Functional Data Structures

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Abstract

A collection of verified functional data structures. The emphasis is on conciseness of algorithms and succinctness of proofs, more in the style of a textbook than a library of efficient algorithms.

For more details see [13].

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1 Sorting

theory *Sorting*
imports
Complex_Main
HOL−Library.Multiset
HOL−Library.Time_Commands
begin

hide_const *List.insort*

declare *Let_def* [*simp*]

1.1 Insertion Sort

fun *insort1* :: *'a*::*linorder* ⇒ *'a list* ⇒ *'a list* **where**
insort1 x [] = [*x*] |
insort1 x (*y*#*ys*) =
(*if x* ≤ *y then x*#*y*#*ys else y*#(*insort1 x ys*))

fun *insort* :: *'a*::*linorder list* ⇒ *'a list* **where**
insort [] = [] |
insort (*x*#*xs*) = *insort1 x* (*insort xs*)

1.1.1 Functional Correctness

lemma *mset_insort1*: *mset* (*insort1 x xs*) = {#*x*#} + *mset xs*
by (*induction xs*) *auto*

lemma *mset_insort*: *mset* (*insort xs*) = *mset xs*
by (*induction xs*) (*auto simp*: *mset_insort1*)

lemma *set_insort1*: *set* (*insort1 x xs*) = {*x*} ∪ *set xs*
by(*simp add*: *mset_insort1 flip*: *set_mset_mset*)

lemma *sorted_insort1*: *sorted* (*insort1 a xs*) = *sorted xs*
by (*induction xs*) (*auto simp*: *set_insort1*)

lemma *sorted_insort*: *sorted* (*insort xs*)
by (*induction xs*) (*auto simp*: *sorted_insort1*)

1.1.2 Time Complexity

time_fun *insort1*
time_fun *insort*

lemma *T_insort1_length*: *T_insort1 x xs* ≤ *length xs + 1*
by (*induction xs*) *auto*

lemma *length_insort1*: *length* (*insort1 x xs*) = *length xs + 1*
by (*induction xs*) *auto*

lemma *length_insort*: *length* (*insort xs*) = *length xs*
by (*metis Sorting.mset_insort size_mset*)

lemma *T_insort_length*: *T_insort xs* ≤ (*length xs + 1*) ^ *2*
proof(*induction xs*)
case *Nil* **show** *?case* **by** *simp*
next
case (*Cons x xs*)
have *T_insort* (*x#xs*) = *T_insort xs + T_insort1 x* (*insort xs*) + *1* **by** *simp*
also have ... ≤ (*length xs + 1*) ^ *2 + T_insort1 x* (*insort xs*) + *1*
using *Cons.IH* **by** *simp*
also have ... ≤ (*length xs + 1*) ^ *2 + length xs + 1 + 1*
using *T_insort1_length*[*of x insort xs*] **by** (*simp add*: *length_insort*)
also have ... ≤ (*length*(*x#xs*) + *1*) ^ *2*
by (*simp add*: *power2_eq_square*)
finally show *?case* **.**
qed

1.2 Merge Sort

fun *merge* :: *'a*::*linorder list* ⇒ *'a list* ⇒ *'a list* **where**
merge [] *ys* = *ys* |
merge xs [] = *xs* |
merge (*x#xs*) (*y#ys*) = (*if x* ≤ *y then x # merge xs* (*y#ys*) *else y # merge* (*x#xs*) *ys*)

fun *msort* :: *'a*::*linorder list* ⇒ *'a list* **where**
msort xs = (*let n* = *length xs in*
if n ≤ *1 then xs*
else merge (*msort* (*take* (*n div 2*) *xs*)) (*msort* (*drop* (*n div 2*) *xs*)))

declare *msort.simps* [*simp del*]

1.2.1 Functional Correctness

lemma *mset_merge*: *mset*(*merge xs ys*) = *mset xs + mset ys*

```
  by(induction xs ys rule: merge.induct) auto

lemma mset_msort: mset (msort xs) = mset xs
proof(induction xs rule: msort.induct)
  case (1 xs)
  let ?n = length xs
  let ?ys = take (?n div 2) xs
  let ?zs = drop (?n div 2) xs
  show ?case
  proof cases
    assume ?n ≤ 1
    thus ?thesis by(simp add: msort.simps[of xs])
  next
    assume ¬ ?n ≤ 1
    hence mset (msort xs) = mset (msort ?ys) + mset (msort ?zs)
      by(simp add: msort.simps[of xs] mset_merge)
    also have ... = mset ?ys + mset ?zs
      using ‹¬ ?n ≤ 1› by(simp add: 1.IH)
   also have ... = mset (?ys @ ?zs) by (simp del: append_take_drop_id)
    also have ... = mset xs by simp
    finally show ?thesis .
  qed
qed
```

Via the previous lemma or directly:

```
lemma set_merge: set(merge xs ys) = set xs ∪ set ys
  by (metis mset_merge set_mset_mset set_mset_union)

lemma set(merge xs ys) = set xs ∪ set ys
  by(induction xs ys rule: merge.induct) (auto)

lemma sorted_merge: sorted (merge xs ys) ⟷ (sorted xs ∧ sorted ys)
  by(induction xs ys rule: merge.induct) (auto simp: set_merge)

lemma sorted_msort: sorted (msort xs)
proof(induction xs rule: msort.induct)
  case (1 xs)
  let ?n = length xs
  show ?case
  proof cases
    assume ?n ≤ 1
    thus ?thesis by(simp add: msort.simps[of xs] sorted01)
  next
    assume ¬ ?n ≤ 1
```

```
    thus ?thesis using 1.IH
      by(simp add: sorted_merge msort.simps[of xs])
  qed
qed
```

1.2.2 Time Complexity

We only count the number of comparisons between list elements.

```
fun C_merge :: 'a::linorder list ⇒ 'a list ⇒ nat where
  C_merge [] ys = 0 |
  C_merge xs [] = 0 |
  C_merge (x#xs) (y#ys) = 1 + (if x ≤ y then C_merge xs (y#ys) else
C_merge (x#xs) ys)

lemma C_merge_ub: C_merge xs ys ≤ length xs + length ys
  by (induction xs ys rule: C_merge.induct) auto

fun C_msort :: 'a::linorder list ⇒ nat where
  C_msort xs =
  (let n = length xs;
       ys = take (n div 2) xs;
       zs = drop (n div 2) xs
   in if n ≤ 1 then 0
      else C_msort ys + C_msort zs + C_merge (msort ys) (msort zs))

declare C_msort.simps [simp del]

lemma length_merge: length(merge xs ys) = length xs + length ys
  by (induction xs ys rule: merge.induct) auto

lemma length_msort: length(msort xs) = length xs
proof (induction xs rule: msort.induct)
  case (1 xs)
  show ?case
    by (auto simp: msort.simps [of xs] 1 length_merge)
qed
```

Why structured proof? To have the name "xs" to specialize msort.simps with xs to ensure that msort.simps cannot be used recursively. Also works without this precaution, but that is just luck.

```
lemma C_msort_le: length xs = 2^k ⟹ C_msort xs ≤ k * 2^k
proof(induction k arbitrary: xs)
  case 0 thus ?case by (simp add: C_msort.simps)
next
```

```
  case (Suc k)
  let ?n = length xs
  let ?ys = take (?n div 2) xs
  let ?zs = drop (?n div 2) xs
  show ?case
  proof (cases ?n ≤ 1)
    case True
    thus ?thesis by(simp add: C_msort.simps)
  next
    case False
    have C_msort(xs) =
      C_msort ?ys + C_msort ?zs + C_merge (msort ?ys) (msort ?zs)
      by (simp add: C_msort.simps msort.simps)
    also have ... ≤ C_msort ?ys + C_msort ?zs + length ?ys + length
?zs
        using C_merge_ub[of msort ?ys msort ?zs] length_msort[of ?ys]
length_msort[of ?zs]
      by arith
    also have ... ≤ k * 2^k + C_msort ?zs + length ?ys + length ?zs
      using Suc.IH[of ?ys] Suc.prems by simp
    also have ... ≤ k * 2^k + k * 2^k + length ?ys + length ?zs
      using Suc.IH[of ?zs] Suc.prems by simp
    also have ... = 2 * k * 2^k + 2 * 2 ^ k
      using Suc.prems by simp
    finally show ?thesis by simp
  qed
qed


lemma C_msort_log: length xs = 2^k ⟹ C_msort xs ≤ length xs * log
2 (length xs)
  using C_msort_le[of xs k]
  by (metis log2_of_power_eq mult.commute of_nat_mono of_nat_mult)
```

