‖− (*r* ∩ *b*) *c q* ⟧ ⟹ ⊢ (*AnnAwait r b c*) *q*

| *AnnConseq*: ⟦⊢ *c q*; *q* ⊆ *q*′ ⟧ ⟹ ⊢ *c q*′

| *Parallel*: ⟦ ∀ *i*<*length Ts*. ∃ *c q*. *Ts*!*i* = (*Some c*, *q*) ∧ ⊢ *c q*; *interfree Ts* ⟧
⟹ ‖− (⋂ *i*∈{*i*. *i*<*length Ts*}. *pre*(*the*(*com*(*Ts*!*i*))))
Parallel Ts
(⋂ *i*∈{*i*. *i*<*length Ts*}. *post*(*Ts*!*i*))

| *Basic*: ‖− {*s*. *f s* ∈*q*} (*Basic f*) *q*

| *Seq*: ⟦ ‖− *p c1 r*; ‖− *r c2 q* ⟧ ⟹ ‖− *p* (*Seq c1 c2*) *q*

| *Cond*: ⟦ ‖− (*p* ∩ *b*) *c1 q*; ‖− (*p* ∩ −*b*) *c2 q* ⟧ ⟹ ‖− *p* (*Cond b c1 c2*) *q*

| *While*: ⟦ ‖− (*p* ∩ *b*) *c p* ⟧ ⟹ ‖− *p* (*While b i c*) (*p* ∩ −*b*)

| *Conseq*: ⟦ *p*′ ⊆ *p*; ‖− *p c q* ; *q* ⊆ *q*′ ⟧ ⟹ ‖− *p*′ *c q*′

1.5 Soundness

lemmas [*cong del*] = *if-weak-cong*

lemmas *ann-hoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct2*]
lemmas *oghoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct1*]

lemmas *AnnBasic* = *oghoare-ann-hoare.AnnBasic*
lemmas *AnnSeq* = *oghoare-ann-hoare.AnnSeq*
lemmas *AnnCond1* = *oghoare-ann-hoare.AnnCond1*
lemmas *AnnCond2* = *oghoare-ann-hoare.AnnCond2*
lemmas *AnnWhile* = *oghoare-ann-hoare.AnnWhile*
lemmas *AnnAwait* = *oghoare-ann-hoare.AnnAwait*
lemmas *AnnConseq* = *oghoare-ann-hoare.AnnConseq*

lemmas *Parallel* = *oghoare-ann-hoare.Parallel*
lemmas *Basic* = *oghoare-ann-hoare.Basic*
lemmas *Seq* = *oghoare-ann-hoare.Seq*
lemmas *Cond* = *oghoare-ann-hoare.Cond*
lemmas *While* = *oghoare-ann-hoare.While*
lemmas *Conseq* = *oghoare-ann-hoare.Conseq*

1.5.1 Soundness of the System for Atomic Programs

lemma *Basic-ntran* [*rule-format*]:
(*Basic f*, *s*) −*Pn*→ (*Parallel Ts*, *t*) ⟶ *All-None Ts* ⟶ *t* = *f s*
apply(*induct n*)
apply(*simp* (*no-asm*))
apply(*fast dest*: *relpow-Suc-D2 Parallel-empty-lemma elim*: *transition-cases*)
done

lemma *SEM-fwhile*: *SEM S* (*p* ∩ *b*) ⊆ *p* ⟹ *SEM* (*fwhile b S k*) *p* ⊆ (*p* ∩ −*b*)
apply (*induct k*)
apply(*simp* (*no-asm*) *add*: *L3-5v-lemma3*)
apply(*simp* (*no-asm*) *add*: *L3-5iv L3-5ii Parallel-empty*)
apply(*rule conjI*)
apply (*blast dest*: *L3-5i*)
apply(*simp add*: *SEM-def sem-def id-def*)
apply (*auto dest*: *Basic-ntran rtrancl-imp-UN-relpow*)
apply *blast*
done

lemma *atom-hoare-sound* [*rule-format*]:
‖− *p c q* ⟶ *atom-com*(*c*) ⟶ ‖= *p c q*
apply (*unfold com-validity-def*)
apply(*rule oghoare-induct*)
apply *simp-all*
— Basic
apply(*simp add*: *SEM-def sem-def*)
apply(*fast dest*: *rtrancl-imp-UN-relpow Basic-ntran*)
— Seq
apply(*rule impI*)
apply(*rule subset-trans*)
prefer *2* **apply** *simp*
apply(*simp add*: *L3-5ii L3-5i*)

— Cond
 apply(*simp add*: *L3-5iv*)
— While
 apply (*force simp add*: *L3-5v dest*: *SEM-fwhile*)
— Conseq
apply(*force simp add*: *SEM-def sem-def*)
done

1.5.2 Soundness of the System for Component Programs

inductive-cases *ann-transition-cases*:
$(None,s) -1\to (c', s')$
$(Some\ (AnnBasic\ r\ f),s) -1\to (c', s')$
$(Some\ (AnnSeq\ c1\ c2),\ s) -1\to (c', s')$
$(Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\to (c', s')$
$(Some\ (AnnCond2\ r\ b\ c),\ s) -1\to (c', s')$
$(Some\ (AnnWhile\ r\ b\ I\ c),\ s) -1\to (c', s')$
$(Some\ (AnnAwait\ r\ b\ c),s) -1\to (c', s')$

Strong Soundness for Component Programs:

lemma *ann-hoare-case-analysis* [*rule-format*]: $\vdash C\ q' \longrightarrow$
$((\forall r\ f.\ C = AnnBasic\ r\ f \longrightarrow (\exists q.\ r \subseteq \{s.\ f\ s \in q\} \wedge q \subseteq q')) \wedge$
$(\forall c0\ c1.\ C = AnnSeq\ c0\ c1 \longrightarrow (\exists q.\ q \subseteq q' \wedge \vdash c0\ pre\ c1 \wedge \vdash c1\ q)) \wedge$
$(\forall r\ b\ c1\ c2.\ C = AnnCond1\ r\ b\ c1\ c2 \longrightarrow (\exists q.\ q \subseteq q' \wedge$
$r \cap b \subseteq pre\ c1 \wedge \vdash c1\ q \wedge r \cap -b \subseteq pre\ c2 \wedge \vdash c2\ q)) \wedge$
$(\forall r\ b\ c.\ C = AnnCond2\ r\ b\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \cap b \subseteq pre\ c \wedge \vdash c\ q \wedge r \cap -b \subseteq q)) \wedge$
$(\forall r\ i\ b\ c.\ C = AnnWhile\ r\ b\ i\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre\ c \wedge \vdash c\ i \wedge i \cap -b \subseteq q)) \wedge$
$(\forall r\ b\ c.\ C = AnnAwait\ r\ b\ c \longrightarrow (\exists q.\ q \subseteq q' \wedge \Vdash (r \cap b)\ c\ q)))$
apply(*rule ann-hoare-induct*)
apply *simp-all*
 apply(*rule-tac x=q* **in** *exI*,*simp*)+
apply(*rule conjI*,*clarify*,*simp*,*clarify*,*rule-tac x=qa* **in** *exI*,*fast*)+
apply(*clarify*,*simp*,*clarify*,*rule-tac x=qa* **in** *exI*,*fast*)
done

lemma *Help*: $(transition \cap \{(x,y).\ True\}) = (transition)$
apply *force*
done

lemma *Strong-Soundness-aux-aux* [*rule-format*]:
$(co,\ s) -1\to (co',\ t) \longrightarrow (\forall c.\ co = Some\ c \longrightarrow s\in pre\ c \longrightarrow$
$(\forall q.\ \vdash c\ q \longrightarrow (if\ co' = None\ then\ t\in q\ else\ t \in pre(the\ co') \wedge \vdash (the\ co')\ q\ )))$
apply(*rule ann-transition-transition.induct* [*THEN conjunct1*])
apply *simp-all*
— Basic
  apply *clarify*
  apply(*frule ann-hoare-case-analysis*)

```
apply force
— Seq
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply(fast intro: AnnConseq)
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply clarify
 apply(rule conjI)
 apply force
 apply(rule AnnSeq,simp)
 apply(fast intro: AnnConseq)
— Cond1
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply(fast intro: AnnConseq)
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply(fast intro: AnnConseq)
— Cond2
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply(fast intro: AnnConseq)
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply(fast intro: AnnConseq)
— While
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply force
 apply clarify
 apply(frule ann-hoare-case-analysis,simp)
 apply auto
 apply(rule AnnSeq)
 apply simp
 apply(rule AnnWhile)
 apply simp-all
— Await
apply(frule ann-hoare-case-analysis,simp)
apply clarify
apply(drule atom-hoare-sound)
 apply simp
apply(simp add: com-validity-def SEM-def sem-def)
apply(simp add: Help All-None-def)
apply force
done

lemma Strong-Soundness-aux: ⟦ (Some c, s) −∗→ (co, t); s ∈ pre c; ⊢ c q ⟧
  ⟹ if co = None then t ∈ q else t ∈ pre (the co) ∧ ⊢ (the co) q
```

apply(*erule rtrancl-induct2*)
apply *simp*
apply(*case-tac a*)
apply(*fast elim*: *ann-transition-cases*)
apply(*erule Strong-Soundness-aux-aux*)
apply *simp*
apply *simp-all*
done

lemma *Strong-Soundness*: $[\![\ (Some\ c,\ s) -*\rightarrow (co,\ t);\ s \in pre\ c;\ \vdash c\ q\ ]\!]$
$\Longrightarrow if\ co = None\ then\ t{\in}q\ else\ t \in pre\ (the\ co)$
apply(*force dest*:*Strong-Soundness-aux*)
done

lemma *ann-hoare-sound*: $\vdash c\ q \Longrightarrow \models c\ q$
apply (*unfold ann-com-validity-def ann-SEM-def ann-sem-def*)
apply *clarify*
apply(*drule Strong-Soundness*)
apply *simp-all*
done

1.5.3 Soundness of the System for Parallel Programs

lemma *Parallel-length-post-P1*: $(Parallel\ Ts,s)\ -P1\rightarrow\ (R',\ t) \Longrightarrow$
$(\exists Rs.\ R' = (Parallel\ Rs) \wedge (length\ Rs) = (length\ Ts) \wedge$
$(\forall i.\ i{<}length\ Ts \longrightarrow post(Rs\ !\ i) = post(Ts\ !\ i)))$
apply(*erule transition-cases*)
apply *simp*
apply *clarify*
apply(*case-tac i=ia*)
apply *simp+*
done

lemma *Parallel-length-post-PStar*: $(Parallel\ Ts,s)\ -P{*}\rightarrow\ (R',t) \Longrightarrow$
$(\exists Rs.\ R' = (Parallel\ Rs) \wedge (length\ Rs) = (length\ Ts) \wedge$
$(\forall i.\ i{<}length\ Ts \longrightarrow post(Ts\ !\ i) = post(Rs\ !\ i)))$
apply(*erule rtrancl-induct2*)
apply(*simp-all*)
apply *clarify*
apply *simp*
apply(*drule Parallel-length-post-P1*)
apply *auto*
done

lemma *assertions-lemma*: $pre\ c \in assertions\ c$
apply(*rule ann-com-com.induct* [*THEN conjunct1*])
apply *auto*
done

lemma *interfree-aux1* [*rule-format*]:
  (*c*,*s*) −*1*→ (*r*,*t*) ⟶ (*interfree-aux*(*c1*, *q1*, *c*) ⟶ *interfree-aux*(*c1*, *q1*, *r*))
apply (*rule ann-transition-transition.induct* [*THEN conjunct1*])
apply(*safe*)
prefer *13*
apply (*rule TrueI*)
apply (*simp-all add*:*interfree-aux-def*)
apply *force*+
done

lemma *interfree-aux2* [*rule-format*]:
  (*c*,*s*) −*1*→ (*r*,*t*) ⟶ (*interfree-aux*(*c*, *q*, *a*) ⟶ *interfree-aux*(*r*, *q*, *a*))
apply (*rule ann-transition-transition.induct* [*THEN conjunct1*])
apply(*force simp add*:*interfree-aux-def*)+
done

lemma *interfree-lemma*: ⟦ (*Some c*, *s*) −*1*→ (*r*, *t*);*interfree Ts* ; *i*<*length Ts*;
           Ts!*i* = (*Some c*, *q*) ⟧ ⟹ *interfree* (*Ts*[*i*:= (*r*, *q*)])
apply(*simp add*: *interfree-def*)
apply *clarify*
apply(*case-tac i*=*j*)
 apply(*drule-tac t* = *ia* **in** *not-sym*)
 apply *simp-all*
apply(*force elim*: *interfree-aux1*)
apply(*force elim*: *interfree-aux2 simp add*:*nth-list-update*)
done

Strong Soundness Theorem for Parallel Programs:

lemma *Parallel-Strong-Soundness-Seq-aux*:
  ⟦*interfree Ts*; *i*<*length Ts*; *com*(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*) ⟧
  ⟹ *interfree* (*Ts*[*i*:=(*Some c0*, *pre c1*)])
apply(*simp add*: *interfree-def*)
apply *clarify*
apply(*case-tac i*=*j*)
 apply(*force simp add*: *nth-list-update interfree-aux-def*)
apply(*case-tac i*=*ia*)
 apply(*erule-tac x*=*ia* **in** *allE*)
 apply(*force simp add*:*interfree-aux-def assertions-lemma*)
apply *simp*
done

lemma *Parallel-Strong-Soundness-Seq* [*rule-format* (*no-asm*)]:
 ⟦ ∀ *i*<*length Ts*. (*if com*(*Ts*!*i*) = *None then b* ∈ *post*(*Ts*!*i*)
  else b ∈ *pre*(*the*(*com*(*Ts*!*i*))) ∧ ⊢ *the*(*com*(*Ts*!*i*)) *post*(*Ts*!*i*));
  com(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*); *i*<*length Ts*; *interfree Ts* ⟧ ⟹
 (∀ *ia*<*length Ts*. (*if com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*) = *None*
  then b ∈ *post*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)
 else b ∈ *pre*(*the*(*com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*))) ∧
 ⊢ *the*(*com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)) *post*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)))

```
  ∧ interfree (Ts[i:= (Some c0, pre c1)])
apply(rule conjI)
 apply safe
 apply(case-tac i=ia)
  apply simp
  apply(force dest: ann-hoare-case-analysis)
 apply simp
apply(fast elim: Parallel-Strong-Soundness-Seq-aux)
done

lemma Parallel-Strong-Soundness-aux-aux [rule-format]:
 (Some c, b) −1→ (co, t) ⟶
  (∀ Ts. i<length Ts ⟶ com(Ts ! i) = Some c ⟶
  (∀ i<length Ts. (if com(Ts ! i) = None then b∈post(Ts!i)
  else b∈pre(the(com(Ts!i))) ∧ ⊢ the(com(Ts!i)) post(Ts!i))) ⟶
 interfree Ts ⟶
  (∀ j. j<length Ts ∧ i≠j ⟶ (if com(Ts!j) = None then t∈post(Ts!j)
  else t∈pre(the(com(Ts!j))) ∧ ⊢ the(com(Ts!j)) post(Ts!j))) )
apply(rule ann-transition-transition.induct [THEN conjunct1])
apply safe
prefer 11
apply(rule TrueI)
apply simp-all
— Basic
   apply(erule-tac x = i in all-dupE, erule (1) notE impE)
   apply(erule-tac x = j in allE , erule (1) notE impE)
   apply(simp add: interfree-def)
   apply(erule-tac x = j in allE,simp)
   apply(erule-tac x = i in allE,simp)
   apply(drule-tac t = i in not-sym)
   apply(case-tac com(Ts ! j)=None)
    apply(force intro: converse-rtrancl-into-rtrancl
         simp add: interfree-aux-def com-validity-def SEM-def sem-def All-None-def)
   apply(simp add:interfree-aux-def)
   apply clarify
   apply simp
   apply(erule-tac x=pre y in ballE)
    apply(force intro: converse-rtrancl-into-rtrancl
          simp add: com-validity-def SEM-def sem-def All-None-def)
   apply(simp add:assertions-lemma)
— Seqs
  apply(erule-tac x = Ts[i:=(Some c0, pre c1)] in allE)
  apply(drule  Parallel-Strong-Soundness-Seq,simp+)
 apply(erule-tac x = Ts[i:=(Some c0, pre c1)] in allE)
 apply(drule  Parallel-Strong-Soundness-Seq,simp+)
— Await
apply(rule-tac x = i in allE , assumption , erule (1) notE impE)
apply(erule-tac x = j in allE , erule (1) notE impE)
apply(simp add: interfree-def)
```

```
apply(erule-tac x = j in allE,simp)
apply(erule-tac x = i in allE,simp)
apply(drule-tac t = i in not-sym)
apply(case-tac com(Ts ! j)=None)
 apply(force intro: converse-rtrancl-into-rtrancl simp add: interfree-aux-def
        com-validity-def SEM-def sem-def All-None-def Help)
apply(simp add:interfree-aux-def)
apply clarify
apply simp
apply(erule-tac x=pre y in ballE)
 apply(force intro: converse-rtrancl-into-rtrancl
       simp add: com-validity-def SEM-def sem-def All-None-def Help)
apply(simp add:assertions-lemma)
done

lemma Parallel-Strong-Soundness-aux [rule-format]:
 ⟦(Ts',s) −P∗→ (Rs',t);  Ts' = (Parallel Ts); interfree Ts;
 ∀ i. i<length Ts ⟶ (∃ c q. (Ts ! i) = (Some c, q) ∧ s∈(pre c) ∧ ⊢ c q ) ⟧ ⟹
  ∀ Rs. Rs' = (Parallel Rs) ⟶ (∀ j. j<length Rs ⟶
  (if com(Rs ! j) = None then t∈post(Ts ! j)
  else t∈pre(the(com(Rs ! j))) ∧ ⊢ the(com(Rs ! j)) post(Ts ! j))) ∧ interfree Rs
apply(erule rtrancl-induct2)
 apply clarify
— Base
 apply force
— Induction step
apply clarify
apply(drule Parallel-length-post-PStar)
apply clarify
apply (ind-cases (Parallel Ts, s) −P1→ (Parallel Rs, t) for Ts s Rs t)
apply(rule conjI)
 apply clarify
 apply(case-tac i=j)
  apply(simp split del:if-split)
  apply(erule Strong-Soundness-aux-aux,simp+)
   apply force
  apply force
 apply(simp split del: if-split)
 apply(erule Parallel-Strong-Soundness-aux-aux)
 apply(simp-all add: split del:if-split)
 apply force
apply(rule interfree-lemma)
apply simp-all
done

lemma Parallel-Strong-Soundness:
 ⟦(Parallel Ts, s) −P∗→ (Parallel Rs, t); interfree Ts; j<length Rs;
  ∀ i. i<length Ts ⟶ (∃ c q. Ts ! i = (Some c, q) ∧ s∈pre c ∧ ⊢ c q) ⟧ ⟹
  if com(Rs ! j) = None then t∈post(Ts ! j) else t∈pre (the(com(Rs ! j)))
```

```
apply(drule  Parallel-Strong-Soundness-aux)
apply simp+
done

lemma oghoare-sound [rule-format]: ‖− p c q ⟶ ‖= p c q
apply (unfold com-validity-def)
apply(rule oghoare-induct)
apply(rule TrueI)+
— Parallel
      apply(simp add: SEM-def sem-def)
      apply(clarify, rename-tac x y i Ts′)
      apply(frule Parallel-length-post-PStar)
      apply clarify
      apply(drule-tac j=i in Parallel-Strong-Soundness)
         apply clarify
        apply simp
       apply force
      apply simp
      apply(erule-tac V = ∀ i. P i for P in thin-rl)
      apply(drule-tac s = length Rs in sym)
      apply(erule allE, erule impE, assumption)
      apply(force dest: nth-mem simp add: All-None-def)
— Basic
    apply(simp add: SEM-def sem-def)
    apply(force dest: rtrancl-imp-UN-relpow Basic-ntran)
— Seq
   apply(rule subset-trans)
    prefer 2 apply assumption
   apply(simp add: L3-5ii L3-5i)
— Cond
  apply(simp add: L3-5iv)
— While
 apply(simp add: L3-5v)
 apply (blast dest: SEM-fwhile)
— Conseq
apply(auto simp add: SEM-def sem-def)
done

end
```

1.6 Generation of Verification Conditions

```
theory OG-Tactics
imports OG-Hoare
begin

lemmas ann-hoare-intros=AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq
lemmas oghoare-intros=Parallel Basic Seq Cond While Conseq
```

```
lemma ParallelConseqRule:
 ⟦ p ⊆ (⋂ i∈{i. i<length Ts}. pre(the(com(Ts ! i))));
  ||− (⋂ i∈{i. i<length Ts}. pre(the(com(Ts ! i))))
      (Parallel Ts)
     (⋂ i∈{i. i<length Ts}. post(Ts ! i));
  (⋂ i∈{i. i<length Ts}. post(Ts ! i)) ⊆ q ⟧
  ⟹ ||− p (Parallel Ts) q
apply (rule Conseq)
prefer 2
 apply fast
apply assumption+
done

lemma SkipRule: p ⊆ q ⟹ ||− p (Basic id) q
apply(rule oghoare-intros)
  prefer 2 apply(rule Basic)
 prefer 2 apply(rule subset-refl)
apply(simp add:Id-def)
done

lemma BasicRule: p ⊆ {s. (f s)∈q} ⟹ ||− p (Basic f) q
apply(rule oghoare-intros)
  prefer 2 apply(rule oghoare-intros)
 prefer 2 apply(rule subset-refl)
apply assumption
done

lemma SeqRule: ⟦ ||− p c1 r; ||− r c2 q ⟧ ⟹ ||− p (Seq c1 c2) q
apply(rule Seq)
apply fast+
done

lemma CondRule:
 ⟦ p ⊆ {s. (s∈b ⟶ s∈w) ∧ (s∉b ⟶ s∈w')}; ||− w c1 q; ||− w' c2 q ⟧
  ⟹ ||− p (Cond b c1 c2) q
apply(rule Cond)
 apply(rule Conseq)
 prefer 4 apply(rule Conseq)
apply simp-all
apply force+
done

lemma WhileRule: ⟦ p ⊆ i; ||− (i ∩ b) c i ; (i ∩ (−b)) ⊆ q ⟧
        ⟹ ||− p (While b i c) q
apply(rule Conseq)
 prefer 2 apply(rule While)
apply assumption+
done
```

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is {*s. True*}:

lemma *AnnatomRule*:
  ⟦ *atom-com*(*c*); ∥− *r c q* ⟧ ⟹ ⊢ (*AnnAwait r* {*s. True*} *c*) *q*
apply(*rule AnnAwait*)
apply *simp-all*
done

lemma *AnnskipRule*:
  r ⊆ *q* ⟹ ⊢ (*AnnBasic r id*) *q*
apply(*rule AnnBasic*)
apply *simp*
done

lemma *AnnwaitRule*:
  ⟦ (*r* ∩ *b*) ⊆ *q* ⟧ ⟹ ⊢ (*AnnAwait r b* (*Basic id*)) *q*
apply(*rule AnnAwait*)
 apply *simp*
apply(*rule BasicRule*)
apply *simp*
done

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

lemma *interfree-aux-rule1*: *interfree-aux*(*co*, *q*, *None*)
by(*simp add*:*interfree-aux-def*)

lemma *interfree-aux-rule2*:
  ∀ (*R*,*r*)∈(*atomics a*). ∥− (*q* ∩ *R*) *r q* ⟹ *interfree-aux*(*None*, *q*, *Some a*)
apply(*simp add*:*interfree-aux-def*)
apply(*force elim*:*oghoare-sound*)
done

lemma *interfree-aux-rule3*:
  (∀ (*R*, *r*)∈(*atomics a*). ∥− (*q* ∩ *R*) *r q* ∧ (∀ *p*∈(*assertions c*). ∥− (*p* ∩ *R*) *r p*))
  ⟹ *interfree-aux*(*Some c*, *q*, *Some a*)
apply(*simp add*:*interfree-aux-def*)
apply(*force elim*:*oghoare-sound*)
done

lemma *AnnBasic-assertions*:
  ⟦*interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧ ⟹
    interfree-aux(*Some* (*AnnBasic r f*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnSeq-assertions*:
  ⟦ *interfree-aux*(*Some c1*, *q*, *Some a*); *interfree-aux*(*Some c2*, *q*, *Some a*)⟧⟹
   interfree-aux(*Some* (*AnnSeq c1 c2*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond1-assertions*:
  ⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c1*, *q*, *Some a*);
  interfree-aux(*Some c2*, *q*, *Some a*)⟧⟹
  interfree-aux(*Some*(*AnnCond1 r b c1 c2*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond2-assertions*:
  ⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c*, *q*, *Some a*)⟧⟹
  interfree-aux(*Some* (*AnnCond2 r b c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnWhile-assertions*:
  ⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *i*, *Some a*);
  interfree-aux(*Some c*, *q*, *Some a*)⟧⟹
  interfree-aux(*Some* (*AnnWhile r b i c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnAwait-assertions*:
  ⟦ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)⟧⟹
  interfree-aux(*Some* (*AnnAwait r b c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnBasic-atomics*:
  ‖− ($q \cap r$) (*Basic f*) *q* ⟹ *interfree-aux*(*None*, *q*, *Some* (*AnnBasic r f*))
by(*simp add*: *interfree-aux-def oghoare-sound*)

lemma *AnnSeq-atomics*:
  ⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
  interfree-aux(*Any*, *q*, *Some* (*AnnSeq a1 a2*))
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond1-atomics*:
  ⟦ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)⟧⟹
   interfree-aux(*Any*, *q*, *Some* (*AnnCond1 r b a1 a2*))
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond2-atomics*:

interfree-aux (Any, q, Some a)⟹ interfree-aux(Any, q, Some (AnnCond2 r b a))
by(simp add: interfree-aux-def)

lemma AnnWhile-atomics: interfree-aux (Any, q, Some a)
⟹ interfree-aux(Any, q, Some (AnnWhile r b i a))
by(simp add: interfree-aux-def)

lemma Annatom-atomics:
‖− (q ∩ r) a q ⟹ interfree-aux (None, q, Some (AnnAwait r {x. True} a))
by(simp add: interfree-aux-def oghoare-sound)

lemma AnnAwait-atomics:
‖− (q ∩ (r ∩ b)) a q ⟹ interfree-aux (None, q, Some (AnnAwait r b a))
by(simp add: interfree-aux-def oghoare-sound)

definition interfree-swap :: ('a ann-triple-op ∗ ('a ann-triple-op) list) ⇒ bool **where**
interfree-swap == λ(x, xs). ∀ y∈set xs. interfree-aux (com x, post x, com y)
∧ interfree-aux(com y, post y, com x)

lemma interfree-swap-Empty: interfree-swap (x, [])
by(simp add:interfree-swap-def)

lemma interfree-swap-List:
⟦ interfree-aux (com x, post x, com y);
interfree-aux (com y, post y ,com x); interfree-swap (x, xs) ⟧
⟹ interfree-swap (x, y#xs)
by(simp add:interfree-swap-def)

lemma interfree-swap-Map: ∀ k. i≤k ∧ k<j ⟶ interfree-aux (com x, post x, c k)
∧ interfree-aux (c k, Q k, com x)
⟹ interfree-swap (x, map (λk. (c k, Q k)) [i..<j])
by(force simp add: interfree-swap-def less-diff-conv)

lemma interfree-Empty: interfree []
by(simp add:interfree-def)

lemma interfree-List:
⟦ interfree-swap(x, xs); interfree xs ⟧ ⟹ interfree (x#xs)
apply(simp add:interfree-def interfree-swap-def)
apply clarify
apply(case-tac i)
apply(case-tac j)
apply simp-all
apply(case-tac j,simp+)
done

lemma interfree-Map:
(∀ i j. a≤i ∧ i<b ∧ a≤j ∧ j<b ∧ i≠j ⟶ interfree-aux (c i, Q i, c j))

$\Longrightarrow$ *interfree* (*map* ($\lambda k.$ (*c k*, *Q k*)) [*a*..<*b*])
by(*force simp add*: *interfree-def less-diff-conv*)

definition *map-ann-hoare* :: (('*a ann-com-op* $*$ '*a assn*) *list*) $\Rightarrow$ *bool* ([$\vdash$] - [*0*] *45*)
where
[$\vdash$] *Ts* == ($\forall$ *i*<*length Ts*. $\exists$ *c q*. *Ts*!*i*=(*Some c*, *q*) $\wedge$ $\vdash$ *c q*)

lemma *MapAnnEmpty*: [$\vdash$] []
by(*simp add*:*map-ann-hoare-def*)

lemma *MapAnnList*: $[\![$ $\vdash$ *c q* ; [$\vdash$] *xs* $]\!]$ $\Longrightarrow$ [$\vdash$] (*Some c*,*q*)#*xs*
apply(*simp add*:*map-ann-hoare-def*)
apply *clarify*
apply(*case-tac i*,*simp*+)
done

lemma *MapAnnMap*:
$\forall k.$ *i*$\le$*k* $\wedge$ *k*<*j* $\longrightarrow$ $\vdash$ (*c k*) (*Q k*) $\Longrightarrow$ [$\vdash$] *map* ($\lambda k.$ (*Some* (*c k*), *Q k*)) [*i*..<*j*]
apply(*simp add*: *map-ann-hoare-def less-diff-conv*)
done

lemma *ParallelRule*:$[\![$ [$\vdash$] *Ts* ; *interfree Ts* $]\!]$
$\Longrightarrow$ $\|-$ ($\bigcap$ *i*$\in${*i*. *i*<*length Ts*}. *pre*(*the*(*com*(*Ts*!*i*))))
Parallel Ts
($\bigcap$ *i*$\in${*i*. *i*<*length Ts*}. *post*(*Ts*!*i*))
apply(*rule Parallel*)
apply(*simp add*:*map-ann-hoare-def*)
apply *simp*
done

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

lemma *Compl-Collect*: $-$(*Collect b*) = {*x*. $\neg$(*b x*)}
by *fast*

lemma *list-length*: *length* []=*0* *length* (*x*#*xs*) = *Suc*(*length xs*)
by *simp-all*
lemma *list-lemmas*: *length* []=*0* *length* (*x*#*xs*) = *Suc*(*length xs*)
(*x*#*xs*) ! *0* = *x* (*x*#*xs*) ! *Suc n* = *xs* ! *n*
by *simp-all*
lemma *le-Suc-eq-insert*: {*i*. *i* <*Suc n*} = *insert n* {*i*. *i*< *n*}
by *auto*
lemmas *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*
lemmas *my-simp-list* = *list-lemmas fst-conv snd-conv*
not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc nat.inject
Collect-mem-eq ball-simps option.simps primrecdef-list
lemmas *ParallelConseq-list* = *INTER-eq Collect-conj-eq length-map length-upt length-append*

ML ‹
fun before-interfree-simp-tac ctxt =
simp-tac (put-simpset HOL-basic-ss ctxt addsimps [@{*thm com.simps*}, @{*thm post.simps*}])

fun interfree-simp-tac ctxt =
asm-simp-tac (put-simpset HOL-ss ctxt
addsimps [@{*thm split*}, @{*thm ball-Un*}, @{*thm ball-empty*}] @ @{*thms my-simp-list*})

fun ParallelConseq ctxt =
simp-tac (put-simpset HOL-basic-ss ctxt
addsimps @{*thms ParallelConseq-list*} @ @{*thms my-simp-list*})
›

The following tactic applies *tac* to each conjunct in a subgoal of the form $A \Longrightarrow a1 \wedge a2 \wedge .. \wedge an$ returning n subgoals, one for each conjunct:

ML ‹
fun conjI-Tac ctxt tac i st = st |>
((*EVERY* [*resolve-tac ctxt* [*conjI*] *i*,
conjI-Tac ctxt tac (*i+1*),
tac i]) *ORELSE* (*tac i*))
›

Tactic for the generation of the verification conditions

The tactic basically uses two subtactics:

HoareRuleTac is called at the level of parallel programs, it uses the ParallelTac to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. ⟨ ⟩ and *AWAIT b THEN - END*, and at each interference freedom test.

AnnHoareRuleTac is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter precond, see NOTE).

NOTE: precond(::bool) informs if the subgoal has the form $\Vert\!-$ *?p c q*, in this case we have precond=False and the generated verification condition would have the form *?p* ⊆ . . . which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

ML ‹

fun WlpTac ctxt i = resolve-tac ctxt @{*thms SeqRule*} *i THEN HoareRuleTac ctxt false* (*i + 1*)

```
and HoareRuleTac ctxt precond i st = st |>
  ( (WlpTac ctxt i THEN HoareRuleTac ctxt precond i)
    ORELSE
    (FIRST[resolve-tac ctxt @{thms SkipRule} i,
           resolve-tac ctxt @{thms BasicRule} i,
           EVERY[resolve-tac ctxt @{thms ParallelConseqRule} i,
                 ParallelConseq ctxt (i+2),
                 ParallelTac ctxt (i+1),
                 ParallelConseq ctxt i],
           EVERY[resolve-tac ctxt @{thms CondRule} i,
                 HoareRuleTac ctxt false (i+2),
                 HoareRuleTac ctxt false (i+1)],
           EVERY[resolve-tac ctxt @{thms WhileRule} i,
                 HoareRuleTac ctxt true (i+1)],
           K all-tac i ]
     THEN (if precond then (K all-tac i) else resolve-tac ctxt @{thms subset-refl}
i)))

and AnnWlpTac ctxt i = resolve-tac ctxt @{thms AnnSeq} i THEN AnnHoareRule-
Tac ctxt (i + 1)
and AnnHoareRuleTac ctxt i st = st |>
  ( (AnnWlpTac ctxt i THEN AnnHoareRuleTac ctxt i )
   ORELSE
    (FIRST[(resolve-tac ctxt @{thms AnnskipRule} i),
           EVERY[resolve-tac ctxt @{thms AnnatomRule} i,
                 HoareRuleTac ctxt true (i+1)],
           (resolve-tac ctxt @{thms AnnwaitRule} i),
           resolve-tac ctxt @{thms AnnBasic} i,
           EVERY[resolve-tac ctxt @{thms AnnCond1} i,
                 AnnHoareRuleTac ctxt (i+3),
                 AnnHoareRuleTac ctxt (i+1)],
           EVERY[resolve-tac ctxt @{thms AnnCond2} i,
                 AnnHoareRuleTac ctxt (i+1)],
           EVERY[resolve-tac ctxt @{thms AnnWhile} i,
                 AnnHoareRuleTac ctxt (i+2)],
           EVERY[resolve-tac ctxt @{thms AnnAwait} i,
                 HoareRuleTac ctxt true (i+1)],
           K all-tac i]))

and ParallelTac ctxt i = EVERY[resolve-tac ctxt @{thms ParallelRule} i,
                               interfree-Tac ctxt (i+1),
                               MapAnn-Tac ctxt i]

and MapAnn-Tac ctxt i st = st |>
  (FIRST[resolve-tac ctxt @{thms MapAnnEmpty} i,
         EVERY[resolve-tac ctxt @{thms MapAnnList} i,
               MapAnn-Tac ctxt (i+1),
               AnnHoareRuleTac ctxt i],
         EVERY[resolve-tac ctxt @{thms MapAnnMap} i,
```

```
          resolve-tac ctxt @{thms allI} i,
          resolve-tac ctxt @{thms impI} i,
          AnnHoareRuleTac ctxt i]])

and interfree-swap-Tac ctxt i st = st |>
  (FIRST[resolve-tac ctxt @{thms interfree-swap-Empty} i,
      EVERY[resolve-tac ctxt @{thms interfree-swap-List} i,
          interfree-swap-Tac ctxt (i+2),
          interfree-aux-Tac ctxt (i+1),
          interfree-aux-Tac ctxt i ],
      EVERY[resolve-tac ctxt @{thms interfree-swap-Map} i,
          resolve-tac ctxt @{thms allI} i,
          resolve-tac ctxt @{thms impI} i,
          conjI-Tac ctxt (interfree-aux-Tac ctxt) i]])

and interfree-Tac ctxt i st = st |>
  (FIRST[resolve-tac ctxt @{thms interfree-Empty} i,
      EVERY[resolve-tac ctxt @{thms interfree-List} i,
          interfree-Tac ctxt (i+1),
          interfree-swap-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms interfree-Map} i,
          resolve-tac ctxt @{thms allI} i,
          resolve-tac ctxt @{thms allI} i,
          resolve-tac ctxt @{thms impI} i,
          interfree-aux-Tac ctxt i ]])

and interfree-aux-Tac ctxt i = (before-interfree-simp-tac ctxt i ) THEN
    (FIRST[resolve-tac ctxt @{thms interfree-aux-rule1} i,
        dest-assertions-Tac ctxt i])

and dest-assertions-Tac ctxt i st = st |>
  (FIRST[EVERY[resolve-tac ctxt @{thms AnnBasic-assertions} i,
          dest-atomics-Tac ctxt (i+1),
          dest-atomics-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms AnnSeq-assertions} i,
          dest-assertions-Tac ctxt (i+1),
          dest-assertions-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms AnnCond1-assertions} i,
          dest-assertions-Tac ctxt (i+2),
          dest-assertions-Tac ctxt (i+1),
          dest-atomics-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms AnnCond2-assertions} i,
          dest-assertions-Tac ctxt (i+1),
          dest-atomics-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms AnnWhile-assertions} i,
          dest-assertions-Tac ctxt (i+2),
          dest-atomics-Tac ctxt (i+1),
          dest-atomics-Tac ctxt i],
      EVERY[resolve-tac ctxt @{thms AnnAwait-assertions} i,
```

```
        dest-atomics-Tac ctxt (i+1),
        dest-atomics-Tac ctxt i],
     dest-atomics-Tac ctxt i])

and dest-atomics-Tac ctxt i st = st |>
  (FIRST[EVERY[resolve-tac ctxt @{thms AnnBasic-atomics} i,
        HoareRuleTac ctxt true i],
     EVERY[resolve-tac ctxt @{thms AnnSeq-atomics} i,
        dest-atomics-Tac ctxt (i+1),
        dest-atomics-Tac ctxt i],
     EVERY[resolve-tac ctxt @{thms AnnCond1-atomics} i,
        dest-atomics-Tac ctxt (i+1),
        dest-atomics-Tac ctxt i],
     EVERY[resolve-tac ctxt @{thms AnnCond2-atomics} i,
        dest-atomics-Tac ctxt i],
     EVERY[resolve-tac ctxt @{thms AnnWhile-atomics} i,
        dest-atomics-Tac ctxt i],
     EVERY[resolve-tac ctxt @{thms Annatom-atomics} i,
        HoareRuleTac ctxt true i],
     EVERY[resolve-tac ctxt @{thms AnnAwait-atomics} i,
        HoareRuleTac ctxt true i],
        K all-tac i])
›
```

The final tactic is given the name *oghoare*:

```
ML ‹
fun oghoare-tac ctxt = SUBGOAL (fn (-, i) => HoareRuleTac ctxt true i)
›
```

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

```
ML ‹
fun annhoare-tac ctxt = SUBGOAL (fn (-, i) => AnnHoareRuleTac ctxt i)

fun interfree-aux-tac ctxt = SUBGOAL (fn (-, i) => interfree-aux-Tac ctxt i)
›
```

The so defined ML tactics are then “exported” to be used in Isabelle proofs.

```
method-setup oghoare = ‹
  Scan.succeed (SIMPLE-METHOD' o oghoare-tac)›
  verification condition generator for the oghoare logic

method-setup annhoare = ‹
  Scan.succeed (SIMPLE-METHOD' o annhoare-tac)›
  verification condition generator for the ann-hoare logic
```

method-setup *interfree-aux* = ‹
Scan.succeed (*SIMPLE-METHOD′ o interfree-aux-tac*)›
verification condition generator for interference freedom tests

Tactics useful for dealing with the generated verification conditions:

method-setup *conjI-tac* = ‹
Scan.succeed (*fn ctxt* => *SIMPLE-METHOD′* (*conjI-Tac ctxt* (*K all-tac*)))›
verification condition generator for interference freedom tests

ML ‹
```
fun disjE-Tac ctxt tac i st = st |>
      ( (EVERY [eresolve-tac ctxt [disjE] i,
          disjE-Tac ctxt tac (i+1),
          tac i]) ORELSE (tac i) )
```
›

method-setup *disjE-tac* = ‹
Scan.succeed (*fn ctxt* => *SIMPLE-METHOD′* (*disjE-Tac ctxt* (*K all-tac*)))›
verification condition generator for interference freedom tests

end

1.7 Concrete Syntax

theory *Quote-Antiquote* **imports** *Main* **begin**

syntax
```
-quote     :: 'b ⇒ ('a ⇒ 'b)              ((«-») [0] 1000)
-antiquote :: ('a ⇒ 'b) ⇒ 'b              (´- [1000] 1000)
-Assert    :: 'a ⇒ 'a set                 (({|-|}) [0] 1000)
```

translations
```
{|b|} ⇀ CONST Collect «b»
```

parse-translation ‹
```
let
  fun quote-tr [t] = Syntax-Trans.quote-tr syntax-const ‹-antiquote› t
    | quote-tr ts = raise TERM (quote-tr, ts);
in [(syntax-const ‹-quote›, K quote-tr)] end
```
›

end
theory *OG-Syntax*
imports *OG-Tactics Quote-Antiquote*
begin

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

abbreviation *Skip* :: *'a com* (*SKIP 63*)
where *SKIP* ≡ *Basic id*

abbreviation *AnnSkip* :: *'a assn* ⇒ *'a ann-com* (*-//SKIP* [*90*] *63*)
where *r SKIP* ≡ *AnnBasic r id*

notation
Seq (-,,/ - [*55*, *56*] *55*) **and**
AnnSeq (-;;/ - [*60*,*61*] *60*)

syntax
-Assign :: *idt* ⇒ *'b* ⇒ *'a com* ((´- :=/ -) [*70*, *65*] *61*)
-AnnAssign :: *'a assn* ⇒ *idt* ⇒ *'b* ⇒ *'a com* ((- ´- :=/ -) [*90*,*70*,*65*] *61*)

translations
´x := a ⇀ *CONST Basic* «´(*-update-name x* (λ-. *a*))»
r ´x := a ⇀ *CONST AnnBasic r* «´(*-update-name x* (λ-. *a*))»

syntax
-AnnCond1 :: *'a assn* ⇒ *'a bexp* ⇒ *'a ann-com* ⇒ *'a ann-com* ⇒ *'a ann-com*
(*- //IF - /THEN - /ELSE - /FI* [*90*,*0*,*0*,*0*] *61*)
-AnnCond2 :: *'a assn* ⇒ *'a bexp* ⇒ *'a ann-com* ⇒ *'a ann-com*
(*- //IF - /THEN - /FI* [*90*,*0*,*0*] *61*)
-AnnWhile :: *'a assn* ⇒ *'a bexp* ⇒ *'a assn* ⇒ *'a ann-com* ⇒ *'a ann-com*
(*- //WHILE - /INV - //DO -//OD* [*90*,*0*,*0*,*0*] *61*)
-AnnAwait :: *'a assn* ⇒ *'a bexp* ⇒ *'a com* ⇒ *'a ann-com*
(*- //AWAIT - /THEN /- /END* [*90*,*0*,*0*] *61*)
-AnnAtom :: *'a assn* ⇒ *'a com* ⇒ *'a ann-com* (-//⟨-⟩ [*90*,*0*] *61*)
-AnnWait :: *'a assn* ⇒ *'a bexp* ⇒ *'a ann-com* (*-//WAIT - END* [*90*,*0*] *61*)

-Cond :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ⇒ *'a com*
((*0IF -/ THEN -/ ELSE -/ FI*) [*0*, *0*, *0*] *61*)
-Cond2 :: *'a bexp* ⇒ *'a com* ⇒ *'a com* (*IF - THEN - FI* [*0*,*0*] *56*)
-While-inv :: *'a bexp* ⇒ *'a assn* ⇒ *'a com* ⇒ *'a com*
((*0WHILE -/ INV - //DO - /OD*) [*0*, *0*, *0*] *61*)
-While :: *'a bexp* ⇒ *'a com* ⇒ *'a com*
((*0WHILE - //DO - /OD*) [*0*, *0*] *61*)

translations
IF b THEN c1 ELSE c2 FI ⇀ *CONST Cond* ⦃*b*⦄ *c1 c2*
IF b THEN c FI ⇌ *IF b THEN c ELSE SKIP FI*
WHILE b INV i DO c OD ⇀ *CONST While* ⦃*b*⦄ *i c*
WHILE b DO c OD ⇌ *WHILE b INV CONST undefined DO c OD*

r IF b THEN c1 ELSE c2 FI ⇀ *CONST AnnCond1 r* ⦃*b*⦄ *c1 c2*
r IF b THEN c FI ⇀ *CONST AnnCond2 r* ⦃*b*⦄ *c*
r WHILE b INV i DO c OD ⇀ *CONST AnnWhile r* ⦃*b*⦄ *i c*
r AWAIT b THEN c END ⇀ *CONST AnnAwait r* ⦃*b*⦄ *c*
r ⟨*c*⟩ ⇌ *r AWAIT CONST True THEN c END*

```
  r WAIT b END ⇌ r AWAIT b THEN SKIP END

nonterminal prgs

syntax
  -PAR :: prgs ⇒ 'a                (COBEGIN//-//COEND [57] 56)
  -prg :: ['a, 'a] ⇒ prgs         (-//- [60, 90] 57)
  -prgs :: ['a, 'a, prgs] ⇒ prgs  (-//-//‖//- [60,90,57] 57)

  -prg-scheme :: ['a, 'a, 'a, 'a, 'a] ⇒ prgs
                (SCHEME [- ≤ - < -] -// - [0,0,0,60, 90] 57)

translations
  -prg c q ⇌ [(CONST Some c, q)]
  -prgs c q ps ⇌ (CONST Some c, q) # ps
  -PAR ps ⇌ CONST Parallel ps

  -prg-scheme j i k c q ⇌ CONST map (λi. (CONST Some c, q)) [j..<k]

print-translation ‹
  let
    fun quote-tr' f (t :: ts) =
          Term.list-comb (f $ Syntax-Trans.quote-tr' syntax-const ‹-antiquote› t,
ts)
      | quote-tr' - - = raise Match;

    fun annquote-tr' f (r :: t :: ts) =
          Term.list-comb (f $ r $ Syntax-Trans.quote-tr' syntax-const ‹-antiquote›
t, ts)
      | annquote-tr' - - = raise Match;

    val assert-tr' = quote-tr' (Syntax.const syntax-const ‹-Assert›);

    fun bexp-tr' name ((Const (const-syntax ‹Collect›, -) $ t) :: ts) =
          quote-tr' (Syntax.const name) (t :: ts)
      | bexp-tr' - - = raise Match;

    fun annbexp-tr' name (r :: (Const (const-syntax ‹Collect›, -) $ t) :: ts) =
          annquote-tr' (Syntax.const name) (r :: t :: ts)
      | annbexp-tr' - - = raise Match;

    fun assign-tr' (Abs (x, -, f $ k $ Bound 0) :: ts) =
        quote-tr' (Syntax.const syntax-const ‹-Assign› $ Syntax-Trans.update-name-tr'
f)
            (Abs (x, dummyT, Syntax-Trans.const-abs-tr' k) :: ts)
      | assign-tr' - = raise Match;

    fun annassign-tr' (r :: Abs (x, -, f $ k $ Bound 0) :: ts) =
        quote-tr' (Syntax.const syntax-const ‹-AnnAssign› $ r $ Syntax-Trans.update-name-tr'
```

```
f)
          (Abs (x, dummyT, Syntax-Trans.const-abs-tr' k) :: ts)
    | annassign-tr' - = raise Match;

  fun Parallel-PAR [(Const (const-syntax ‹Cons›, -) $
          (Const (const-syntax ‹Pair›, -) $ (Const (const-syntax ‹Some›,-) $ t1
) $ t2) $
            Const (const-syntax ‹Nil›, -))] = Syntax.const syntax-const ‹-prg› $
t1 $ t2
    | Parallel-PAR [(Const (const-syntax ‹Cons›, -) $
          (Const (const-syntax ‹Pair›, -) $ (Const (const-syntax ‹Some›, -) $
t1) $ t2) $ ts)] =
          Syntax.const syntax-const ‹-prgs› $ t1 $ t2 $ Parallel-PAR [ts]
    | Parallel-PAR - = raise Match;

  fun Parallel-tr' ts = Syntax.const syntax-const ‹-PAR› $ Parallel-PAR ts;
 in
  [(const-syntax ‹Collect›, K assert-tr'),
   (const-syntax ‹Basic›, K assign-tr'),
   (const-syntax ‹Cond›, K (bexp-tr' syntax-const ‹-Cond›)),
   (const-syntax ‹While›, K (bexp-tr' syntax-const ‹-While-inv›)),
   (const-syntax ‹AnnBasic›, K annassign-tr'),
   (const-syntax ‹AnnWhile›, K (annbexp-tr' syntax-const ‹-AnnWhile›)),
   (const-syntax ‹AnnAwait›, K (annbexp-tr' syntax-const ‹-AnnAwait›)),
   (const-syntax ‹AnnCond1›, K (annbexp-tr' syntax-const ‹-AnnCond1›)),
   (const-syntax ‹AnnCond2›, K (annbexp-tr' syntax-const ‹-AnnCond2›))]
 end
›
```

end

1.8 Examples

theory *OG-Examples* **imports** *OG-Syntax* **begin**

1.8.1 Mutual Exclusion

Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

```
record Petersons-mutex-1 =
 pr1 :: nat
 pr2 :: nat
 in1 :: bool
 in2 :: bool
 hold :: nat

lemma Petersons-mutex-1:
  ∥− {´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 }
```

```
  COBEGIN ⦃´pr1=0 ∧ ¬´in1⦄
  WHILE True INV ⦃´pr1=0 ∧ ¬´in1⦄
  DO
  ⦃´pr1=0 ∧ ¬´in1⦄ ⟨ ´in1:=True,,´pr1:=1 ⟩;;
  ⦃´pr1=1 ∧ ´in1⦄  ⟨ ´hold:=1,,´pr1:=2 ⟩;;
  ⦃´pr1=2 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)⦄
  AWAIT (¬´in2 ∨ ¬(´hold=1)) THEN ´pr1:=3 END;;
  ⦃´pr1=3 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)⦄
   ⟨´in1:=False,,´pr1:=0⟩
  OD ⦃´pr1=0 ∧ ¬´in1⦄
  ‖
  ⦃´pr2=0 ∧ ¬´in2⦄
  WHILE True INV ⦃´pr2=0 ∧ ¬´in2⦄
  DO
  ⦃´pr2=0 ∧ ¬´in2⦄ ⟨ ´in2:=True,,´pr2:=1 ⟩;;
  ⦃´pr2=1 ∧ ´in2⦄ ⟨  ´hold:=2,,´pr2:=2 ⟩;;
  ⦃´pr2=2 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))⦄
  AWAIT (¬´in1 ∨ ¬(´hold=2)) THEN ´pr2:=3  END;;
  ⦃´pr2=3 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))⦄
    ⟨´in2:=False,,´pr2:=0⟩
  OD ⦃´pr2=0 ∧ ¬´in2⦄
  COEND
  ⦃´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2⦄
```
apply *oghoare*
— 104 verification conditions.
apply *auto*
done

Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

record *Busy-wait-mutex* =
```
 flag1 :: bool
 flag2 :: bool
 turn  :: nat
 after1 :: bool
 after2 :: bool
```

lemma *Busy-wait-mutex*:
```
 ‖−  ⦃True⦄
  ´flag1:=False,, ´flag2:=False,,
  COBEGIN ⦃¬´flag1⦄
        WHILE True
        INV ⦃¬´flag1⦄
        DO ⦃¬´flag1⦄ ⟨ ´flag1:=True,,´after1:=False ⟩;;
           ⦃´flag1 ∧ ¬´after1⦄ ⟨ ´turn:=1,,´after1:=True ⟩;;
           ⦃´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)⦄
            WHILE ¬(´flag2 ⟶ ´turn=2)
```

```
        INV {|´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)|}
        DO {|´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)|} SKIP OD;;
       {|´flag1 ∧ ´after1 ∧ (´flag2 ∧ ´after2 ⟶ ´turn=2)|}
        ´flag1:=False
     OD
    {|False|}
 ‖
   {|¬´flag2|}
     WHILE True
     INV {|¬´flag2|}
     DO {|¬´flag2|} ⟨ ´flag2:=True,,´after2:=False ⟩;;
       {|´flag2 ∧ ¬´after2|} ⟨ ´turn:=2,,´after2:=True ⟩;;
       {|´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)|}
        WHILE ¬(´flag1 ⟶ ´turn=1)
        INV {|´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)|}
        DO {|´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)|} SKIP OD;;
       {|´flag2 ∧ ´after2 ∧ (´flag1 ∧ ´after1 ⟶ ´turn=1)|}
        ´flag2:=False
     OD
    {|False|}
 COEND
 {|False|}
apply oghoare
— 122 vc
apply auto
done
```

Peterson's Algorithm III: A Solution using Semaphores

```
record Semaphores-mutex =
 out :: bool
 who :: nat

lemma Semaphores-mutex:
 ‖- {|i≠j|}
  ´out:=True ,,
  COBEGIN {|i≠j|}
      WHILE True INV {|i≠j|}
      DO {|i≠j|} AWAIT ´out THEN ´out:=False,, ´who:=i END;;
         {|¬´out ∧ ´who=i ∧ i≠j|} ´out:=True OD
      {|False|}
  ‖
      {|i≠j|}
      WHILE True INV {|i≠j|}
      DO {|i≠j|} AWAIT ´out THEN ´out:=False,,´who:=j END;;
         {|¬´out ∧ ´who=j ∧ i≠j|} ´out:=True OD
      {|False|}
  COEND
  {|False|}
```

apply *oghoare*
— 38 vc
apply *auto*
done

Peterson's Algorithm III: Parameterized version:

lemma *Semaphores-parameterized-mutex*:
 0<n ⟹ ∥− ⦃*True*⦄
  ´*out*:=*True* ,,
 COBEGIN
  SCHEME [*0*≤ *i*< *n*]
    ⦃*True*⦄
     WHILE True INV ⦃*True*⦄
      DO ⦃*True*⦄ *AWAIT* ´*out THEN* ´*out*:=*False*,, ´*who*:=*i END*;;
         ⦃¬´*out* ∧ ´*who*=*i*⦄ ´*out*:=*True OD*
    ⦃*False*⦄
 COEND
  ⦃*False*⦄
apply *oghoare*
— 20 vc
apply *auto*
done

The Ticket Algorithm

record *Ticket-mutex* =
 num :: *nat*
 nextv :: *nat*
 turn :: *nat list*
 index :: *nat*

lemma *Ticket-mutex*:
⟦ *0<n*; *I*=«*n*=*length* ´*turn* ∧ *0*<´*nextv* ∧ (∀ *k l*. *k*<*n* ∧ *l*<*n* ∧ *k*≠*l*
    ⟶ ´*turn*!*k* < ´*num* ∧ (´*turn*!*k* =*0* ∨ ´*turn*!*k*≠´*turn*!*l*))» ⟧
   ⟹ ∥− ⦃*n*=*length* ´*turn*⦄
   ´*index*:= *0*,,
   WHILE ´*index* < *n INV* ⦃*n*=*length* ´*turn* ∧ (∀ *i*<´*index*. ´*turn*!*i*=*0*)⦄
    DO ´*turn*:= ´*turn*[´*index*:=*0*],, ´*index*:=´*index* +*1 OD*,,
  ´*num*:=*1* ,, ´*nextv*:=*1* ,,
 COBEGIN
  SCHEME [*0*≤ *i*< *n*]
    ⦃´*I*⦄
     WHILE True INV ⦃´*I*⦄
      DO ⦃´*I*⦄ ⟨ ´*turn* :=´*turn*[*i*:=´*num*],, ´*num*:=´*num*+*1* ⟩;;
         ⦃´*I*⦄ *WAIT* ´*turn*!*i*=´*nextv END*;;
         ⦃´*I* ∧ ´*turn*!*i*=´*nextv*⦄ ´*nextv*:=´*nextv*+*1*
      OD
    ⦃*False*⦄
 COEND

{|*False*|}
apply *oghoare*
— 35 vc
apply *simp-all*
— 16 vc
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
— 11 vc
apply *simp-all*
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
— 10 subgoals left
apply(*erule less-SucE*)
 apply *simp*
apply *simp*
— 9 subgoals left
apply(*case-tac i=k*)
 apply *force*
apply *simp*
apply(*case-tac i=l*)
 apply *force*
apply *force*
— 8 subgoals left
prefer *8*
apply *force*
apply *force*
— 6 subgoals left
prefer *6*
apply(*erule-tac x=j* **in** *allE*)
apply *fastforce*
— 5 subgoals left
prefer *5*
apply(*case-tac* [!] *j=k*)
— 10 subgoals left
apply *simp-all*
apply(*erule-tac x=k* **in** *allE*)
apply *force*
— 9 subgoals left
apply(*case-tac j=l*)
 apply *simp*
 apply(*erule-tac x=k* **in** *allE*)
 apply(*erule-tac x=k* **in** *allE*)
 apply(*erule-tac x=l* **in** *allE*)
 apply *force*
apply(*erule-tac x=k* **in** *allE*)
apply(*erule-tac x=k* **in** *allE*)
apply(*erule-tac x=l* **in** *allE*)
apply *force*
— 8 subgoals left
apply *force*
apply(*case-tac j=l*)

```
 apply simp
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply force
apply force
apply force
— 5 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
apply force
— 3 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
apply force
— 1 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
done
```

1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

Apt and Olderog. "Verification of sequential and concurrent Programs" page 294:

```
record Zero-search =
  turn :: nat
  found :: bool
  x :: nat
  y :: nat

lemma Zero-search:
  ⟦I1= « a≤´x ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
      ∧ (¬´found ∧ a<´ x ⟶ f(´x)≠0) » ;
    I2= «´y≤a+1 ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
      ∧ (¬´found ∧ ´y≤a ⟶ f(´y)≠0) » ⟧ ⟹
  ∥− {|∃ u. f(u)=0|}
  ´turn:=1,, ´found:= False,,
  ´x:=a,, ´y:=a+1 ,,
  COBEGIN {|´I1|}
```

```
        WHILE ¬´found
        INV {|´I1|}
        DO {|a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)|}
           WAIT ´turn=1 END;;
           {|a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)|}
           ´turn:=2;;
           {|a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)|}
           ⟨ ´x:=´x+1,,
             IF f(´x)=0 THEN ´found:=True ELSE SKIP FI⟩
        OD;;
        {|´I1 ∧ ´found|}
        ´turn:=2
        {|´I1 ∧ ´found|}
  ‖
       {|´I2|}
        WHILE ¬´found
        INV {|´I2|}
        DO {|´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)|}
           WAIT ´turn=2 END;;
           {|´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)|}
           ´turn:=1;;
           {|´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)|}
           ⟨ ´y:=(´y − 1),,
             IF f(´y)=0 THEN ´found:=True ELSE SKIP FI⟩
        OD;;
        {|´I2 ∧ ´found|}
        ´turn:=1
        {|´I2 ∧ ´found|}
  COEND
  {|f(´x)=0 ∨ f(´y)=0|}
```

apply *oghoare*

— 98 verification conditions

apply *auto*

— auto takes about 3 minutes !!

done

Easier Version: without AWAIT. Apt and Olderog. page 256:

lemma *Zero-Search-2*:

```
⟦I1=« a≤´x ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
    ∧ (¬´found ∧ a<´x ⟶ f(´x)≠0)»;
 I2= «´y≤a+1 ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
    ∧ (¬´found ∧ ´y≤a ⟶ f(´y)≠0)»⟧ ⟹
  ‖− {|∃ u. f(u)=0|}
  ´found:= False,,
  ´x:=a,, ´y:=a+1,,
  COBEGIN {|´I1|}
        WHILE ¬´found
        INV {|´I1|}
        DO {|a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)|}
```

$\langle$ ´x:=´x+1,,IF f(´x)=0 THEN ´found:=True ELSE SKIP FI$\rangle$
OD
{|´I1 ∧ ´found|}
‖
{|´I2|}
WHILE ¬´found
INV {|´I2|}
DO {|´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)|}
$\langle$ ´y:=(´y − 1),,IF f(´y)=0 THEN ´found:=True ELSE SKIP FI$\rangle$
OD
{|´I2 ∧ ´found|}
COEND
{|f(´x)=0 ∨ f(´y)=0|}
apply *oghoare*
— 20 vc
apply *auto*
— auto takes aprox. 2 minutes.
done

1.8.3 Producer/Consumer

Previous lemmas

lemma *nat-lemma2*: ⟦ b = m∗(n::nat) + t; a = s∗n + u; t=u; b−a < n ⟧ ⟹ m ≤ s
proof −
assume b = m∗(n::nat) + t a = s∗n + u t=u
hence (m − s) ∗ n = b − a **by** (*simp add*: *diff-mult-distrib*)
also assume . . . < n
finally have m − s < 1 **by** *simp*
thus *?thesis* **by** *arith*
qed

lemma *mod-lemma*: ⟦ (c::nat) ≤ a; a < b; b − c < n ⟧ ⟹ b mod n ≠ a mod n
apply(*subgoal-tac* b=b div n∗n + b mod n)
prefer *2* **apply** (*simp add*: *div-mult-mod-eq* [*symmetric*])
apply(*subgoal-tac* a=a div n∗n + a mod n)
prefer *2*
apply(*simp add*: *div-mult-mod-eq* [*symmetric*])
apply(*subgoal-tac* b − a ≤ b − c)
prefer *2* **apply** *arith*
apply(*drule le-less-trans*)
back
apply *assumption*
apply(*frule less-not-refl2*)
apply(*drule less-imp-le*)
apply (*drule-tac* m = a **and** k = n **in** *div-le-mono*)
apply(*safe*)
apply(*frule-tac* b = b **and** a = a **and** n = n **in** *nat-lemma2*, *assumption*, *assumption*)

apply *assumption*
apply(*drule order-antisym*, *assumption*)
apply(*rotate-tac* −*3*)
apply(*simp*)
done

Producer/Consumer Algorithm

record *Producer-consumer* =
  ins :: *nat*
  outs :: *nat*
  li :: *nat*
  lj :: *nat*
  vx :: *nat*
  vy :: *nat*
  buffer :: *nat list*
  b :: *nat list*

The whole proof takes aprox. 4 minutes.

lemma *Producer-consumer*:

```
 ⟦INIT= «0<length a ∧ 0<length ´buffer ∧ length ´b=length a» ;
   I= «(∀ k<´ins. ´outs≤k ⟶ (a ! k) = ´buffer ! (k mod (length ´buffer))) ∧
          ´outs≤´ins ∧ ´ins−´outs≤length ´buffer» ;
   I1= «´I ∧ ´li≤length a» ;
   p1= «´I1 ∧ ´li=´ins» ;
   I2 = «´I ∧ (∀ k<´lj. (a ! k)=(´b ! k)) ∧ ´lj≤length a» ;
   p2 = «´I2 ∧ ´lj=´outs» ⟧ ⟹
 ‖− ⦃´INIT⦄
´ins:=0,, ´outs:=0,, ´li:=0,, ´lj:=0,,
 COBEGIN ⦃´p1 ∧ ´INIT⦄
   WHILE ´li <length a
     INV ⦃´p1 ∧ ´INIT⦄
   DO ⦃´p1 ∧ ´INIT ∧ ´li<length a⦄
       ´vx:= (a ! ´li);;
     ⦃´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)⦄
        WAIT ´ins−´outs < length ´buffer END;;
     ⦃´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)
        ∧ ´ins−´outs < length ´buffer⦄
       ´buffer:=(list-update ´buffer (´ins mod (length ´buffer)) ´vx);;
     ⦃´p1 ∧ ´INIT ∧ ´li<length a
        ∧ (a ! ´li)=(´buffer ! (´ins mod (length ´buffer)))
        ∧ ´ins−´outs <length ´buffer⦄
       ´ins:=´ins+1;;
     ⦃´I1 ∧ ´INIT ∧ (´li+1)=´ins ∧ ´li<length a⦄
       ´li:=´li+1
   OD
  ⦃´p1 ∧ ´INIT ∧ ´li=length a⦄
  ‖
  ⦃´p2 ∧ ´INIT⦄
```

WHILE ´lj < length a
  INV ⦃*´p2* ∧ *´INIT*⦄
 DO ⦃*´p2* ∧ *´lj<length a* ∧ *´INIT*⦄
     WAIT ´outs<´ins END;;
   ⦃*´p2* ∧ *´lj<length a* ∧ *´outs<´ins* ∧ *´INIT*⦄
    ´vy:=(*´buffer* ! (*´outs mod* (*length ´buffer*)));;
   ⦃*´p2* ∧ *´lj<length a* ∧ *´outs<´ins* ∧ *´vy*=(*a* ! *´lj*) ∧ *´INIT*⦄
    ´outs:=*´outs+1*;;
   ⦃*´I2* ∧ (*´lj+1*)=*´outs* ∧ *´lj<length a* ∧ *´vy*=(*a* ! *´lj*) ∧ *´INIT*⦄
    ´b:=(*list-update ´b ´lj ´vy*);;
   ⦃*´I2* ∧ (*´lj+1*)=*´outs* ∧ *´lj<length a* ∧ (*a* ! *´lj*)=(*´b* ! *´lj*) ∧ *´INIT*⦄
    ´lj:=*´lj+1*
 OD
⦃*´p2* ∧ *´lj=length a* ∧ *´INIT*⦄
COEND
⦃ ∀ *k<length a*. (*a* ! *k*)=(*´b* ! *k*)⦄
apply *oghoare*
— 138 vc
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
— 112 subgoals left
apply(*simp-all* (*no-asm*))
— 43 subgoals left
apply(*tactic* ‹*ALLGOALS* (*conjI-Tac* ***context*** (*K all-tac*))›)
— 419 subgoals left
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
— 99 subgoals left
apply(*simp-all only*:*length-0-conv* [*THEN sym*])
— 20 subgoals left
apply (*simp-all del*:*length-0-conv length-greater-0-conv add*: *nth-list-update mod-lemma*)
— 9 subgoals left
apply (*force simp add*:*less-Suc-eq*)
apply(*hypsubst-thin*, *drule sym*)
apply (*force simp add*:*less-Suc-eq*)+
done

1.8.4 Parameterized Examples

Set Elements of an Array to Zero

record *Example1* =
 a :: *nat* ⇒ *nat*

lemma *Example1*:
‖− ⦃*True*⦄
 COBEGIN SCHEME [*0≤i<n*] ⦃*True*⦄ *´a*:=*´a* (*i*:=*0*) ⦃*´a i=0*⦄ *COEND*
⦃∀ *i* < *n*. *´a i* = *0*⦄
apply *oghoare*
apply *simp-all*
done

Same example with lists as auxiliary variables.

```
record Example1-list =
  A :: nat list
lemma Example1-list:
 ||− {|n < length ´A|}
   COBEGIN
     SCHEME [0≤i<n] {|n < length ´A|} ´A:=´A[i:=0] {|´A!i=0|}
   COEND
    {|∀ i < n. ´A!i = 0|}
apply oghoare
apply force+
done
```

Increment a Variable in Parallel

First some lemmas about summation properties.

```
lemma Example2-lemma2-aux: !!b. j<n ⟹
 (∑ i=0..<n. (b i::nat)) =
 (∑ i=0..<j. b i) + b j + (∑ i=0..<n−(Suc j) . b (Suc j + i))
apply(induct n)
 apply simp-all
apply(simp add:less-Suc-eq)
 apply(auto)
apply(subgoal-tac n − j = Suc(n− Suc j))
  apply simp
apply arith
done

lemma Example2-lemma2-aux2:
  !!b. j≤ s ⟹ (∑ i::nat=0..<j. (b (s:=t)) i) = (∑ i=0..<j. b i)
apply(induct j)
 apply simp-all
done

lemma Example2-lemma2:
 !!b. [[j<n; b j=0]] ⟹ Suc (∑ i::nat=0..<n. b i)=(∑ i=0..<n. (b (j := Suc 0))
i)
apply(frule-tac b=(b (j:=(Suc 0))) in Example2-lemma2-aux)
apply(erule-tac  t=sum (b(j := (Suc 0))) {0..<n} in ssubst)
apply(frule-tac b=b in Example2-lemma2-aux)
apply(erule-tac  t=sum b {0..<n} in ssubst)
apply(subgoal-tac Suc (sum b {0..<j} + b j + (∑ i=0..<n − Suc j. b (Suc j +
i)))=(sum b {0..<j} + Suc (b j) + (∑ i=0..<n − Suc j. b (Suc j + i))))
apply(rotate-tac −1)
apply(erule ssubst)
apply(subgoal-tac j≤j)
 apply(drule-tac b=b and t=(Suc 0) in Example2-lemma2-aux2)
apply(rotate-tac −1)
```

```
apply(erule ssubst)
apply simp-all
done


record Example2 =
 c :: nat ⇒ nat
 x :: nat

lemma Example-2: 0<n ⟹
 ∥− {|´x=0 ∧ (∑ i=0..<n. ´c i)=0|}
 COBEGIN
   SCHEME [0≤i<n]
  {|´x=(∑ i=0..<n. ´c i) ∧ ´c i=0|}
   ⟨ ´x:=´x+(Suc 0),, ´c:=´c (i:=(Suc 0)) ⟩
  {|´x=(∑ i=0..<n. ´c i) ∧ ´c i=(Suc 0)|}
 COEND
 {|´x=n|}
apply oghoare
apply (simp-all cong del: sum.cong-simp)
apply (tactic ‹ALLGOALS (clarify-tac context)›)
apply (simp-all cong del: sum.cong-simp)
   apply(erule (1) Example2-lemma2)
  apply(erule (1) Example2-lemma2)
 apply(erule (1) Example2-lemma2)
apply(simp)
done

end
```

Chapter 2

Case Study: Single and Multi-Mutator Garbage Collection Algorithms

2.1 Formalization of the Memory

theory *Graph* **imports** *Main* **begin**

datatype *node* = *Black* | *White*

type-synonym *nodes* = *node list*
type-synonym *edge* = *nat* × *nat*
type-synonym *edges* = *edge list*

consts *Roots* :: *nat set*

definition *Proper-Roots* :: *nodes* ⇒ *bool* **where**
Proper-Roots M ≡ *Roots*≠{} ∧ *Roots* ⊆ {*i*. *i*<*length M*}

definition *Proper-Edges* :: (*nodes* × *edges*) ⇒ *bool* **where**
Proper-Edges ≡ (λ(*M*,*E*). ∀ *i*<*length E*. *fst*(*E*!*i*)<*length M* ∧ *snd*(*E*!*i*)<*length M*)

definition *BtoW* :: (*edge* × *nodes*) ⇒ *bool* **where**
BtoW ≡ (λ(*e*,*M*). (*M*!*fst e*)=*Black* ∧ (*M*!*snd e*)≠*Black*)

definition *Blacks* :: *nodes* ⇒ *nat set* **where**
Blacks M ≡ {*i*. *i*<*length M* ∧ *M*!*i*=*Black*}

definition *Reach* :: *edges* ⇒ *nat set* **where**
Reach E ≡ {*x*. (∃ *path*. *1*<*length path* ∧ *path*!(*length path* − *1*)∈*Roots* ∧ *x*=*path*!*0*
∧ (∀ *i*<*length path* − *1*. (∃ *j*<*length E*. *E*!*j*=(*path*!(*i*+*1*), *path*!*i*))))
∨ *x*∈*Roots*}

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

2.1.1 Proofs about Graphs

lemmas *Graph-defs= Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def*
declare *Graph-defs* [*simp*]