1.3 Bottom-Up Merge Sort

```
fun merge_adj :: ('a::linorder) list list ⇒ 'a list list where
  merge_adj [] = [] |
  merge_adj [xs] = [xs] |
  merge_adj (xs # ys # zss) = merge xs ys # merge_adj zss
```

For the termination proof of *merge_all* below.

```
lemma length_merge_adjacent[simp]: length (merge_adj xs) = (length xs
+ 1) div 2
```

by (*induction xs rule*: *merge_adj.induct*) *auto*

fun *merge_all* :: ('*a*::*linorder*) *list list* ⇒ '*a list* **where**
merge_all [] = [] |
merge_all [*xs*] = *xs* |
merge_all xss = *merge_all* (*merge_adj xss*)

definition *msort_bu* :: ('*a*::*linorder*) *list* ⇒ '*a list* **where**
msort_bu xs = *merge_all* (*map* (λ*x*. [*x*]) *xs*)

1.3.1 Functional Correctness

abbreviation *mset_mset* :: '*a list list* ⇒ '*a multiset* **where**
mset_mset xss ≡ ∑# (*image_mset mset* (*mset xss*))

lemma *mset_merge_adj*:
mset_mset (*merge_adj xss*) = *mset_mset xss*
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *mset_merge*)

lemma *mset_merge_all*:
mset (*merge_all xss*) = *mset_mset xss*
by(*induction xss rule*: *merge_all.induct*) (*auto simp*: *mset_merge mset_merge_adj*)

lemma *mset_msort_bu*: *mset* (*msort_bu xs*) = *mset xs*
by(*simp add*: *msort_bu_def mset_merge_all multiset.map_comp comp_def*)

lemma *sorted_merge_adj*:
∀ *xs* ∈ *set xss*. *sorted xs* ⟹ ∀ *xs* ∈ *set* (*merge_adj xss*). *sorted xs*
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *sorted_merge*)

lemma *sorted_merge_all*:
∀ *xs* ∈ *set xss*. *sorted xs* ⟹ *sorted* (*merge_all xss*)
by (*induction xss rule*: *merge_all.induct*) (*auto simp add*: *sorted_merge_adj*)

lemma *sorted_msort_bu*: *sorted* (*msort_bu xs*)
by(*simp add*: *msort_bu_def sorted_merge_all*)

1.3.2 Time Complexity

fun *C_merge_adj* :: ('*a*::*linorder*) *list list* ⇒ *nat* **where**
C_merge_adj [] = *0* |
C_merge_adj [*xs*] = *0* |
C_merge_adj (*xs* # *ys* # *zss*) = *C_merge xs ys* + *C_merge_adj zss*

fun *C_merge_all* :: ('*a*::*linorder*) *list list* ⇒ *nat* **where**
C_merge_all [] = *0* |
C_merge_all [*xs*] = *0* |
C_merge_all xss = *C_merge_adj xss* + *C_merge_all* (*merge_adj xss*)

definition *C_msort_bu* :: ('*a*::*linorder*) *list* ⇒ *nat* **where**
C_msort_bu xs = *C_merge_all* (*map* (λ*x*. [*x*]) *xs*)

lemma *length_merge_adj*:
⟦ *even*(*length xss*); ∀ *xs* ∈ *set xss*. *length xs* = *m* ⟧
⟹ ∀ *xs* ∈ *set* (*merge_adj xss*). *length xs* = *2*∗*m*
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *length_merge*)

lemma *C_merge_adj*: ∀ *xs* ∈ *set xss*. *length xs* = *m* ⟹ *C_merge_adj xss* ≤ *m* ∗ *length xss*
proof(*induction xss rule*: *C_merge_adj.induct*)
case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp*
next
case (*3 x y*) **thus** *?case* **using** *C_merge_ub*[*of x y*] **by** (*simp add*: *algebra_simps*)
qed

lemma *C_merge_all*: ⟦ ∀ *xs* ∈ *set xss*. *length xs* = *m*; *length xss* = *2*^*k* ⟧
⟹ *C_merge_all xss* ≤ *m* ∗ *k* ∗ *2*^*k*
proof (*induction xss arbitrary*: *k m rule*: *C_merge_all.induct*)
case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp*
next
case (*3 xs ys xss*)
let *?xss* = *xs* # *ys* # *xss*
let *?xss2* = *merge_adj ?xss*
obtain *k'* **where** *k'*: *k* = *Suc k'* **using** *3.prems*(*2*)
by (*metis length_Cons nat.inject nat_power_eq_Suc_0_iff nat.exhaust*)
have *even* (*length ?xss*) **using** *3.prems*(*2*) *k'* **by** *auto*
from *length_merge_adj*[*OF this 3.prems*(*1*)]
have ∗: ∀ *x* ∈ *set*(*merge_adj ?xss*). *length x* = *2*∗*m* **.**
have ∗∗: *length ?xss2* = *2* ^ *k'* **using** *3.prems*(*2*) *k'* **by** *auto*
have *C_merge_all ?xss* = *C_merge_adj ?xss* + *C_merge_all ?xss2* **by** *simp*
also have ... ≤ *m* ∗ *2*^*k* + *C_merge_all ?xss2*
using *3.prems*(*2*) *C_merge_adj*[*OF 3.prems*(*1*)] **by** (*auto simp*: *alge-*

bra_simps)
also have ... $\leq m * 2\hat{}k + (2*m) * k' * 2\hat{}k'$
using *3.IH*[*OF* $*$**] **by** *simp*
also have ... $= m * k * 2\hat{}k$
using k' **by** (*simp add*: *algebra_simps*)
finally show *?case* **.**
qed

corollary *C_msort_bu*: *length xs* $= 2 \hat{} k \Longrightarrow$ *C_msort_bu xs* $\leq k * 2 \hat{}$ *k*
using *C_merge_all*[*of map* ($\lambda x.$ [*x*]) *xs 1*] **by** (*simp add*: *C_msort_bu_def*)

1.4 Quicksort

fun *quicksort* :: ('*a*::*linorder*) *list* $\Rightarrow$ '*a list* **where**
quicksort [] = [] |
quicksort (*x*#*xs*) = *quicksort* (*filter* ($\lambda y.$ *y* < *x*) *xs*) @ [*x*] @ *quicksort* (*filter* ($\lambda y.$ *x* $\leq$ *y*) *xs*)

lemma *mset_quicksort*: *mset* (*quicksort xs*) = *mset xs*
by (*induction xs rule*: *quicksort.induct*) (*auto simp*: *not_le*)

lemma *set_quicksort*: *set* (*quicksort xs*) = *set xs*
by(*rule mset_eq_setD*[*OF mset_quicksort*])

lemma *sorted_quicksort*: *sorted* (*quicksort xs*)
proof (*induction xs rule*: *quicksort.induct*)
qed (*auto simp*: *sorted_append set_quicksort*)

1.5 Insertion Sort w.r.t. Keys and Stability

hide_const *List.insort_key*

fun *insort1_key* :: ('*a* $\Rightarrow$ '*k*::*linorder*) $\Rightarrow$ '*a* $\Rightarrow$ '*a list* $\Rightarrow$ '*a list* **where**
insort1_key f x [] = [*x*] |
insort1_key f x (*y* # *ys*) = (*if f x* $\leq$ *f y then x* # *y* # *ys else y* # *insort1_key f x ys*)

fun *insort_key* :: ('*a* $\Rightarrow$ '*k*::*linorder*) $\Rightarrow$ '*a list* $\Rightarrow$ '*a list* **where**
insort_key f [] = [] |
insort_key f (*x* # *xs*) = *insort1_key f x* (*insort_key f xs*)

1.5.1 Standard functional correctness

lemma *mset_insort1_key*: *mset* (*insort1_key f x xs*) = {#*x*#} + *mset xs*

by(*induction xs*) *simp_all*

lemma *mset_insort_key*: *mset* (*insort_key f xs*) = *mset xs*
by(*induction xs*) (*simp_all add*: *mset_insort1_key*)

lemma *set_insort1_key*: *set* (*insort1_key f x xs*) = {*x*} ∪ *set xs*
by (*induction xs*) *auto*

lemma *sorted_insort1_key*: *sorted* (*map f* (*insort1_key f a xs*)) = *sorted* (*map f xs*)
by(*induction xs*)(*auto simp*: *set_insort1_key*)

lemma *sorted_insort_key*: *sorted* (*map f* (*insort_key f xs*))
by(*induction xs*)(*simp_all add*: *sorted_insort1_key*)