Graph 1

```
lemma Graph1-aux [rule-format]:
  ⟦ Roots⊆Blacks M; ∀ i<length E. ¬BtoW(E!i,M)⟧
  ⟹ 1< length path ⟶ (path!(length path − 1))∈Roots ⟶
  (∀ i<length path − 1. (∃ j. j < length E ∧ E!j=(path!(Suc i), path!i)))
  ⟶ M!(path!0) = Black
apply(induct-tac path)
 apply force
apply clarify
apply simp
apply(case-tac list)
 apply force
apply simp
apply(rename-tac lista)
apply(rotate-tac −2)
apply(erule-tac x = 0 in all-dupE)
apply simp
apply clarify
apply(erule allE , erule (1) notE impE)
apply simp
apply(erule mp)
apply(case-tac lista)
 apply force
apply simp
apply(erule mp)
apply clarify
apply(erule-tac x = Suc i in allE)
apply force
done

lemma Graph1:
  ⟦Roots⊆Blacks M; Proper-Edges(M, E); ∀ i<length E. ¬BtoW(E!i,M) ⟧
  ⟹ Reach E⊆Blacks M
apply (unfold Reach-def)
apply simp
apply clarify
apply(erule disjE)
 apply clarify
```

```
 apply(rule conjI)
  apply(subgoal-tac 0< length path - Suc 0)
   apply(erule allE , erule (1) notE impE)
   apply force
  apply simp
 apply(rule Graph1-aux)
apply auto
done
```

Graph 2

```
lemma Ex-first-occurrence [rule-format]:
  P (n::nat) ⟶ (∃ m. P m ∧ (∀ i. i<m ⟶ ¬ P i))
apply(rule nat-less-induct)
apply clarify
apply(case-tac ∀ m. m<n ⟶ ¬ P m)
apply auto
done

lemma Compl-lemma: (n::nat)≤l ⟹ (∃ m. m≤l ∧ n=l - m)
apply(rule-tac x = l - n in exI)
apply arith
done

lemma Ex-last-occurrence:
  ⟦P (n::nat); n≤l⟧ ⟹ (∃ m. P (l - m) ∧ (∀ i. i<m ⟶ ¬P (l - i)))
apply(drule Compl-lemma)
apply clarify
apply(erule Ex-first-occurrence)
done

lemma Graph2:
  ⟦T ∈ Reach E; R<length E⟧ ⟹ T ∈ Reach (E[R:=(fst(E!R), T)])
apply (unfold Reach-def)
apply clarify
apply simp
apply(case-tac ∀ z<length path. fst(E!R)≠path!z)
 apply(rule-tac x = path in exI)
 apply simp
 apply clarify
 apply(erule allE , erule (1) notE impE)
 apply clarify
 apply(rule-tac x = j in exI)
 apply(case-tac j=R)
  apply(erule-tac x = Suc i in allE)
  apply simp
 apply (force simp add:nth-list-update)
apply simp
apply(erule exE)
```

apply(*subgoal-tac* $z \leq$ *length path* $-$ *Suc 0*)
prefer *2* **apply** *arith*
apply(*drule-tac* $P = \lambda m.$ *m<length path* $\wedge$ *fst*(*E*!*R*)=*path*!*m* **in** *Ex-last-occurrence*)
apply *assumption*
apply *clarify*
apply *simp*
apply(*rule-tac* $x =$ (*path*!*0*)#(*drop* (*length path* $-$ *Suc m*) *path*) **in** *exI*)
apply *simp*
apply(*case-tac* *length path* $-$ (*length path* $-$ *Suc m*))
apply *arith*
apply *simp*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + *nat* $\leq$ *length path*)
prefer *2* **apply** *arith*
apply(*subgoal-tac* *length path* $-$ *Suc m* + *nat* = *length path* $-$ *Suc 0*)
prefer *2* **apply** *arith*
apply *clarify*
apply(*case-tac* *i*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + *nata* $\leq$ *length path*)
prefer *2* **apply** *arith*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + (*Suc nata*) $\leq$ *length path*)
prefer *2* **apply** *arith*
apply *simp*
apply(*erule-tac* $x =$ *length path* $-$ *Suc m* + *nata* **in** *allE*)
apply *simp*
apply *clarify*
apply(*rule-tac* $x = j$ **in** *exI*)
apply(*case-tac* *R*=*j*)
prefer *2* **apply** *force*
apply *simp*
apply(*drule-tac* $t =$ *path* ! (*length path* $-$ *Suc m*) **in** *sym*)
apply *simp*
apply(*case-tac* *length path* $-$ *Suc 0* < *m*)
apply(*subgoal-tac* (*length path* $-$ *Suc m*)=*0*)
prefer *2* **apply** *arith*
apply(*simp del*: *diff-is-0-eq*)
apply(*subgoal-tac* *Suc nata*$\leq$*nat*)
prefer *2* **apply** *arith*
apply(*drule-tac* $n =$ *Suc nata* **in** *Compl-lemma*)
apply *clarify*
subgoal using [[*linarith-split-limit* = *0*]] **by** *force*
apply(*drule leI*)
apply(*subgoal-tac* *Suc* (*length path* $-$ *Suc m* + *nata*)=(*length path* $-$ *Suc 0*) $-$ (*m* $-$ *Suc nata*))
apply(*erule-tac* $x = m -$ (*Suc nata*) **in** *allE*)
apply(*case-tac* *m*)
apply *simp*
apply *simp*

apply *simp*
done

Graph 3

declare *min.absorb1* [*simp*] *min.absorb2* [*simp*]

lemma *Graph3*:
  $[\![T{\in}Reach\ E;\ R{<}length\ E]\!] \Longrightarrow Reach(E[R{:=}(fst(E!R),T)]) \subseteq Reach\ E$
apply (*unfold Reach-def*)
apply *clarify*
apply *simp*
apply(*case-tac* $\exists\, i{<}length\ path - 1.\ (fst(E!R),T){=}(path!(Suc\ i),path!i)$)
— the changed edge is part of the path
 apply(*erule exE*)
 apply(*drule-tac* $P = \lambda i.\ i{<}length\ path - 1 \wedge (fst(E!R),T){=}(path!Suc\ i,path!i)$
in *Ex-first-occurrence*)
 apply *clarify*
 apply(*erule disjE*)
— T is NOT a root
  apply *clarify*
  apply(*rule-tac* $x = (take\ m\ path)@patha$ **in** *exI*)
  apply(*subgoal-tac* $\neg(length\ path{\le}m)$)
   prefer *2* **apply** *arith*
  apply(*simp*)
  apply(*rule conjI*)
   apply(*subgoal-tac* $\neg(m + length\ patha - 1 < m)$)
    prefer *2* **apply** *arith*
   apply(*simp add*: *nth-append*)
  apply(*rule conjI*)
   apply(*case-tac m*)
    apply *force*
   apply(*case-tac path*)
    apply *force*
   apply *force*
  apply *clarify*
  apply(*case-tac* $Suc\ i{\le}m$)
   apply(*erule-tac* $x = i$ **in** *allE*)
   apply *simp*
   apply *clarify*
   apply(*rule-tac* $x = j$ **in** *exI*)
   apply(*case-tac* $Suc\ i{<}m$)
    apply(*simp add*: *nth-append*)
    apply(*case-tac* $R{=}j$)
     apply(*simp add*: *nth-list-update*)
     apply(*case-tac* $i{=}m$)
      apply *force*
     apply(*erule-tac* $x = i$ **in** *allE*)
     apply *force*

```
apply(force simp add: nth-list-update)
apply(simp add: nth-append)
apply(subgoal-tac i=m − 1)
prefer 2 apply arith
apply(case-tac R=j)
apply(erule-tac x = m − 1 in allE)
apply(simp add: nth-list-update)
apply(force simp add: nth-list-update)
apply(simp add: nth-append)
apply(rotate-tac −4)
apply(erule-tac x = i − m in allE)
apply(subgoal-tac Suc (i − m)=(Suc i − m) )
prefer 2 apply arith
apply simp
— T is a root
apply(case-tac m=0)
apply force
apply(rule-tac x = take (Suc m) path in exI)
apply(subgoal-tac ¬(length path≤Suc m) )
prefer 2 apply arith
apply clarsimp
apply(erule-tac x = i in allE)
apply simp
apply clarify
apply(case-tac R=j)
apply(force simp add: nth-list-update)
apply(force simp add: nth-list-update)
— the changed edge is not part of the path
apply(rule-tac x = path in exI)
apply simp
apply clarify
apply(erule-tac x = i in allE)
apply clarify
apply(case-tac R=j)
apply(erule-tac x = i in allE)
apply simp
apply(force simp add: nth-list-update)
done
```

Graph 4

```
lemma Graph4:
  ⟦T ∈ Reach E; Roots⊆Blacks M; I≤length E; T<length M; R<length E;
  ∀ i<I. ¬BtoW(E!i,M); R<I; M!fst(E!R)=Black; M!T≠Black⟧ ⟹
  (∃ r. I≤r ∧ r<length E ∧ BtoW(E[R:=(fst(E!R),T)]!r,M))
apply (unfold Reach-def)
apply simp
apply(erule disjE)
prefer 2 apply force
```

apply *clarify*
— there exist a black node in the path to T
apply(*case-tac* $\exists m<length\ path.\ M!(path!m)=Black$)
apply(*erule exE*)
apply(*drule-tac* $P = \lambda m.\ m<length\ path \wedge M!(path!m)=Black$ **in** *Ex-first-occurrence*)
apply *clarify*
apply(*case-tac ma*)
apply *force*
apply *simp*
apply(*case-tac length path*)
apply *force*
apply *simp*
apply(*erule-tac* $P = \lambda i.\ i < nata \longrightarrow P\ i$ **and** $x = nat$ **for** P **in** *allE*)
apply *simp*
apply *clarify*
apply(*erule-tac* $P = \lambda i.\ i < Suc\ nat \longrightarrow P\ i$ **and** $x = nat$ **for** P **in** *allE*)
apply *simp*
apply(*case-tac* $j<I$)
apply(*erule-tac* $x = j$ **in** *allE*)
apply *force*
apply(*rule-tac* $x = j$ **in** *exI*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*rotate-tac* -1)
apply(*erule-tac* $x = length\ path - 1$ **in** *allE*)
apply(*case-tac length path*)
apply *force*
apply *force*
done

declare *min.absorb1* [*simp del*] *min.absorb2* [*simp del*]

Graph 5

lemma *Graph5*:
$[\![\ T \in Reach\ E\ ;\ Roots \subseteq Blacks\ M;\ \forall i<R.\ \neg BtoW(E!i,M);\ T<length\ M;$
$R<length\ E;\ M!fst(E!R)=Black;\ M!snd(E!R)=Black;\ M!T \neq Black]\!]$
$\Longrightarrow (\exists r.\ R<r \wedge r<length\ E \wedge BtoW(E[R:=(fst(E!R),T)]!r,M))$
apply (*unfold Reach-def*)
apply *simp*
apply(*erule disjE*)
prefer *2* **apply** *force*
apply *clarify*
— there exist a black node in the path to T
apply(*case-tac* $\exists m<length\ path.\ M!(path!m)=Black$)
apply(*erule exE*)
apply(*drule-tac* $P = \lambda m.\ m<length\ path \wedge M!(path!m)=Black$ **in** *Ex-first-occurrence*)
apply *clarify*
apply(*case-tac ma*)

apply *force*
apply *simp*
apply(*case-tac length path*)
apply *force*
apply *simp*
apply(*erule-tac* $P = \lambda i.\ i < nata \longrightarrow P\ i$ **and** $x = nat$ **for** P **in** *allE*)
apply *simp*
apply *clarify*
apply(*erule-tac* $P = \lambda i.\ i < Suc\ nat \longrightarrow P\ i$ **and** $x = nat$ **for** P **in** *allE*)
apply *simp*
apply(*case-tac* $j \leq R$)
apply(*drule le-imp-less-or-eq* [*of* - *R*])
apply(*erule disjE*)
apply(*erule allE* , *erule* (*1*) *notE impE*)
apply *force*
apply *force*
apply(*rule-tac* $x = j$ **in** *exI*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*rotate-tac* -1)
apply(*erule-tac* $x = length\ path - 1$ **in** *allE*)
apply(*case-tac length path*)
apply *force*
apply *force*
done

Other lemmas about graphs

lemma *Graph6*:
$[\![Proper\text{-}Edges(M,E);\ R<length\ E\ ;\ T<length\ M]\!] \Longrightarrow Proper\text{-}Edges(M,E[R:=(fst(E!R),T)])$
apply (*unfold Proper-Edges-def*)
apply(*force simp add*: *nth-list-update*)
done

lemma *Graph7*:
$[\![Proper\text{-}Edges(M,E)]\!] \Longrightarrow Proper\text{-}Edges(M[T:=a],E)$
apply (*unfold Proper-Edges-def*)
apply *force*
done

lemma *Graph8*:
$[\![Proper\text{-}Roots(M)]\!] \Longrightarrow Proper\text{-}Roots(M[T:=a])$
apply (*unfold Proper-Roots-def*)
apply *force*
done

Some specific lemmata for the verification of garbage collection algorithms.

lemma *Graph9*: $j<length\ M \Longrightarrow Blacks\ M \subseteq Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)

apply(*force simp add*: *nth-list-update*)
done

lemma *Graph10* [*rule-format* (*no-asm*)]: $\forall\, i.\ M!i{=}a \longrightarrow M[i{:=}a]{=}M$
apply(*induct-tac M*)
apply *auto*
apply(*case-tac i*)
apply *auto*
done

lemma *Graph11* [*rule-format* (*no-asm*)]:
$[\![\ M!j{\neq}Black;j{<}length\ M]\!] \Longrightarrow Blacks\ M \subset Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*rule psubsetI*)
apply(*force simp add*: *nth-list-update*)
apply *safe*
apply(*erule-tac* $c = j$ **in** *equalityCE*)
apply *auto*
done

lemma *Graph12*: $[\![a{\subseteq}Blacks\ M;j{<}length\ M]\!] \Longrightarrow a{\subseteq}Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*force simp add*: *nth-list-update*)
done

lemma *Graph13*: $[\![a{\subset}\ Blacks\ M;j{<}length\ M]\!] \Longrightarrow a \subset Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*erule psubset-subset-trans*)
apply(*force simp add*: *nth-list-update*)
done

declare *Graph-defs* [*simp del*]

end

2.2 The Single Mutator Case

theory *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

declare *psubsetE* [*rule del*]

Declaration of variables:

record *gar-coll-state* =
M :: *nodes*
E :: *edges*
bc :: *nat set*
obc :: *nat set*
Ma :: *nodes*
ind :: *nat*

k :: *nat*
z :: *bool*

2.2.1 The Mutator

The mutator first redirects an arbitrary edge R from an arbitrary accessible node towards an arbitrary accessible node T. It then colors the new target T black.

We declare the arbitrarily selected node and edge as constants:

consts R :: *nat* T :: *nat*

The following predicate states, given a list of nodes m and a list of edges e, the conditions under which the selected edge R and node T are valid:

definition *Mut-init* :: *gar-coll-state* ⇒ *bool* **where**
Mut-init ≡ « T ∈ *Reach* ´*E* ∧ R < *length* ´*E* ∧ T < *length* ´*M* »

For the mutator we consider two modules, one for each action. An auxiliary variable ´*z* is set to false if the mutator has already redirected an edge but has not yet colored the new target.

definition *Redirect-Edge* :: *gar-coll-state ann-com* **where**
Redirect-Edge ≡ {|´*Mut-init* ∧ ´*z*|} ⟨´*E*:=´*E*[*R*:=(*fst*(´*E*!*R*), *T*)],, ´*z*:= (¬´*z*)⟩

definition *Color-Target* :: *gar-coll-state ann-com* **where**
Color-Target ≡ {|´*Mut-init* ∧ ¬´*z*|} ⟨´*M*:=´*M*[*T*:=*Black*],, ´*z*:= (¬´*z*)⟩

definition *Mutator* :: *gar-coll-state ann-com* **where**
Mutator ≡
{|´*Mut-init* ∧ ´*z*|}
WHILE True INV {|´*Mut-init* ∧ ´*z*|}
DO Redirect-Edge ;; *Color-Target OD*

Correctness of the mutator

lemmas *mutator-defs* = *Mut-init-def Redirect-Edge-def Color-Target-def*

lemma *Redirect-Edge*:
⊢ *Redirect-Edge pre*(*Color-Target*)
apply (*unfold mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply(*force elim*:*Graph2*)
done

lemma *Color-Target*:
⊢ *Color-Target* {|´*Mut-init* ∧ ´*z*|}
apply (*unfold mutator-defs*)
apply *annhoare*

apply(*simp-all*)
done

lemma *Mutator*:
⊢ *Mutator* {|*False*|}
apply(*unfold Mutator-def*)
apply *annhoare*
apply(*simp-all add*:*Redirect-Edge Color-Target*)
apply(*simp add*:*mutator-defs*)
done

2.2.2 The Collector

A constant *M-init* is used to give ´*Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *gar-coll-state* ⇒ *bool* **where**
Proper-M-init ≡ « *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* »

definition *Proper* :: *gar-coll-state* ⇒ *bool* **where**
Proper ≡ « *Proper-Roots* ´*M* ∧ *Proper-Edges*(´*M*, ´*E*) ∧ ´*Proper-M-init* »

definition *Safe* :: *gar-coll-state* ⇒ *bool* **where**
Safe ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

lemmas *collector-defs* = *Proper-M-init-def Proper-def Safe-def*

Blackening the roots

definition *Blacken-Roots* :: *gar-coll-state ann-com* **where**
Blacken-Roots ≡
{|´*Proper*|}
´*ind*:=*0*;;
{|´*Proper* ∧ ´*ind*=*0*|}
WHILE ´*ind*<*length* ´*M*
INV {|´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M*|}
DO {|´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
IF ´*ind*∈*Roots THEN*
{|´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧
´*ind*∈*Roots*|}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{|´*Proper* ∧ (∀ *i*<´*ind*+*1*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
´*ind*:=´*ind*+*1*
OD

lemma *Blacken-Roots*:
⊢ *Blacken-Roots* {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M*|}
apply (*unfold Blacken-Roots-def*)

apply *annhoare*
apply(*simp-all add*:*collector-defs Graph-defs*)
apply *safe*
apply(*simp-all add*:*nth-list-update*)
 apply (*erule less-SucE*)
  apply *simp+*
 apply *force*
apply *force*
done

Propagating black

definition *PBInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
 PBInv ≡ « λ*ind*. ´*obc* < *Blacks* ´*M* ∨ (∀ *i* <*ind*. ¬*BtoW* (´*E*!*i*, ´*M*) ∨
 (¬´*z* ∧ *i*=*R* ∧ (*snd*(´*E*!*R*)) = *T* ∧ (∃ *r*. *ind* ≤ *r* ∧ *r* < *length* ´*E* ∧ *BtoW*(´*E*!*r*,´*M*))))»

definition *Propagate-Black-aux* :: *gar-coll-state ann-com* **where**
 Propagate-Black-aux ≡
 {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*|}
 ´*ind*:=*0*;;
 {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M* ∧ ´*ind*=*0*|}
 WHILE ´*ind*<*length* ´*E*
  INV {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ ´*PBInv* ´*ind* ∧ ´*ind*≤*length* ´*E*|}
 DO {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E*|}
  IF ´*M*!(*fst* (´*E*!´*ind*)) = *Black THEN*
   {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
    ∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black*|}
   ´*M*:=´*M*[*snd*(´*E*!´*ind*):=*Black*];;
   {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
    ∧ ´*PBInv* (´*ind* + *1*) ∧ ´*ind*<*length* ´*E*|}
   ´*ind*:=´*ind*+*1*
  FI
 OD

lemma *Propagate-Black-aux*:
 ⊢ *Propagate-Black-aux*
 {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
  ∧ (´*obc* < *Blacks* ´*M* ∨ ´*Safe*)|}
apply (*unfold Propagate-Black-aux-def PBInv-def collector-defs*)
apply *annhoare*
apply(*simp-all add*:*Graph6 Graph7 Graph8 Graph12*)
   apply *force*
  apply *force*
 apply *force*
— 4 subgoals left
apply *clarify*
apply(*simp add*:*Proper-Edges-def Proper-Roots-def Graph6 Graph7 Graph8 Graph12*)

```
apply (erule disjE)
 apply(rule disjI1)
 apply(erule Graph13)
 apply force
apply (case-tac M x ! snd (E x ! ind x)=Black)
 apply (simp add: Graph10 BtoW-def)
 apply (rule disjI2)
 apply clarify
 apply (erule less-SucE)
  apply (erule-tac x=i in allE , erule (1) notE impE)
  apply simp
  apply clarify
  apply (drule-tac y = r in le-imp-less-or-eq)
  apply (erule disjE)
   apply (subgoal-tac Suc (ind x)≤r)
    apply fast
   apply arith
  apply fast
 apply fast
apply(rule disjI1)
apply(erule subset-psubset-trans)
apply(erule Graph11)
apply fast
— 3 subgoals left
apply force
apply force
— last
apply clarify
apply simp
apply(subgoal-tac ind x = length (E x))
 apply (simp)
 apply(drule Graph1)
   apply simp
  apply clarify
  apply(erule allE, erule impE, assumption)
  apply force
 apply force
apply arith
done
```

Refining propagating black

```
definition Auxk :: gar-coll-state ⇒ bool where
  Auxk ≡ «´k<length ´M ∧ (´M!´k≠Black ∨ ¬BtoW(´E!´ind, ´M) ∨
          ´obc<Blacks ´M ∨ (¬´z ∧ ´ind=R ∧ snd(´E!R)=T
          ∧ (∃ r. ´ind<r ∧ r<length ´E ∧ BtoW(´E!r, ´M))))»

definition Propagate-Black :: gar-coll-state ann-com where
  Propagate-Black ≡
```

```
{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M|}
´ind:=0;;
{|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧ ´ind=0|}
WHILE ´ind<length ´E
 INV {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ ´PBInv ´ind ∧ ´ind≤length ´E|}
DO {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ ´PBInv ´ind ∧ ´ind<length ´E|}
 IF (´M!(fst (´E!´ind)))=Black THEN
  {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black|}
   ´k:=(snd(´E!´ind));;
  {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black
    ∧ ´Auxk|}
   ⟨´M:=´M[´k:=Black],, ´ind:=´ind+1⟩
 ELSE {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ ´PBInv ´ind ∧ ´ind<length ´E|}
       ⟨IF (´M!(fst (´E!´ind)))≠Black THEN ´ind:=´ind+1 FI⟩
 FI
OD
```

lemma *Propagate-Black*:

```
 ⊢ Propagate-Black
 {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
   ∧ ( ´obc < Blacks ´M ∨ ´Safe)|}
apply (unfold Propagate-Black-def  PBInv-def Auxk-def collector-defs)
apply annhoare
apply(simp-all add: Graph6 Graph7 Graph8 Graph12)
      apply force
     apply force
    apply force
— 5 subgoals left
apply clarify
apply(simp add:BtoW-def Proper-Edges-def)
— 4 subgoals left
apply clarify
apply(simp add:Proper-Edges-def Graph6 Graph7 Graph8 Graph12)
apply (erule disjE)
 apply (rule disjI1)
 apply (erule psubset-subset-trans)
 apply (erule Graph9)
apply (case-tac M x!k x=Black)
 apply (case-tac M x ! snd (E x ! ind x)=Black)
  apply (simp add: Graph10 BtoW-def)
  apply (rule disjI2)
  apply clarify
  apply (erule less-SucE)
   apply (erule-tac x=i in allE , erule (1) notE impE)
```

```
   apply simp
   apply clarify
   apply (drule-tac y = r in le-imp-less-or-eq)
   apply (erule disjE)
    apply (subgoal-tac Suc (ind x)≤r)
     apply fast
    apply arith
   apply fast
  apply fast
 apply (simp add: Graph10 BtoW-def)
 apply (erule disjE)
  apply (erule disjI1)
 apply clarify
 apply (erule less-SucE)
  apply force
 apply simp
 apply (subgoal-tac Suc R≤r)
  apply fast
 apply arith
apply(rule disjI1)
apply(erule subset-psubset-trans)
apply(erule Graph11)
apply fast
— 2 subgoals left
apply clarify
apply(simp add:Proper-Edges-def Graph6 Graph7 Graph8 Graph12)
apply (erule disjE)
 apply fast
apply clarify
apply (erule less-SucE)
 apply (erule-tac x=i in allE , erule (1) notE impE)
 apply simp
 apply clarify
 apply (drule-tac y = r in le-imp-less-or-eq)
 apply (erule disjE)
  apply (subgoal-tac Suc (ind x)≤r)
   apply fast
  apply arith
 apply (simp add: BtoW-def)
apply (simp add: BtoW-def)
— last
apply clarify
apply simp
apply(subgoal-tac ind x = length (E x))
 apply (simp)
 apply(drule Graph1)
   apply simp
  apply clarify
  apply(erule allE, erule impE, assumption)
```

```
  apply force
 apply force
apply arith
done
```

Counting black nodes

```
definition CountInv :: gar-coll-state ⇒ nat ⇒ bool where
  CountInv ≡ « λind. {i. i<ind ∧ ´Ma!i=Black}⊆´bc »

definition Count :: gar-coll-state ann-com where
  Count ≡
  {|´Proper ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}|}
  ´ind:=0;;
  {|´Proper ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
   ∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}
   ∧ ´ind=0|}
   WHILE ´ind<length ´M
     INV {|´Proper ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ ´CountInv ´ind
          ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind≤length ´M|}
   DO {|´Proper ∧ Roots⊆Blacks ´M
        ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ length ´Ma=length ´M ∧ ´CountInv ´ind
        ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}
       IF ´M!´ind=Black
          THEN {|´Proper ∧ Roots⊆Blacks ´M
                ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
                ∧ length ´Ma=length ´M ∧ ´CountInv ´ind
             ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
          ´bc:=insert ´ind ´bc
       FI;;
      {|´Proper ∧ Roots⊆Blacks ´M
        ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ length ´Ma=length ´M ∧ ´CountInv (´ind+1)
        ∧ ( ´obc < Blacks ´Ma ∨ ´Safe) ∧ ´ind<length ´M|}
      ´ind:=´ind+1
   OD

lemma Count:
  ⊢ Count
  {|´Proper ∧ Roots⊆Blacks ´M
   ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆´bc ∧ ´bc⊆Blacks ´M ∧ length ´Ma=length
´M
   ∧ (´obc < Blacks ´Ma ∨ ´Safe)|}
```

apply(*unfold Count-def*)
apply *annhoare*
apply(*simp-all add*:*CountInv-def Graph6 Graph7 Graph8 Graph12 Blacks-def collector-defs*)
      apply *force*
     apply *force*
    apply *force*
   apply *clarify*
   apply *simp*
   apply(*fast elim*:*less-SucE*)
  apply *clarify*
  apply *simp*
  apply(*fast elim*:*less-SucE*)
 apply *force*
apply *force*
done

Appending garbage nodes to the free list

axiomatization *Append-to-free* :: *nat* × *edges* ⇒ *edges*
where
  Append-to-free0: *length* (*Append-to-free* (*i*, *e*)) = *length e* **and**
  Append-to-free1: *Proper-Edges* (*m*, *e*)
      ⟹ *Proper-Edges* (*m*, *Append-to-free*(*i*, *e*)) **and**
  Append-to-free2: *i* ∉ *Reach e*
    ⟹ *n* ∈ *Reach* (*Append-to-free*(*i*, *e*)) = (*n* = *i* ∨ *n* ∈ *Reach e*)

definition *AppendInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
  AppendInv ≡ «λ*ind*. ∀ *i*<*length* ´*M*. *ind*≤*i* ⟶ *i*∈*Reach* ´*E* ⟶ ´*M*!*i*=*Black*»

definition *Append* :: *gar-coll-state ann-com* **where**
   Append ≡
  {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe*|}
  ´*ind*:=*0*;;
  {|´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe* ∧ ´*ind*=*0*|}
    WHILE ´*ind*<*length* ´*M*
      INV {|´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*≤*length* ´*M*|}
    DO {|´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M*|}
       IF ´*M*!´*ind*=*Black* *THEN*
          {|´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*M*!´*ind*=*Black*|}
          ´*M*:=´*M*[´*ind*:=*White*]
       ELSE {|´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∉*Reach* ´*E*|}
              ´*E*:=*Append-to-free*(´*ind*,´*E*)
       FI;;
     {|´*Proper* ∧ ´*AppendInv* (´*ind*+*1*) ∧ ´*ind*<*length* ´*M*|}
       ´*ind*:=´*ind*+*1*
    OD

lemma *Append*:

```
 ⊢ Append {|´Proper|}
apply(unfold Append-def AppendInv-def)
apply annhoare
apply(simp-all add:collector-defs Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1
Graph12)
       apply(force simp:Blacks-def nth-list-update)
      apply force
     apply force
    apply(force simp add:Graph-defs)
   apply force
  apply clarify
  apply simp
  apply(rule conjI)
   apply (erule Append-to-free1)
  apply clarify
  apply (drule-tac n = i in Append-to-free2)
  apply force
 apply force
apply force
done
```

Correctness of the Collector

```
definition Collector :: gar-coll-state ann-com where
  Collector ≡
{|´Proper|}
 WHILE True INV {|´Proper|}
 DO
  Blacken-Roots;;
  {|´Proper ∧ Roots⊆Blacks ´M|}
   ´obc:={};;
  {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={}|}
   ´bc:=Roots;;
  {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots|}
   ´Ma:=M-init;;
  {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´Ma=M-init|}
   WHILE ´obc≠´bc
     INV {|´Proper ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆´bc ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe)|}
   DO {|´Proper ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M|}
       ´obc:=´bc;;
       Propagate-Black;;
      {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ (´obc < Blacks ´M ∨ ´Safe)|}
       ´Ma:=´M;;
      {|´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´Ma
        ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧ length ´Ma=length ´M
        ∧ ( ´obc < Blacks ´Ma ∨ ´Safe)|}
```

```
        ´bc:={};;
        Count
    OD;;
   Append
  OD

lemma Collector:
 ⊢ Collector {|False|}
apply(unfold Collector-def)
apply annhoare
apply(simp-all add: Blacken-Roots Propagate-Black Count Append)
apply(simp-all add:Blacken-Roots-def Propagate-Black-def Count-def Append-def
collector-defs)
   apply (force simp add: Proper-Roots-def)
  apply force
 apply force
apply clarify
apply (erule disjE)
apply(simp add:psubsetI)
 apply(force dest:subset-antisym)
done
```

2.2.3 Interference Freedom

```
lemmas modules = Redirect-Edge-def Color-Target-def Blacken-Roots-def
                 Propagate-Black-def Count-def Append-def
lemmas Invariants = PBInv-def Auxk-def CountInv-def AppendInv-def
lemmas abbrev = collector-defs mutator-defs Invariants

lemma interfree-Blacken-Roots-Redirect-Edge:
 interfree-aux (Some Blacken-Roots, {}, Some Redirect-Edge)
apply (unfold modules)
apply interfree-aux
apply safe
apply (simp-all add:Graph6 Graph12 abbrev)
done

lemma interfree-Redirect-Edge-Blacken-Roots:
  interfree-aux (Some Redirect-Edge, {}, Some Blacken-Roots)
apply (unfold modules)
apply interfree-aux
apply safe
apply(simp add:abbrev)+
done

lemma interfree-Blacken-Roots-Color-Target:
  interfree-aux (Some Blacken-Roots, {}, Some Color-Target)
apply (unfold modules)
apply interfree-aux
```

apply *safe*
apply(*simp-all add*:*Graph7 Graph8 nth-list-update abbrev*)
done

lemma *interfree-Color-Target-Blacken-Roots*:
  interfree-aux (*Some Color-Target*, {}, *Some Blacken-Roots*)
apply (*unfold modules*)
apply *interfree-aux*
apply *safe*
apply(*simp add*:*abbrev*)+
done

lemma *interfree-Propagate-Black-Redirect-Edge*:
  interfree-aux (*Some Propagate-Black*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
— 11 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*erule disjE*, *erule disjI1*, *rule disjI2*, *rule allI*, (*rule impI*)+, *case-tac R*=*i*, *rule conjI*, *erule sym*)
 apply(*erule Graph4*)
   apply(*simp*)+
  apply (*simp add*:*BtoW-def*)
 apply (*simp add*:*BtoW-def*)
apply(*rule conjI*)
 apply (*force simp add*:*BtoW-def*)
apply(*erule Graph4*)
   apply *simp*+
— 7 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*erule disjE*, *erule disjI1*, *rule disjI2*, *rule allI*, (*rule impI*)+, *case-tac R*=*i*, *rule conjI*, *erule sym*)
 apply(*erule Graph4*)
   apply(*simp*)+
  apply (*simp add*:*BtoW-def*)
 apply (*simp add*:*BtoW-def*)
apply(*rule conjI*)
 apply (*force simp add*:*BtoW-def*)
apply(*erule Graph4*)
   apply *simp*+
— 6 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*rule conjI*)

```
 apply(erule disjE, erule disjI1, rule disjI2, rule allI, (rule impI)+, case-tac R=i,
rule conjI, erule sym)
  apply(erule Graph4)
    apply(simp)+
   apply (simp add:BtoW-def)
  apply (simp add:BtoW-def)
 apply(rule conjI)
  apply (force simp add:BtoW-def)
 apply(erule Graph4)
    apply simp+
apply(simp add:BtoW-def nth-list-update)
apply force
— 5 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
— 4 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
apply(rule conjI)
 apply(erule disjE, erule disjI1, rule disjI2, rule allI, (rule impI)+, case-tac R=i,
rule conjI, erule sym)
  apply(erule Graph4)
    apply(simp)+
   apply (simp add:BtoW-def)
  apply (simp add:BtoW-def)
 apply(rule conjI)
  apply (force simp add:BtoW-def)
 apply(erule Graph4)
    apply simp+
apply(rule conjI)
 apply(simp add:nth-list-update)
 apply force
apply(rule impI, rule impI, erule disjE, erule disjI1, case-tac R = (ind x) ,case-tac
M x ! T = Black)
  apply(force simp add:BtoW-def)
 apply(case-tac M x !snd (E x ! ind x)=Black)
  apply(rule disjI2)
  apply simp
  apply (erule Graph5)
  apply simp+
 apply(force simp add:BtoW-def)
apply(force simp add:BtoW-def)
— 3 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
— 2 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
apply(erule disjE, erule disjI1, rule disjI2, rule allI, (rule impI)+, case-tac R=i,
rule conjI, erule sym)
 apply clarify
 apply(erule Graph4)
   apply(simp)+
```

```
  apply (simp add:BtoW-def)
 apply (simp add:BtoW-def)
apply(rule conjI)
 apply (force simp add:BtoW-def)
apply(erule Graph4)
   apply simp+
done
```

lemma *interfree-Redirect-Edge-Propagate-Black*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Propagate-Black*)

```
apply (unfold modules )
apply interfree-aux
apply(clarify, simp add:abbrev)+
done
```

lemma *interfree-Propagate-Black-Color-Target*:
interfree-aux (*Some Propagate-Black*, {}, *Some Color-Target*)

```
apply (unfold modules )
apply interfree-aux
— 11 subgoals left
apply(clarify, simp add:abbrev Graph7 Graph8 Graph12)+
apply(erule conjE)+
apply(erule disjE,rule disjI1,erule psubset-subset-trans,erule Graph9,
     case-tac M x!T=Black, rule disjI2,rotate-tac −1, simp add: Graph10, clarify,
      erule allE, erule impE, assumption, erule impE, assumption,
       simp add:BtoW-def, rule disjI1, erule subset-psubset-trans, erule Graph11,
force)
— 7 subgoals left
apply(clarify, simp add:abbrev Graph7 Graph8 Graph12)
apply(erule conjE)+
apply(erule disjE,rule disjI1,erule psubset-subset-trans,erule Graph9,
     case-tac M x!T=Black, rule disjI2,rotate-tac −1, simp add: Graph10, clarify,
      erule allE, erule impE, assumption, erule impE, assumption,
       simp add:BtoW-def, rule disjI1, erule subset-psubset-trans, erule Graph11,
force)
— 6 subgoals left
apply(clarify, simp add:abbrev Graph7 Graph8 Graph12)
apply clarify
apply (rule conjI)
 apply(erule disjE,rule disjI1,erule psubset-subset-trans,erule Graph9,
     case-tac M x!T=Black, rule disjI2,rotate-tac −1, simp add: Graph10, clarify,
      erule allE, erule impE, assumption, erule impE, assumption,
       simp add:BtoW-def, rule disjI1, erule subset-psubset-trans, erule Graph11,
force)
apply(simp add:nth-list-update)
— 5 subgoals left
apply(clarify, simp add:abbrev Graph7 Graph8 Graph12)
— 4 subgoals left
apply(clarify, simp add:abbrev Graph7 Graph8 Graph12)
```

apply (*rule conjI*)
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x! *T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
apply(*rule conjI*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*,*rule impI*, *case-tac M x*! *T*=*Black*,*rotate-tac* −*1*, *force simp add*:
BtoW-def Graph10,
erule subset-psubset-trans, *erule Graph11*, *force*)
— 3 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
— 2 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x! *T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
— 3 subgoals left
apply(*simp add*:*abbrev*)
done

lemma *interfree-Color-Target-Propagate-Black*:
interfree-aux (*Some Color-Target*, {}, *Some Propagate-Black*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev*)+
done

lemma *interfree-Count-Redirect-Edge*:
interfree-aux (*Some Count*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
— 9 subgoals left
apply(*simp-all add*:*abbrev Graph6 Graph12*)
— 6 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*,
erule disjE,*erule disjI1*,*rule disjI2*,*rule subset-trans*, *erule Graph3*,*force*,*force*)+
done

lemma *interfree-Redirect-Edge-Count*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Count*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*,*simp add*:*abbrev*)+
apply(*simp add*:*abbrev*)
done

lemma *interfree-Count-Color-Target*:
interfree-aux (*Some Count*, {}, *Some Color-Target*)
apply (*unfold modules*)
apply *interfree-aux*
— 9 subgoals left
apply(*simp-all add*:*abbrev Graph7 Graph8 Graph12*)
— 6 subgoals left
apply(*clarify*,*simp add*:*abbrev Graph7 Graph8 Graph12*,
erule disjE, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)+
— 2 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply(*rule conjI*)
apply(*erule disjE*, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)
apply(*simp add*:*nth-list-update*)
— 1 subgoal left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*,
erule disjE, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)
done

lemma *interfree-Color-Target-Count*:
interfree-aux (*Some Color-Target*, {}, *Some Count*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev*)+
apply(*simp add*:*abbrev*)
done

lemma *interfree-Append-Redirect-Edge*:
interfree-aux (*Some Append*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*simp-all add*:*abbrev Graph6 Append-to-free0 Append-to-free1 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Append-to-free0 Append-to-free1 Graph12*,
force dest:*Graph3*)+
done

lemma *interfree-Redirect-Edge-Append*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Append*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
apply (*force simp add*: *Append-to-free2*)
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
done

lemma *interfree-Append-Color-Target*:
interfree-aux (*Some Append*, {}, *Some Color-Target*)
apply (*unfold modules*)

apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12 nth-list-update*)+
apply(*simp add*:*abbrev Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12 nth-list-update*)
done

lemma *interfree-Color-Target-Append*:
interfree-aux (*Some Color-Target*, {}, *Some Append*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
apply (*force simp add*: *Append-to-free2*)
apply(*clarify*,*simp add*:*abbrev Append-to-free0*)+
done

lemmas *collector-mutator-interfree* =
interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target
interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target
interfree-Count-Redirect-Edge interfree-Count-Color-Target
interfree-Append-Redirect-Edge interfree-Append-Color-Target
interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots
interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black
interfree-Redirect-Edge-Count interfree-Color-Target-Count
interfree-Redirect-Edge-Append interfree-Color-Target-Append

Interference freedom Collector-Mutator

lemma *interfree-Collector-Mutator*:
interfree-aux (*Some Collector*, {}, *Some Mutator*)
apply(*unfold Collector-def Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*collector-mutator-interfree*)
apply(*unfold modules collector-defs Mut-init-def*)
apply(*tactic* ‹*TRYALL* (*interfree-aux-tac* ***context***)›)
— 32 subgoals left
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 20 subgoals left
apply(*tactic*‹*TRYALL* (*clarify-tac* ***context***)›)
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* ***context*** [*disjE*])›)
apply *simp-all*
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
resolve-tac ***context*** [*subset-trans*],
eresolve-tac ***context*** @{*thms Graph3*},
force-tac ***context***,
assume-tac ***context***])›)
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
eresolve-tac ***context*** [*subset-trans*],

resolve-tac ***context*** @{*thms Graph9*},
force-tac ***context***])›)
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI1*],
eresolve-tac ***context*** @{*thms psubset-subset-trans*},
resolve-tac ***context*** @{*thms Graph9*},
force-tac ***context***])›)
done

Interference freedom Mutator-Collector

lemma *interfree-Mutator-Collector*:
interfree-aux (*Some Mutator*, {}, *Some Collector*)
apply(*unfold Collector-def Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*collector-mutator-interfree*)
apply(*unfold modules collector-defs Mut-init-def*)
apply(*tactic* ‹*TRYALL* (*interfree-aux-tac* ***context***)›)
— 64 subgoals left
apply(*simp-all add*:*nth-list-update Invariants Append-to-free0*)+
apply(*tactic*‹*TRYALL* (*clarify-tac* ***context***)›)
— 4 subgoals left
apply *force*
apply(*simp add*:*Append-to-free2*)
apply *force*
apply(*simp add*:*Append-to-free2*)
done

The Garbage Collection algorithm

In total there are 289 verification conditions.

lemma *Gar-Coll*:
‖− {|´*Proper* ∧ ´*Mut-init* ∧ ´*z*|}
COBEGIN
Collector
{|*False*|}
‖
Mutator
{|*False*|}
COEND
{|*False*|}
apply *oghoare*
apply(*force simp add*: *Mutator-def Collector-def modules*)
apply(*rule Collector*)
apply(*rule Mutator*)
apply(*simp add*:*interfree-Collector-Mutator*)
apply(*simp add*:*interfree-Mutator-Collector*)
apply *force*
done

end

2.3 The Multi-Mutator Case

theory *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

record *mut* =
  Z :: *bool*
  R :: *nat*
  T :: *nat*

Declaration of variables:

record *mul-gar-coll-state* =
  M :: *nodes*
  E :: *edges*
  bc :: *nat set*
  obc :: *nat set*
  Ma :: *nodes*
  ind :: *nat*
  k :: *nat*
  q :: *nat*
  l :: *nat*
  Muts :: *mut list*

2.3.1 The Mutators

definition *Mul-mut-init* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
  Mul-mut-init ≡ « λ*n*. *n*=*length* ´*Muts* ∧ (∀ *i*<*n*. *R* (´*Muts*!*i*)<*length* ´*E*
      ∧ *T* (´*Muts*!*i*)<*length* ´*M*) »

definition *Mul-Redirect-Edge* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
  Mul-Redirect-Edge j n ≡
  ⦃´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)⦄
  ⟨*IF T*(´*Muts*!*j*) ∈ *Reach* ´*E THEN*
  ´*E*:= ´*E*[*R* (´*Muts*!*j*):= (*fst* (´*E*!*R*(´*Muts*!*j*)), *T* (´*Muts*!*j*))] *FI*,,
  ´*Muts*:= ´*Muts*[*j*:= (´*Muts*!*j*) ⦇*Z*:=*False*⦈]⟩

definition *Mul-Color-Target* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
  Mul-Color-Target j n ≡
  ⦃´*Mul-mut-init n* ∧ ¬ *Z* (´*Muts*!*j*)⦄
  ⟨´*M*:=´*M*[*T* (´*Muts*!*j*):=*Black*],, ´*Muts*:=´*Muts*[*j*:= (´*Muts*!*j*) ⦇*Z*:=*True*⦈]⟩

definition *Mul-Mutator* :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com* **where**
  Mul-Mutator j n ≡
  ⦃´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)⦄
  WHILE True
    INV ⦃´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)⦄

DO Mul-Redirect-Edge j n ;;
  Mul-Color-Target j n
OD

lemmas *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

Correctness of the proof outline of one mutator

lemma *Mul-Redirect-Edge*: $0 \leq j \wedge j < n \Longrightarrow$
 $\vdash$ *Mul-Redirect-Edge j n*
  pre(*Mul-Color-Target j n*)
apply (*unfold mul-mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply *clarify*
apply(*simp add*:*nth-list-update*)
done

lemma *Mul-Color-Target*: $0 \leq j \wedge j < n \Longrightarrow$
 $\vdash$ *Mul-Color-Target j n*
  {| ´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)|}
apply (*unfold mul-mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply *clarify*
apply(*simp add*:*nth-list-update*)
done

lemma *Mul-Mutator*: $0 \leq j \wedge j < n \Longrightarrow$
 $\vdash$ *Mul-Mutator j n* {|*False*|}
apply(*unfold Mul-Mutator-def*)
apply *annhoare*
apply(*simp-all add*:*Mul-Redirect-Edge Mul-Color-Target*)
apply(*simp add*:*mul-mutator-defs Mul-Redirect-Edge-def*)
done

Interference freedom between mutators

lemma *Mul-interfree-Redirect-Edge-Redirect-Edge*:
 $[\![0 \leq i;\ i < n;\ 0 \leq j;\ j < n;\ i \neq j]\!] \Longrightarrow$
 interfree-aux (*Some* (*Mul-Redirect-Edge i n*),{}, *Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemma *Mul-interfree-Redirect-Edge-Color-Target*:
 $[\![0 \leq i;\ i < n;\ 0 \leq j;\ j < n;\ i \neq j]\!] \Longrightarrow$
 interfree-aux (*Some*(*Mul-Redirect-Edge i n*),{},*Some*(*Mul-Color-Target j n*))

apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemma *Mul-interfree-Color-Target-Redirect-Edge*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*nth-list-update*)
done

lemma *Mul-interfree-Color-Target-Color-Target*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Color-Target j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemmas *mul-mutator-interfree* =
Mul-interfree-Redirect-Edge-Redirect-Edge Mul-interfree-Redirect-Edge-Color-Target
Mul-interfree-Color-Target-Redirect-Edge Mul-interfree-Color-Target-Color-Target

lemma *Mul-interfree-Mutator-Mutator*: $[\![i < n;\ j < n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator i n*), {}, *Some* (*Mul-Mutator j n*))
apply(*unfold Mul-Mutator-def*)
apply(*interfree-aux*)
apply(*simp-all add*:*mul-mutator-interfree*)
apply(*simp-all add*: *mul-mutator-defs*)
apply(*tactic* ‹*TRYALL* (*interfree-aux-tac* ***context***)›)
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
apply (*simp-all add*:*nth-list-update*)
done

Modular Parameterized Mutators

lemma *Mul-Parameterized-Mutators*: $0<n \Longrightarrow$
$\|-$ $\{\!|$*´Mul-mut-init n* $\wedge$ ($\forall$ *i*<*n*. *Z* (*´Muts*!*i*))$|\!\}$
COBEGIN
SCHEME [$0 \leq j < n$]
Mul-Mutator j n
$\{\!|$*False*$|\!\}$
COEND
$\{\!|$*False*$|\!\}$

apply *oghoare*
apply(*force simp add*:*Mul-Mutator-def mul-mutator-defs nth-list-update*)
apply(*erule Mul-Mutator*)
apply(*simp add*:*Mul-interfree-Mutator-Mutator*)
apply(*force simp add*:*Mul-Mutator-def mul-mutator-defs nth-list-update*)
done

2.3.2 The Collector

definition *Queue* :: *mul-gar-coll-state* ⇒ *nat* **where**
Queue ≡ « *length* (*filter* (λ*i*. ¬ *Z i* ∧ ´*M*!(*T i*) ≠ *Black*) ´*Muts*) »

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *mul-gar-coll-state* ⇒ *bool* **where**
Proper-M-init ≡ « *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* »

definition *Mul-Proper* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
Mul-Proper ≡ « λ*n*. *Proper-Roots* ´*M* ∧ *Proper-Edges* (´*M*, ´*E*) ∧ ´*Proper-M-init* ∧ *n*=*length* ´*Muts* »

definition *Safe* :: *mul-gar-coll-state* ⇒ *bool* **where**
Safe ≡ « *Reach* ´*E* ⊆ *Blacks* ´*M* »

lemmas *mul-collector-defs* = *Proper-M-init-def Mul-Proper-def Safe-def*

Blackening Roots

definition *Mul-Blacken-Roots* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Blacken-Roots n ≡
{|´*Mul-Proper n*|}
´*ind*:=*0*;;
{|´*Mul-Proper n* ∧ ´*ind*=*0*|}
WHILE ´*ind*<*length* ´*M*
INV {|´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M*|}
DO {|´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
IF ´*ind*∈*Roots THEN*
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*|}
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
{|´*Mul-Proper n* ∧ (∀ *i*<´*ind*+*1*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*|}
´*ind*:=´*ind*+*1*
OD

lemma *Mul-Blacken-Roots*:
⊢ *Mul-Blacken-Roots n*
{|´*Mul-Proper n* ∧ *Roots* ⊆ *Blacks* ´*M*|}
apply (*unfold Mul-Blacken-Roots-def*)

apply *annhoare*
apply(*simp-all add*:*mul-collector-defs Graph-defs*)
apply *safe*
apply(*simp-all add*:*nth-list-update*)
 apply (*erule less-SucE*)
  apply *simp+*
 apply *force*
apply *force*
done

Propagating Black

definition *Mul-PBInv* :: *mul-gar-coll-state* ⇒ *bool* **where**
 Mul-PBInv ≡ «´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue*
  ∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*)) ∧ ´*l*≤´*Queue*»

definition *Mul-Auxk* :: *mul-gar-coll-state* ⇒ *bool* **where**
 Mul-Auxk ≡ «´*l*<´*Queue* ∨ ´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨ ´*obc*⊂*Blacks* ´*M*»

definition *Mul-Propagate-Black* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
 Mul-Propagate-Black n ≡
 ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
 ∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*)⦄
 ´*ind*:=*0*;;
 ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
  ∧ ´*obc*⊆*Blacks* ´*M* ∧ *Blacks* ´*M*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
  ∧ (´*Safe* ∨ ´*l*≤´*Queue* ∨ ´*obc*⊂*Blacks* ´*M*) ∧ ´*ind*=*0*⦄
 WHILE ´*ind*<*length* ´*E*
 INV ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
   ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ ´*Mul-PBInv* ∧ ´*ind*≤*length* ´*E*⦄
 DO ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
  ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
  ∧ ´*Mul-PBInv* ∧ ´*ind*<*length* ´*E*⦄
  IF ´*M*!(*fst* (´*E*!´*ind*))=*Black THEN*
  ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
   ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ ´*Mul-PBInv* ∧ (´*M*!*fst*(´*E*!´*ind*))=*Black* ∧ ´*ind*<*length* ´*E*⦄
  ´*k*:=*snd*(´*E*!´*ind*);;
  ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
   ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ (´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue* ∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*))
    ∧ ´*l*≤´*Queue* ∧ ´*Mul-Auxk*) ∧ ´*k*<*length* ´*M* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black*
   ∧ ´*ind*<*length* ´*E*⦄
  ⟨´*M*:=´*M*[´*k*:=*Black*],,´*ind*:=´*ind*+*1*⟩
  ELSE ⦃´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
   ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ ´*Mul-PBInv* ∧ ´*ind*<*length* ´*E*⦄

⟨*IF ´M*!(*fst* (*´E*!*´ind*))≠*Black THEN ´ind*:=*´ind*+*1 FI*⟩ *FI*
OD

lemma *Mul-Propagate-Black*:
⊢ *Mul-Propagate-Black n*
{|*´Mul-Proper n* ∧ *Roots*⊆*Blacks ´M* ∧ *´obc*⊆*Blacks ´M* ∧ *´bc*⊆*Blacks ´M*
∧ (*´Safe* ∨ *´obc*⊂*Blacks ´M* ∨ *´l*<*´Queue* ∧ (*´l*≤*´Queue* ∨ *´obc*⊂*Blacks ´M*))|}
apply(*unfold Mul-Propagate-Black-def*)
apply *annhoare*
apply(*simp-all add*:*Mul-PBInv-def mul-collector-defs Mul-Auxk-def Graph6 Graph7 Graph8 Graph12 mul-collector-defs Queue-def*)
— 8 subgoals left
apply *force*
apply *force*
apply *force*
apply(*force simp add*:*BtoW-def Graph-defs*)
— 4 subgoals left
apply *clarify*
apply(*simp add*: *mul-collector-defs Graph12 Graph6 Graph7 Graph8*)
apply(*disjE-tac*)
 apply(*simp-all add*:*Graph12 Graph13*)
 apply(*case-tac M x*! *k x*=*Black*)
  apply(*simp add*: *Graph10*)
 apply(*rule disjI2*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*, *force*)
apply(*case-tac M x*! *k x*=*Black*)
 apply(*simp add*: *Graph10 BtoW-def*)
 apply(*rule disjI2*, *clarify*, *erule less-SucE*, *force*)
 apply(*case-tac M x*!*snd*(*E x*! *ind x*)=*Black*)
  apply(*force*)
 apply(*force*)
apply(*rule disjI2*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*, *force*)
— 2 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
 apply (*simp-all*)
apply *clarify*
apply(*erule less-SucE*)
 apply *force*
apply (*simp add*:*BtoW-def*)
— 1 subgoal left
apply *clarify*
apply *simp*
apply(*disjE-tac*)
apply (*simp-all*)
apply(*rule disjI1* , *rule Graph1*)
 apply *simp-all*
done

Counting Black Nodes

definition *Mul-CountInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
Mul-CountInv ≡ « λ*ind*. {*i*. *i*<*ind* ∧ ´*Ma*!*i*=*Black*}⊆´*bc* »

definition *Mul-Count* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**

```
  Mul-Count n ≡
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
    ∧ ´q<n+1 ∧ ´bc={}|}
  ´ind:=0;;
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
    ∧ ´q<n+1 ∧ ´bc={} ∧ ´ind=0|}
  WHILE ´ind<length ´M
     INV {|´Mul-Proper n ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
         ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
          ∧ ´q<n+1 ∧ ´ind≤length ´M|}
  DO {|´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
       ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
       ∧ ´q<n+1 ∧ ´ind<length ´M|}
     IF ´M!´ind=Black
     THEN {|´Mul-Proper n ∧ Roots⊆Blacks ´M
           ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
           ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
            ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
           ∧ ´q<n+1 ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
          ´bc:=insert ´ind ´bc
     FI;;
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv (´ind+1)
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M|}
  ´ind:=´ind+1
  OD
```

lemma *Mul-Count*:

```
 ⊢ Mul-Count n
  {|´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
```

$\wedge$ *length ´Ma=length ´M* $\wedge$ *Blacks ´Ma*$\subseteq$*´bc*
$\wedge$ (*´Safe* $\vee$ *´obc*$\subset$*Blacks ´Ma* $\vee$ *´l*<*´q* $\wedge$ (*´q*$\leq$*´Queue* $\vee$ *´obc*$\subset$*Blacks ´M*))
$\wedge$ *´q*<*n*+*1*|}
apply (*unfold Mul-Count-def*)
apply *annhoare*
apply(*simp-all add*:*Mul-CountInv-def mul-collector-defs Mul-Auxk-def Graph6 Graph7 Graph8 Graph12 mul-collector-defs Queue-def*)
— 7 subgoals left
apply *force*
apply *force*
apply *force*
— 4 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
apply *simp-all*
apply(*simp add*:*Blacks-def*)
apply *clarify*
apply(*erule less-SucE*)
back
apply *force*
apply *force*
— 3 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
apply *simp-all*
apply *clarify*
apply(*erule less-SucE*)
back
apply *force*
apply *simp*
apply(*rotate-tac −1*)
apply (*force simp add*:*Blacks-def*)
— 2 subgoals left
apply *force*
— 1 subgoal left
apply *clarify*
apply(*drule-tac x* = *ind x* **in** *le-imp-less-or-eq*)
apply (*simp-all add*:*Blacks-def*)
done

Appending garbage nodes to the free list

axiomatization *Append-to-free* :: *nat* × *edges* ⇒ *edges*
where
Append-to-free0: *length* (*Append-to-free* (*i*, *e*)) = *length e* **and**
Append-to-free1: *Proper-Edges* (*m*, *e*)
⟹ *Proper-Edges* (*m*, *Append-to-free*(*i*, *e*)) **and**

Append-to-free2: $i \notin Reach\ e$
$\Longrightarrow n \in Reach\ (Append\text{-}to\text{-}free(i,\ e)) = (\ n = i \lor n \in Reach\ e)$

definition *Mul-AppendInv* :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool* **where**
Mul-AppendInv ≡ « λ*ind*. (∀ *i*. *ind*≤*i* ⟶ *i*<*length* ´*M* ⟶ *i*∈*Reach* ´*E* ⟶
´*M*!*i*=*Black*)»

definition *Mul-Append* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**

```
Mul-Append n ≡
{|´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe|}
´ind:=0;;
{|´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´Safe ∧ ´ind=0|}
WHILE ´ind<length ´M
  INV {|´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind≤length ´M|}
DO {|´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M|}
    IF ´M!´ind=Black THEN
  {|´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´M!´ind=Black|}
    ´M:=´M[´ind:=White]
    ELSE
    {|´Mul-Proper n ∧ ´Mul-AppendInv ´ind ∧ ´ind<length ´M ∧ ´ind∉Reach
´E|}
    ´E:=Append-to-free(´ind,´E)
    FI;;
{|´Mul-Proper n ∧ ´Mul-AppendInv (´ind+1) ∧ ´ind<length ´M|}
 ´ind:=´ind+1
OD
```

lemma *Mul-Append*:
⊢ *Mul-Append n*
{|´*Mul-Proper n*|}
apply(*unfold Mul-Append-def*)
apply *annhoare*
apply(*simp-all add*: *mul-collector-defs Mul-AppendInv-def*
Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Graph-defs*)
apply *force*
apply(*force simp add*:*Append-to-free1 Append-to-free2*)
apply *force*
apply *force*
done

Collector

definition *Mul-Collector* :: *nat* ⇒ *mul-gar-coll-state ann-com* **where**
Mul-Collector n ≡
{|´*Mul-Proper n*|}

```
WHILE True INV ⦃´Mul-Proper n⦄
DO
Mul-Blacken-Roots n ;;
⦃´Mul-Proper n ∧ Roots⊆Blacks ´M⦄
 ´obc:={};;
⦃´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={}⦄
 ´bc:=Roots;;
⦃´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots⦄
 ´l:=0;;
⦃´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´l=0⦄
 WHILE ´l<n+1
   INV ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧
        (´Safe ∨ (´l≤´Queue ∨ ´bc⊂Blacks ´M) ∧ ´l<n+1)⦄
 DO ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´l≤´Queue ∨ ´bc⊂Blacks ´M)⦄
    ´obc:=´bc;;
    Mul-Propagate-Black n;;
    ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
      ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M))⦄
    ´bc:={};;
    ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
      ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M)) ∧ ´bc={}⦄
      ⟨ ´Ma:=´M,, ´q:=´Queue ⟩;;
    Mul-Count n;;
    ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
      ∧ ´q<n+1⦄
    IF ´obc=´bc THEN
    ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
      ∧ ´q<n+1 ∧ ´obc=´bc⦄
    ´l:=´l+1
    ELSE ⦃´Mul-Proper n ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
         ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
          ∧ ´q<n+1 ∧ ´obc≠´bc⦄
         ´l:=0 FI
 OD;;
 Mul-Append n
OD
```

lemmas *mul-modules* = *Mul-Redirect-Edge-def Mul-Color-Target-def*
Mul-Blacken-Roots-def Mul-Propagate-Black-def
Mul-Count-def Mul-Append-def

lemma *Mul-Collector*:
$\vdash$ *Mul-Collector n*
$\{\!|$*False*$|\!\}$
apply(*unfold Mul-Collector-def*)
apply *annhoare*
apply(*simp-all only*:*pre.simps Mul-Blacken-Roots*
Mul-Propagate-Black Mul-Count Mul-Append)
apply(*simp-all add*:*mul-modules*)
apply(*simp-all add*:*mul-collector-defs Queue-def*)
apply *force*
apply *force*
apply *force*
apply (*force simp add*: *less-Suc-eq-le*)
apply *force*
apply (*force dest*:*subset-antisym*)
apply *force*
apply *force*
apply *force*
done

2.3.3 Interference Freedom

lemma *le-length-filter-update*[*rule-format*]:
$\forall\, i.\ (\neg P\ (list!i) \vee P\ j) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) \leq length(filter\ P\ (list[i{:=}j]))$
apply(*induct-tac list*)
apply(*simp*)
apply(*clarify*)
apply(*case-tac i*)
apply(*simp*)
apply(*simp*)
done

lemma *less-length-filter-update* [*rule-format*]:
$\forall\, i.\ P\ j \wedge \neg(P\ (list!i)) \wedge i{<}length\ list$
$\longrightarrow length(filter\ P\ list) < length(filter\ P\ (list[i{:=}j]))$
apply(*induct-tac list*)
apply(*simp*)
apply(*clarify*)
apply(*case-tac i*)
apply(*simp*)
apply(*simp*)
done

lemma *Mul-interfree-Blacken-Roots-Redirect-Edge*: ⟦*0*≤*j*; *j*<*n*⟧ ⟹
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*Graph6 Graph9 Graph12 nth-list-update mul-mutator-defs mul-collector-defs*)
done

lemma *Mul-interfree-Redirect-Edge-Blacken-Roots*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Blacken-Roots n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Blacken-Roots-Color-Target*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some* (*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs nth-list-update Graph7 Graph8 Graph9 Graph12*)
done

lemma *Mul-interfree-Color-Target-Blacken-Roots*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some* (*Mul-Blacken-Roots n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Propagate-Black-Redirect-Edge*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs Mul-PBInv-def nth-list-update Graph6*)
— 7 subgoals left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)

```
— 6 subgoals left
apply clarify
apply(disjE-tac)
  apply(simp-all add:Graph6)
 apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(rule impI,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def less-Suc-eq-le
le-length-filter-update)
— 5 subgoals left
apply clarify
apply(disjE-tac)
  apply(simp-all add:Graph6)
 apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp
add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply(rule conjI)
  apply(rule impI,(rule disjI2)+,rule conjI)
   apply clarify
   apply(case-tac R (Muts x! j)=i)
    apply (force simp add: nth-list-update BtoW-def)
   apply (force simp add: nth-list-update)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(rule impI,(rule disjI2)+, erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
 apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
— 4 subgoals left
apply clarify
apply(disjE-tac)
  apply(simp-all add:Graph6)
 apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp
add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply(rule conjI)
```

```
  apply(rule impI,(rule disjI2)+,rule conjI)
   apply clarify
   apply(case-tac R (Muts x! j)=i)
    apply (force simp add: nth-list-update BtoW-def)
   apply (force simp add: nth-list-update)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(rule impI,(rule disjI2)+, erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
 apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
— 3 subgoals left
apply clarify
apply(disjE-tac)
  apply(simp-all add:Graph6)
  apply (rule impI)
   apply(rule conjI)
    apply(rule disjI1,rule subset-trans,erule Graph3,simp,simp)
   apply(case-tac R (Muts x ! j)= ind x)
    apply(simp add:nth-list-update)
   apply(simp add:nth-list-update)
  apply(case-tac R (Muts x ! j)= ind x)
   apply(simp add:nth-list-update)
  apply(simp add:nth-list-update)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule conjI)
   apply(rule impI)
   apply(rule conjI)
    apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
    apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
   apply(case-tac R (Muts x ! j)= ind x)
    apply(simp add:nth-list-update)
   apply(simp add:nth-list-update)
  apply(rule impI)
  apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(rule conjI)
  apply(rule impI)
   apply(rule conjI)
    apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
    apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
   apply(case-tac R (Muts x ! j)= ind x)
    apply(simp add:nth-list-update)
   apply(simp add:nth-list-update)
  apply(rule impI)
  apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
```

apply(*erule conjE*)
apply(*rule conjI*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule impI*,*rule conjI*,(*rule disjI2*)+,*rule conjI*)
apply *clarify*
apply(*case-tac R* (*Muts x*! *j*)=*i*)
apply (*force simp add*: *nth-list-update BtoW-def*)
apply (*force simp add*: *nth-list-update*)
apply(*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*,*rule conjI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
apply(*case-tac R* (*Muts x*! *j*)=*ind x*)
apply (*force simp add*: *nth-list-update*)
apply (*force simp add*: *nth-list-update*)
apply(*rule impI*, (*rule disjI2*)+, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
— 2 subgoals left
apply *clarify*
apply(*rule conjI*)
apply(*disjE-tac*)
apply(*simp-all add*:*Mul-Auxk-def Graph6*)
apply (*rule impI*)
apply(*rule conjI*)
apply(*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule impI*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)

```
apply(rule impI)
apply(rule conjI)
 apply(erule conjE)+
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply((rule disjI2)+,rule conjI)
   apply clarify
   apply(case-tac R (Muts x! j)=i)
    apply (force simp add: nth-list-update BtoW-def)
   apply (force simp add: nth-list-update)
  apply(rule conjI)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
  apply(rule impI)
  apply(case-tac R (Muts x ! j)= ind x)
   apply(simp add:nth-list-update BtoW-def)
  apply (simp  add:nth-list-update)
  apply(rule impI)
  apply simp
  apply(disjE-tac)
   apply(rule disjI1, erule less-le-trans)
   apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
  apply force
 apply(rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
 apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
 apply(case-tac R (Muts x ! j)= ind x)
  apply(simp add:nth-list-update)
 apply(simp add:nth-list-update)
apply(disjE-tac)
apply simp-all
apply(conjI-tac)
 apply(rule impI)
 apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(erule conjE)+
apply(rule impI,(rule disjI2)+,rule conjI)
 apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule impI)+
apply simp
apply(disjE-tac)
 apply(rule disjI1, erule less-le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply force
— 1 subgoal left
apply clarify
apply(disjE-tac)
  apply(simp-all add:Graph6)
 apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp
add:Queue-def less-Suc-eq-le le-length-filter-update)
```

```
apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule less-le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply(rule conjI)
  apply(rule impI,(rule disjI2)+,rule conjI)
   apply clarify
   apply(case-tac R (Muts x! j)=i)
    apply (force simp add: nth-list-update BtoW-def)
   apply (force simp add: nth-list-update)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(rule impI,(rule disjI2)+, erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
 apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
done

lemma Mul-interfree-Redirect-Edge-Propagate-Black: ⟦0≤j; j<n⟧⟹
  interfree-aux (Some(Mul-Redirect-Edge j n ),{},Some (Mul-Propagate-Black n))
apply (unfold mul-modules)
apply interfree-aux
apply safe
apply(simp-all add:mul-mutator-defs nth-list-update)
done

lemma Mul-interfree-Propagate-Black-Color-Target: ⟦0≤j; j<n⟧⟹
  interfree-aux (Some(Mul-Propagate-Black n),{},Some (Mul-Color-Target j n ))
apply (unfold mul-modules)
apply interfree-aux
apply(simp-all add: mul-collector-defs mul-mutator-defs)
— 7 subgoals left
apply clarify
apply (simp add:Graph7 Graph8 Graph12)
apply(disjE-tac)
  apply(simp add:Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply((rule disjI2)+,erule subset-psubset-trans, erule Graph11, simp)
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 6 subgoals left
apply clarify
apply (simp add:Graph7 Graph8 Graph12)
apply(disjE-tac)
  apply(simp add:Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
```

```
  apply(rule disjI2,rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply((rule disjI2)+,erule subset-psubset-trans, erule Graph11, simp)
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 5 subgoals left
apply clarify
apply (simp add:mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12)
apply(disjE-tac)
   apply(simp add:Graph7 Graph8 Graph12)
  apply(rule disjI2,rule disjI1, erule psubset-subset-trans,simp add:Graph9)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply(rule disjI2,rule disjI1,erule subset-psubset-trans, erule Graph11, simp)
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply((rule disjI2)+)
 apply (rule conjI)
  apply(simp add:Graph10)
 apply(erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
apply(rule disjI2,rule disjI1,erule subset-psubset-trans, erule Graph11, simp)
— 4 subgoals left
apply clarify
apply (simp add:mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12)
apply(disjE-tac)
   apply(simp add:Graph7 Graph8 Graph12)
  apply(rule disjI2,rule disjI1, erule psubset-subset-trans,simp add:Graph9)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply(rule disjI2,rule disjI1,erule subset-psubset-trans, erule Graph11, simp)
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply((rule disjI2)+)
 apply (rule conjI)
  apply(simp add:Graph10)
 apply(erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
apply(rule disjI2,rule disjI1,erule subset-psubset-trans, erule Graph11, simp)
— 3 subgoals left
apply clarify
apply (simp add:mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply(simp add:Graph10)
 apply(disjE-tac)
  apply simp-all
  apply(rule disjI2, rule disjI2, rule disjI1,erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
```

apply(*erule conjE*)
apply((*rule disjI2*)+,*erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
apply (*force simp add*:*nth-list-update*)
— 2 subgoals left
apply *clarify*
apply(*simp add*:*Mul-Auxk-def Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*simp add*:*Graph10*)
apply(*disjE-tac*)
apply *simp-all*
apply(*rule disjI2*, *rule disjI2*, *rule disjI1*,*erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*erule conjE*)+
apply((*rule disjI2*)+,*rule conjI*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule impI*)+)
apply *simp*
apply(*erule disjE*)
apply(*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply *force*
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
apply (*force simp add*:*nth-list-update*)
— 1 subgoal left
apply *clarify*
apply (*simp add*:*mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*simp add*:*Graph10*)
apply(*disjE-tac*)
apply *simp-all*
apply(*rule disjI2*, *rule disjI2*, *rule disjI1*,*erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*erule conjE*)
apply((*rule disjI2*)+,*erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
done

lemma *Mul-interfree-Color-Target-Propagate-Black*: $[\![0 \leq j;\ j<n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some*(*Mul-Propagate-Black n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

```
lemma Mul-interfree-Count-Redirect-Edge: [[0≤j; j<n]] ⟹
  interfree-aux (Some(Mul-Count n ),{},Some(Mul-Redirect-Edge j n))
apply (unfold mul-modules)
apply interfree-aux
— 9 subgoals left
apply(simp add:mul-mutator-defs mul-collector-defs Mul-CountInv-def Graph6)
apply clarify
apply disjE-tac
   apply(simp add:Graph6)
  apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
 apply(simp add:Graph6)
apply clarify
apply disjE-tac
 apply(simp add:Graph6)
 apply(rule conjI)
  apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(simp add:Graph6)
— 8 subgoals left
apply(simp add:mul-mutator-defs nth-list-update)
— 7 subgoals left
apply(simp add:mul-mutator-defs mul-collector-defs)
apply clarify
apply disjE-tac
   apply(simp add:Graph6)
  apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
 apply(simp add:Graph6)
apply clarify
apply disjE-tac
 apply(simp add:Graph6)
 apply(rule conjI)
  apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1,erule le-trans,force simp add:Queue-def
less-Suc-eq-le le-length-filter-update)
apply(simp add:Graph6)
— 6 subgoals left
apply(simp add:mul-mutator-defs mul-collector-defs Mul-CountInv-def)
apply clarify
apply disjE-tac
   apply(simp add:Graph6 Queue-def)
  apply(rule impI,rule disjI1,rule subset-trans,erule Graph3,simp,simp)
 apply(simp add:Graph6)
apply clarify
apply disjE-tac
 apply(simp add:Graph6)
```

apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 5 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 4 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 3 subgoals left
apply(*simp add*:*mul-mutator-defs nth-list-update*)
— 2 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*

apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 1 subgoal left
apply(*simp add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Redirect-Edge-Count*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Count n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Count-Color-Target*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-defs mul-mutator-defs Mul-CountInv-def*)
— 6 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 5 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)

apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 4 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 3 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 2 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12 nth-list-update*)
apply (*simp add*: *Graph7 Graph8 Graph12 nth-list-update*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*rule conjI*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *nth-list-update*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*rule conjI*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
apply (*simp add*: *nth-list-update*)