1.5.2 Stability

lemma *insort1_is_Cons*: ∀ *x*∈*set xs*. *f a* ≤ *f x* ⟹ *insort1_key f a xs* = *a* # *xs*
by (*cases xs*) *auto*

lemma *filter_insort1_key_neg*:
¬ *P x* ⟹ *filter P* (*insort1_key f x xs*) = *filter P xs*
by (*induction xs*) *simp_all*

lemma *filter_insort1_key_pos*:
sorted (*map f xs*) ⟹ *P x* ⟹ *filter P* (*insort1_key f x xs*) = *insort1_key f x* (*filter P xs*)
by (*induction xs*) (*auto*, *subst insort1_is_Cons*, *auto*)

lemma *sort_key_stable*: *filter* (λ*y*. *f y* = *k*) (*insort_key f xs*) = *filter* (λ*y*. *f y* = *k*) *xs*
proof (*induction xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons a xs*)
thus *?case*
proof (*cases f a* = *k*)
case *False* **thus** *?thesis* **by** (*simp add*: *Cons.IH filter_insort1_key_neg*)
next
case *True*
have *filter* (λ*y*. *f y* = *k*) (*insort_key f* (*a* # *xs*))
= *filter* (λ*y*. *f y* = *k*) (*insort1_key f a* (*insort_key f xs*)) **by** *simp*

also have ... = *insort1_key f a* (*filter* (λ*y*. *f y* = *k*) (*insort_key f xs*))
by (*simp add*: *True filter_insort1_key_pos sorted_insort_key*)
also have ... = *insort1_key f a* (*filter* (λ*y*. *f y* = *k*) *xs*) **by** (*simp add*: *Cons.IH*)
also have ... = *a* # (*filter* (λ*y*. *f y* = *k*) *xs*) **by**(*simp add*: *True insort1_is_Cons*)
also have ... = *filter* (λ*y*. *f y* = *k*) (*a* # *xs*) **by** (*simp add*: *True*)
finally show *?thesis* **.**
qed
qed

1.6 Uniqueness of Sorting

lemma *sorting_unique*:
assumes *mset ys* = *mset xs sorted xs sorted ys*
shows *xs* = *ys*
using *assms*
proof (*induction xs arbitrary*: *ys*)
case (*Cons x xs ys′*)
obtain *y ys* **where** *ys′*: *ys′* = *y* # *ys*
using *Cons.prems* **by** (*cases ys′*) *auto*
have *x* = *y*
using *Cons.prems* **unfolding** *ys′*
proof (*induction x y arbitrary*: *xs ys rule*: *linorder_wlog*)
case (*le x y xs ys*)
have *x* ∈# *mset* (*x* # *xs*)
by *simp*
also have *mset* (*x* # *xs*) = *mset* (*y* # *ys*)
using *le* **by** *simp*
finally show *x* = *y*
using *le* **by** *auto*
qed (*simp_all add*: *eq_commute*)
thus *?case*
using *Cons.prems Cons.IH*[*of ys*] **by** (*auto simp*: *ys′*)
qed *auto*

end

2 Creating Almost Complete Trees

theory *Balance*
imports
HOL−Library.Tree_Real

begin

fun *bal* :: *nat* ⇒ *'a list* ⇒ *'a tree* ∗ *'a list* **where**
bal n xs = (*if n=0 then* (*Leaf*,*xs*) *else*
(*let m* = *n div 2*;
(*l*, *ys*) = *bal m xs*;
(*r*, *zs*) = *bal* (*n*−*1*−*m*) (*tl ys*)
in (*Node l* (*hd ys*) *r*, *zs*)))

declare *bal.simps*[*simp del*]
declare *Let_def*[*simp*]

definition *bal_list* :: *nat* ⇒ *'a list* ⇒ *'a tree* **where**
bal_list n xs = *fst* (*bal n xs*)

definition *balance_list* :: *'a list* ⇒ *'a tree* **where**
balance_list xs = *bal_list* (*length xs*) *xs*

definition *bal_tree* :: *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
bal_tree n t = *bal_list n* (*inorder t*)

definition *balance_tree* :: *'a tree* ⇒ *'a tree* **where**
balance_tree t = *bal_tree* (*size t*) *t*

lemma *bal_simps*:
bal 0 xs = (*Leaf*, *xs*)
n > *0* ⟹
bal n xs =
(*let m* = *n div 2*;
(*l*, *ys*) = *bal m xs*;
(*r*, *zs*) = *bal* (*n*−*1*−*m*) (*tl ys*)
in (*Node l* (*hd ys*) *r*, *zs*))
by(*simp_all add*: *bal.simps*)

lemma *bal_inorder*:
⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*zs*) ⟧
⟹ *xs* = *inorder t* @ *zs* ∧ *size t* = *n*
proof(*induction n arbitrary*: *xs t zs rule*: *less_induct*)
case (*less n*) **show** *?case*
proof *cases*
assume *n* = *0* **thus** *?thesis* **using** *less.prems* **by** (*simp add*: *bal_simps*)
next
assume [*arith*]: *n* ≠ *0*
let *?m* = *n div 2* **let** *?m'* = *n* − *1* − *?m*

from *less.prems*(*2*) **obtain** *l r ys* **where**
b1: *bal ?m xs* = (*l*,*ys*) **and**
b2: *bal ?m′* (*tl ys*) = (*r*,*zs*) **and**
t: *t* = ⟨*l*, *hd ys*, *r*⟩
by(*auto simp*: *bal_simps split*: *prod.splits*)
have *IH1*: *xs* = *inorder l* @ *ys* ∧ *size l* = *?m*
using *b1 less.prems*(*1*) **by**(*intro less.IH*) *auto*
have *IH2*: *tl ys* = *inorder r* @ *zs* ∧ *size r* = *?m′*
using *b2 IH1 less.prems*(*1*) **by**(*intro less.IH*) *auto*
show *?thesis* **using** *t IH1 IH2 less.prems*(*1*) *hd_Cons_tl*[*of ys*] **by** *fastforce*
qed
qed

corollary *inorder_bal_list*[*simp*]:
n ≤ *length xs* ⟹ *inorder*(*bal_list n xs*) = *take n xs*
unfolding *bal_list_def*
by (*metis* (*mono_tags*) *prod.collapse*[*of bal n xs*] *append_eq_conv_conj bal_inorder length_inorder*)

corollary *inorder_balance_list*[*simp*]: *inorder*(*balance_list xs*) = *xs*
by(*simp add*: *balance_list_def*)

corollary *inorder_bal_tree*:
n ≤ *size t* ⟹ *inorder*(*bal_tree n t*) = *take n* (*inorder t*)
by(*simp add*: *bal_tree_def*)

corollary *inorder_balance_tree*[*simp*]: *inorder*(*balance_tree t*) = *inorder t*
by(*simp add*: *balance_tree_def inorder_bal_tree*)

The length/size lemmas below do not require the precondition $n \leq$ *length xs* (or $n \leq$ *size t*) that they come with. They could take advantage of the fact that *bal xs n* yields a result even if *length xs* < *n*. In that case the result will contain one or more occurrences of *hd* []. However, this is counter-intuitive and does not reflect the execution in an eager functional language.

lemma *bal_length*: ⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*zs*) ⟧ ⟹ *length zs* = *length xs* − *n*
using *bal_inorder* **by** *fastforce*

corollary *size_bal_list*[*simp*]: *n* ≤ *length xs* ⟹ *size*(*bal_list n xs*) = *n*
unfolding *bal_list_def* **using** *bal_inorder prod.exhaust_sel* **by** *blast*

corollary *size_balance_list*[*simp*]: *size*(*balance_list xs*) = *length xs*
by (*simp add*: *balance_list_def*)

corollary *size_bal_tree*[*simp*]: $n \le size\ t \Longrightarrow size(bal\_tree\ n\ t) = n$
by(*simp add*: *bal_tree_def*)

corollary *size_balance_tree*[*simp*]: $size(balance\_tree\ t) = size\ t$
by(*simp add*: *balance_tree_def*)