— 1 subgoal left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
done

lemma *Mul-interfree-Color-Target-Count*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Count n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Append-Redirect-Edge*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
apply(*simp-all add*:*Graph6 Append-to-free0 Append-to-free1 mul-collector-defs mul-mutator-defs Mul-AppendInv-def*)
apply(*erule-tac x*=*j* **in** *allE*, *force dest*:*Graph3*)+
done

lemma *Mul-interfree-Redirect-Edge-Append*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Append n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
apply(*simp-all add*:*mul-collector-defs Append-to-free0 Mul-AppendInv-def mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Append-Color-Target*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs Mul-AppendInv-def Graph7*

Graph8 Append-to-free0 Append-to-free1
Graph12 nth-list-update)
done

lemma *Mul-interfree-Color-Target-Append*: ⟦*0*≤*j*; *j*<*n*⟧⟹
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Append n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ‹*ALLGOALS* (*clarify-tac* ***context***)›)
apply(*simp-all add*: *mul-mutator-defs nth-list-update*)
apply(*simp add*:*Mul-AppendInv-def Append-to-free0*)
done

Interference freedom Collector-Mutator

lemmas *mul-collector-mutator-interfree* =
Mul-interfree-Blacken-Roots-Redirect-Edge Mul-interfree-Blacken-Roots-Color-Target
Mul-interfree-Propagate-Black-Redirect-Edge Mul-interfree-Propagate-Black-Color-Target
Mul-interfree-Count-Redirect-Edge Mul-interfree-Count-Color-Target
Mul-interfree-Append-Redirect-Edge Mul-interfree-Append-Color-Target
Mul-interfree-Redirect-Edge-Blacken-Roots Mul-interfree-Color-Target-Blacken-Roots
Mul-interfree-Redirect-Edge-Propagate-Black Mul-interfree-Color-Target-Propagate-Black
Mul-interfree-Redirect-Edge-Count Mul-interfree-Color-Target-Count
Mul-interfree-Redirect-Edge-Append Mul-interfree-Color-Target-Append

lemma *Mul-interfree-Collector-Mutator*: *j*<*n* ⟹
interfree-aux (*Some* (*Mul-Collector n*), {}, *Some* (*Mul-Mutator j n*))
apply(*unfold Mul-Collector-def Mul-Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-mutator-interfree*)
apply(*unfold mul-modules mul-collector-defs mul-mutator-defs*)
apply(*tactic* ‹*TRYALL* (*interfree-aux-tac* ***context***)›)
— 42 subgoals left
apply (*clarify*,*simp add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)+
— 24 subgoals left
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 14 subgoals left
apply(*tactic* ‹*TRYALL* (*clarify-tac* ***context***)›)
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
apply(*tactic* ‹*TRYALL* (*resolve-tac* **context** [*conjI*])›)
apply(*tactic* ‹*TRYALL* (*resolve-tac* **context** [*impI*])›)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* **context** [*disjE*])›)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* **context** [*conjE*])›)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* **context** [*disjE*])›)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* **context** [*disjE*])›)
— 72 subgoals left
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 35 subgoals left

apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI1*],
resolve-tac ***context*** [*subset-trans*],
eresolve-tac ***context*** @{*thms Graph3*},
force-tac ***context***,
assume-tac ***context***])›)
— 28 subgoals left
apply(*tactic* ‹*TRYALL* (*eresolve-tac* ***context*** [*conjE*])›)
apply(*tactic* ‹*TRYALL* (*eresolve-tac* ***context*** [*disjE*])›)
— 34 subgoals left
apply(*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac* [!] *M x*!(*T* (*Muts x* ! *j*))=*Black*)
apply(*simp-all add*:*Graph10*)
— 47 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*REPEAT o resolve-tac* ***context*** [*disjI2*],
eresolve-tac ***context*** @{*thms subset-psubset-trans*},
eresolve-tac ***context*** @{*thms Graph11*},
force-tac ***context***])›)
— 41 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
resolve-tac ***context*** [*disjI1*],
eresolve-tac ***context*** @{*thms le-trans*},
force-tac (***context*** *addsimps* @{*thms Queue-def less-Suc-eq-le le-length-filter-update*})])›)
— 35 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
resolve-tac ***context*** [*disjI1*],
eresolve-tac ***context*** @{*thms psubset-subset-trans*},
resolve-tac ***context*** @{*thms Graph9*},
force-tac ***context***])›)
— 31 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
resolve-tac ***context*** [*disjI1*],
eresolve-tac ***context*** @{*thms subset-psubset-trans*},
eresolve-tac ***context*** @{*thms Graph11*},
force-tac ***context***])›)
— 29 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*REPEAT o resolve-tac* ***context*** [*disjI2*],
eresolve-tac ***context*** @{*thms subset-psubset-trans*},
eresolve-tac ***context*** @{*thms subset-psubset-trans*},
eresolve-tac ***context*** @{*thms Graph11*},
force-tac ***context***])›)
— 25 subgoals left
apply(*tactic* ‹*TRYALL*(*EVERY'*[*resolve-tac* ***context*** [*disjI2*],
resolve-tac ***context*** [*disjI2*],
resolve-tac ***context*** [*disjI1*],
eresolve-tac ***context*** @{*thms le-trans*},
force-tac (***context*** *addsimps* @{*thms Queue-def less-Suc-eq-le le-length-filter-update*})])›)

— 10 subgoals left
apply(*rule disjI2*,*rule disjI2*,*rule conjI*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*, *rule disjI1*, *rule less-imp-le*, *erule less-le-trans*, *force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)+
done

Interference freedom Mutator-Collector

lemma *Mul-interfree-Mutator-Collector*: $j < n \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator j n*), {}, *Some* (*Mul-Collector n*))
apply(*unfold Mul-Collector-def Mul-Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-mutator-interfree*)
apply(*unfold mul-modules mul-collector-defs mul-mutator-defs*)
apply(*tactic* ‹*TRYALL* (*interfree-aux-tac* ***context***)›)
— 76 subgoals left
apply (*clarsimp simp add*: *nth-list-update*)+
— 56 subgoals left
apply (*clarsimp simp add*: *Mul-AppendInv-def Append-to-free0 nth-list-update*)+
done

The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

lemma *Mul-Gar-Coll*:
‖− {|´*Mul-Proper n* ∧ ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))|}
COBEGIN
Mul-Collector n
{|*False*|}
‖
SCHEME [*0*≤ *j*< *n*]
Mul-Mutator j n
{|*False*|}
COEND
{|*False*|}
apply *oghoare*
— Strengthening the precondition
apply(*rule Int-greatest*)
apply (*case-tac n*)
apply(*force simp add*: *Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
apply(*simp add*: *Mul-Mutator-def mul-collector-defs mul-mutator-defs nth-append*)
apply *force*
apply *clarify*
apply(*case-tac i*)
apply(*simp add*:*Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
apply(*simp add*: *Mul-Mutator-def mul-mutator-defs mul-collector-defs nth-append nth-map-upt*)
— Collector
apply(*rule Mul-Collector*)

— Mutator
apply(*erule Mul-Mutator*)
— Interference freedom
apply(*simp add*:*Mul-interfree-Collector-Mutator*)
apply(*simp add*:*Mul-interfree-Mutator-Collector*)
apply(*simp add*:*Mul-interfree-Mutator-Mutator*)
— Weakening of the postcondition
apply(*case-tac n*)
apply(*simp add*:*Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
apply(*simp add*:*Mul-Mutator-def mul-mutator-defs mul-collector-defs nth-append*)
done

end

Chapter 3

The Rely-Guarantee Method

3.1 Abstract Syntax

theory *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (bexp) as sets of states. Syntax of commands *com* and parallel commands *par-com*.

type-synonym *'a bexp* = *'a set*

datatype *'a com* =
  Basic 'a ⇒'a
| *Seq 'a com 'a com*
| *Cond 'a bexp 'a com 'a com*
| *While 'a bexp 'a com*
| *Await 'a bexp 'a com*

type-synonym *'a par-com* = *'a com option list*

end

3.2 Operational Semantics

theory *RG-Tran*
imports *RG-Com*
begin

3.2.1 Semantics of Component Programs

Environment transitions

type-synonym *'a conf* = ((*'a com*) *option*) × *'a*

inductive-set
  etran :: (*'a conf* × *'a conf*) *set*
  and *etran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*e*→ - [*81*,*81*] *80*)

where
P $-e\rightarrow$ $Q \equiv (P,Q) \in$ *etran*
| *Env*: (P, s) $-e\rightarrow$ (P, t)

lemma *etranE*: c $-e\rightarrow$ $c' \Longrightarrow (\bigwedge P\ s\ t.\ c = (P, s) \Longrightarrow c' = (P, t) \Longrightarrow Q) \Longrightarrow Q$
by (*induct c*, *induct c'*, *erule etran.cases*, *blast*)

Component transitions

inductive-set
ctran :: ('*a conf* × '*a conf*) *set*
and *ctran'* :: '*a conf* ⇒ '*a conf* ⇒ *bool* (- $-c\rightarrow$ - [*81*,*81*] *80*)
and *ctrans* :: '*a conf* ⇒ '*a conf* ⇒ *bool* (- $-c{*}\rightarrow$ - [*81*,*81*] *80*)
where
P $-c\rightarrow$ $Q \equiv (P,Q) \in$ *ctran*
| P $-c{*}\rightarrow$ $Q \equiv (P,Q) \in ctran^*$

| *Basic*: (*Some*(*Basic f*), *s*) $-c\rightarrow$ (*None*, *f s*)

| *Seq1*: (*Some P0*, *s*) $-c\rightarrow$ (*None*, *t*) ⟹ (*Some*(*Seq P0 P1*), *s*) $-c\rightarrow$ (*Some P1*, *t*)

| *Seq2*: (*Some P0*, *s*) $-c\rightarrow$ (*Some P2*, *t*) ⟹ (*Some*(*Seq P0 P1*), *s*) $-c\rightarrow$ (*Some*(*Seq P2 P1*), *t*)

| *CondT*: *s*∈*b* ⟹ (*Some*(*Cond b P1 P2*), *s*) $-c\rightarrow$ (*Some P1*, *s*)
| *CondF*: *s*∉*b* ⟹ (*Some*(*Cond b P1 P2*), *s*) $-c\rightarrow$ (*Some P2*, *s*)

| *WhileF*: *s*∉*b* ⟹ (*Some*(*While b P*), *s*) $-c\rightarrow$ (*None*, *s*)
| *WhileT*: *s*∈*b* ⟹ (*Some*(*While b P*), *s*) $-c\rightarrow$ (*Some*(*Seq P* (*While b P*)), *s*)

| *Await*: ⟦*s*∈*b*; (*Some P*, *s*) $-c{*}\rightarrow$ (*None*, *t*)⟧ ⟹ (*Some*(*Await b P*), *s*) $-c\rightarrow$ (*None*, *t*)

monos *rtrancl-mono*

3.2.2 Semantics of Parallel Programs

type-synonym '*a par-conf* = ('*a par-com*) × '*a*

inductive-set
par-etran :: ('*a par-conf* × '*a par-conf*) *set*
and *par-etran'* :: ['*a par-conf*,'*a par-conf*] ⇒ *bool* (- $-pe\rightarrow$ - [*81*,*81*] *80*)
where
P $-pe\rightarrow$ $Q \equiv (P,Q) \in$ *par-etran*
| *ParEnv*: (*Ps*, *s*) $-pe\rightarrow$ (*Ps*, *t*)

inductive-set
par-ctran :: ('*a par-conf* × '*a par-conf*) *set*
and *par-ctran'* :: ['*a par-conf*,'*a par-conf*] ⇒ *bool* (- $-pc\rightarrow$ - [*81*,*81*] *80*)

where
  $P\ {-pc\rightarrow}\ Q \equiv (P,Q) \in \textit{par-ctran}$
| *ParComp*: $[\![i<\textit{length}\ Ps;\ (Ps!i,\ s)\ {-c\rightarrow}\ (r,\ t)]\!] \Longrightarrow (Ps,\ s)\ {-pc\rightarrow}\ (Ps[i:=r],\ t)$

lemma *par-ctranE*: $c\ {-pc\rightarrow}\ c' \Longrightarrow$
  $(\bigwedge i\ Ps\ s\ r\ t.\ c = (Ps,\ s) \Longrightarrow c' = (Ps[i := r],\ t) \Longrightarrow i < \textit{length}\ Ps \Longrightarrow$
    $(Ps\ !\ i,\ s)\ {-c\rightarrow}\ (r,\ t) \Longrightarrow P) \Longrightarrow P$
  by (*induct c*, *induct c'*, *erule par-ctran.cases*, *blast*)

3.2.3 Computations

Sequential computations

type-synonym *'a confs = 'a conf list*

inductive-set *cptn* :: *'a confs set*
where
  CptnOne: $[(P,s)] \in \textit{cptn}$
| *CptnEnv*: $(P,\ t)\#xs \in \textit{cptn} \Longrightarrow (P,s)\#(P,t)\#xs \in \textit{cptn}$
| *CptnComp*: $[\![(P,s)\ {-c\rightarrow}\ (Q,t);\ (Q,\ t)\#xs \in \textit{cptn}\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs \in \textit{cptn}$

definition *cp* :: *('a com) option* ⇒ *'a* ⇒ *('a confs) set* **where**
  $cp\ P\ s \equiv \{l.\ l!0=(P,s) \land l \in \textit{cptn}\}$

Parallel computations

type-synonym *'a par-confs = 'a par-conf list*

inductive-set *par-cptn* :: *'a par-confs set*
where
  ParCptnOne: $[(P,s)] \in \textit{par-cptn}$
| *ParCptnEnv*: $(P,t)\#xs \in \textit{par-cptn} \Longrightarrow (P,s)\#(P,t)\#xs \in \textit{par-cptn}$
| *ParCptnComp*: $[\![\ (P,s)\ {-pc\rightarrow}\ (Q,t);\ (Q,t)\#xs \in \textit{par-cptn}\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs$
$\in \textit{par-cptn}$

definition *par-cp* :: *'a par-com* ⇒ *'a* ⇒ *('a par-confs) set* **where**
  $\textit{par-cp}\ P\ s \equiv \{l.\ l!0=(P,s) \land l \in \textit{par-cptn}\}$

3.2.4 Modular Definition of Computation

definition *lift* :: *'a com* ⇒ *'a conf* ⇒ *'a conf* **where**
  $\textit{lift}\ Q \equiv \lambda(P,\ s).\ (\textit{if}\ P{=}\textit{None then}\ (\textit{Some}\ Q,s)\ \textit{else}\ (\textit{Some}(\textit{Seq}\ (\textit{the}\ P)\ Q),\ s))$

inductive-set *cptn-mod* :: *('a confs) set*
where
  CptnModOne: $[(P,\ s)] \in \textit{cptn-mod}$
| *CptnModEnv*: $(P,\ t)\#xs \in \textit{cptn-mod} \Longrightarrow (P,\ s)\#(P,\ t)\#xs \in \textit{cptn-mod}$
| *CptnModNone*: $[\![(\textit{Some}\ P,\ s)\ {-c\rightarrow}\ (\textit{None},\ t);\ (\textit{None},\ t)\#xs \in \textit{cptn-mod}\ ]\!] \Longrightarrow$
$(\textit{Some}\ P,s)\#(\textit{None},\ t)\#xs \in \textit{cptn-mod}$

```
| CptnModCondT: ⟦(Some P0, s)#ys ∈ cptn-mod; s ∈ b ⟧ ⟹ (Some(Cond b P0
P1), s)#(Some P0, s)#ys ∈ cptn-mod
| CptnModCondF: ⟦(Some P1, s)#ys ∈ cptn-mod; s ∉ b ⟧ ⟹ (Some(Cond b P0
P1), s)#(Some P1, s)#ys ∈ cptn-mod
| CptnModSeq1: ⟦(Some P0, s)#xs ∈ cptn-mod; zs=map (lift P1) xs ⟧
                ⟹ (Some(Seq P0 P1), s)#zs ∈ cptn-mod
| CptnModSeq2:
  ⟦(Some P0, s)#xs ∈ cptn-mod; fst(last ((Some P0, s)#xs)) = None;
  (Some P1, snd(last ((Some P0, s)#xs)))#ys ∈ cptn-mod;
  zs=(map (lift P1) xs)@ys ⟧ ⟹ (Some(Seq P0 P1), s)#zs ∈ cptn-mod

| CptnModWhile1:
  ⟦ (Some P, s)#xs ∈ cptn-mod; s ∈ b; zs=map (lift (While b P)) xs ⟧
  ⟹ (Some(While b P), s)#(Some(Seq P (While b P)), s)#zs ∈ cptn-mod
| CptnModWhile2:
  ⟦ (Some P, s)#xs ∈ cptn-mod; fst(last ((Some P, s)#xs))=None; s ∈ b;
  zs=(map (lift (While b P)) xs)@ys;
  (Some(While b P), snd(last ((Some P, s)#xs)))#ys ∈ cptn-mod⟧
  ⟹ (Some(While b P), s)#(Some(Seq P (While b P)), s)#zs ∈ cptn-mod
```

3.2.5 Equivalence of Both Definitions.

```
lemma last-length: ((a#xs)!(length xs))=last (a#xs)
  by (induct xs) auto

lemma div-seq [rule-format]: list ∈ cptn-mod ⟹
 (∀ s P Q zs. list=(Some (Seq P Q), s)#zs ⟶
  (∃ xs. (Some P, s)#xs ∈ cptn-mod  ∧ (zs=(map (lift Q) xs) ∨
  ( fst(((Some P, s)#xs)!length xs)=None ∧
  (∃ ys. (Some Q, snd(((Some P, s)#xs)!length xs))#ys ∈ cptn-mod
  ∧ zs=(map (lift (Q)) xs)@ys)))))
apply(erule cptn-mod.induct)
apply simp-all
    apply clarify
    apply(force intro:CptnModOne)
   apply clarify
   apply(erule-tac x=Pa in allE)
   apply(erule-tac x=Q in allE)
   apply simp
   apply clarify
   apply(erule disjE)
    apply(rule-tac x=(Some Pa,t)#xsa in exI)
    apply(rule conjI)
     apply clarify
     apply(erule CptnModEnv)
    apply(rule disjI1)
    apply(simp add:lift-def)
   apply clarify
   apply(rule-tac x=(Some Pa,t)#xsa in exI)
```

```
   apply(rule conjI)
    apply(erule CptnModEnv)
   apply(rule disjI2)
   apply(rule conjI)
    apply(case-tac xsa,simp,simp)
   apply(rule-tac x=ys in exI)
   apply(rule conjI)
    apply simp
   apply(simp add:lift-def)
  apply clarify
  apply(erule ctran.cases,simp-all)
 apply clarify
 apply(rule-tac x=xs in exI)
 apply simp
 apply clarify
apply(rule-tac x=xs in exI)
apply(simp add: last-length)
done

lemma cptn-onlyif-cptn-mod-aux [rule-format]:
  ∀s Q t xs.((Some a, s), Q, t) ∈ ctran ⟶ (Q, t) # xs ∈ cptn-mod
  ⟶ (Some a, s) # (Q, t) # xs ∈ cptn-mod
  supply [[simproc del: defined-all]]
apply(induct a)
apply simp-all
— basic
apply clarify
apply(erule ctran.cases,simp-all)
apply(rule CptnModNone,rule Basic,simp)
apply clarify
apply(erule ctran.cases,simp-all)
— Seq1
apply(rule-tac xs=[(None,ta)] in CptnModSeq2)
  apply(erule CptnModNone)
  apply(rule CptnModOne)
 apply simp
apply simp
apply(simp add:lift-def)
— Seq2
apply(erule-tac x=sa in allE)
apply(erule-tac x=Some P2 in allE)
apply(erule allE,erule impE, assumption)
apply(drule div-seq,simp)
apply clarify
apply(erule disjE)
 apply clarify
 apply(erule allE,erule impE, assumption)
 apply(erule-tac CptnModSeq1)
 apply(simp add:lift-def)
```

```
apply clarify
apply(erule allE,erule impE, assumption)
apply(erule-tac CptnModSeq2)
  apply (simp add:last-length)
 apply (simp add:last-length)
apply(simp add:lift-def)
— Cond
apply clarify
apply(erule ctran.cases,simp-all)
apply(force elim: CptnModCondT)
apply(force elim: CptnModCondF)
— While
apply  clarify
apply(erule ctran.cases,simp-all)
apply(rule CptnModNone,erule WhileF,simp)
apply(drule div-seq,force)
apply clarify
apply (erule disjE)
 apply(force elim:CptnModWhile1)
apply clarify
apply(force simp add:last-length elim:CptnModWhile2)
— await
apply clarify
apply(erule ctran.cases,simp-all)
apply(rule CptnModNone,erule Await,simp+)
done

lemma cptn-onlyif-cptn-mod [rule-format]: c ∈ cptn ⟹ c ∈ cptn-mod
apply(erule cptn.induct)
  apply(rule CptnModOne)
 apply(erule CptnModEnv)
apply(case-tac P)
 apply simp
 apply(erule ctran.cases,simp-all)
apply(force elim:cptn-onlyif-cptn-mod-aux)
done

lemma lift-is-cptn: c∈cptn ⟹ map (lift P) c ∈ cptn
apply(erule cptn.induct)
  apply(force simp add:lift-def CptnOne)
 apply(force intro:CptnEnv simp add:lift-def)
apply(force simp add:lift-def intro:CptnComp Seq2 Seq1 elim:ctran.cases)
done

lemma cptn-append-is-cptn [rule-format]:
∀ b a. b#c1∈cptn ⟶  a#c2∈cptn ⟶ (b#c1)!length c1=a ⟶ b#c1@c2∈cptn
apply(induct c1)
 apply simp
apply clarify
```

apply(*erule cptn.cases*,*simp-all*)
apply(*force intro*:*CptnEnv*)
apply(*force elim*:*CptnComp*)
done

lemma *last-lift*: ⟦*xs*≠[]; *fst*(*xs*!(*length xs* − (*Suc 0*)))=*None*⟧
⟹ *fst*((*map* (*lift P*) *xs*)!(*length* (*map* (*lift P*) *xs*)− (*Suc 0*)))=(*Some P*)
by (*cases* (*xs* ! (*length xs* − (*Suc 0*)))) (*simp add*:*lift-def*)

lemma *last-fst* [*rule-format*]: *P*((*a*#*x*)!*length x*) ⟶ ¬*P a* ⟶ *P* (*x*!(*length x* − (*Suc 0*)))
by (*induct x*) *simp-all*

lemma *last-fst-esp*:
fst(((*Some a*,*s*)#*xs*)!(*length xs*))=*None* ⟹ *fst*(*xs*!(*length xs* − (*Suc 0*)))=*None*
apply(*erule last-fst*)
apply *simp*
done

lemma *last-snd*: *xs*≠[] ⟹
snd(((*map* (*lift P*) *xs*))!(*length* (*map* (*lift P*) *xs*) − (*Suc 0*)))=*snd*(*xs*!(*length xs* − (*Suc 0*)))
by (*cases* (*xs* ! (*length xs* − (*Suc 0*)))) (*simp-all add*:*lift-def*)

lemma *Cons-lift*: (*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)
by (*simp add*:*lift-def*)

lemma *Cons-lift-append*:
(*Some* (*Seq P Q*), *s*) # (*map* (*lift Q*) *xs*) @ *ys* = *map* (*lift Q*) ((*Some P*, *s*) # *xs*)@ *ys*
by (*simp add*:*lift-def*)

lemma *lift-nth*: *i*<*length xs* ⟹ *map* (*lift Q*) *xs* ! *i* = *lift Q* (*xs*! *i*)
by (*simp add*:*lift-def*)

lemma *snd-lift*: *i*< *length xs* ⟹ *snd*(*lift Q* (*xs* ! *i*))= *snd* (*xs* ! *i*)
by (*cases xs*!*i*) (*simp add*:*lift-def*)

lemma *cptn-if-cptn-mod*: *c* ∈ *cptn-mod* ⟹ *c* ∈ *cptn*
apply(*erule cptn-mod.induct*)
apply(*rule CptnOne*)
apply(*erule CptnEnv*)
apply(*erule CptnComp*,*simp*)
apply(*rule CptnComp*)
apply(*erule CondT*,*simp*)
apply(*rule CptnComp*)
apply(*erule CondF*,*simp*)
— Seq1

```
apply(erule cptn.cases,simp-all)
  apply(rule CptnOne)
 apply clarify
 apply(drule-tac P=P1 in lift-is-cptn)
 apply(simp add:lift-def)
 apply(rule CptnEnv,simp)
apply clarify
apply(simp add:lift-def)
apply(rule conjI)
 apply clarify
 apply(rule CptnComp)
  apply(rule Seq1,simp)
 apply(drule-tac P=P1 in lift-is-cptn)
 apply(simp add:lift-def)
apply clarify
apply(rule CptnComp)
 apply(rule Seq2,simp)
apply(drule-tac P=P1 in lift-is-cptn)
apply(simp add:lift-def)
— Seq2
apply(rule cptn-append-is-cptn)
  apply(drule-tac P=P1 in lift-is-cptn)
  apply(simp add:lift-def)
 apply simp
apply(simp split: if-split-asm)
apply(frule-tac P=P1 in last-lift)
 apply(rule last-fst-esp)
 apply (simp add:last-length)
apply(simp add:Cons-lift lift-def split-def last-conv-nth)
— While1
apply(rule CptnComp)
 apply(rule WhileT,simp)
apply(drule-tac P=While b P in lift-is-cptn)
apply(simp add:lift-def)
— While2
apply(rule CptnComp)
 apply(rule WhileT,simp)
apply(rule cptn-append-is-cptn)
  apply(drule-tac P=While b P in lift-is-cptn)
  apply(simp add:lift-def)
 apply simp
apply(simp split: if-split-asm)
apply(frule-tac P=While b P in last-lift)
 apply(rule last-fst-esp,simp add:last-length)
apply(simp add:Cons-lift lift-def split-def last-conv-nth)
done

theorem cptn-iff-cptn-mod: (c ∈ cptn) = (c ∈ cptn-mod)
apply(rule iffI)
```

```
apply(erule cptn-onlyif-cptn-mod)
apply(erule cptn-if-cptn-mod)
done
```

3.3 Validity of Correctness Formulas

3.3.1 Validity for Component Programs.

```
type-synonym 'a rgformula =
  'a com × 'a set × ('a × 'a) set × ('a × 'a) set × 'a set

definition assum :: ('a set × ('a × 'a) set) ⇒ ('a confs) set where
  assum ≡ λ(pre, rely). {c. snd(c!0) ∈ pre ∧ (∀ i. Suc i<length c ⟶
            c!i −e→ c!(Suc i) ⟶ (snd(c!i), snd(c!Suc i)) ∈ rely)}

definition comm :: (('a × 'a) set × 'a set) ⇒ ('a confs) set where
  comm ≡ λ(guar, post). {c. (∀ i. Suc i<length c ⟶
            c!i −c→ c!(Suc i) ⟶ (snd(c!i), snd(c!Suc i)) ∈ guar) ∧
            (fst (last c) = None ⟶ snd (last c) ∈ post)}

definition com-validity :: 'a com ⇒ 'a set ⇒ ('a × 'a) set ⇒ ('a × 'a) set ⇒ 'a
set ⇒ bool
                 (⊨ - sat [-, -, -, -] [60,0,0,0,0] 45) where
  ⊨ P sat [pre, rely, guar, post] ≡
   ∀ s. cp (Some P) s ∩ assum(pre, rely) ⊆ comm(guar, post)
```

3.3.2 Validity for Parallel Programs.

```
definition All-None :: (('a com) option) list ⇒ bool where
  All-None xs ≡ ∀ c∈set xs. c=None

definition par-assum :: ('a set × ('a × 'a) set) ⇒ ('a par-confs) set where
  par-assum ≡ λ(pre, rely). {c. snd(c!0) ∈ pre ∧ (∀ i. Suc i<length c ⟶
           c!i −pe→ c!Suc i ⟶ (snd(c!i), snd(c!Suc i)) ∈ rely)}

definition par-comm :: (('a × 'a) set × 'a set) ⇒ ('a par-confs) set where
  par-comm ≡ λ(guar, post). {c. (∀ i. Suc i<length c ⟶
        c!i −pc→ c!Suc i ⟶ (snd(c!i), snd(c!Suc i)) ∈ guar) ∧
        (All-None (fst (last c)) ⟶ snd( last c) ∈ post)}

definition par-com-validity :: 'a  par-com ⇒ 'a set ⇒ ('a × 'a) set ⇒ ('a × 'a)
set
⇒ 'a set ⇒ bool  (⊨ - SAT [-, -, -, -] [60,0,0,0,0] 45) where
  ⊨ Ps SAT [pre, rely, guar, post] ≡
   ∀ s. par-cp Ps s ∩ par-assum(pre, rely) ⊆ par-comm(guar, post)
```

3.3.3 Compositionality of the Semantics

Definition of the conjoin operator

definition *same-length* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-length c clist ≡ (∀ *i<length clist. length*(*clist*!*i*)=*length c*)

definition *same-state* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-state c clist ≡ (∀ *i <length clist.* ∀ *j<length c. snd*(*c*!*j*) = *snd*((*clist*!*i*)!*j*))

definition *same-program* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
same-program c clist ≡ (∀ *j<length c. fst*(*c*!*j*) = *map* (λ*x. fst*(*nth x j*)) *clist*)

definition *compat-label* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* **where**
compat-label c clist ≡ (∀ *j. Suc j<length c* ⟶
(*c*!*j* −*pc*→ *c*!*Suc j* ∧ (∃ *i<length clist.* (*clist*!*i*)!*j* −*c*→ (*clist*!*i*)! *Suc j* ∧
(∀ *l<length clist. l≠i* ⟶ (*clist*!*l*)!*j* −*e*→ (*clist*!*l*)! *Suc j*))) ∨
(*c*!*j* −*pe*→ *c*!*Suc j* ∧ (∀ *i<length clist.* (*clist*!*i*)!*j* −*e*→ (*clist*!*i*)! *Suc j*)))

definition *conjoin* :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* (- ∝ - [*65*,*65*] *64*)
where
c ∝ *clist* ≡ (*same-length c clist*) ∧ (*same-state c clist*) ∧ (*same-program c clist*)
∧ (*compat-label c clist*)

Some previous lemmas

lemma *list-eq-if* [*rule-format*]:
∀ *ys. xs=ys* ⟶ (*length xs* = *length ys*) ⟶ (∀ *i<length xs. xs*!*i*=*ys*!*i*)
by (*induct xs*) *auto*

lemma *list-eq*: (*length xs* = *length ys* ∧ (∀ *i<length xs. xs*!*i*=*ys*!*i*)) = (*xs*=*ys*)
apply(*rule iffI*)
apply *clarify*
apply(*erule nth-equalityI*)
apply *simp+*
done

lemma *nth-tl*: ⟦ *ys*!*0*=*a*; *ys*≠[] ⟧ ⟹ *ys*=(*a*#(*tl ys*))
by (*cases ys*) *simp-all*

lemma *nth-tl-if* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P ys* ⟶ *P* (*a*#(*tl ys*))
by (*induct ys*) *simp-all*

lemma *nth-tl-onlyif* [*rule-format*]: *ys*≠[] ⟶ *ys*!*0*=*a* ⟶ *P* (*a*#(*tl ys*)) ⟶ *P ys*
by (*induct ys*) *simp-all*

lemma *seq-not-eq1*: *Seq c1 c2*≠*c1*
by (*induct c1*) *auto*

lemma *seq-not-eq2*: *Seq c1 c2*≠*c2*

by (*induct c2*) *auto*

lemma *if-not-eq1*: *Cond b c1 c2* $\neq$*c1*
  by (*induct c1*) *auto*

lemma *if-not-eq2*: *Cond b c1 c2*$\neq$*c2*
  by (*induct c2*) *auto*

lemmas *seq-and-if-not-eq* [*simp*] = *seq-not-eq1 seq-not-eq2*
seq-not-eq1 [*THEN not-sym*] *seq-not-eq2* [*THEN not-sym*]
if-not-eq1 if-not-eq2 if-not-eq1 [*THEN not-sym*] *if-not-eq2* [*THEN not-sym*]

lemma *prog-not-eq-in-ctran-aux*:
  assumes *c*: (P,s) $-c\rightarrow$ (Q,t)
  shows $P\neq Q$ **using** *c*
  by (*induct x1* $\equiv$ (P,s) *x2* $\equiv$ (Q,t) *arbitrary*: *P s Q t*) *auto*

lemma *prog-not-eq-in-ctran* [*simp*]: $\neg$ (P,s) $-c\rightarrow$ (P,t)
apply *clarify*
apply(*drule prog-not-eq-in-ctran-aux*)
apply *simp*
done

lemma *prog-not-eq-in-par-ctran-aux* [*rule-format*]: (P,s) $-pc\rightarrow$ (Q,t) $\Longrightarrow$ $(P\neq Q)$
apply(*erule par-ctran.induct*)
apply(*drule prog-not-eq-in-ctran-aux*)
apply *clarify*
apply(*drule list-eq-if*)
 apply *simp-all*
apply *force*
done

lemma *prog-not-eq-in-par-ctran* [*simp*]: $\neg$ (P,s) $-pc\rightarrow$ (P,t)
apply *clarify*
apply(*drule prog-not-eq-in-par-ctran-aux*)
apply *simp*
done

lemma *tl-in-cptn*: $[\![$ *a#xs* $\in$*cptn*; *xs*$\neq$[] $]\!]$ $\Longrightarrow$ *xs*$\in$*cptn*
  by (*force elim*: *cptn.cases*)

lemma *tl-zero*[*rule-format*]:
  P (*ys*!*Suc j*) $\longrightarrow$ *Suc j*<*length ys* $\longrightarrow$ *ys*$\neq$[] $\longrightarrow$ *P* (*tl*(*ys*)!*j*)
  by (*induct ys*) *simp-all*

3.3.4 The Semantics is Compositional

lemma *aux-if* [*rule-format*]:
  $\forall$ *xs s clist*. (*length clist* = *length xs* $\wedge$ ($\forall$ *i*<*length xs*. (*xs*!*i*,*s*)#*clist*!*i* $\in$ *cptn*)

∧ ((xs, s)#ys ∝ map (λi. (fst i,s)#snd i) (zip xs clist))
⟶ (xs, s)#ys ∈ par-cptn)
apply(*induct ys*)
apply(*clarify*)
apply(*rule ParCptnOne*)
apply(*clarify*)
apply(*simp add:conjoin-def compat-label-def*)
apply *clarify*
apply(*erule-tac x=0* **and** *P=λj. H j ⟶ (P j ∨ Q j)* **for** *H P Q* **in** *all-dupE*, *simp*)
apply(*erule disjE*)
— first step is a Component step
apply *clarify*
apply *simp*
apply(*subgoal-tac a=(xs[i:=(fst(clist!i!0))])*)
apply(*subgoal-tac b=snd(clist!i!0)*,*simp*)
prefer *2*
apply(*simp add: same-state-def*)
apply(*erule-tac x=i* **in** *allE*,*erule impE*,*assumption*,
erule-tac x=1 **and** *P=λj. (H j) ⟶ (snd (d j))=(snd (e j))* **for** *H d e* **in** *allE*, *simp*)
prefer *2*
apply(*simp add:same-program-def*)
apply(*erule-tac x=1* **and** *P=λj. H j ⟶ (fst (s j))=(t j)* **for** *H s t* **in** *allE*,*simp*)
apply(*rule nth-equalityI*,*simp*)
apply *clarify*
apply(*case-tac i=ia*,*simp*,*simp*)
apply(*erule-tac x=ia* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*)
apply(*drule-tac t=i* **in** *not-sym*,*simp*)
apply(*erule etranE*,*simp*)
apply(*rule ParCptnComp*)
apply(*erule ParComp*,*simp*)
— applying the induction hypothesis
apply(*erule-tac x=xs[i := fst (clist ! i ! 0)]* **in** *allE*)
apply(*erule-tac x=snd (clist ! i ! 0)* **in** *allE*)
apply(*erule mp*)
apply(*rule-tac x=map tl clist* **in** *exI*,*simp*)
apply(*rule conjI*,*clarify*)
apply(*case-tac i=ia*,*simp*)
apply(*rule nth-tl-if*)
apply(*force simp add:same-length-def length-Suc-conv*)
apply *simp*
apply(*erule allE*,*erule impE*,*assumption*,*erule tl-in-cptn*)
apply(*force simp add:same-length-def length-Suc-conv*)
apply(*rule nth-tl-if*)
apply(*force simp add:same-length-def length-Suc-conv*)
apply(*simp add:same-state-def*)
apply(*erule-tac x=ia* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P=λj. H j ⟶ (snd (d j))=(snd (e j))* **for** *H d e* **in** *allE*)

```
   apply(erule-tac x=ia and P=λj. H j ⟶ I j ⟶ J j for H I J in allE)
   apply(drule-tac t=i  in not-sym,simp)
   apply(erule etranE,simp)
  apply(erule allE,erule impE,assumption,erule tl-in-cptn)
  apply(force simp add:same-length-def length-Suc-conv)
 apply(simp add:same-length-def same-state-def)
 apply(rule conjI)
  apply clarify
  apply(case-tac j,simp,simp)
  apply(erule-tac x=ia in allE, erule impE, assumption,
       erule-tac x=Suc(Suc nat) and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for
H d e in allE,simp)
  apply(force simp add:same-length-def length-Suc-conv)
 apply(rule conjI)
  apply(simp add:same-program-def)
  apply clarify
  apply(case-tac j,simp)
   apply(rule nth-equalityI,simp)
   apply clarify
   apply(case-tac i=ia,simp,simp)
  apply(erule-tac x=Suc(Suc nat) and P=λj. H j ⟶ (fst (s j))=(t j) for H s t
in allE,simp)
  apply(rule nth-equalityI,simp,simp)
  apply(force simp add:length-Suc-conv)
 apply(rule allI,rule impI)
 apply(erule-tac x=Suc j and P=λj. H j ⟶ (I j ∨ J j) for H I J in allE,simp)
 apply(erule disjE)
  apply clarify
  apply(rule-tac x=ia in exI,simp)
  apply(case-tac i=ia,simp)
   apply(rule conjI)
    apply(force simp add: length-Suc-conv)
   apply clarify
   apply(erule-tac x=l and P=λj. H j ⟶ I j ⟶ J j for H I J in allE,erule
impE,assumption)
   apply(erule-tac x=l and P=λj. H j ⟶ I j ⟶ J j for H I J in allE,erule
impE,assumption)
   apply simp
   apply(case-tac j,simp)
    apply(rule tl-zero)
      apply(erule-tac x=l in allE, erule impE, assumption,
            erule-tac x=1 and P=λj.  (H j) ⟶ (snd (d j))=(snd (e j)) for H d e
in allE,simp)
      apply(force elim:etranE intro:Env)
     apply force
    apply force
   apply simp
   apply(rule tl-zero)
    apply(erule tl-zero)
```

apply *force*
apply *force*
apply *force*
apply *force*
apply(*rule conjI*,*simp*)
apply(*rule nth-tl-if*)
apply *force*
apply(*erule-tac x=ia* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P=λj. H j ⟶ (snd (d j))=(snd (e j))* **for** *H d e* **in** *allE*)
apply(*erule-tac x=ia* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*)
apply(*drule-tac t=i* **in** *not-sym*,*simp*)
apply(*erule etranE*,*simp*)
apply(*erule tl-zero*)
apply *force*
apply *force*
apply *clarify*
apply(*case-tac i=l*,*simp*)
apply(*rule nth-tl-if*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*,*force*)
apply *simp*
apply(*erule-tac P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*,*erule impE*,*assumption*,*erule impE*,*assumption*)
apply(*erule tl-zero*,*force*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*,*force*)
apply(*rule nth-tl-if*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*,*force*)
apply(*erule-tac x=l* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P=λj. H j ⟶ (snd (d j))=(snd (e j))* **for** *H d e* **in** *allE*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*,*erule impE*, *assumption*,*simp*)
apply(*erule etranE*,*simp*)
apply(*rule tl-zero*)
apply *force*
apply *force*
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*,*force*)
apply(*rule disjI2*)
apply(*case-tac j*,*simp*)
apply *clarify*
apply(*rule tl-zero*)
apply(*erule-tac x=ia* **and** *P=λj. H j ⟶ I j∈etran* **for** *H I* **in** *allE*,*erule impE*, *assumption*)
apply(*case-tac i=ia*,*simp*,*simp*)
apply(*erule-tac x=ia* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P=λj. H j ⟶ (snd (d j))=(snd (e j))* **for** *H d e* **in** *allE*)
apply(*erule-tac x=ia* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*,*erule*

```
impE, assumption,simp)
    apply(force elim:etranE intro:Env)
   apply force
   apply(erule-tac x=ia and P=λj. H j ⟶ (length (s j) = t) for H s t in
allE,force)
 apply simp
 apply clarify
 apply(rule tl-zero)
   apply(rule tl-zero,force)
    apply force
    apply(erule-tac x=ia and P=λj. H j ⟶ (length (s j) = t) for H s t in
allE,force)
  apply force
 apply(erule-tac x=ia and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
— first step is an environmental step
apply clarify
apply(erule par-etran.cases)
apply simp
apply(rule ParCptnEnv)
apply(erule-tac x=Ps in allE)
apply(erule-tac x=t in allE)
apply(erule mp)
apply(rule-tac x=map tl clist in exI,simp)
apply(rule conjI)
 apply clarify
 apply(erule-tac x=i and P=λj. H j ⟶ I j ∈ cptn for H I in allE,simp)
 apply(erule cptn.cases)
   apply(simp add:same-length-def)
  apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
  apply(simp add:same-state-def)
  apply(erule-tac x=i  in allE, erule impE, assumption,
   erule-tac x=1 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in allE,simp)
 apply(erule-tac x=i and P=λj. H j ⟶ J j ∈etran for H J in allE,simp)
 apply(erule etranE,simp)
apply(simp add:same-state-def same-length-def)
apply(rule conjI,clarify)
 apply(case-tac j,simp,simp)
 apply(erule-tac x=i  in allE, erule impE, assumption,
       erule-tac x=Suc(Suc nat) and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H
d e in allE,simp)
 apply(rule tl-zero)
   apply(simp)
  apply force
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
apply(rule conjI)
 apply(simp add:same-program-def)
 apply clarify
 apply(case-tac j,simp)
  apply(rule nth-equalityI,simp)
```

```
  apply clarify
  apply simp
 apply(erule-tac x=Suc(Suc nat) and P=λj. H j ⟶ (fst (s j))=(t j) for H s t
in allE,simp)
 apply(rule nth-equalityI,simp,simp)
 apply(force simp add:length-Suc-conv)
apply(rule allI,rule impI)
apply(erule-tac x=Suc j and P=λj. H j ⟶ (I j ∨ J j) for H I J in allE,simp)
apply(erule disjE)
 apply clarify
 apply(rule-tac x=i in exI,simp)
 apply(rule conjI)
  apply(erule-tac x=i and P=λi. H i ⟶ J i ∈etran for H J in allE, erule impE,
assumption)
  apply(erule etranE,simp)
  apply(erule-tac x=i  in allE, erule impE, assumption,
        erule-tac x=1 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE,simp)
  apply(rule nth-tl-if)
   apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
  apply simp
 apply(erule tl-zero,force)
  apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
 apply clarify
 apply(erule-tac x=l and P=λi. H i ⟶ J i ∈etran for H J in allE, erule impE,
assumption)
 apply(erule etranE,simp)
 apply(erule-tac x=l  in allE, erule impE, assumption,
        erule-tac x=1 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE,simp)
 apply(rule nth-tl-if)
   apply(erule-tac x=l and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
  apply simp
  apply(rule tl-zero,force)
  apply force
 apply(erule-tac x=l and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
apply(rule disjI2)
apply simp
apply clarify
apply(case-tac j,simp)
 apply(rule tl-zero)
    apply(erule-tac x=i and P=λi. H i ⟶ J i ∈etran for H J in allE, erule
impE, assumption)
    apply(erule-tac x=i and P=λi. H i ⟶ J i ∈etran for H J in allE, erule
impE, assumption)
   apply(force elim:etranE intro:Env)
  apply force
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
apply simp
```

```
apply(rule tl-zero)
  apply(rule tl-zero,force)
   apply force
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
 apply force
apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
done

lemma aux-onlyif [rule-format]: ∀xs s. (xs, s)#ys ∈ par-cptn ⟶
  (∃clist. (length clist = length xs) ∧
  (xs, s)#ys ∝ map (λi. (fst i,s)#(snd i)) (zip xs clist) ∧
  (∀i<length xs. (xs!i,s)#(clist!i) ∈ cptn))
  supply [[simproc del: defined-all]]
apply(induct ys)
 apply(clarify)
 apply(rule-tac x=map (λi. []) [0..<length xs] in exI)
 apply(simp add: conjoin-def same-length-def same-state-def same-program-def com-
pat-label-def)
 apply(rule conjI)
  apply(rule nth-equalityI,simp,simp)
 apply(force intro: cptn.intros)
apply(clarify)
apply(erule par-cptn.cases,simp)
 apply simp
 apply(erule-tac x=xs in allE)
 apply(erule-tac x=t in allE,simp)
 apply clarify
 apply(rule-tac x=(map (λj. (P!j, t)#(clist!j)) [0..<length P]) in exI,simp)
 apply(rule conjI)
  prefer 2
  apply clarify
  apply(rule CptnEnv,simp)
 apply(simp add:conjoin-def same-length-def same-state-def)
 apply (rule conjI)
  apply clarify
  apply(case-tac j,simp,simp)
 apply(rule conjI)
  apply(simp add:same-program-def)
  apply clarify
  apply(case-tac j,simp)
   apply(rule nth-equalityI,simp,simp)
  apply simp
  apply(rule nth-equalityI,simp,simp)
 apply(simp add:compat-label-def)
 apply clarify
 apply(case-tac j,simp)
  apply(simp add:ParEnv)
  apply clarify
  apply(simp add:Env)
```

```
 apply simp
 apply(erule-tac x=nat in allE,erule impE, assumption)
 apply(erule disjE,simp)
  apply clarify
  apply(rule-tac x=i in exI,simp)
 apply force
apply(erule par-ctran.cases,simp)
apply(erule-tac x=Ps[i:=r] in allE)
apply(erule-tac x=ta in allE,simp)
apply clarify
apply(rule-tac x=(map (λj. (Ps!j, ta)#(clist!j)) [0..<length Ps]) [i:=((r, ta)#(clist!i))]
in exI,simp)
apply(rule conjI)
 prefer 2
 apply clarify
 apply(case-tac i=ia,simp)
  apply(erule CptnComp)
  apply(erule-tac x=ia and P=λj. H j ⟶ (I j ∈ cptn) for H I in allE,simp)
 apply simp
 apply(erule-tac x=ia in allE)
 apply(rule CptnEnv,simp)
apply(simp add:conjoin-def)
apply (rule conjI)
 apply(simp add:same-length-def)
 apply clarify
 apply(case-tac i=ia,simp,simp)
apply(rule conjI)
 apply(simp add:same-state-def)
 apply clarify
 apply(case-tac j, simp, simp (no-asm-simp))
 apply(case-tac i=ia,simp,simp)
apply(rule conjI)
 apply(simp add:same-program-def)
 apply clarify
 apply(case-tac j,simp)
  apply(rule nth-equalityI,simp,simp)
 apply simp
 apply(rule nth-equalityI,simp,simp)
 apply(erule-tac x=nat and P=λj. H j ⟶ (fst (a j))=((b j)) for H a b in allE)
 apply(case-tac nat)
  apply clarify
  apply(case-tac i=ia,simp,simp)
 apply clarify
 apply(case-tac i=ia,simp,simp)
apply(simp add:compat-label-def)
apply clarify
apply(case-tac j)
 apply(rule conjI,simp)
  apply(erule ParComp,assumption)
```

```
  apply clarify
  apply(rule-tac x=i in exI,simp)
 apply clarify
 apply(rule Env)
apply simp
apply(erule-tac x=nat and P=λj. H j ⟶ (P j ∨ Q j) for H P Q in allE,simp)
apply(erule disjE)
 apply clarify
 apply(rule-tac x=ia in exI,simp)
 apply(rule conjI)
  apply(case-tac i=ia,simp,simp)
 apply clarify
 apply(case-tac i=l,simp)
  apply(case-tac l=ia,simp,simp)
 apply(erule-tac x=l in allE,erule impE,assumption,erule impE, assumption,simp)
 apply simp
 apply(erule-tac x=l in allE,erule impE,assumption,erule impE, assumption,simp)
apply clarify
apply(erule-tac x=ia and P=λj. H j ⟶ (P j)∈etran for H P in allE, erule
impE, assumption)
apply(case-tac i=ia,simp,simp)
done

lemma one-iff-aux: xs≠[] ⟹ (∀ys. ((xs, s)#ys ∈ par-cptn) =
 (∃clist. length clist= length xs ∧
 ((xs, s)#ys ∝ map (λi. (fst i,s)#(snd i)) (zip xs clist)) ∧
 (∀i<length xs. (xs!i,s)#(clist!i) ∈ cptn))) =
 (par-cp (xs) s = {c. ∃clist. (length clist)=(length xs) ∧
 (∀i<length clist. (clist!i) ∈ cp(xs!i) s) ∧ c ∝ clist})
apply (rule iffI)
 apply(rule subset-antisym)
  apply(rule subsetI)
  apply(clarify)
  apply(simp add:par-cp-def cp-def)
  apply(case-tac x)
   apply(force elim:par-cptn.cases)
  apply simp
  apply(rename-tac a list)
  apply(erule-tac x=list in allE)
  apply clarify
  apply simp
  apply(rule-tac x=map (λi. (fst i, s) # snd i) (zip xs clist) in exI,simp)
 apply(rule subsetI)
 apply(clarify)
 apply(case-tac x)
  apply(erule-tac x=0 in allE)
 apply(simp add:cp-def conjoin-def same-length-def same-program-def same-state-def
compat-label-def)
  apply clarify
```

```
 apply(erule cptn.cases,force,force,force)
apply(simp add:par-cp-def conjoin-def  same-length-def same-program-def same-state-def
compat-label-def)
apply clarify
apply(erule-tac x=0 and P=λj. H j ⟶ (length (s j) = t) for H s t in all-dupE)
apply(subgoal-tac a = xs)
 apply(subgoal-tac b = s,simp)
  prefer 3
  apply(erule-tac x=0 and P=λj. H j ⟶ (fst (s j))=((t j)) for H s t in allE)
  apply (simp add:cp-def)
  apply(rule nth-equalityI,simp,simp)
 prefer 2
 apply(erule-tac x=0 in allE)
 apply (simp add:cp-def)
 apply(erule-tac x=0 and P=λj. H j ⟶ (∀ i. T i ⟶ (snd (d j i))=(snd (e j
i))) for H T d e in allE,simp)
 apply(erule-tac x=0 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE,simp)
apply(erule-tac x=list in allE)
apply(rule-tac x=map tl clist in exI,simp)
apply(rule conjI)
 apply clarify
 apply(case-tac j,simp)
  apply(erule-tac x=i  in allE, erule impE, assumption,
        erule-tac x=0 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE,simp)
 apply(erule-tac x=i  in allE, erule impE, assumption,
       erule-tac x=Suc nat and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e
in allE)
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
 apply(case-tac clist!i,simp,simp)
apply(rule conjI)
 apply clarify
 apply(rule nth-equalityI,simp,simp)
 apply(case-tac j)
  apply clarify
  apply(erule-tac x=i in allE)
  apply(simp add:cp-def)
 apply clarify
 apply simp
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
 apply(case-tac clist!i,simp,simp)
apply(thin-tac H = (∃ i. J i) for H J)
apply(rule conjI)
 apply clarify
 apply(erule-tac x=j in allE,erule impE, assumption,erule disjE)
  apply clarify
  apply(rule-tac x=i in exI,simp)
  apply(case-tac j,simp)
```

apply(*rule conjI*)
apply(*erule-tac x=i* **in** *allE*)
apply(*simp add*:*cp-def*)
apply(*erule-tac x=i* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*i*,*simp*,*simp*)
apply *clarify*
apply(*erule-tac x=l* **in** *allE*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*)
apply *clarify*
apply(*simp add*:*cp-def*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*l*,*simp*,*simp*)
apply *simp*
apply(*rule conjI*)
apply(*erule-tac x=i* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*i*,*simp*,*simp*)
apply *clarify*
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ I j ⟶ J j* **for** *H I J* **in** *allE*)
apply(*erule-tac x=l* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*l*,*simp*,*simp*)
apply *clarify*
apply(*erule-tac x=i* **in** *allE*)
apply(*simp add*:*cp-def*)
apply(*erule-tac x=i* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*i*,*simp*)
apply(*rule nth-tl-if*,*simp*,*simp*)
apply(*erule-tac x=i* **and** *P=λj. H j ⟶ (P j)∈etran* **for** *H P* **in** *allE*, *erule impE*, *assumption*,*simp*)
apply(*simp add*:*cp-def*)
apply *clarify*
apply(*rule nth-tl-if*)
apply(*erule-tac x=i* **and** *P=λj. H j ⟶ (length (s j) = t)* **for** *H s t* **in** *allE*)
apply(*case-tac clist*!*i*,*simp*,*simp*)
apply *force*
apply *force*
apply *clarify*
apply(*rule iffI*)
apply(*simp add*:*par-cp-def*)
apply(*erule-tac c=(xs, s) # ys* **in** *equalityCE*)
apply *simp*
apply *clarify*
apply(*rule-tac x=map tl clist* **in** *exI*)
apply *simp*
apply (*rule conjI*)
apply(*simp add*:*conjoin-def cp-def*)
apply(*rule conjI*)
apply *clarify*
apply(*unfold same-length-def*)
apply *clarify*

```
apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,simp)
apply(rule conjI)
 apply(simp add:same-state-def)
 apply clarify
 apply(erule-tac x=i in allE, erule impE, assumption,
   erule-tac x=j and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in allE)
 apply(case-tac j,simp)
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
 apply(case-tac clist!i,simp,simp)
apply(rule conjI)
 apply(simp add:same-program-def)
 apply clarify
 apply(rule nth-equalityI,simp,simp)
 apply(case-tac j,simp)
 apply clarify
 apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
 apply(case-tac clist!i,simp,simp)
apply clarify
apply(simp add:compat-label-def)
apply(rule allI,rule impI)
apply(erule-tac x=j in allE,erule impE, assumption)
apply(erule disjE)
 apply clarify
 apply(rule-tac x=i in exI,simp)
 apply(rule conjI)
  apply(erule-tac x=i in allE)
  apply(case-tac j,simp)
   apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
   apply(case-tac clist!i,simp,simp)
  apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
  apply(case-tac clist!i,simp,simp)
 apply clarify
 apply(erule-tac x=l and P=λj. H j ⟶ I j ⟶ J j for H I J in allE)
 apply(erule-tac x=l and P=λj. H j ⟶ (length (s j) = t) for H s t in allE)
 apply(case-tac clist!l,simp,simp)
 apply(erule-tac x=l in allE,simp)
apply(rule disjI2)
apply clarify
apply(rule tl-zero)
  apply(case-tac j,simp,simp)
  apply(rule tl-zero,force)
   apply force
   apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
 apply force
apply(erule-tac x=i and P=λj. H j ⟶ (length (s j) = t) for H s t in allE,force)
apply clarify
apply(erule-tac x=i in allE)
```

```
  apply(simp add:cp-def)
  apply(rule nth-tl-if)
    apply(simp add:conjoin-def)
    apply clarify
    apply(simp add:same-length-def)
    apply(erule-tac x=i in allE,simp)
   apply simp
  apply simp
 apply simp
apply clarify
apply(erule-tac c=(xs, s) # ys in equalityCE)
 apply(simp add:par-cp-def)
apply simp
apply(erule-tac x=map (λi. (fst i, s) # snd i) (zip xs clist) in allE)
apply simp
apply clarify
apply(simp add:cp-def)
done

theorem one: xs≠[] ⟹
 par-cp xs s = {c. ∃ clist. (length clist)=(length xs) ∧
              (∀ i<length clist. (clist!i) ∈ cp(xs!i) s) ∧ c ∝ clist}
apply(frule one-iff-aux)
apply(drule sym)
apply(erule iffD2)
apply clarify
apply(rule iffI)
 apply(erule aux-onlyif)
apply clarify
apply(force intro:aux-if)
done

end
```

3.4 The Proof System

```
theory RG-Hoare imports RG-Tran begin
```

3.4.1 Proof System for Component Programs

```
declare Un-subset-iff [simp del] sup.bounded-iff [simp del]

definition stable :: 'a set ⇒ ('a × 'a) set ⇒ bool where
  stable ≡ λf g. (∀ x y. x ∈ f ⟶ (x, y) ∈ g ⟶ y ∈ f)

inductive
  rghoare :: ['a com, 'a set, ('a × 'a) set, ('a × 'a) set, 'a set] ⇒ bool
    (⊢ - sat [-, -, -, -] [60,0,0,0,0] 45)
where
```

Basic: $[\![$ *pre* $\subseteq$ {*s*. *f s* $\in$ *post*}; {(*s*,*t*). *s* $\in$ *pre* $\wedge$ (*t*=*f s* $\vee$ *t*=*s*)} $\subseteq$ *guar*;
stable pre rely; *stable post rely* $]\!]$
$\Longrightarrow \vdash$ *Basic f sat* [*pre*, *rely*, *guar*, *post*]

| *Seq*: $[\![$ $\vdash$ *P sat* [*pre*, *rely*, *guar*, *mid*]; $\vdash$ *Q sat* [*mid*, *rely*, *guar*, *post*] $]\!]$
$\Longrightarrow \vdash$ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]

| *Cond*: $[\![$ *stable pre rely*; $\vdash$ *P1 sat* [*pre* $\cap$ *b*, *rely*, *guar*, *post*];
$\vdash$ *P2 sat* [*pre* $\cap$ $-$*b*, *rely*, *guar*, *post*]; $\forall$ *s*. (*s*,*s*)$\in$*guar* $]\!]$
$\Longrightarrow \vdash$ *Cond b P1 P2 sat* [*pre*, *rely*, *guar*, *post*]

| *While*: $[\![$ *stable pre rely*; (*pre* $\cap$ $-$*b*) $\subseteq$ *post*; *stable post rely*;
$\vdash$ *P sat* [*pre* $\cap$ *b*, *rely*, *guar*, *pre*]; $\forall$ *s*. (*s*,*s*)$\in$*guar* $]\!]$
$\Longrightarrow \vdash$ *While b P sat* [*pre*, *rely*, *guar*, *post*]

| *Await*: $[\![$ *stable pre rely*; *stable post rely*;
$\forall$ *V*. $\vdash$ *P sat* [*pre* $\cap$ *b* $\cap$ {*V*}, {(*s*, *t*). *s* = *t*},
UNIV, {*s*. (*V*, *s*) $\in$ *guar*} $\cap$ *post*] $]\!]$
$\Longrightarrow \vdash$ *Await b P sat* [*pre*, *rely*, *guar*, *post*]

| *Conseq*: $[\![$ *pre* $\subseteq$ *pre'*; *rely* $\subseteq$ *rely'*; *guar'* $\subseteq$ *guar*; *post'* $\subseteq$ *post*;
$\vdash$ *P sat* [*pre'*, *rely'*, *guar'*, *post'*] $]\!]$
$\Longrightarrow \vdash$ *P sat* [*pre*, *rely*, *guar*, *post*]

definition *Pre* :: *'a rgformula* $\Rightarrow$ *'a set* **where**
Pre x $\equiv$ *fst*(*snd x*)

definition *Post* :: *'a rgformula* $\Rightarrow$ *'a set* **where**
Post x $\equiv$ *snd*(*snd*(*snd*(*snd x*)))

definition *Rely* :: *'a rgformula* $\Rightarrow$ (*'a* $\times$ *'a*) *set* **where**
Rely x $\equiv$ *fst*(*snd*(*snd x*))

definition *Guar* :: *'a rgformula* $\Rightarrow$ (*'a* $\times$ *'a*) *set* **where**
Guar x $\equiv$ *fst*(*snd*(*snd*(*snd x*)))

definition *Com* :: *'a rgformula* $\Rightarrow$ *'a com* **where**
Com x $\equiv$ *fst x*

3.4.2 Proof System for Parallel Programs

type-synonym *'a par-rgformula* =
(*'a rgformula*) *list* $\times$ *'a set* $\times$ (*'a* $\times$ *'a*) *set* $\times$ (*'a* $\times$ *'a*) *set* $\times$ *'a set*

inductive
par-rghoare :: (*'a rgformula*) *list* $\Rightarrow$ *'a set* $\Rightarrow$ (*'a* $\times$ *'a*) *set* $\Rightarrow$ (*'a* $\times$ *'a*) *set* $\Rightarrow$ *'a set* $\Rightarrow$ *bool*
($\vdash$ - *SAT* [-, -, -, -] [*60*,*0*,*0*,*0*,*0*] *45*)
where

Parallel:
⟦ ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j*<*length xs* ∧ *j*≠*i*}. *Guar*(*xs*!*j*)) ⊆ *Rely*(*xs*!*i*);
(⋃*j*∈{*j*. *j*<*length xs*}. *Guar*(*xs*!*j*)) ⊆ *guar*;
pre ⊆ (⋂*i*∈{*i*. *i*<*length xs*}. *Pre*(*xs*!*i*));
(⋂*i*∈{*i*. *i*<*length xs*}. *Post*(*xs*!*i*)) ⊆ *post*;
∀ *i*<*length xs*. ⊢ *Com*(*xs*!*i*) *sat* [*Pre*(*xs*!*i*),*Rely*(*xs*!*i*),*Guar*(*xs*!*i*),*Post*(*xs*!*i*)] ⟧
⟹ ⊢ *xs SAT* [*pre*, *rely*, *guar*, *post*]

3.5 Soundness

Some previous lemmas

lemma *tl-of-assum-in-assum*:
(*P*, *s*) # (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*) ⟹ *stable pre rely*
⟹ (*P*, *t*) # *xs* ∈ *assum* (*pre*, *rely*)
apply(*simp add*:*assum-def*)
apply *clarify*
apply(*rule conjI*)
apply(*erule-tac x=0* **in** *allE*)
apply(*simp* (*no-asm-use*)*only*:*stable-def*)
apply(*erule allE*,*erule allE*,*erule impE*,*assumption*,*erule mp*)
apply(*simp add*:*Env*)
apply *clarify*
apply(*erule-tac x=Suc i* **in** *allE*)
apply *simp*
done

lemma *etran-in-comm*:
(*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*) ⟹ (*P*, *s*) # (*P*, *t*) # *xs* ∈ *comm*(*guar*, *post*)
apply(*simp add*:*comm-def*)
apply *clarify*
apply(*case-tac i*,*simp+*)
done

lemma *ctran-in-comm*:
⟦(*s*, *s*) ∈ *guar*; (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)⟧
⟹ (*P*, *s*) # (*Q*, *s*) # *xs* ∈ *comm*(*guar*, *post*)
apply(*simp add*:*comm-def*)
apply *clarify*
apply(*case-tac i*,*simp+*)
done

lemma *takecptn-is-cptn* [*rule-format*, *elim*!]:
∀ *j*. *c* ∈ *cptn* ⟶ *take* (*Suc j*) *c* ∈ *cptn*
apply(*induct c*)
apply(*force elim*: *cptn.cases*)
apply *clarify*
apply(*case-tac j*)
apply *simp*

apply(*rule CptnOne*)
apply *simp*
apply(*force intro*:*cptn.intros elim*:*cptn.cases*)
done

lemma *dropcptn-is-cptn* [*rule-format*,*elim*!]:
$\forall j<length\ c.\ c \in cptn \longrightarrow drop\ j\ c \in cptn$
apply(*induct c*)
apply(*force elim*: *cptn.cases*)
apply *clarify*
apply(*case-tac j*,*simp*+)
apply(*erule cptn.cases*)
apply *simp*
apply *force*
apply *force*
done

lemma *takepar-cptn-is-par-cptn* [*rule-format*,*elim*]:
$\forall j.\ c \in par\text{-}cptn \longrightarrow take\ (Suc\ j)\ c \in par\text{-}cptn$
apply(*induct c*)
apply(*force elim*: *cptn.cases*)
apply *clarify*
apply(*case-tac j*,*simp*)
apply(*rule ParCptnOne*)
apply(*force intro*:*par-cptn.intros elim*:*par-cptn.cases*)
done

lemma *droppar-cptn-is-par-cptn* [*rule-format*]:
$\forall j<length\ c.\ c \in par\text{-}cptn \longrightarrow drop\ j\ c \in par\text{-}cptn$
apply(*induct c*)
apply(*force elim*: *par-cptn.cases*)
apply *clarify*
apply(*case-tac j*,*simp*+)
apply(*erule par-cptn.cases*)
apply *simp*
apply *force*
apply *force*
done

lemma *tl-of-cptn-is-cptn*: $[\![x \# xs \in cptn;\ xs \neq []]\!] \Longrightarrow xs\ \in cptn$
apply(*subgoal-tac 1 < length (x # xs)*)
apply(*drule dropcptn-is-cptn*,*simp*+)
done

lemma *not-ctran-None* [*rule-format*]:
$\forall s.\ (None,\ s)\#xs \in cptn \longrightarrow (\forall i<length\ xs.\ ((None,\ s)\#xs)!i\ -e\rightarrow\ xs!i)$
apply(*induct xs*,*simp*+)
apply *clarify*
apply(*erule cptn.cases*,*simp*)

```
 apply simp
 apply(case-tac i,simp)
  apply(rule Env)
 apply simp
apply(force elim:ctran.cases)
done

lemma cptn-not-empty [simp]:[] ∉ cptn
apply(force elim:cptn.cases)
done

lemma etran-or-ctran [rule-format]:
  ∀ m i. x∈cptn ⟶ m ≤ length x
   ⟶ (∀ i. Suc i < m ⟶ ¬ x!i −c→ x!Suc i) ⟶ Suc i < m
   ⟶ x!i −e→ x!Suc i
  supply [[simproc del: defined-all]]
apply(induct x,simp)
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp)
  apply(rule Env)
 apply simp
 apply(erule-tac x=m − 1 in allE)
 apply(case-tac m,simp,simp)
 apply(subgoal-tac (∀ i. Suc i < nata ⟶ (((P, t) # xs) ! i, xs ! i) ∉ ctran))
  apply force
 apply clarify
   apply(erule-tac x=Suc ia in allE,simp)
apply(erule-tac x=0 and P=λj. H j ⟶ (J j) ∉ ctran for H J in allE,simp)
done

lemma etran-or-ctran2 [rule-format]:
  ∀ i. Suc i<length x ⟶ x∈cptn ⟶ (x!i −c→ x!Suc i ⟶ ¬ x!i −e→ x!Suc i)
  ∨ (x!i −e→ x!Suc i ⟶ ¬ x!i −c→ x!Suc i)
apply(induct x)
 apply simp
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp+)
apply(case-tac i,simp)
 apply(force elim:etran.cases)
apply simp
done

lemma etran-or-ctran2-disjI1:
  ⟦ x∈cptn; Suc i<length x; x!i −c→ x!Suc i⟧ ⟹ ¬ x!i −e→ x!Suc i
by(drule etran-or-ctran2,simp-all)

lemma etran-or-ctran2-disjI2:
```

```
  ⟦ x∈cptn; Suc i<length x; x!i −e→ x!Suc i⟧ ⟹ ¬ x!i −c→ x!Suc i
by(drule etran-or-ctran2,simp-all)

lemma not-ctran-None2 [rule-format]:
  ⟦ (None, s) # xs ∈cptn; i<length xs⟧ ⟹ ¬ ((None, s) # xs) ! i −c→ xs ! i
apply(frule not-ctran-None,simp)
apply(case-tac i,simp)
 apply(force elim:etranE)
apply simp
apply(rule etran-or-ctran2-disjI2,simp-all)
apply(force intro:tl-of-cptn-is-cptn)
done

lemma Ex-first-occurrence [rule-format]: P (n::nat) ⟶ (∃ m. P m ∧ (∀ i<m. ¬
P i))
apply(rule nat-less-induct)
apply clarify
apply(case-tac ∀ m. m<n ⟶ ¬ P m)
apply auto
done

lemma stability [rule-format]:
  ∀ j k. x ∈ cptn ⟶ stable p rely ⟶ j≤k ⟶ k<length x ⟶ snd(x!j)∈p  ⟶
  (∀ i. (Suc i)<length x ⟶
          (x!i −e→ x!(Suc i)) ⟶ (snd(x!i), snd(x!(Suc i))) ∈ rely) ⟶
  (∀ i. j≤i ∧ i<k ⟶ x!i −e→ x!Suc i) ⟶ snd(x!k)∈p ∧ fst(x!j)=fst(x!k)
  supply [[simproc del: defined-all]]
apply(induct x)
 apply clarify
 apply(force elim:cptn.cases)
apply clarify
apply(erule cptn.cases,simp)
 apply simp
 apply(case-tac k,simp,simp)
 apply(case-tac j,simp)
  apply(erule-tac x=0 in allE)
  apply(erule-tac x=nat and P=λj. (0≤j) ⟶ (J j) for J in allE,simp)
  apply(subgoal-tac t∈p)
   apply(subgoal-tac (∀ i. i < length xs ⟶ ((P, t) # xs) ! i −e→ xs ! i ⟶ (snd
(((P, t) # xs) ! i), snd (xs ! i)) ∈ rely))
    apply clarify
   apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j)∈etran for H J in allE,simp)
   apply clarify
   apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j) ⟶ (T j)∈rely for H J T
in allE,simp)
  apply(erule-tac x=0 and P=λj. (H j) ⟶ (J j)∈etran ⟶ T j for H J T in
allE,simp)
  apply(simp(no-asm-use) only:stable-def)
  apply(erule-tac x=s in allE)
```

```
  apply(erule-tac x=t in allE)
  apply simp
  apply(erule mp)
  apply(erule mp)
  apply(rule Env)
 apply simp
 apply(erule-tac x=nata in allE)
 apply(erule-tac x=nat and P=λj. (s≤j) ⟶ (J j) for s J in allE,simp)
 apply(subgoal-tac (∀ i. i < length xs ⟶ ((P, t) # xs) ! i −e→ xs ! i ⟶ (snd
(((P, t) # xs) ! i), snd (xs ! i)) ∈ rely))
  apply clarify
  apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j)∈etran for H J in allE,simp)
 apply clarify
 apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j) ⟶ (T j)∈rely for H J T
in allE,simp)
apply(case-tac k,simp,simp)
apply(case-tac j)
 apply(erule-tac x=0 and P=λj. (H j) ⟶ (J j)∈etran for H J in allE,simp)
 apply(erule etran.cases,simp)
apply(erule-tac x=nata in allE)
apply(erule-tac x=nat and P=λj. (s≤j) ⟶ (J j) for s J in allE,simp)
apply(subgoal-tac (∀ i. i < length xs ⟶ ((Q, t) # xs) ! i −e→ xs ! i ⟶ (snd
(((Q, t) # xs) ! i), snd (xs ! i)) ∈ rely))
 apply clarify
 apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j)∈etran for H J in allE,simp)
apply clarify
apply(erule-tac x=Suc i and P=λj. (H j) ⟶ (J j) ⟶ (T j)∈rely for H J T in
allE,simp)
done
```