lemma *min_height_bal*:
$[\![\ n \le length\ xs;\ bal\ n\ xs = (t,zs)\ ]\!] \Longrightarrow min\_height\ t = nat(\lfloor log\ 2\ (n + 1)\rfloor)$
proof(*induction n arbitrary*: *xs t zs rule*: *less_induct*)
case (*less n*)
show *?case*
proof *cases*
assume $n = 0$ **thus** *?thesis* **using** *less.prems*(*2*) **by** (*simp add*: *bal_simps*)
next
assume [*arith*]: $n \neq 0$
let $?m = n\ div\ 2$ **let** $?m' = n - 1 - ?m$
from *less.prems* **obtain** *l r ys* **where**
b1: $bal\ ?m\ xs = (l,ys)$ **and**
b2: $bal\ ?m'\ (tl\ ys) = (r,zs)$ **and**
t: $t = \langle l,\ hd\ ys,\ r\rangle$
by(*auto simp*: *bal_simps split*: *prod.splits*)
let $?hl = nat\ (floor(log\ 2\ (?m + 1)))$
let $?hr = nat\ (floor(log\ 2\ (?m' + 1)))$
have *IH1*: $min\_height\ l = ?hl$ **using** *less.IH*[*OF _ _ b1*] *less.prems*(*1*)
by *simp*
have *IH2*: $min\_height\ r = ?hr$
using *less.prems*(*1*) *bal_length*[*OF _ b1*] *b2* **by**(*intro less.IH*) *auto*
have $(n+1)\ div\ 2 \ge 1$ **by** *arith*
hence *0*: $log\ 2\ ((n+1)\ div\ 2) \ge 0$ **by** *simp*
have $?m' \le ?m$ **by** *arith*
hence *le*: $?hr \le ?hl$ **by**(*simp add*: *nat_mono floor_mono*)
have $min\_height\ t = min\ ?hl\ ?hr + 1$ **by** (*simp add*: *t IH1 IH2*)
also have $\ldots = ?hr + 1$ **using** *le* **by** (*simp add*: *min_absorb2*)
also have $?m' + 1 = (n+1)\ div\ 2$ **by** *linarith*
also have $nat\ (floor(log\ 2\ ((n+1)\ div\ 2))) + 1$
$= nat\ (floor(log\ 2\ ((n+1)\ div\ 2) + 1))$
using *0* **by** *linarith*
also have $\ldots = nat\ (floor(log\ 2\ (n + 1)))$
using *floor_log2_div2*[*of n+1*] **by** (*simp add*: *log_mult*)
finally show *?thesis* **.**
qed
qed

lemma *height_bal*:
  ⟦ $n \leq$ *length xs*; *bal n xs* = (*t*,*zs*) ⟧ ⟹ *height t* = *nat* ⌈*log 2* (*n* + *1*)⌉
proof(*induction n arbitrary*: *xs t zs rule*: *less_induct*)
  case (*less n*) **show** *?case*
  proof *cases*
    assume *n* = *0* **thus** *?thesis*
      using *less.prems* **by** (*simp add*: *bal_simps*)
  next
    assume [*arith*]: *n* ≠ *0*
    let *?m* = *n div 2* **let** *?m'* = *n* − *1* − *?m*
    from *less.prems* **obtain** *l r ys* **where**
      b1: *bal ?m xs* = (*l*,*ys*) **and**
      b2: *bal ?m'* (*tl ys*) = (*r*,*zs*) **and**
      t: *t* = ⟨*l*, *hd ys*, *r*⟩
      by(*auto simp*: *bal_simps split*: *prod.splits*)
    let *?hl* = *nat* ⌈*log 2* (*?m* + *1*)⌉
    let *?hr* = *nat* ⌈*log 2* (*?m'* + *1*)⌉
    have *IH1*: *height l* = *?hl* **using** *less.IH*[*OF* _ _ *b1*] *less.prems*(*1*) **by** *simp*
    have *IH2*: *height r* = *?hr*
      using *b2 bal_length*[*OF* _ *b1*] *less.prems*(*1*) **by**(*intro less.IH*) *auto*
    have *0*: *log 2* (*?m* + *1*) ≥ *0* **by** *simp*
    have *?m'* ≤ *?m* **by** *arith*
    hence *le*: *?hr* ≤ *?hl*
      by(*simp add*: *nat_mono ceiling_mono del*: *nat_ceiling_le_eq*)
    have *height t* = *max ?hl ?hr* + *1* **by** (*simp add*: *t IH1 IH2*)
    also have ... = *?hl* + *1* **using** *le* **by** (*simp add*: *max_absorb1*)
    also have ... = *nat* ⌈*log 2* (*?m* + *1*) + *1*⌉ **using** *0* **by** *linarith*
    also have ... = *nat* ⌈*log 2* (*n* + *1*)⌉
      using *ceiling_log2_div2*[*of n*+*1*] **by** (*simp*)
    finally show *?thesis* **.**
  qed
qed

lemma *acomplete_bal*:
  assumes *n* ≤ *length xs bal n xs* = (*t*,*ys*) **shows** *acomplete t*
unfolding *acomplete_def*
using *height_bal*[*OF assms*] *min_height_bal*[*OF assms*]
by *linarith*

lemma *height_bal_list*:
  n ≤ *length xs* ⟹ *height* (*bal_list n xs*) = *nat* ⌈*log 2* (*n* + *1*)⌉
unfolding *bal_list_def* **by** (*metis height_bal prod.collapse*)

lemma *height_balance_list*:
height (*balance_list xs*) = *nat* ⌈*log 2* (*length xs* + *1*)⌉
by (*simp add*: *balance_list_def height_bal_list*)

corollary *height_bal_tree*:
n ≤ *size t* ⟹ *height* (*bal_tree n t*) = *nat*⌈*log 2* (*n* + *1*)⌉
unfolding *bal_list_def bal_tree_def*
by (*metis bal_list_def height_bal_list length_inorder*)

corollary *height_balance_tree*:
height (*balance_tree t*) = *nat*⌈*log 2* (*size t* + *1*)⌉
by (*simp add*: *bal_tree_def balance_tree_def height_bal_list*)

corollary *acomplete_bal_list*[*simp*]: *n* ≤ *length xs* ⟹ *acomplete* (*bal_list n xs*)
unfolding *bal_list_def* **by** (*metis acomplete_bal prod.collapse*)

corollary *acomplete_balance_list*[*simp*]: *acomplete* (*balance_list xs*)
by (*simp add*: *balance_list_def*)

corollary *acomplete_bal_tree*[*simp*]: *n* ≤ *size t* ⟹ *acomplete* (*bal_tree n t*)
by (*simp add*: *bal_tree_def*)

corollary *acomplete_balance_tree*[*simp*]: *acomplete* (*balance_tree t*)
by (*simp add*: *balance_tree_def*)

lemma *wbalanced_bal*: ⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*ys*) ⟧ ⟹ *wbalanced t*
proof(*induction n arbitrary*: *xs t ys rule*: *less_induct*)
case (*less n*)
show *?case*
proof *cases*
assume *n* = *0*
thus *?thesis* **using** *less.prems*(*2*) **by**(*simp add*: *bal_simps*)
next
assume [*arith*]: *n* ≠ *0*
with *less.prems* **obtain** *l ys r zs* **where**
rec1: *bal* (*n div 2*) *xs* = (*l*, *ys*) **and**
rec2: *bal* (*n* − *1* − *n div 2*) (*tl ys*) = (*r*, *zs*) **and**
t: *t* = ⟨*l*, *hd ys*, *r*⟩
by(*auto simp add*: *bal_simps split*: *prod.splits*)
have *l*: *wbalanced l* **using** *less.IH*[*OF _ _ rec1*] *less.prems*(*1*) **by** *linarith*
have *wbalanced r*

using *rec1 rec2 bal_length*[*OF* _ *rec1*] *less.prems*(*1*) **by**(*intro less.IH*) *auto*
with *l t bal_length*[*OF* _ *rec1*] *less.prems*(*1*) *bal_inorder*[*OF* _ *rec1*] *bal_inorder*[*OF* _ *rec2*]
show *?thesis* **by** *auto*
qed
qed