3.5.1 Soundness of the System for Component Programs

Soundness of the Basic rule

```
lemma unique-ctran-Basic [rule-format]:
  ∀ s i. x ∈ cptn ⟶ x ! 0 = (Some (Basic f), s) ⟶
  Suc i<length x ⟶ x!i −c→ x!Suc i ⟶
  (∀ j. Suc j<length x ⟶ i≠j ⟶ x!j −e→ x!Suc j)
apply(induct x,simp)
apply simp
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp+)
 apply clarify
 apply(case-tac j,simp)
  apply(rule Env)
 apply simp
apply clarify
apply simp
apply(case-tac i)
```

```
 apply(case-tac j,simp,simp)
 apply(erule ctran.cases,simp-all)
 apply(force elim: not-ctran-None)
apply(ind-cases ((Some (Basic f), sa), Q, t) ∈ ctran for sa Q t)
apply simp
apply(drule-tac i=nat in not-ctran-None,simp)
apply(erule etranE,simp)
done

lemma exists-ctran-Basic-None [rule-format]:
  ∀ s i. x ∈ cptn ⟶ x ! 0 = (Some (Basic f), s)
  ⟶ i<length x ⟶ fst(x!i)=None ⟶ (∃ j<i. x!j −c→ x!Suc j)
apply(induct x,simp)
apply simp
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp,simp)
 apply(erule-tac x=nat in allE,simp)
 apply clarify
 apply(rule-tac x=Suc j in exI,simp,simp)
apply clarify
apply(case-tac i,simp,simp)
apply(rule-tac x=0 in exI,simp)
done

lemma Basic-sound:
   ⟦pre ⊆ {s. f s ∈ post}; {(s, t). s ∈ pre ∧ t = f s} ⊆ guar;
  stable pre rely; stable post rely⟧
  ⟹ ⊨ Basic f sat [pre, rely, guar, post]
  supply [[simproc del: defined-all]]
apply(unfold com-validity-def)
apply clarify
apply(simp add:comm-def)
apply(rule conjI)
 apply clarify
 apply(simp add:cp-def assum-def)
 apply clarify
 apply(frule-tac j=0 and k=i and p=pre in stability)
      apply simp-all
   apply(erule-tac x=ia in allE,simp)
  apply(erule-tac i=i and f=f in unique-ctran-Basic,simp-all)
 apply(erule subsetD,simp)
 apply(case-tac x!i)
 apply clarify
 apply(drule-tac s=Some (Basic f) in sym,simp)
 apply(thin-tac ∀ j. H j for H)
 apply(force elim:ctran.cases)
apply clarify
apply(simp add:cp-def)
```

```
apply clarify
apply(frule-tac i=length x − 1 and f=f in exists-ctran-Basic-None,simp+)
  apply(case-tac x,simp+)
  apply(rule last-fst-esp,simp add:last-length)
 apply (case-tac x,simp+)
apply(simp add:assum-def)
apply clarify
apply(frule-tac j=0 and k=j and p=pre in stability)
      apply simp-all
  apply(erule-tac x=i in allE,simp)
 apply(erule-tac i=j and f=f in unique-ctran-Basic,simp-all)
apply(case-tac x!j)
apply clarify
apply simp
apply(drule-tac s=Some (Basic f) in sym,simp)
apply(case-tac x!Suc j,simp)
apply(rule ctran.cases,simp)
apply(simp-all)
apply(drule-tac c=sa in subsetD,simp)
apply clarify
apply(frule-tac j=Suc j and k=length x − 1 and p=post in stability,simp-all)
 apply(case-tac x,simp+)
 apply(erule-tac x=i in allE)
apply(erule-tac i=j and f=f in unique-ctran-Basic,simp-all)
  apply arith+
apply(case-tac x)
apply(simp add:last-length)+
done
```

Soundness of the Await rule

```
lemma unique-ctran-Await [rule-format]:
  ∀ s i. x ∈ cptn ⟶ x ! 0 = (Some (Await b c), s) ⟶
  Suc i<length x ⟶ x!i −c→ x!Suc i ⟶
  (∀ j. Suc j<length x ⟶ i≠j ⟶ x!j −e→ x!Suc j)
apply(induct x,simp+)
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp+)
 apply clarify
 apply(case-tac j,simp)
  apply(rule Env)
 apply simp
apply clarify
apply simp
apply(case-tac i)
 apply(case-tac j,simp,simp)
 apply(erule ctran.cases,simp-all)
 apply(force elim: not-ctran-None)
```

```
apply(ind-cases ((Some (Await b c), sa), Q, t) ∈ ctran for sa Q t,simp)
apply(drule-tac i=nat in not-ctran-None,simp)
apply(erule etranE,simp)
done

lemma exists-ctran-Await-None [rule-format]:
  ∀ s i.  x ∈ cptn ⟶ x ! 0 = (Some (Await b c), s)
  ⟶ i<length x ⟶ fst(x!i)=None ⟶ (∃ j<i. x!j −c→ x!Suc j)
apply(induct x,simp+)
apply clarify
apply(erule cptn.cases,simp)
 apply(case-tac i,simp+)
 apply(erule-tac x=nat in allE,simp)
 apply clarify
 apply(rule-tac x=Suc j in exI,simp,simp)
apply clarify
apply(case-tac i,simp,simp)
apply(rule-tac x=0 in exI,simp)
done

lemma Star-imp-cptn:
  (P, s) −c∗→ (R, t) ⟹ ∃ l ∈ cp P s. (last l)=(R, t)
  ∧ (∀ i. Suc i<length l ⟶ l!i −c→ l!Suc i)
apply (erule converse-rtrancl-induct2)
 apply(rule-tac x=[(R,t)] in bexI)
  apply simp
 apply(simp add:cp-def)
 apply(rule CptnOne)
apply clarify
apply(rule-tac x=(a, b)#l in bexI)
 apply (rule conjI)
  apply(case-tac l,simp add:cp-def)
  apply(simp add:last-length)
 apply clarify
apply(case-tac i,simp)
apply(simp add:cp-def)
apply force
apply(simp add:cp-def)
 apply(case-tac l)
 apply(force elim:cptn.cases)
apply simp
apply(erule CptnComp)
apply clarify
done

lemma Await-sound:
  ⟦stable pre rely; stable post rely;
  ∀ V. ⊢ P sat [pre ∩ b ∩ {s. s = V}, {(s, t). s = t},
                UNIV, {s. (V, s) ∈ guar} ∩ post] ∧
```

```
  ⊨ P sat [pre ∩ b ∩ {s. s = V}, {(s, t). s = t},
                UNIV, {s. (V, s) ∈ guar} ∩ post] ⟧
  ⟹ ⊨ Await b P sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(simp add:comm-def)
apply(rule conjI)
 apply clarify
 apply(simp add:cp-def assum-def)
 apply clarify
 apply(frule-tac j=0 and k=i and p=pre in stability,simp-all)
   apply(erule-tac x=ia in allE,simp)
  apply(subgoal-tac x∈ cp (Some(Await b P)) s)
  apply(erule-tac i=i in unique-ctran-Await,force,simp-all)
  apply(simp add:cp-def)
— here starts the different part.
 apply(erule ctran.cases,simp-all)
 apply(drule Star-imp-cptn)
 apply clarify
 apply(erule-tac x=sa in allE)
 apply clarify
 apply(erule-tac x=sa in allE)
 apply(drule-tac c=l in subsetD)
  apply (simp add:cp-def)
  apply clarify
  apply(erule-tac x=ia and P=λi. H i ⟶ (J i, I i)∈ctran for H J I in allE,simp)
  apply(erule etranE,simp)
 apply simp
apply clarify
apply(simp add:cp-def)
apply clarify
apply(frule-tac i=length x − 1 in exists-ctran-Await-None,force)
  apply (case-tac x,simp+)
 apply(rule last-fst-esp,simp add:last-length)
 apply(case-tac x, simp+)
apply clarify
apply(simp add:assum-def)
apply clarify
apply(frule-tac j=0 and k=j and p=pre in stability,simp-all)
  apply(erule-tac x=i in allE,simp)
 apply(erule-tac i=j in unique-ctran-Await,force,simp-all)
apply(case-tac x!j)
apply clarify
apply simp
apply(drule-tac s=Some (Await b P) in sym,simp)
apply(case-tac x!Suc j,simp)
apply(rule ctran.cases,simp)
apply(simp-all)
apply(drule Star-imp-cptn)
```

```
apply clarify
apply(erule-tac x=sa in allE)
apply clarify
apply(erule-tac x=sa in allE)
apply(drule-tac c=l in subsetD)
 apply (simp add:cp-def)
 apply clarify
 apply(erule-tac x=i and P=λi. H i ⟶ (J i, I i)∈ctran for H J I in allE,simp)
 apply(erule etranE,simp)
apply simp
apply clarify
apply(frule-tac j=Suc j and k=length x − 1 and p=post in stability,simp-all)
 apply(case-tac x,simp+)
 apply(erule-tac x=i in allE)
apply(erule-tac i=j in unique-ctran-Await,force,simp-all)
 apply arith+
apply(case-tac x)
apply(simp add:last-length)+
done
```

Soundness of the Conditional rule

```
lemma Cond-sound:
  ⟦ stable pre rely; ⊨ P1 sat [pre ∩ b, rely, guar, post];
  ⊨ P2 sat [pre ∩ − b, rely, guar, post]; ∀ s. (s,s)∈guar⟧
  ⟹ ⊨ (Cond b P1 P2) sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(simp add:cp-def comm-def)
apply(case-tac ∃ i. Suc i<length x ∧ x!i −c→ x!Suc i)
 prefer 2
 apply simp
 apply clarify
 apply(frule-tac j=0 and k=length x − 1 and p=pre in stability,simp+)
     apply(case-tac x,simp+)
    apply(simp add:assum-def)
   apply(simp add:assum-def)
  apply(erule-tac m=length x in etran-or-ctran,simp+)
 apply(case-tac x, (simp add:last-length)+)
apply(erule exE)
apply(drule-tac n=i and P=λi. H i ∧ (J i, I i) ∈ ctran for H J I in Ex-first-occurrence)
apply clarify
apply (simp add:assum-def)
apply(frule-tac j=0 and k=m and p=pre in stability,simp+)
 apply(erule-tac m=Suc m in etran-or-ctran,simp+)
apply(erule ctran.cases,simp-all)
 apply(erule-tac x=sa in allE)
 apply(drule-tac c=drop (Suc m) x in subsetD)
  apply simp
```

```
  apply clarify
 apply simp
 apply clarify
 apply(case-tac i≤m)
  apply(drule le-imp-less-or-eq)
  apply(erule disjE)
   apply(erule-tac x=i in allE, erule impE, assumption)
   apply simp+
 apply(erule-tac x=i − (Suc m) and P=λj. H j ⟶ J j ⟶ (I j)∈guar for H J
I in allE)
 apply(subgoal-tac (Suc m)+(i − Suc m) ≤ length x)
  apply(subgoal-tac (Suc m)+Suc (i − Suc m) ≤ length x)
   apply(rotate-tac −2)
   apply simp
  apply arith
 apply arith
apply(case-tac length (drop (Suc m) x),simp)
apply(erule-tac x=sa in allE)
back
apply(drule-tac c=drop (Suc m) x in subsetD,simp)
 apply clarify
apply simp
apply clarify
apply(case-tac i≤m)
 apply(drule le-imp-less-or-eq)
 apply(erule disjE)
  apply(erule-tac x=i in allE, erule impE, assumption)
  apply simp
 apply simp
apply(erule-tac x=i − (Suc m) and P=λj. H j ⟶ J j ⟶ (I j)∈guar for H J I
in allE)
apply(subgoal-tac (Suc m)+(i − Suc m) ≤ length x)
 apply(subgoal-tac (Suc m)+Suc (i − Suc m) ≤ length x)
  apply(rotate-tac −2)
  apply simp
 apply arith
apply arith
done
```

Soundness of the Sequential rule

```
inductive-cases Seq-cases [elim!]: (Some (Seq P Q), s) −c→ t

lemma last-lift-not-None: fst ((lift Q) ((x#xs)!(length xs))) ≠ None
apply(subgoal-tac length xs<length (x # xs))
 apply(drule-tac Q=Q in  lift-nth)
 apply(erule ssubst)
 apply (simp add:lift-def)
 apply(case-tac (x # xs) ! length xs,simp)
```

```
apply simp
done

lemma Seq-sound1 [rule-format]:
  x∈ cptn-mod ⟹ ∀ s P. x !0=(Some (Seq P Q), s) ⟶
  (∀ i<length x. fst(x!i)≠Some Q) ⟶
  (∃ xs∈ cp (Some P) s. x=map (lift Q) xs)
  supply [[simproc del: defined-all]]
apply(erule cptn-mod.induct)
apply(unfold cp-def)
apply safe
apply simp-all
    apply(simp add:lift-def)
    apply(rule-tac x=[(Some Pa, sa)] in exI,simp add:CptnOne)
   apply(subgoal-tac (∀ i < Suc (length xs). fst (((Some (Seq Pa Q), t) # xs) ! i)
≠ Some Q))
    apply clarify
    apply(rule-tac x=(Some Pa, sa) #(Some Pa, t) # zs in exI,simp)
    apply(rule conjI,erule CptnEnv)
    apply(simp (no-asm-use) add:lift-def)
   apply clarify
   apply(erule-tac x=Suc i in allE, simp)
  apply(ind-cases ((Some (Seq Pa Q), sa), None, t) ∈ ctran for Pa sa t)
 apply(rule-tac x=(Some P, sa) # xs in exI, simp add:cptn-iff-cptn-mod lift-def)
apply(erule-tac x=length xs in allE, simp)
apply(simp only:Cons-lift-append)
apply(subgoal-tac length xs < length ((Some P, sa) # xs))
 apply(simp only :nth-append length-map last-length nth-map)
 apply(case-tac last((Some P, sa) # xs))
 apply(simp add:lift-def)
apply simp
done

lemma Seq-sound2 [rule-format]:
  x ∈ cptn ⟹ ∀ s P i. x!0=(Some (Seq P Q), s) ⟶ i<length x
  ⟶ fst(x!i)=Some Q ⟶
  (∀ j<i. fst(x!j)≠(Some Q)) ⟶
  (∃ xs ys. xs ∈ cp (Some P) s ∧ length xs=Suc i
   ∧ ys ∈ cp (Some Q) (snd(xs !i)) ∧ x=(map (lift Q) xs)@tl ys)
  supply [[simproc del: defined-all]]
apply(erule cptn.induct)
apply(unfold cp-def)
apply safe
apply simp-all
 apply(case-tac i,simp+)
 apply(erule allE,erule impE,assumption,simp)
 apply clarify
 apply(subgoal-tac (∀ j < nat. fst (((Some (Seq Pa Q), t) # xs) ! j) ≠ Some
Q),clarify)
```

```
  prefer 2
  apply force
 apply(case-tac xsa,simp,simp)
 apply(rename-tac list)
 apply(rule-tac x=(Some Pa, sa) #(Some Pa, t) # list in exI,simp)
 apply(rule conjI,erule CptnEnv)
 apply(simp (no-asm-use) add:lift-def)
 apply(rule-tac x=ys in exI,simp)
apply(ind-cases ((Some (Seq Pa Q), sa), t) ∈ ctran for Pa sa t)
 apply simp
 apply(rule-tac x=(Some Pa, sa)#[(None, ta)] in exI,simp)
 apply(rule conjI)
  apply(drule-tac xs=[] in CptnComp,force simp add:CptnOne,simp)
 apply(case-tac i, simp+)
 apply(case-tac nat,simp+)
 apply(rule-tac x=(Some Q,ta)#xs in exI,simp add:lift-def)
 apply(case-tac nat,simp+)
 apply(force)
apply(case-tac i, simp+)
apply(case-tac nat,simp+)
apply(erule-tac x=Suc nata in allE,simp)
apply clarify
apply(subgoal-tac (∀j<Suc nata. fst (((Some (Seq P2 Q), ta) # xs) ! j) ≠ Some
Q),clarify)
 prefer 2
 apply clarify
 apply force
apply(rule-tac x=(Some Pa, sa)#(Some P2, ta)#(tl xsa) in exI,simp)
apply(rule conjI,erule CptnComp)
apply(rule nth-tl-if,force,simp+)
apply(rule-tac x=ys in exI,simp)
apply(rule conjI)
apply(rule nth-tl-if,force,simp+)
 apply(rule tl-zero,simp+)
 apply force
apply(rule conjI,simp add:lift-def)
apply(subgoal-tac lift Q (Some P2, ta) =(Some (Seq P2 Q), ta))
 apply(simp add:Cons-lift del:list.map)
 apply(rule nth-tl-if)
   apply force
  apply simp+
apply(simp add:lift-def)
done


lemma last-lift-not-None2: fst ((lift Q) (last (x#xs))) ≠ None
apply(simp only:last-length [THEN sym])
apply(subgoal-tac length xs<length (x # xs))
 apply(drule-tac Q=Q in  lift-nth)
```

```
 apply(erule ssubst)
 apply (simp add:lift-def)
 apply(case-tac (x # xs) ! length xs,simp)
apply simp
done

lemma Seq-sound:
  ⟦⊨ P sat [pre, rely, guar, mid]; ⊨ Q sat [mid, rely, guar, post]⟧
  ⟹ ⊨ Seq P Q sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(case-tac ∃ i<length x. fst(x!i)=Some Q)
 prefer 2
 apply (simp add:cp-def cptn-iff-cptn-mod)
 apply clarify
 apply(frule-tac Seq-sound1,force)
  apply force
 apply clarify
 apply(erule-tac x=s in allE,simp)
 apply(drule-tac c=xs in subsetD,simp add:cp-def cptn-iff-cptn-mod)
  apply(simp add:assum-def)
  apply clarify
   apply(erule-tac P=λj. H j ⟶ J j ⟶ I j for H J I in allE,erule impE,
assumption)
  apply(simp add:snd-lift)
  apply(erule mp)
  apply(force elim:etranE intro:Env simp add:lift-def)
 apply(simp add:comm-def)
 apply(rule conjI)
  apply clarify
   apply(erule-tac P=λj. H j ⟶ J j ⟶ I j for H J I in allE,erule impE,
assumption)
  apply(simp add:snd-lift)
  apply(erule mp)
  apply(case-tac (xs!i))
  apply(case-tac (xs! Suc i))
  apply(case-tac fst(xs!i))
   apply(erule-tac x=i in allE, simp add:lift-def)
  apply(case-tac fst(xs!Suc i))
   apply(force simp add:lift-def)
  apply(force simp add:lift-def)
 apply clarify
 apply(case-tac xs,simp add:cp-def)
 apply clarify
 apply (simp del:list.map)
 apply (rename-tac list)
 apply(subgoal-tac (map (lift Q) ((a, b) # list))≠[])
  apply(drule last-conv-nth)
  apply (simp del:list.map)
```

```
  apply(simp only:last-lift-not-None)
 apply simp
— ∃ i<length x. fst (x ! i) = Some Q
apply(erule exE)
apply(drule-tac n=i and P=λi. i < length x ∧ fst (x ! i) = Some Q in Ex-first-occurrence)
apply clarify
apply (simp add:cp-def)
 apply clarify
 apply(frule-tac i=m in Seq-sound2,force)
  apply simp+
apply clarify
apply(simp add:comm-def)
apply(erule-tac x=s in allE)
apply(drule-tac c=xs in subsetD,simp)
 apply(case-tac xs=[],simp)
 apply(simp add:cp-def assum-def nth-append)
 apply clarify
 apply(erule-tac x=i in allE)
  back
 apply(simp add:snd-lift)
 apply(erule mp)
 apply(force elim:etranE intro:Env simp add:lift-def)
apply simp
apply clarify
apply(erule-tac x=snd(xs!m) in allE)
apply(drule-tac c=ys in subsetD,simp add:cp-def assum-def)
 apply(case-tac xs≠[])
 apply(drule last-conv-nth,simp)
 apply(rule conjI)
  apply(erule mp)
  apply(case-tac xs!m)
  apply(case-tac fst(xs!m),simp)
  apply(simp add:lift-def nth-append)
 apply clarify
 apply(erule-tac x=m+i in allE)
 back
 back
 apply(case-tac ys,(simp add:nth-append)+)
 apply (case-tac i, (simp add:snd-lift)+)
  apply(erule mp)
  apply(case-tac xs!m)
  apply(force elim:etran.cases intro:Env simp add:lift-def)
 apply simp
apply simp
apply clarify
apply(rule conjI,clarify)
 apply(case-tac i<m,simp add:nth-append)
  apply(simp add:snd-lift)
  apply(erule allE, erule impE, assumption, erule mp)
```

```
  apply(case-tac (xs ! i))
  apply(case-tac (xs ! Suc i))
  apply(case-tac fst(xs ! i),force simp add:lift-def)
  apply(case-tac fst(xs ! Suc i))
   apply (force simp add:lift-def)
  apply (force simp add:lift-def)
 apply(erule-tac x=i−m in allE)
 back
 back
 apply(subgoal-tac Suc (i − m) < length ys,simp)
  prefer 2
  apply arith
 apply(simp add:nth-append snd-lift)
 apply(rule conjI,clarify)
  apply(subgoal-tac i=m)
   prefer 2
   apply arith
  apply clarify
  apply(simp add:cp-def)
  apply(rule tl-zero)
    apply(erule mp)
    apply(case-tac lift Q (xs!m),simp add:snd-lift)
    apply(case-tac xs!m,case-tac fst(xs!m),simp add:lift-def snd-lift)
     apply(case-tac ys,simp+)
    apply(simp add:lift-def)
   apply simp
  apply force
 apply clarify
 apply(rule tl-zero)
   apply(rule tl-zero)
     apply (subgoal-tac i−m=Suc(i−Suc m))
      apply simp
      apply(erule mp)
      apply(case-tac ys,simp+)
   apply force
  apply arith
 apply force
apply clarify
apply(case-tac (map (lift Q) xs @ tl ys)≠[])
 apply(drule last-conv-nth)
 apply(simp add: snd-lift nth-append)
 apply(rule conjI,clarify)
  apply(case-tac ys,simp+)
 apply clarify
 apply(case-tac ys,simp+)
done
```

Soundness of the While rule

lemma *last-append*[*rule-format*]:
 ∀*xs*. *ys*≠[] ⟶ ((*xs*@*ys*)!(*length* (*xs*@*ys*) − (*Suc 0*)))=(*ys*!(*length ys* − (*Suc 0*)))
apply(*induct ys*)
 apply *simp*
apply *clarify*
apply (*simp add*:*nth-append*)
done

lemma *assum-after-body*:
 ⟦ ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*];
 (*Some P*, *s*) # *xs* ∈ *cptn-mod*; *fst* (*last* ((*Some P*, *s*) # *xs*)) = *None*; *s* ∈ *b*;
 (*Some* (*While b P*), *s*) # (*Some* (*Seq P* (*While b P*)), *s*) #
  map (*lift* (*While b P*)) *xs* @ *ys* ∈ *assum* (*pre*, *rely*)⟧
 ⟹ (*Some* (*While b P*), *snd* (*last* ((*Some P*, *s*) # *xs*))) # *ys* ∈ *assum* (*pre*, *rely*)
apply(*simp add*:*assum-def com-validity-def cp-def cptn-iff-cptn-mod*)
apply *clarify*
apply(*erule-tac x*=*s* **in** *allE*)
apply(*drule-tac c*=(*Some P*, *s*) # *xs* **in** *subsetD*,*simp*)
 apply *clarify*
 apply(*erule-tac x*=*Suc i* **in** *allE*)
 apply *simp*
 apply(*simp add*:*Cons-lift-append nth-append snd-lift del*:*list*.*map*)
 apply(*erule mp*)
 apply(*erule etranE*,*simp*)
 apply(*case-tac fst*(((*Some P*, *s*) # *xs*) ! *i*))
  apply(*force intro*:*Env simp add*:*lift-def*)
 apply(*force intro*:*Env simp add*:*lift-def*)
apply(*rule conjI*)
 apply *clarify*
 apply(*simp add*:*comm-def last-length*)
apply *clarify*
apply(*rule conjI*)
 apply(*simp add*:*comm-def*)
apply *clarify*
apply(*erule-tac x*=*Suc*(*length xs* + *i*) **in** *allE*,*simp*)
apply(*case-tac i*, *simp add*:*nth-append Cons-lift-append snd-lift last-conv-nth lift-def split-def*)
apply(*simp add*:*Cons-lift-append nth-append snd-lift*)
done

lemma *While-sound-aux* [*rule-format*]:
 ⟦ *pre* ∩ − *b* ⊆ *post*; ⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*, *s*) ∈ *guar*;
  stable pre rely; *stable post rely*; *x* ∈ *cptn-mod* ⟧
 ⟹ ∀ *s xs*. *x*=(*Some*(*While b P*),*s*)#*xs* ⟶ *x*∈*assum*(*pre*, *rely*) ⟶ *x* ∈ *comm* (*guar*, *post*)
 supply [[*simproc del*: *defined-all*]]
apply(*erule cptn-mod*.*induct*)
apply *safe*

apply (*simp-all del*:*last.simps*)
— 5 subgoals left
apply(*simp add*:*comm-def*)
— 4 subgoals left
apply(*rule etran-in-comm*)
apply(*erule mp*)
apply(*erule tl-of-assum-in-assum*,*simp*)
— While-None
apply(*ind-cases* ((*Some* (*While b P*), *s*), *None*, *t*) ∈ *ctran* **for** *s t*)
apply(*simp add*:*comm-def*)
apply(*simp add*:*cptn-iff-cptn-mod* [*THEN sym*])
apply(*rule conjI*,*clarify*)
 apply(*force simp add*:*assum-def*)
apply *clarify*
apply(*rule conjI*, *clarify*)
 apply(*case-tac i*,*simp*,*simp*)
 apply(*force simp add*:*not-ctran-None2*)
apply(*subgoal-tac* ∀ *i*. *Suc i* < *length* ((*None*, *t*) # *xs*) ⟶ (((*None*, *t*) # *xs*) ! *i*, ((*None*, *t*) # *xs*) ! *Suc i*)∈ *etran*)
 prefer *2*
 apply *clarify*
 apply(*rule-tac m*=*length* ((*None*, *s*) # *xs*) **in** *etran-or-ctran*,*simp*+)
 apply(*erule not-ctran-None2*,*simp*)
 apply *simp*+
apply(*frule-tac j*=*0* **and** *k*=*length* ((*None*, *s*) # *xs*) − *1* **and** *p*=*post* **in** *stability*,*simp*+)
   apply(*force simp add*:*assum-def subsetD*)
  apply(*simp add*:*assum-def*)
  apply *clarify*
  apply(*erule-tac x*=*i* **in** *allE*,*simp*)
  apply(*erule-tac x*=*Suc i* **in** *allE*,*simp*)
 apply *simp*
apply *clarify*
apply (*simp add*:*last-length*)
— WhileOne
apply(*rule ctran-in-comm*,*simp*)
apply(*simp add*:*Cons-lift del*:*list.map*)
apply(*simp add*:*comm-def del*:*list.map*)
apply(*rule conjI*)
 apply *clarify*
 apply(*case-tac fst*(((*Some P*, *sa*) # *xs*) ! *i*))
  apply(*case-tac* ((*Some P*, *sa*) # *xs*) ! *i*)
  apply (*simp add*:*lift-def*)
  apply(*ind-cases* (*Some* (*While b P*), *ba*) −*c*→ *t* **for** *ba t*)
   apply *simp*
  apply *simp*
 apply(*simp add*:*snd-lift del*:*list.map*)
 apply(*simp only*:*com-validity-def cp-def cptn-iff-cptn-mod*)
 apply(*erule-tac x*=*sa* **in** *allE*)

```
 apply(drule-tac c=(Some P, sa) # xs in subsetD)
  apply (simp add:assum-def del:list.map)
  apply clarify
  apply(erule-tac x=Suc ia in allE,simp add:snd-lift del:list.map)
  apply(erule mp)
  apply(case-tac fst(((Some P, sa) # xs) ! ia))
   apply(erule etranE,simp add:lift-def)
   apply(rule Env)
  apply(erule etranE,simp add:lift-def)
  apply(rule Env)
 apply (simp add:comm-def del:list.map)
 apply clarify
 apply(erule allE,erule impE,assumption)
 apply(erule mp)
 apply(case-tac ((Some P, sa) # xs) ! i)
 apply(case-tac xs!i)
 apply(simp add:lift-def)
 apply(case-tac fst(xs!i))
  apply force
 apply force
— last=None
apply clarify
apply(subgoal-tac (map (lift (While b P)) ((Some P, sa) # xs))≠[])
 apply(drule last-conv-nth)
 apply (simp del:list.map)
 apply(simp only:last-lift-not-None)
apply simp
— WhileMore
apply(rule ctran-in-comm,simp del:last.simps)
— metiendo la hipotesis antes de dividir la conclusion.
apply(subgoal-tac (Some (While b P), snd (last ((Some P, sa) # xs))) # ys ∈
assum (pre, rely))
 apply (simp del:last.simps)
 prefer 2
 apply(erule assum-after-body)
  apply (simp del:last.simps)+
— lo de antes.
apply(simp add:comm-def del:list.map last.simps)
apply(rule conjI)
 apply clarify
 apply(simp only:Cons-lift-append)
 apply(case-tac i<length xs)
  apply(simp add:nth-append del:list.map last.simps)
  apply(case-tac fst(((Some P, sa) # xs) ! i))
   apply(case-tac ((Some P, sa) # xs) ! i)
   apply (simp add:lift-def del:last.simps)
   apply(ind-cases (Some (While b P), ba) −c→ t for ba t)
    apply simp
   apply simp
```

```
  apply(simp add:snd-lift del:list.map last.simps)
  apply(thin-tac ∀ i. i < length ys ⟶ P i for P)
  apply(simp only:com-validity-def cp-def cptn-iff-cptn-mod)
  apply(erule-tac x=sa in allE)
  apply(drule-tac c=(Some P, sa) # xs in subsetD)
   apply (simp add:assum-def del:list.map last.simps)
   apply clarify
    apply(erule-tac x=Suc ia in allE,simp add:nth-append snd-lift del:list.map
last.simps, erule mp)
   apply(case-tac fst(((Some P, sa) # xs) ! ia))
    apply(erule etranE,simp add:lift-def)
    apply(rule Env)
   apply(erule etranE,simp add:lift-def)
   apply(rule Env)
  apply (simp add:comm-def del:list.map)
  apply clarify
  apply(erule allE,erule impE,assumption)
  apply(erule mp)
  apply(case-tac ((Some P, sa) # xs) ! i)
  apply(case-tac xs!i)
  apply(simp add:lift-def)
  apply(case-tac fst(xs!i))
   apply force
 apply force
— i ≥ length xs
apply(subgoal-tac i−length xs <length ys)
 prefer 2
 apply arith
apply(erule-tac x=i−length xs in allE,clarify)
apply(case-tac i=length xs)
 apply (simp add:nth-append snd-lift del:list.map last.simps)
 apply(simp add:last-length del:last.simps)
 apply(erule mp)
 apply(case-tac last((Some P, sa) # xs))
 apply(simp add:lift-def del:last.simps)
— i>length xs
apply(case-tac i−length xs)
 apply arith
apply(simp add:nth-append del:list.map last.simps)
apply(rotate-tac −3)
apply(subgoal-tac i− Suc (length xs)=nat)
 prefer 2
 apply arith
apply simp
— last=None
apply clarify
apply(case-tac ys)
 apply(simp add:Cons-lift del:list.map last.simps)
 apply(subgoal-tac (map (lift (While b P)) ((Some P, sa) # xs))≠[])
```

```
  apply(drule last-conv-nth)
  apply (simp del:list.map)
  apply(simp only:last-lift-not-None)
 apply simp
apply(subgoal-tac ((Some (Seq P (While b P)), sa) # map (lift (While b P)) xs
@ ys)≠[])
 apply(drule last-conv-nth)
 apply (simp del:list.map last.simps)
 apply(simp add:nth-append del:last.simps)
 apply(rename-tac a list)
 apply(subgoal-tac ((Some (While b P), snd (last ((Some P, sa) # xs))) # a #
list)≠[])
  apply(drule last-conv-nth)
  apply (simp del:list.map last.simps)
 apply simp
apply simp
done

lemma While-sound:
  ⟦stable pre rely; pre ∩ − b ⊆ post; stable post rely;
    ⊨ P sat [pre ∩ b, rely, guar, pre]; ∀ s. (s,s)∈guar⟧
  ⟹ ⊨ While b P sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(erule-tac xs=tl x in While-sound-aux)
 apply(simp add:com-validity-def)
 apply force
 apply simp-all
apply(simp add:cptn-iff-cptn-mod cp-def)
apply(simp add:cp-def)
apply clarify
apply(rule nth-equalityI)
 apply simp-all
 apply(case-tac x,simp+)
apply(case-tac i,simp+)
apply(case-tac x,simp+)
done
```

Soundness of the Rule of Consequence

```
lemma Conseq-sound:
  ⟦pre ⊆ pre'; rely ⊆ rely'; guar' ⊆ guar; post' ⊆ post;
  ⊨ P sat [pre', rely', guar', post']⟧
  ⟹ ⊨ P sat [pre, rely, guar, post]
apply(simp add:com-validity-def assum-def comm-def)
apply clarify
apply(erule-tac x=s in allE)
apply(drule-tac c=x in subsetD)
 apply force
```

apply *force*
done

Soundness of the system for sequential component programs

theorem *rgsound*:
 $\vdash P\ sat\ [pre,\ rely,\ guar,\ post] \Longrightarrow \models P\ sat\ [pre,\ rely,\ guar,\ post]$
apply(*erule rghoare.induct*)
 apply(*force elim*:*Basic-sound*)
 apply(*force elim*:*Seq-sound*)
 apply(*force elim*:*Cond-sound*)
 apply(*force elim*:*While-sound*)
 apply(*force elim*:*Await-sound*)
apply(*erule Conseq-sound*,*simp+*)
done

3.5.2 Soundness of the System for Parallel Programs

definition *ParallelCom* :: $('a\ rgformula)\ list \Rightarrow 'a\ par\text{-}com$ **where**
 $ParallelCom\ Ps \equiv map\ (Some \circ fst)\ Ps$

lemma *two*:
 $[\![\ \forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.\ Guar\ (xs\ !\ j))$
 $\subseteq Rely\ (xs\ !\ i);$
 $pre \subseteq (\bigcap i{\in}\{i.\ i < length\ xs\}.\ Pre\ (xs\ !\ i));$
 $\forall i{<}length\ xs.$
 $\models Com\ (xs\ !\ i)\ sat\ [Pre\ (xs\ !\ i),\ Rely\ (xs\ !\ i),\ Guar\ (xs\ !\ i),\ Post\ (xs\ !\ i)];$
 $length\ xs{=}length\ clist;\ x \in par\text{-}cp\ (ParallelCom\ xs)\ s;\ x{\in}par\text{-}assum(pre,\ rely);$
 $\forall i{<}length\ clist.\ clist!i{\in}cp\ (Some(Com(xs!i)))\ s;\ x \propto clist\ ]\!]$
 $\Longrightarrow \forall j\ i.\ i{<}length\ clist \wedge Suc\ j{<}length\ x \longrightarrow (clist!i!j)\ {-c\rightarrow}\ (clist!i!Suc\ j)$
 $\longrightarrow (snd(clist!i!j),\ snd(clist!i!Suc\ j)) \in Guar(xs!i)$
apply(*unfold par-cp-def*)
apply (*rule ccontr*)
— By contradiction:
apply *simp*
apply(*erule exE*)
— the first c-tran that does not satisfy the guarantee-condition is from σ-i at step m.
apply(*drule-tac* $n{=}j$ **and** $P{=}\lambda j.\ \exists i.\ H\ i\ j$ **for** H **in** *Ex-first-occurrence*)
apply(*erule exE*)
apply *clarify*
— σ-$i \in A(pre,\ rely\text{-}1)$
apply(*subgoal-tac* $take\ (Suc\ (Suc\ m))\ (clist!i) \in assum(Pre(xs!i),\ Rely(xs!i))$)
— but this contradicts $\models \sigma$-i *sat* [*pre-i*,*rely-i*,*guar-i*,*post-i*]
 apply(*erule-tac* $x{=}i$ **and** $P{=}\lambda i.\ H\ i \longrightarrow \models (J\ i)\ sat\ [I\ i,K\ i,M\ i,N\ i]$ **for** $H\ J\ I\ K\ M\ N$ **in** *allE*,*erule impE*,*assumption*)
 apply(*simp add*:*com-validity-def*)
 apply(*erule-tac* $x{=}s$ **in** *allE*)
 apply(*simp add*:*cp-def comm-def*)
 apply(*drule-tac* $c{=}take\ (Suc\ (Suc\ m))\ (clist\ !\ i)$ **in** *subsetD*)

apply *simp*
apply (*blast intro*: *takecptn-is-cptn*)
apply *simp*
apply *clarify*
apply(*erule-tac x*=*m* **and** *P*=λ*j*. *I j* ∧ *J j* ⟶ *H j* **for** *I J H* **in** *allE*)
apply (*simp add*:*conjoin-def same-length-def*)
apply(*simp add*:*assum-def*)
apply(*rule conjI*)
apply(*erule-tac x*=*i* **and** *P*=λ*j*. *H j* ⟶ *I j* ∈*cp* (*K j*) (*J j*) **for** *H I K J* **in** *allE*)
apply(*simp add*:*cp-def par-assum-def*)
apply(*drule-tac c*=*s* **in** *subsetD*,*simp*)
apply *simp*
apply *clarify*
apply(*erule-tac x*=*i* **and** *P*=λ*j*. *H j* ⟶ *M* ∪ ⋃((*T j*) ‘ (*S j*)) ⊆ (*L j*) **for** *H M S T L* **in** *allE*)
apply *simp*
apply(*erule subsetD*)
apply *simp*
apply(*simp add*:*conjoin-def compat-label-def*)
apply *clarify*
apply(*erule-tac x*=*ia* **and** *P*=λ*j*. *H j* ⟶ (*P j*) ∨ *Q j* **for** *H P Q* **in** *allE*,*simp*)
— each etran in σ-*1*[*0*. . .*m*] corresponds to
apply(*erule disjE*)
— a c-tran in some σ-{*ib*}
apply *clarify*
apply(*case-tac i*=*ib*,*simp*)
apply(*erule etranE*,*simp*)
apply(*erule-tac x*=*ib* **and** *P*=λ*i*. *H i* ⟶ (*I i*) ∨ (*J i*) **for** *H I J* **in** *allE*)
apply (*erule etranE*)
apply(*case-tac ia*=*m*,*simp*)
apply *simp*
apply(*erule-tac x*=*ia* **and** *P*=λ*j*. *H j* ⟶ (∀ *i*. *P i j*) **for** *H P* **in** *allE*)
apply(*subgoal-tac ia*<*m*,*simp*)
prefer *2*
apply *arith*
apply(*erule-tac x*=*ib* **and** *P*=λ*j*. (*I j*, *H j*) ∈ *ctran* ⟶ *P i j* **for** *I H P* **in** *allE*,*simp*)
apply(*simp add*:*same-state-def*)
apply(*erule-tac x*=*i* **and** *P*=λ*j*. (*T j*) ⟶ (∀ *i*. (*H j i*) ⟶ (*snd* (*d j i*))=(*snd* (*e j i*))) **for** *T H d e* **in** *all-dupE*)
apply(*erule-tac x*=*ib* **and** *P*=λ*j*. (*T j*) ⟶ (∀ *i*. (*H j i*) ⟶ (*snd* (*d j i*))=(*snd* (*e j i*))) **for** *T H d e* **in** *allE*,*simp*)
— or an e-tran in σ, therefore it satisfies *rely* ∨ *guar*-{*ib*}
apply (*force simp add*:*par-assum-def same-state-def*)
done

lemma *three* [*rule-format*]:
⟦ *xs*≠[]; ∀ *i*<*length xs*. *rely* ∪ (⋃*j*∈{*j*. *j* < *length xs* ∧ *j* ≠ *i*}. *Guar* (*xs* ! *j*))

```
  ⊆ Rely (xs ! i);
  pre ⊆ (⋂ i∈{i. i < length xs}. Pre (xs ! i));
  ∀ i<length xs.
   ⊨ Com (xs ! i) sat [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];
  length xs=length clist; x ∈ par-cp (ParallelCom xs) s; x ∈ par-assum(pre, rely);
 ∀ i<length clist. clist!i∈cp (Some(Com(xs!i))) s; x ∝ clist ⟧
 ⟹ ∀ j i. i<length clist ∧ Suc j<length x ⟶ (clist!i!j) −e→ (clist!i!Suc j)
 ⟶ (snd(clist!i!j), snd(clist!i!Suc j)) ∈ rely ∪ (⋃ j∈{j. j < length xs ∧ j ≠ i}.
Guar (xs ! j))
apply(drule two)
 apply simp-all
apply clarify
apply(simp add:conjoin-def compat-label-def)
apply clarify
apply(erule-tac x=j and P=λj. H j ⟶ (J j ∧ (∃ i. P i j)) ∨ I j for H J P I in
allE,simp)
apply(erule disjE)
 prefer 2
 apply(force simp add:same-state-def par-assum-def)
apply clarify
apply(case-tac i=ia,simp)
 apply(erule etranE,simp)
apply(erule-tac x=ia and P=λi. H i ⟶ (I i) ∨ (J i) for H I J in allE,simp)
apply(erule-tac x=j and P=λj. ∀ i. S j i ⟶ (I j i, H j i) ∈ ctran ⟶ P i j for
S I H P in allE)
apply(erule-tac x=ia and P=λj. S j ⟶ (I j, H j) ∈ ctran ⟶ P j for S I H P
in allE)
apply(simp add:same-state-def)
apply(erule-tac x=i and P=λj. T j ⟶ (∀ i. H j i ⟶ (snd (d j i))=(snd (e j
i))) for T H d e in all-dupE)
apply(erule-tac x=ia and P=λj. T j ⟶ (∀ i. H j i ⟶ (snd (d j i))=(snd (e j
i))) for T H d e in allE,simp)
done

lemma four:
 ⟦xs≠[]; ∀ i < length xs. rely ∪ (⋃ j∈{j. j < length xs ∧ j ≠ i}. Guar (xs ! j))
   ⊆ Rely (xs ! i);
  (⋃ j∈{j. j < length xs}. Guar (xs ! j)) ⊆ guar;
  pre ⊆ (⋂ i∈{i. i < length xs}. Pre (xs ! i));
  ∀ i < length xs.
   ⊨ Com (xs ! i) sat [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];
  x ∈ par-cp (ParallelCom xs) s; x ∈ par-assum (pre, rely); Suc i < length x;
  x ! i −pc→ x ! Suc i⟧
 ⟹ (snd (x ! i), snd (x ! Suc i)) ∈ guar
apply(simp add: ParallelCom-def)
apply(subgoal-tac (map (Some ∘ fst) xs)≠[])
 prefer 2
 apply simp
apply(frule rev-subsetD)
```

```
 apply(erule one [THEN equalityD1])
apply(erule subsetD)
apply simp
apply clarify
apply(drule-tac pre=pre and rely=rely and  x=x and s=s and xs=xs and clist=clist
in two)
apply(assumption+)
     apply(erule sym)
    apply(simp add:ParallelCom-def)
   apply assumption
  apply(simp add:Com-def)
 apply assumption
apply(simp add:conjoin-def same-program-def)
apply clarify
apply(erule-tac x=i and P=λj. H j ⟶ fst(I j)=(J j) for H I J in all-dupE)
apply(erule-tac x=Suc i and P=λj. H j ⟶ fst(I j)=(J j) for H I J in allE)
apply(erule par-ctranE,simp)
apply(erule-tac x=i and P=λj. ∀i. S j i ⟶ (I j i, H j i) ∈ ctran ⟶ P i j for
S I H P in allE)
apply(erule-tac x=ia and P=λj. S j ⟶ (I j, H j) ∈ ctran ⟶ P j for S I H P
in allE)
apply(rule-tac x=ia in exI)
apply(simp add:same-state-def)
apply(erule-tac x=ia and P=λj. T j ⟶ (∀i. H j i ⟶ (snd (d j i))=(snd (e j
i))) for T H d e in all-dupE,simp)
apply(erule-tac x=ia and P=λj. T j ⟶ (∀i. H j i ⟶ (snd (d j i))=(snd (e j
i))) for T H d e in allE,simp)
apply(erule-tac x=i and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
all-dupE)
apply(erule-tac x=i and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
all-dupE,simp)
apply(erule-tac x=Suc i and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE,simp)
apply(erule mp)
apply(subgoal-tac r=fst(clist ! ia ! Suc i),simp)
apply(drule-tac i=ia in list-eq-if)
back
apply simp-all
done

lemma parcptn-not-empty [simp]:[] ∉ par-cptn
apply(force elim:par-cptn.cases)
done

lemma five:
  ⟦xs≠[]; ∀i<length xs. rely ∪ (⋃j∈{j. j < length xs ∧ j ≠ i}. Guar (xs ! j))
   ⊆ Rely (xs ! i);
   pre ⊆ (⋂i∈{i. i < length xs}. Pre (xs ! i));
   (⋂i∈{i. i < length xs}. Post (xs ! i)) ⊆ post;
```

```
  ∀ i < length xs.
   ⊨ Com (xs ! i) sat [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];
   x ∈ par-cp (ParallelCom xs) s; x ∈ par-assum (pre, rely);
  All-None (fst (last x)) ⟧ ⟹ snd (last x) ∈ post
apply(simp add: ParallelCom-def)
apply(subgoal-tac (map (Some ∘ fst) xs)≠[])
 prefer 2
 apply simp
apply(frule rev-subsetD)
 apply(erule one [THEN equalityD1])
apply(erule subsetD)
apply simp
apply clarify
apply(subgoal-tac ∀ i<length clist. clist!i∈assum(Pre(xs!i), Rely(xs!i)))
 apply(erule-tac x=xa and P=λi. H i ⟶ ⊨ (J i) sat [I i,K i,M i,N i] for H J
I K M N in allE,erule impE,assumption)
 apply(simp add:com-validity-def)
 apply(erule-tac x=s in allE)
 apply(erule-tac x=xa and P=λj. H j ⟶ (I j) ∈ cp (J j) s for H I J in allE,simp)
 apply(drule-tac c=clist!xa in subsetD)
  apply (force simp add:Com-def)
 apply(simp add:comm-def conjoin-def same-program-def del:last.simps)
 apply clarify
 apply(erule-tac x=length x - 1 and P=λj. H j ⟶ fst(I j)=(J j) for H I J in
allE)
 apply (simp add:All-None-def same-length-def)
 apply(erule-tac x=xa and P=λj. H j ⟶ length(J j)=(K j) for H J K in allE)
 apply(subgoal-tac length x - 1 < length x,simp)
  apply(case-tac x≠[])
   apply(simp add: last-conv-nth)
   apply(erule-tac x=clist!xa in ballE)
    apply(simp add:same-state-def)
    apply(subgoal-tac clist!xa≠[])
     apply(simp add: last-conv-nth)
    apply(case-tac x)
     apply (force simp add:par-cp-def)
    apply (force simp add:par-cp-def)
   apply force
  apply (force simp add:par-cp-def)
 apply(case-tac x)
  apply (force simp add:par-cp-def)
 apply (force simp add:par-cp-def)
apply clarify
apply(simp add:assum-def)
apply(rule conjI)
 apply(simp add:conjoin-def same-state-def par-cp-def)
 apply clarify
 apply(erule-tac x=i and P=λj. T j ⟶ (∀ i. H j i ⟶ (snd (d j i))=(snd (e j
i))) for T H d e in allE,simp)
```

```
 apply(erule-tac x=0 and P=λj. H j ⟶ (snd (d j))=(snd (e j)) for H d e in
allE)
 apply(case-tac x,simp+)
 apply (simp add:par-assum-def)
 apply clarify
 apply(drule-tac c=snd (clist ! i ! 0) in subsetD)
 apply assumption
 apply simp
apply clarify
apply(erule-tac x=i in all-dupE)
apply(rule subsetD, erule mp, assumption)
apply(erule-tac pre=pre and rely=rely and x=x and s=s in  three)
 apply(erule-tac x=ib in allE,erule mp)
 apply simp-all
 apply(simp add:ParallelCom-def)
 apply(force simp add:Com-def)
apply(simp add:conjoin-def same-length-def)
done

lemma ParallelEmpty [rule-format]:
  ∀ i s. x ∈ par-cp (ParallelCom []) s ⟶
  Suc i < length x ⟶ (x ! i, x ! Suc i) ∉ par-ctran
apply(induct-tac x)
 apply(simp add:par-cp-def ParallelCom-def)
apply clarify
apply(case-tac list,simp,simp)
apply(case-tac i)
 apply(simp add:par-cp-def ParallelCom-def)
 apply(erule par-ctranE,simp)
apply(simp add:par-cp-def ParallelCom-def)
apply clarify
apply(erule par-cptn.cases,simp)
 apply simp
apply(erule par-ctranE)
back
apply simp
done

theorem par-rgsound:
 ⊢ c SAT [pre, rely, guar, post] ⟹
   ⊨ (ParallelCom c) SAT [pre, rely, guar, post]
apply(erule par-rghoare.induct)
apply(case-tac xs,simp)
 apply(simp add:par-com-validity-def par-comm-def)
 apply clarify
 apply(case-tac post=UNIV,simp)
  apply clarify
  apply(drule ParallelEmpty)
   apply assumption
```

```
  apply simp
 apply clarify
 apply simp
apply(subgoal-tac xs≠[])
 prefer 2
 apply simp
apply(rename-tac a list)
apply(thin-tac xs = a # list)
apply(simp add:par-com-validity-def par-comm-def)
apply clarify
apply(rule conjI)
 apply clarify
 apply(erule-tac pre=pre and rely=rely and guar=guar and x=x and s=s and
xs=xs in four)
       apply(assumption+)
     apply clarify
     apply (erule allE, erule impE, assumption,erule rgsound)
    apply(assumption+)
apply clarify
apply(erule-tac pre=pre and rely=rely and post=post and x=x and s=s and
xs=xs in five)
      apply(assumption+)
   apply clarify
   apply (erule allE, erule impE, assumption,erule rgsound)
  apply(assumption+)
done

end
```

3.6 Concrete Syntax

```
theory RG-Syntax
imports RG-Hoare Quote-Antiquote
begin

abbreviation Skip :: 'a com  (SKIP)
  where SKIP ≡ Basic id

notation Seq  ((-;;/ -) [60,61] 60)

syntax
  -Assign    :: idt ⇒ 'b ⇒ 'a com                    ((´- :=/ -) [70, 65] 61)
  -Cond       :: 'a bexp ⇒ 'a com ⇒ 'a com ⇒ 'a com   ((0IF -/ THEN -/ ELSE
-/FI) [0, 0, 0] 61)
  -Cond2     :: 'a bexp ⇒ 'a com ⇒ 'a com             ((0IF - THEN - FI) [0,0] 56)
  -While     :: 'a bexp ⇒ 'a com ⇒ 'a com             ((0WHILE - /DO - /OD)  [0,
0] 61)
  -Await     :: 'a bexp ⇒ 'a com ⇒ 'a com             ((0AWAIT - /THEN /- /END)
[0,0] 61)
```

```
-Atom      :: 'a com ⇒ 'a com                  ((⟨-⟩) 61)
-Wait      :: 'a bexp ⇒ 'a com                 ((0WAIT - END) 61)

translations
  ´x := a ⇀ CONST Basic «´(-update-name x (λ-. a))»
  IF b THEN c1 ELSE c2 FI ⇀ CONST Cond {|b|} c1 c2
  IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI
  WHILE b DO c OD ⇀ CONST While {|b|} c
  AWAIT b THEN c END ⇌ CONST Await {|b|} c
  ⟨c⟩ ⇌ AWAIT CONST True THEN c END
  WAIT b END ⇌ AWAIT b THEN SKIP END

nonterminal prgs

syntax
  -PAR        :: prgs ⇒ 'a             (COBEGIN//-//COEND 60)
  -prg        :: 'a ⇒ prgs             (- 57)
  -prgs       :: ['a, prgs] ⇒ prgs     (-//‖//- [60,57] 57)

translations
  -prg a ⇀ [a]
  -prgs a ps ⇀ a # ps
  -PAR ps ⇀ ps

syntax
  -prg-scheme :: ['a, 'a, 'a, 'a] ⇒ prgs  (SCHEME [- ≤ - < -] - [0,0,0,60] 57)

translations
  -prg-scheme j i k c ⇌ (CONST map (λi. c) [j..<k])
```

Translations for variables before and after a transition:

```
syntax
  -before :: id ⇒ 'a (º-)
  -after  :: id ⇒ 'a (ª-)

translations
  ºx ⇌ x ´CONST fst
  ªx ⇌ x ´CONST snd

print-translation ‹
  let
    fun quote-tr' f (t :: ts) =
          Term.list-comb (f $ Syntax-Trans.quote-tr' syntax-const ‹-antiquote› t,
ts)
      | quote-tr' - - = raise Match;

    val assert-tr' = quote-tr' (Syntax.const syntax-const ‹-Assert›);

    fun bexp-tr' name ((Const (const-syntax ‹Collect›, -) $ t) :: ts) =
```

quote-tr′ (*Syntax.const name*) (*t* :: *ts*)
| *bexp-tr′* - - = *raise Match*;

fun assign-tr′ (*Abs* (*x*, -, *f* $ *k* $ *Bound 0*) :: *ts*) =
quote-tr′ (*Syntax.const* ***syntax-const*** ‹*-Assign*› $ *Syntax-Trans.update-name-tr′ f*)
(*Abs* (*x*, *dummyT*, *Syntax-Trans.const-abs-tr′ k*) :: *ts*)
| *assign-tr′* - = *raise Match*;
in
[(***const-syntax*** ‹*Collect*›, *K assert-tr′*),
(***const-syntax*** ‹*Basic*›, *K assign-tr′*),
(***const-syntax*** ‹*Cond*›, *K* (*bexp-tr′* ***syntax-const*** ‹*-Cond*›)),
(***const-syntax*** ‹*While*›, *K* (*bexp-tr′* ***syntax-const*** ‹*-While*›))]
end
›

end

3.7 Examples

theory *RG-Examples*
imports *RG-Syntax*
begin

lemmas *definitions* [*simp*]= *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

3.7.1 Set Elements of an Array to Zero

lemma *le-less-trans2*: ⟦(*j*::*nat*)<*k*; *i*≤ *j*⟧ ⟹ *i*<*k*
by *simp*

lemma *add-le-less-mono*: ⟦ (*a*::*nat*) < *c*; *b*≤*d* ⟧ ⟹ *a* + *b* < *c* + *d*
by *simp*

record *Example1* =
A :: *nat list*

lemma *Example1*:
⊢ *COBEGIN*
SCHEME [*0* ≤ *i* < *n*]
(´*A* := ´*A* [*i* := *0*],
⦃ *n* < *length* ´*A* ⦄,
⦃ *length* ºA = *length* ªA ∧ ºA ! *i* = ªA ! *i* ⦄,
⦃ *length* ºA = *length* ªA ∧ (∀ *j*<*n*. *i* ≠ *j* ⟶ ºA ! *j* = ªA ! *j*) ⦄,
⦃ ´*A* ! *i* = *0* ⦄)
COEND
SAT [⦃ *n* < *length* ´*A* ⦄, ⦃ ºA = ªA ⦄, ⦃ *True* ⦄, ⦃ ∀ *i* < *n*. ´*A* ! *i* = *0* ⦄]
apply(*rule Parallel*)
apply (*auto intro*!: *Basic*)

```
done

lemma Example1-parameterized:
k < t ⟹
 ⊢ COBEGIN
    SCHEME [k∗n≤i<(Suc k)∗n] (´A:=´A[i:=0],
   ⦃t∗n < length ´A⦄,
   ⦃t∗n < length ºA ∧ length ºA=length ªA ∧ ºA!i = ªA!i⦄,
   ⦃t∗n < length ºA ∧ length ºA=length ªA ∧ (∀j<length ºA . i≠j ⟶ ºA!j =
ªA!j)⦄,
   ⦃´A!i=0⦄)
   COEND
 SAT [⦃t∗n < length ´A⦄,
     ⦃t∗n < length ºA ∧ length ºA=length ªA ∧ (∀i<n. ºA!(k∗n+i)=ªA!(k∗n+i))⦄,
      ⦃t∗n < length ºA ∧ length ºA=length ªA ∧
     (∀i<length ºA . (i<k∗n ⟶ ºA!i = ªA!i) ∧ ((Suc k)∗n ≤ i⟶ ºA!i = ªA!i))⦄,
      ⦃∀i<n. ´A!(k∗n+i) = 0⦄]
apply(rule Parallel)
    apply auto
  apply(erule-tac x=k∗n +i in allE)
  apply(subgoal-tac k∗n+i <length (A b))
   apply force
  apply(erule le-less-trans2)
  apply(case-tac t,simp+)
  apply (simp add:add.commute)
  apply(simp add: add-le-mono)
apply(rule Basic)
   apply simp
   apply clarify
   apply (subgoal-tac k∗n+i< length (A x))
    apply simp
   apply(erule le-less-trans2)
   apply(case-tac t,simp+)
   apply (simp add:add.commute)
   apply(rule add-le-mono, auto)
done
```

3.7.2 Increment a Variable in Parallel

Two components

```
record Example2 =
  x :: nat
  c-0 :: nat
  c-1 :: nat

lemma Example2:
 ⊢ COBEGIN
    (⟨ ´x:=´x+1;; ´c-0:=´c-0 + 1 ⟩,
     ⦃´x=´c-0 + ´c-1 ∧ ´c-0=0⦄,
```

```
    {|ºc-0 = ªc-0 ∧
       (ºx=ºc-0 + ºc-1
       ⟶ ªx = ªc-0 + ªc-1)|},
    {|ºc-1 = ªc-1 ∧
        (ºx=ºc-0 + ºc-1
        ⟶ ªx =ªc-0 + ªc-1)|},
    {|´x=´c-0 + ´c-1 ∧ ´c-0=1 |})
 ∥
     (⟨ ´x:=´x+1;; ´c-1:=´c-1+1 ⟩,
    {|´x=´c-0 + ´c-1 ∧ ´c-1=0 |},
    {|ºc-1 = ªc-1 ∧
       (ºx=ºc-0 + ºc-1
       ⟶ ªx = ªc-0 + ªc-1)|},
    {|ºc-0 = ªc-0 ∧
        (ºx=ºc-0 + ºc-1
       ⟶ ªx =ªc-0 + ªc-1)|},
    {|´x=´c-0 + ´c-1 ∧ ´c-1=1|})
 COEND
 SAT [{|´x=0 ∧ ´c-0=0 ∧ ´c-1=0|},
     {|ºx=ªx ∧ ºc-0= ªc-0 ∧ ºc-1=ªc-1|},
     {|True|},
     {|´x=2|}]
apply(rule Parallel)
   apply simp-all
   apply clarify
   apply(case-tac i)
    apply simp
    apply(rule conjI)
     apply clarify
     apply simp
    apply clarify
    apply simp
   apply simp
   apply(rule conjI)
    apply clarify
    apply simp
   apply clarify
   apply simp
   apply(subgoal-tac xa=0)
    apply simp
   apply arith
  apply clarify
  apply(case-tac xaa, simp, simp)
 apply clarify
 apply simp
 apply(erule-tac x=0 in all-dupE)
 apply(erule-tac x=1 in allE,simp)
apply clarify
apply(case-tac i,simp)
```

```
 apply(rule Await)
  apply simp-all
 apply(clarify)
 apply(rule Seq)
  prefer 2
  apply(rule Basic)
   apply simp-all
  apply(rule subset-refl)
 apply(rule Basic)
 apply simp-all
 apply clarify
 apply simp
apply(rule Await)
 apply simp-all
apply(clarify)
apply(rule Seq)
 prefer 2
 apply(rule Basic)
  apply simp-all
 apply(rule subset-refl)
apply(auto intro!: Basic)
done
```

Parameterized

```
lemma Example2-lemma2-aux: j<n ⟹
 (∑ i=0..<n. (b i::nat)) =
 (∑ i=0..<j. b i) + b j + (∑ i=0..<n−(Suc j) . b (Suc j + i))
apply(induct n)
 apply simp-all
apply(simp add:less-Suc-eq)
 apply(auto)
apply(subgoal-tac n − j = Suc(n− Suc j))
  apply simp
apply arith
done

lemma Example2-lemma2-aux2:
  j≤ s ⟹ (∑ i::nat=0..<j. (b (s:=t)) i) = (∑ i=0..<j. b i)
  by (induct j) simp-all

lemma Example2-lemma2:
⟦j<n; b j=0⟧ ⟹ Suc (∑ i::nat=0..<n. b i)=(∑ i=0..<n. (b (j := Suc 0)) i)
apply(frule-tac b=(b (j:=(Suc 0))) in Example2-lemma2-aux)
apply(erule-tac  t=sum (b(j := (Suc 0))) {0..<n} in ssubst)
apply(frule-tac b=b in Example2-lemma2-aux)
apply(erule-tac  t=sum b {0..<n} in ssubst)
apply(subgoal-tac Suc (sum b {0..<j} + b j + (∑ i=0..<n − Suc j. b (Suc j +
i)))=(sum b {0..<j} + Suc (b j) + (∑ i=0..<n − Suc j. b (Suc j + i))))
```

apply(*rotate-tac* −*1*)
apply(*erule ssubst*)
apply(*subgoal-tac* $j \le j$)
apply(*drule-tac* $b=b$ **and** $t=(Suc\ 0)$ **in** *Example2-lemma2-aux2*)
apply(*rotate-tac* −*1*)
apply(*erule ssubst*)
apply *simp-all*
done

lemma *Example2-lemma2-Suc0*: $[\![j<n;\ b\ j=0]\!] \Longrightarrow$
$Suc\ (\sum i::nat=0..< n.\ b\ i)=(\sum i=0..< n.\ (b\ (j:=Suc\ 0))\ i)$
by(*simp add*:*Example2-lemma2*)

record *Example2-parameterized* =
  $C :: nat \Rightarrow nat$
  $y\ :: nat$

lemma *Example2-parameterized*: $0<n \Longrightarrow$
 $\vdash$ *COBEGIN SCHEME* $[0 \le i < n]$
  $(\langle\ ´y:=´y+1;;\ ´C:=´C\ (i:=1)\ \rangle,$
  $\{\!|´y=(\sum i=0..<n.\ ´C\ i) \wedge ´C\ i=0|\!\},$
  $\{\!|^{o}C\ i = {}^{a}C\ i \wedge$
   $(^{o}y=(\sum i=0..<n.\ ^{o}C\ i) \longrightarrow {}^{a}y =(\sum i=0..<n.\ ^{a}C\ i))|\!\},$
  $\{\!|(\forall j<n.\ i\neq j \longrightarrow {}^{o}C\ j = {}^{a}C\ j) \wedge$
   $(^{o}y=(\sum i=0..<n.\ ^{o}C\ i) \longrightarrow {}^{a}y =(\sum i=0..<n.\ ^{a}C\ i))|\!\},$
  $\{\!|´y=(\sum i=0..<n.\ ´C\ i) \wedge ´C\ i=1|\!\})$
  COEND
 SAT $[\{\!|´y=0 \wedge (\sum i=0..<n.\ ´C\ i)=0\ |\!\},\ \{\!|^{o}C={}^{a}C \wedge {}^{o}y={}^{a}y|\!\},\ \{\!|True|\!\},\ \{\!|´y=n|\!\}]$
apply(*rule Parallel*)
apply *force*
apply *force*
apply(*force*)
apply *clarify*
apply *simp*
apply *simp*
apply *clarify*
apply *simp*
apply(*rule Await*)
apply *simp-all*
apply *clarify*
apply(*rule Seq*)
prefer *2*
apply(*rule Basic*)
apply(*rule subset-refl*)
apply *simp+*
apply(*rule Basic*)
apply *simp*
apply *clarify*
apply *simp*

apply(*simp add*:*Example2-lemma2-Suc0 cong*:*if-cong*)
apply *simp-all*
done

3.7.3 Find Least Element

A previous lemma:

lemma *mod-aux* :$[\![i < (n::nat);\ a\ mod\ n = i;\ \ j < a + n;\ j\ mod\ n = i;\ a < j]\!] \Longrightarrow$ *False*
apply(*subgoal-tac a=a div n∗n + a mod n*)
 prefer *2* **apply** (*simp* (*no-asm-use*))
apply(*subgoal-tac j=j div n∗n + j mod n*)
 prefer *2* **apply** (*simp* (*no-asm-use*))
apply *simp*
apply(*subgoal-tac a div n∗n < j div n∗n*)
prefer *2* **apply** *arith*
apply(*subgoal-tac j div n∗n < (a div n + 1)∗n*)
prefer *2* **apply** *simp*
apply (*simp only*:*mult-less-cancel2*)
apply *arith*
done

record *Example3 =*
  X :: nat ⇒ nat
  Y :: nat ⇒ nat

lemma *Example3*: $m\ mod\ n{=}0 \Longrightarrow$
$\vdash$ *COBEGIN*
 SCHEME $[0{\le}i{<}n]$
 (*WHILE* $(\forall j{<}n.\ {}^\prime X\ i < {}^\prime Y\ j)$ *DO*
   IF $P(B!({}^\prime X\ i))$ *THEN* ${}^\prime Y{:=}{}^\prime Y\ (i{:=}{}^\prime X\ i)$
   ELSE ${}^\prime X{:=}\ {}^\prime X\ (i{:=}({}^\prime X\ i){+}\ n)$ *FI*
  OD,
 $\{\!|({}^\prime X\ i)\ mod\ n{=}i \land (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \land ({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \land {}^\prime Y\ i{\le}\ m{+}i)|\!\}$,
 $\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^aY\ j \le {}^oY\ j) \land {}^oX\ i = {}^aX\ i \land$
  ${}^oY\ i = {}^aY\ i|\!\}$,
 $\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^oX\ j = {}^aX\ j \land {}^oY\ j = {}^aY\ j) \land$
  ${}^aY\ i \le {}^oY\ i|\!\}$,
 $\{\!|({}^\prime X\ i)\ mod\ n{=}i \land (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \land ({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \land {}^\prime Y\ i{\le}\ m{+}i) \land (\exists j{<}n.\ {}^\prime Y\ j \le {}^\prime X\ i)\ |\!\})$
 COEND
 SAT $[\{\!|\ \forall i{<}n.\ {}^\prime X\ i{=}i \land {}^\prime Y\ i{=}m{+}i\ |\!\},\{\!|{}^oX{=}{}^aX \land {}^oY{=}{}^aY|\!\},\{\!|True|\!\},$
  $\{\!|\forall i{<}n.\ ({}^\prime X\ i)\ mod\ n{=}i \land (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \land$
   $({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \land {}^\prime Y\ i{\le}\ m{+}i) \land (\exists j{<}n.\ {}^\prime Y\ j \le {}^\prime X\ i)|\!\}]$
apply(*rule Parallel*)
— 5 subgoals left
apply *force+*
apply *clarify*

apply *simp*
apply(*rule While*)
apply *force*
apply *force*
apply *force*
apply (*erule dvdE*)
apply(*rule-tac pre'*={| ´*X i mod n* = *i* ∧ (∀ *j*. *j*<´*X i* ⟶ *j mod n* = *i* ⟶ ¬*P*(*B*!*j*)) ∧ (´*Y i* < *n* ∗ *k* ⟶ *P* (*B*!(´*Y i*))) ∧ ´*X i*<´*Y i*|} **in** *Conseq*)
apply *force*
apply(*rule subset-refl*)+
apply(*rule Cond*)
apply *force*
apply(*rule Basic*)
apply *force*
apply *fastforce*
apply *force*
apply *force*
apply(*rule Basic*)
apply *simp*
apply *clarify*
apply *simp*
apply (*case-tac X x* (*j mod n*) ≤ *j*)
apply (*drule le-imp-less-or-eq*)
apply (*erule disjE*)
apply (*drule-tac j*=*j* **and** *n*=*n* **and** *i*=*j mod n* **and** *a*=*X x* (*j mod n*) **in** *mod-aux*)
apply *auto*
done

Same but with a list as auxiliary variable:

record *Example3-list* =
X :: *nat list*
Y :: *nat list*

lemma *Example3-list*: *m mod n*=*0* ⟹ ⊢ (*COBEGIN SCHEME* [*0*≤*i*<*n*]
(*WHILE* (∀ *j*<*n*. ´*X*!*i* < ´*Y*!*j*) *DO*
IF P(*B*!(´*X*!*i*)) *THEN* ´*Y*:=´*Y*[*i*:=´*X*!*i*] *ELSE* ´*X*:= ´*X*[*i*:=(´*X*!*i*)+ *n*] *FI*
OD,
{|*n*<*length* ´*X* ∧ *n*<*length* ´*Y* ∧ (´*X*!*i*) *mod n*=*i* ∧ (∀ *j*<´*X*!*i*. *j mod n*=*i* ⟶ ¬*P*(*B*!*j*)) ∧ (´*Y*!*i*<*m* ⟶ *P*(*B*!(´*Y*!*i*)) ∧ ´*Y*!*i*≤ *m*+*i*)|},
{|(∀ *j*<*n*. *i*≠*j* ⟶ ª*Y*!*j* ≤ º*Y*!*j*) ∧ º*X*!*i* = ª*X*!*i* ∧
º*Y*!*i* = ª*Y*!*i* ∧ *length* º*X* = *length* ª*X* ∧ *length* º*Y* = *length* ª*Y*|},
{|(∀ *j*<*n*. *i*≠*j* ⟶ º*X*!*j* = ª*X*!*j* ∧ º*Y*!*j* = ª*Y*!*j*) ∧
ª*Y*!*i* ≤ º*Y*!*i* ∧ *length* º*X* = *length* ª*X* ∧ *length* º*Y* = *length* ª*Y*|},
{|(´*X*!*i*) *mod n*=*i* ∧ (∀ *j*<´*X*!*i*. *j mod n*=*i* ⟶ ¬*P*(*B*!*j*)) ∧ (´*Y*!*i*<*m* ⟶ *P*(*B*!(´*Y*!*i*)) ∧ ´*Y*!*i*≤ *m*+*i*) ∧ (∃ *j*<*n*. ´*Y*!*j* ≤ ´*X*!*i*) |}) *COEND*)
SAT [{|*n*<*length* ´*X* ∧ *n*<*length* ´*Y* ∧ (∀ *i*<*n*. ´*X*!*i*=*i* ∧ ´*Y*!*i*=*m*+*i*) |},
{|º*X*=ª*X* ∧ º*Y*=ª*Y*|},
{|*True*|},

```
      {|∀ i<n. (´X!i) mod n=i ∧ (∀ j<´X!i. j mod n=i ⟶ ¬P(B!j)) ∧
        (´Y!i<m ⟶ P(B!(´Y!i)) ∧ ´Y!i≤ m+i) ∧ (∃ j<n. ´Y!j ≤ ´X!i)|}]
  apply (rule Parallel)
apply (auto cong del: image-cong-simp)
apply force
apply (rule While)
    apply force
   apply force
  apply force
  apply (erule dvdE)
 apply(rule-tac pre'={|n<length ´X ∧ n<length ´Y ∧ ´X ! i mod n = i ∧ (∀ j. j
< ´X ! i ⟶ j mod n = i ⟶ ¬ P (B ! j)) ∧ (´Y ! i < n ∗ k ⟶ P (B ! (´Y !
i))) ∧ ´X!i<´Y!i|} in Conseq)
     apply force
    apply(rule subset-refl)+
 apply(rule Cond)
    apply force
   apply(rule Basic)
      apply force
     apply force
    apply force
   apply force
  apply(rule Basic)
     apply simp
     apply clarify
     apply simp
     apply(rule allI)
     apply(rule impI)+
     apply(case-tac X x ! i≤ j)
      apply(drule le-imp-less-or-eq)
      apply(erule disjE)
       apply(drule-tac j=j and n=n and i=i and a=X x ! i in mod-aux)
     apply auto
done

end
theory Hoare-Parallel
imports OG-Examples Gar-Coll Mul-Gar-Coll RG-Examples
begin

end
```

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