An alternative proof via *wbalanced ?t* $\Longrightarrow$ *acomplete ?t*:

lemma $[\![\ n \leq$ *length xs*; *bal n xs* = (*t*,*ys*) $]\!] \Longrightarrow$ *acomplete t*
by(*rule acomplete_if_wbalanced*[*OF wbalanced_bal*])

lemma *wbalanced_bal_list*[*simp*]: $n \leq$ *length xs* $\Longrightarrow$ *wbalanced* (*bal_list n xs*)
by(*simp add*: *bal_list_def*) (*metis prod.collapse wbalanced_bal*)

lemma *wbalanced_balance_list*[*simp*]: *wbalanced* (*balance_list xs*)
by(*simp add*: *balance_list_def*)

lemma *wbalanced_bal_tree*[*simp*]: $n \leq$ *size t* $\Longrightarrow$ *wbalanced* (*bal_tree n t*)
by(*simp add*: *bal_tree_def*)

lemma *wbalanced_balance_tree*: *wbalanced* (*balance_tree t*)
by (*simp add*: *balance_tree_def*)

hide_const (**open**) *bal*

end

3 Three-Way Comparison

theory *Cmp*
imports *Main*
begin

datatype *cmp_val* = *LT* | *EQ* | *GT*

definition *cmp* :: *'a*:: *linorder* $\Rightarrow$ *'a* $\Rightarrow$ *cmp_val* **where**
cmp x y = (*if x* < *y then LT else if x*=*y then EQ else GT*)

lemma
LT[*simp*]: *cmp x y* = *LT* $\longleftrightarrow$ *x* < *y*
and *EQ*[*simp*]: *cmp x y* = *EQ* $\longleftrightarrow$ *x* = *y*
and *GT*[*simp*]: *cmp x y* = *GT* $\longleftrightarrow$ *x* > *y*

by (*auto simp*: *cmp_def*)

lemma *case_cmp_if*[*simp*]: (*case c of EQ* ⇒ *e* | *LT* ⇒ *l* | *GT* ⇒ *g*) =
(*if c* = *LT then l else if c* = *GT then g else e*)
by(*simp split*: *cmp_val.split*)

end

4 Lists Sorted wrt <

theory *Sorted_Less*
imports *Less_False*
begin

hide_const *sorted*

Is a list sorted without duplicates, i.e., wrt <?.

abbreviation *sorted* :: 'a::*linorder list* ⇒ *bool* **where**
sorted ≡ *sorted_wrt* (<)

lemmas *sorted_wrt_Cons* = *sorted_wrt.simps*(*2*)

The definition of *sorted_wrt* relates each element to all the elements after it. This causes a blowup of the formulas. Thus we simplify matters by only comparing adjacent elements.

declare
sorted_wrt.simps(*2*)[*simp del*]
sorted_wrt1[*simp*] *sorted_wrt2*[*OF transp_on_less*, *simp*]

lemma *sorted_cons*: *sorted* (*x*#*xs*) ⟹ *sorted xs*
by(*simp add*: *sorted_wrt_Cons*)

lemma *sorted_cons'*: *ASSUMPTION* (*sorted* (*x*#*xs*)) ⟹ *sorted xs*
by(*rule ASSUMPTION_D* [*THEN sorted_cons*])

lemma *sorted_snoc*: *sorted* (*xs* @ [*y*]) ⟹ *sorted xs*
by(*simp add*: *sorted_wrt_append*)

lemma *sorted_snoc'*: *ASSUMPTION* (*sorted* (*xs* @ [*y*])) ⟹ *sorted xs*
by(*rule ASSUMPTION_D* [*THEN sorted_snoc*])

lemma *sorted_mid_iff*:
sorted(*xs* @ *y* # *ys*) = (*sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))
by(*fastforce simp add*: *sorted_wrt_Cons sorted_wrt_append*)

lemma *sorted_mid_iff2*:
sorted(*x* # *xs* @ *y* # *ys*) =
(*sorted*(*x* # *xs*) ∧ *x* < *y* ∧ *sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))
by(*fastforce simp add*: *sorted_wrt_Cons sorted_wrt_append*)

lemma *sorted_mid_iff'*: *NO_MATCH* [] *ys* ⟹
sorted(*xs* @ *y* # *ys*) = (*sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))
by(*rule sorted_mid_iff*)

lemmas *sorted_lems* = *sorted_mid_iff'* *sorted_mid_iff2* *sorted_cons'* *sorted_snoc'*

Splay trees need two additional *sorted* lemmas:

lemma *sorted_snoc_le*:
ASSUMPTION(*sorted*(*xs* @ [*x*])) ⟹ *x* ≤ *y* ⟹ *sorted* (*xs* @ [*y*])
by (*auto simp add*: *sorted_wrt_append ASSUMPTION_def*)

lemma *sorted_Cons_le*:
ASSUMPTION(*sorted*(*x* # *xs*)) ⟹ *y* ≤ *x* ⟹ *sorted* (*y* # *xs*)
by (*auto simp add*: *sorted_wrt_Cons ASSUMPTION_def*)

end

5 List Insertion and Deletion

theory *List_Ins_Del*
imports *Sorted_Less*
begin

5.1 Elements in a list

lemma *sorted_Cons_iff*:
sorted(*x* # *xs*) = ((∀ *y* ∈ *set xs*. *x* < *y*) ∧ *sorted xs*)
by(*simp add*: *sorted_wrt_Cons*)

lemma *sorted_snoc_iff*:
sorted(*xs* @ [*x*]) = (*sorted xs* ∧ (∀ *y* ∈ *set xs*. *y* < *x*))
by(*simp add*: *sorted_wrt_append*)

lemmas *isin_simps* = *sorted_mid_iff'* *sorted_Cons_iff* *sorted_snoc_iff*

5.2 Inserting into an ordered list without duplicates:

fun *ins_list* :: *'a*::*linorder* ⇒ *'a list* ⇒ *'a list* **where**
ins_list x [] = [*x*] |
ins_list x (*a*#*xs*) =
(*if x* < *a then x*#*a*#*xs else if x*=*a then a*#*xs else a* # *ins_list x xs*)

lemma *set_ins_list*: *set* (*ins_list x xs*) = *set xs* ∪ {*x*}
by(*induction xs*) *auto*

lemma *sorted_ins_list*: *sorted xs* ⟹ *sorted*(*ins_list x xs*)
by(*induction xs rule*: *induct_list012*) *auto*

lemma *ins_list_sorted*: *sorted* (*xs* @ [*a*]) ⟹
ins_list x (*xs* @ *a* # *ys*) =
(*if x* < *a then ins_list x xs* @ (*a*#*ys*) *else xs* @ *ins_list x* (*a*#*ys*))
by(*induction xs*) (*auto simp*: *sorted_lems*)

In principle, *sorted* (*?xs* @ [*?a*]) ⟹ *ins_list ?x* (*?xs* @ *?a* # *?ys*) = (*if ?x* < *?a then ins_list ?x ?xs* @ *?a* # *?ys else ?xs* @ *ins_list ?x* (*?a* # *?ys*)) suffices, but the following two corollaries speed up proofs.

corollary *ins_list_sorted1*: *sorted* (*xs* @ [*a*]) ⟹ *a* ≤ *x* ⟹
ins_list x (*xs* @ *a* # *ys*) = *xs* @ *ins_list x* (*a*#*ys*)
by(*auto simp add*: *ins_list_sorted*)

corollary *ins_list_sorted2*: *sorted* (*xs* @ [*a*]) ⟹ *x* < *a* ⟹
ins_list x (*xs* @ *a* # *ys*) = *ins_list x xs* @ (*a*#*ys*)
by(*auto simp*: *ins_list_sorted*)

lemmas *ins_list_simps* = *sorted_lems ins_list_sorted1 ins_list_sorted2*

Splay trees need two additional *ins_list* lemmas:

lemma *ins_list_Cons*: *sorted* (*x* # *xs*) ⟹ *ins_list x xs* = *x* # *xs*
by (*induction xs*) *auto*

lemma *ins_list_snoc*: *sorted* (*xs* @ [*x*]) ⟹ *ins_list x xs* = *xs* @ [*x*]
by(*induction xs*) (*auto simp add*: *sorted_mid_iff2*)

5.3 Delete one occurrence of an element from a list:

fun *del_list* :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
del_list x [] = [] |
del_list x (*a*#*xs*) = (*if x*=*a then xs else a* # *del_list x xs*)

lemma *del_list_idem*: *x* ∉ *set xs* ⟹ *del_list x xs* = *xs*

by (*induct xs*) *simp_all*

lemma *set_del_list*:
sorted xs $\Longrightarrow$ *set* (*del_list x xs*) = *set xs* − {*x*}
by(*induct xs*) (*auto simp*: *sorted_Cons_iff*)

lemma *sorted_del_list*: *sorted xs* $\Longrightarrow$ *sorted*(*del_list x xs*)
apply(*induction xs rule*: *induct_list012*)
apply *auto*
by (*meson order.strict_trans sorted_Cons_iff*)

lemma *del_list_sorted*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = (*if x* < *a then del_list x xs* @ *a* # *ys else xs* @ *del_list x* (*a* # *ys*))
by(*induction xs*)
(*fastforce simp*: *sorted_lems sorted_Cons_iff intro*!: *del_list_idem*)+

In principle, *sorted* (*?xs* @ *?a* # *?ys*) $\Longrightarrow$ *del_list ?x* (*?xs* @ *?a* # *?ys*) = (*if ?x* < *?a then del_list ?x ?xs* @ *?a* # *?ys else ?xs* @ *del_list ?x* (*?a* # *?ys*)) suffices, but the following corollaries speed up proofs.

corollary *del_list_sorted1*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$ *a* ≤ *x* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = *xs* @ *del_list x* (*a* # *ys*)
by (*auto simp*: *del_list_sorted*)

corollary *del_list_sorted2*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$ *x* < *a* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = *del_list x xs* @ *a* # *ys*
by (*auto simp*: *del_list_sorted*)

corollary *del_list_sorted3*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs*) $\Longrightarrow$ *x* < *b* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs*) = *del_list x* (*xs* @ *a* # *ys*) @ *b* # *zs*
by (*auto simp*: *del_list_sorted sorted_lems*)

corollary *del_list_sorted4*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) $\Longrightarrow$ *x* < *c* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) = *del_list x* (*xs* @ *a* # *ys* @ *b* # *zs*) @ *c* # *us*
by (*auto simp*: *del_list_sorted sorted_lems*)

corollary *del_list_sorted5*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us* @ *d* # *vs*) $\Longrightarrow$ *x* < *d* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us* @ *d* # *vs*) =
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) @ *d* # *vs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemmas *del_list_simps* = *sorted_lems*
del_list_sorted1
del_list_sorted2
del_list_sorted3
del_list_sorted4
del_list_sorted5

Splay trees need two additional *del_list* lemmas:

lemma *del_list_notin_Cons*: *sorted* (*x* # *xs*) ⟹ *del_list x xs* = *xs*
by(*induction xs*)(*fastforce simp*: *sorted_Cons_iff*)+

lemma *del_list_sorted_app*:
sorted(*xs* @ [*x*]) ⟹ *del_list x* (*xs* @ *ys*) = *xs* @ *del_list x ys*
by (*induction xs*) (*auto simp*: *sorted_mid_iff2*)

end

6 Specifications of Set ADT

theory *Set_Specs*
imports *List_Ins_Del*
begin

The basic set interface with traditional *set*-based specification:

locale *Set* =
fixes *empty* :: *'s*
fixes *insert* :: *'a* ⇒ *'s* ⇒ *'s*
fixes *delete* :: *'a* ⇒ *'s* ⇒ *'s*
fixes *isin* :: *'s* ⇒ *'a* ⇒ *bool*
fixes *set* :: *'s* ⇒ *'a set*
fixes *invar* :: *'s* ⇒ *bool*
assumes *set_empty*: *set empty* = {}
assumes *set_isin*: *invar s* ⟹ *isin s x* = (*x* ∈ *set s*)
assumes *set_insert*: *invar s* ⟹ *set*(*insert x s*) = *set s* ∪ {*x*}
assumes *set_delete*: *invar s* ⟹ *set*(*delete x s*) = *set s* − {*x*}
assumes *invar_empty*: *invar empty*
assumes *invar_insert*: *invar s* ⟹ *invar*(*insert x s*)
assumes *invar_delete*: *invar s* ⟹ *invar*(*delete x s*)

lemmas (**in** *Set*) *set_specs* =
set_empty set_isin set_insert set_delete invar_empty invar_insert invar_delete

The basic set interface with *inorder*-based specification:

locale *Set_by_Ordered* =
fixes *empty* :: *'t*
fixes *insert* :: *'a*::*linorder* ⇒ *'t* ⇒ *'t*
fixes *delete* :: *'a* ⇒ *'t* ⇒ *'t*
fixes *isin* :: *'t* ⇒ *'a* ⇒ *bool*
fixes *inorder* :: *'t* ⇒ *'a list*
fixes *inv* :: *'t* ⇒ *bool*
assumes *inorder_empty*: *inorder empty* = []
assumes *isin*: *inv t* ∧ *sorted*(*inorder t*) ⟹
isin t x = (*x* ∈ *set* (*inorder t*))
assumes *inorder_insert*: *inv t* ∧ *sorted*(*inorder t*) ⟹
inorder(*insert x t*) = *ins_list x* (*inorder t*)
assumes *inorder_delete*: *inv t* ∧ *sorted*(*inorder t*) ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
assumes *inorder_inv_empty*: *inv empty*
assumes *inorder_inv_insert*: *inv t* ∧ *sorted*(*inorder t*) ⟹ *inv*(*insert x t*)
assumes *inorder_inv_delete*: *inv t* ∧ *sorted*(*inorder t*) ⟹ *inv*(*delete x t*)

begin

It implements the traditional specification:

definition *set* :: *'t* ⇒ *'a set* **where**
set = *List.set o inorder*

definition *invar* :: *'t* ⇒ *bool* **where**
invar t = (*inv t* ∧ *sorted* (*inorder t*))

sublocale *Set*
empty insert delete isin set invar
proof(*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*auto simp*: *inorder_empty set_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin invar_def set_def*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert set_ins_list set_def invar_def*)
next
case (*4 s x*) **thus** *?case*
by (*auto simp*: *inorder_delete set_del_list invar_def set_def*)
next
case *5* **thus** *?case* **by**(*simp add*: *inorder_empty inorder_inv_empty invar_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *inorder_insert inorder_inv_insert sorted_ins_list*

invar_def)
next
case *7* **thus** *?case* **by** (*auto simp*: *inorder_delete inorder_inv_delete sorted_del_list invar_def*)
qed

end

Set2 = Set with binary operations:

locale *Set2* = *Set*
where *insert* = *insert* **for** *insert* :: *'a* ⇒ *'s* ⇒ *'s* +
fixes *union* :: *'s* ⇒ *'s* ⇒ *'s*
fixes *inter* :: *'s* ⇒ *'s* ⇒ *'s*
fixes *diff* :: *'s* ⇒ *'s* ⇒ *'s*
assumes *set_union*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*union s1 s2*) = *set s1* ∪ *set s2*
assumes *set_inter*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*inter s1 s2*) = *set s1* ∩ *set s2*
assumes *set_diff*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*diff s1 s2*) = *set s1* − *set s2*
assumes *invar_union*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*union s1 s2*)
assumes *invar_inter*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*inter s1 s2*)
assumes *invar_diff*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*diff s1 s2*)

end

7 Unbalanced Tree Implementation of Set

theory *Tree_Set*
imports
HOL−Library.Tree
Cmp
Set_Specs
begin

definition *empty* :: *'a tree* **where**
empty = *Leaf*

fun *isin* :: *'a*::*linorder tree* ⇒ *'a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node l a r*) *x* =
(*case cmp x a of*
LT ⇒ *isin l x* |
EQ ⇒ *True* |

```
    GT ⇒ isin r x)

hide_const (open) insert

fun insert :: 'a::linorder ⇒ 'a tree ⇒ 'a tree where
insert x Leaf = Node Leaf x Leaf |
insert x (Node l a r) =
  (case cmp x a of
     LT ⇒ Node (insert x l) a r |
     EQ ⇒ Node l a r |
     GT ⇒ Node l a (insert x r))
```

Deletion by replacing:

```
fun split_min :: 'a tree ⇒ 'a * 'a tree where
split_min (Node l a r) =
  (if l = Leaf then (a,r) else let (x,l') = split_min l in (x, Node l' a r))

fun delete :: 'a::linorder ⇒ 'a tree ⇒ 'a tree where
delete x Leaf = Leaf |
delete x (Node l a r) =
  (case cmp x a of
     LT ⇒  Node (delete x l) a r |
     GT ⇒  Node l a (delete x r) |
     EQ ⇒ if r = Leaf then l else let (a',r') = split_min r in Node l a' r')
```

Deletion by joining:

```
fun join :: ('a::linorder)tree ⇒ 'a tree ⇒ 'a tree where
join t Leaf = t |
join Leaf t = t |
join (Node t1 a t2) (Node t3 b t4) =
  (case join t2 t3 of
     Leaf ⇒ Node t1 a (Node Leaf b t4) |
     Node u2 x u3 ⇒ Node (Node t1 a u2) x (Node u3 b t4))

fun delete2 :: 'a::linorder ⇒ 'a tree ⇒ 'a tree where
delete2 x Leaf = Leaf |
delete2 x (Node l a r) =
  (case cmp x a of
     LT ⇒  Node (delete2 x l) a r |
     GT ⇒  Node l a (delete2 x r) |
     EQ ⇒ join l r)
```

7.1 Functional Correctness Proofs

```
lemma isin_set: sorted(inorder t) ⟹ isin t x = (x ∈ set (inorder t))
```

by (*induction t*) (*auto simp*: *isin_simps*)

lemma *inorder_insert*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps*)

lemma *split_minD*:
split_min t = (*x*,*t'*) $\Longrightarrow$ *t* $\neq$ *Leaf* $\Longrightarrow$ *x* # *inorder t'* = *inorder t*
by(*induction t arbitrary*: *t' rule*: *split_min.induct*)
(*auto simp*: *sorted_lems split*: *prod.splits if_splits*)

lemma *inorder_delete*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *del_list_simps split_minD split*: *prod.splits*)

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = λ_. *True*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
qed (*rule TrueI*)+

lemma *inorder_join*:
inorder(*join l r*) = *inorder l* @ *inorder r*
by(*induction l r rule*: *join.induct*) (*auto split*: *tree.split*)

lemma *inorder_delete2*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*delete2 x t*) = *del_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *inorder_join del_list_simps*)

interpretation *S2*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete2*
and *inorder* = *inorder* **and** *inv* = λ_. *True*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)

next
case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete2*)
qed (*rule TrueI*)+

end

8 Association List Update and Deletion

theory *AList_Upd_Del*
imports *Sorted_Less*
begin

abbreviation *sorted1 ps* ≡ *sorted*(*map fst ps*)

Define own *map_of* function to avoid pulling in an unknown amount of lemmas implicitly (via the simpset).

hide_const (**open**) *map_of*

fun *map_of* :: (*'a*∗*'b*)*list* ⇒ *'a* ⇒ *'b option* **where**
map_of [] = (λ*x*. *None*) |
map_of ((*a*,*b*)#*ps*) = (λ*x*. *if x*=*a then Some b else map_of ps x*)

Updating an association list:

fun *upd_list* :: *'a*::*linorder* ⇒ *'b* ⇒ (*'a*∗*'b*) *list* ⇒ (*'a*∗*'b*) *list* **where**
upd_list x y [] = [(*x*,*y*)] |
upd_list x y ((*a*,*b*)#*ps*) =
(*if x* < *a then* (*x*,*y*)#(*a*,*b*)#*ps else*
if x = *a then* (*x*,*y*)#*ps else* (*a*,*b*) # *upd_list x y ps*)

fun *del_list* :: *'a*::*linorder* ⇒ (*'a*∗*'b*)*list* ⇒ (*'a*∗*'b*)*list* **where**
del_list x [] = [] |
del_list x ((*a*,*b*)#*ps*) = (*if x* = *a then ps else* (*a*,*b*) # *del_list x ps*)

8.1 Lemmas for *map_of*

lemma *map_of_ins_list*: *map_of* (*upd_list x y ps*) = (*map_of ps*)(*x* := *Some y*)
by(*induction ps*) *auto*

lemma *map_of_append*: *map_of* (*ps* @ *qs*) *x* =

(*case map_of ps x of None ⇒ map_of qs x | Some y ⇒ Some y*)
by(*induction ps*)(*auto*)

lemma *map_of_None*: *sorted* (*x # map fst ps*) ⟹ *map_of ps x = None*
by (*induction ps*) (*fastforce simp*: *sorted_lems sorted_wrt_Cons*)+

lemma *map_of_None2*: *sorted* (*map fst ps @ [x]*) ⟹ *map_of ps x = None*
by (*induction ps*) (*auto simp*: *sorted_lems*)

lemma *map_of_del_list*: *sorted1 ps* ⟹
map_of(*del_list x ps*) = (*map_of ps*)(*x := None*)
by(*induction ps*) (*auto simp*: *map_of_None sorted_lems fun_eq_iff*)

lemma *map_of_sorted_Cons*: *sorted* (*a # map fst ps*) ⟹ *x < a* ⟹
map_of ps x = None
by (*simp add*: *map_of_None sorted_Cons_le*)

lemma *map_of_sorted_snoc*: *sorted* (*map fst ps @ [a]*) ⟹ *a ≤ x* ⟹
map_of ps x = None
by (*simp add*: *map_of_None2 sorted_snoc_le*)

lemmas *map_of_sorteds = map_of_sorted_Cons map_of_sorted_snoc*
lemmas *map_of_simps = sorted_lems map_of_append map_of_sorteds*

8.2 Lemmas for *upd_list*

lemma *sorted_upd_list*: *sorted1 ps* ⟹ *sorted1* (*upd_list x y ps*)
apply(*induction ps*)
apply *simp*
apply(*case_tac ps*)
apply *auto*
done

lemma *upd_list_sorted*: *sorted1* (*ps @ [(a,b)]*) ⟹
upd_list x y (*ps @* (*a,b*) *# qs*) =
(*if x < a then upd_list x y ps @* (*a,b*) *# qs*
else ps @ upd_list x y ((*a,b*) *# qs*))
by(*induction ps*) (*auto simp*: *sorted_lems*)

In principle, *sorted1* (*?ps @ [(?a, ?b)]*) ⟹ *upd_list ?x ?y* (*?ps @* (*?a, ?b*) *# ?qs*) = (*if ?x < ?a then upd_list ?x ?y ?ps @* (*?a, ?b*) *# ?qs else ?ps @ upd_list ?x ?y* ((*?a, ?b*) *# ?qs*)) suffices, but the following two corollaries speed up proofs.

corollary *upd_list_sorted1*: ⟦ *sorted* (*map fst ps* @ [*a*]); *x* < *a* ⟧ ⟹
upd_list x y (*ps* @ (*a*,*b*) # *qs*) = *upd_list x y ps* @ (*a*,*b*) # *qs*
by (*auto simp*: *upd_list_sorted*)

corollary *upd_list_sorted2*: ⟦ *sorted* (*map fst ps* @ [*a*]); *a* ≤ *x* ⟧ ⟹
upd_list x y (*ps* @ (*a*,*b*) # *qs*) = *ps* @ *upd_list x y* ((*a*,*b*) # *qs*)
by (*auto simp*: *upd_list_sorted*)

lemmas *upd_list_simps* = *sorted_lems upd_list_sorted1 upd_list_sorted2*

Splay trees need two additional *upd_list* lemmas:

lemma *upd_list_Cons*:
sorted1 ((*x*,*y*) # *xs*) ⟹ *upd_list x y xs* = (*x*,*y*) # *xs*
by (*induction xs*) *auto*

lemma *upd_list_snoc*:
sorted1 (*xs* @ [(*x*,*y*)]) ⟹ *upd_list x y xs* = *xs* @ [(*x*,*y*)]
by(*induction xs*) (*auto simp add*: *sorted_mid_iff2*)

8.3 Lemmas for *del_list*

lemma *sorted_del_list*: *sorted1 ps* ⟹ *sorted1* (*del_list x ps*)
apply(*induction ps*)
apply *simp*
apply(*case_tac ps*)
apply (*auto simp*: *sorted_Cons_le*)
done

lemma *del_list_idem*: *x* ∉ *set*(*map fst xs*) ⟹ *del_list x xs* = *xs*
by (*induct xs*) *auto*

lemma *del_list_sorted*: *sorted1* (*ps* @ (*a*,*b*) # *qs*) ⟹
del_list x (*ps* @ (*a*,*b*) # *qs*) =
(*if x* < *a then del_list x ps* @ (*a*,*b*) # *qs*
else ps @ *del_list x* ((*a*,*b*) # *qs*))
by(*induction ps*)
(*fastforce simp*: *sorted_lems sorted_wrt_Cons intro*!: *del_list_idem*)+

In principle, *sorted1* (*?ps* @ (*?a*, *?b*) # *?qs*) ⟹ *del_list ?x* (*?ps* @ (*?a*, *?b*) # *?qs*) = (*if ?x* < *?a then del_list ?x ?ps* @ (*?a*, *?b*) # *?qs else ?ps* @ *del_list ?x* ((*?a*, *?b*) # *?qs*)) suffices, but the following corollaries speed up proofs.

corollary *del_list_sorted1*: *sorted1* (*xs* @ (*a*,*b*) # *ys*) ⟹ *a* ≤ *x* ⟹
del_list x (*xs* @ (*a*,*b*) # *ys*) = *xs* @ *del_list x* ((*a*,*b*) # *ys*)

by (*auto simp*: *del_list_sorted*)

lemma *del_list_sorted2*: *sorted1* (*xs* @ (*a*,*b*) # *ys*) ⟹ *x* < *a* ⟹
del_list x (*xs* @ (*a*,*b*) # *ys*) = *del_list x xs* @ (*a*,*b*) # *ys*
by (*auto simp*: *del_list_sorted*)

lemma *del_list_sorted3*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) ⟹ *x* < *b* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) = *del_list x* (*xs* @ (*a*,*a*′) # *ys*) @ (*b*,*b*′) # *zs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemma *del_list_sorted4*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) ⟹ *x* < *c* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) = *del_list x* (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) @ (*c*,*c*′) # *us*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemma *del_list_sorted5*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us* @ (*d*,*d*′) # *vs*) ⟹ *x* < *d* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us* @ (*d*,*d*′) # *vs*) =
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) @ (*d*,*d*′) # *vs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemmas *del_list_simps* = *sorted_lems*
del_list_sorted1
del_list_sorted2
del_list_sorted3
del_list_sorted4
del_list_sorted5

Splay trees need two additional *del_list* lemmas:

lemma *del_list_notin_Cons*: *sorted* (*x* # *map fst xs*) ⟹ *del_list x xs* = *xs*
by(*induction xs*)(*fastforce simp*: *sorted_wrt_Cons*)+

lemma *del_list_sorted_app*:
sorted(*map fst xs* @ [*x*]) ⟹ *del_list x* (*xs* @ *ys*) = *xs* @ *del_list x ys*
by (*induction xs*) (*auto simp*: *sorted_mid_iff2*)

end

9 Specifications of Map ADT

theory *Map_Specs*
imports *AList_Upd_Del*
begin

The basic map interface with $'a \Rightarrow 'b\ option$ based specification:

locale *Map* =
fixes *empty* :: $'m$
fixes *update* :: $'a \Rightarrow 'b \Rightarrow 'm \Rightarrow 'm$
fixes *delete* :: $'a \Rightarrow 'm \Rightarrow 'm$
fixes *lookup* :: $'m \Rightarrow 'a \Rightarrow 'b\ option$
fixes *invar* :: $'m \Rightarrow bool$
assumes *map_empty*: $lookup\ empty = (\lambda\_.\ None)$
and *map_update*: $invar\ m \Longrightarrow lookup(update\ a\ b\ m) = (lookup\ m)(a := Some\ b)$
and *map_delete*: $invar\ m \Longrightarrow lookup(delete\ a\ m) = (lookup\ m)(a := None)$
and *invar_empty*: *invar empty*
and *invar_update*: $invar\ m \Longrightarrow invar(update\ a\ b\ m)$
and *invar_delete*: $invar\ m \Longrightarrow invar(delete\ a\ m)$

lemmas (**in** *Map*) *map_specs* =
map_empty map_update map_delete invar_empty invar_update invar_delete

The basic map interface with *inorder*-based specification:

locale *Map_by_Ordered* =
fixes *empty* :: $'t$
fixes *update* :: $'a{::}linorder \Rightarrow 'b \Rightarrow 't \Rightarrow 't$
fixes *delete* :: $'a \Rightarrow 't \Rightarrow 't$
fixes *lookup* :: $'t \Rightarrow 'a \Rightarrow 'b\ option$
fixes *inorder* :: $'t \Rightarrow ('a * 'b)\ list$
fixes *inv* :: $'t \Rightarrow bool$
assumes *inorder_empty*: $inorder\ empty = []$
and *inorder_lookup*: $inv\ t \wedge sorted1\ (inorder\ t) \Longrightarrow lookup\ t\ a = map\_of\ (inorder\ t)\ a$
and *inorder_update*: $inv\ t \wedge sorted1\ (inorder\ t) \Longrightarrow inorder(update\ a\ b\ t) = upd\_list\ a\ b\ (inorder\ t)$
and *inorder_delete*: $inv\ t \wedge sorted1\ (inorder\ t) \Longrightarrow inorder(delete\ a\ t) = del\_list\ a\ (inorder\ t)$
and *inorder_inv_empty*: *inv empty*
and *inorder_inv_update*: $inv\ t \wedge sorted1\ (inorder\ t) \Longrightarrow inv(update\ a\ b\ t)$
and *inorder_inv_delete*: $inv\ t \wedge sorted1\ (inorder\ t) \Longrightarrow inv(delete\ a\ t)$

begin

It implements the traditional specification:

definition *invar* :: $'t \Rightarrow$ *bool* **where**
invar t == *inv t* $\wedge$ *sorted1* (*inorder t*)

sublocale *Map*
empty update delete lookup invar
proof(*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*auto simp*: *inorder_lookup inorder_empty inorder_inv_empty*)
next
case *2* **thus** *?case*
by(*simp add*: *fun_eq_iff inorder_update inorder_inv_update map_of_ins_list inorder_lookup*
sorted_upd_list invar_def)
next
case *3* **thus** *?case*
by(*simp add*: *fun_eq_iff inorder_delete inorder_inv_delete map_of_del_list inorder_lookup*
sorted_del_list invar_def)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_empty inorder_inv_empty invar_def*)
next
case *5* **thus** *?case* **by**(*simp add*: *inorder_update inorder_inv_update sorted_upd_list invar_def*)
next
case *6* **thus** *?case* **by** (*auto simp*: *inorder_delete inorder_inv_delete sorted_del_list invar_def*)
qed

end

end

10 Unbalanced Tree Implementation of Map

theory *Tree_Map*
imports
Tree_Set
Map_Specs
begin

fun *lookup* :: ($'a$::*linorder*$*'b$) *tree* $\Rightarrow$ $'a \Rightarrow$ $'b$ *option* **where**

```
lookup Leaf x = None |
lookup (Node l (a,b) r) x =
  (case cmp x a of LT ⇒ lookup l x | GT ⇒ lookup r x | EQ ⇒ Some b)

fun update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree ⇒ ('a*'b) tree where
update x y Leaf = Node Leaf (x,y) Leaf |
update x y (Node l (a,b) r) = (case cmp x a of
   LT ⇒ Node (update x y l) (a,b) r |
   EQ ⇒ Node l (x,y) r |
   GT ⇒ Node l (a,b) (update x y r))

fun delete :: 'a::linorder ⇒ ('a*'b) tree ⇒ ('a*'b) tree where
delete x Leaf = Leaf |
delete x (Node l (a,b) r) = (case cmp x a of
  LT ⇒ Node (delete x l) (a,b) r |
  GT ⇒ Node l (a,b) (delete x r) |
  EQ ⇒ if r = Leaf then l else let (ab',r') = split_min r in Node l ab' r')
```

10.1 Functional Correctness Proofs

```
lemma lookup_map_of:
  sorted1(inorder t) ⟹ lookup t x = map_of (inorder t) x
by (induction t) (auto simp: map_of_simps split: option.split)

lemma inorder_update:
  sorted1(inorder t) ⟹ inorder(update a b t) = upd_list a b (inorder t)
by(induction t) (auto simp: upd_list_simps)

lemma inorder_delete:
  sorted1(inorder t) ⟹ inorder(delete x t) = del_list x (inorder t)
by(induction t) (auto simp: del_list_simps split_minD split: prod.splits)

interpretation M: Map_by_Ordered
where empty = empty and lookup = lookup and update = update and
delete = delete
and inorder = inorder and inv = λ_. True
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: lookup_map_of)
next
  case 3 thus ?case by(simp add: inorder_update)
next
  case 4 thus ?case by(simp add: inorder_delete)
```

qed *auto*

end

11 Tree Rotations

theory *Tree_Rotations*
imports *HOL−Library.Tree*
begin

How to transform a tree into a list and into any other tree (with the same *inorder*) by rotations.

fun *is_list* :: *'a tree* ⇒ *bool* **where**
is_list (*Node l _ r*) = (*l* = *Leaf* ∧ *is_list r*) |
is_list Leaf = *True*

Termination proof via measure function. NB *size t* − *rlen t* works for the actual rotation equation but not for the second equation.

fun *rlen* :: *'a tree* ⇒ *nat* **where**
rlen Leaf = *0* |
rlen (*Node l x r*) = *rlen r* + *1*

lemma *rlen_le_size*: *rlen t* ≤ *size t*
by(*induction t*) *auto*

11.1 Without positions

function (*sequential*) *list_of* :: *'a tree* ⇒ *'a tree* **where**
list_of (*Node* (*Node A a B*) *b C*) = *list_of* (*Node A a* (*Node B b C*)) |
list_of (*Node Leaf a A*) = *Node Leaf a* (*list_of A*) |
list_of Leaf = *Leaf*
by *pat_completeness auto*

termination
proof
let *?R* = *measure*(λ*t*. *2*∗*size t* − *rlen t*)
show *wf ?R* **by** (*auto simp add*: *mlex_prod_def*)

fix *A a B b C*
show (*Node A a* (*Node B b C*), *Node* (*Node A a B*) *b C*) ∈ *?R*
using *rlen_le_size*[*of C*] **by**(*simp*)

fix *a A* **show** (*A*, *Node Leaf a A*) ∈ *?R* **using** *rlen_le_size*[*of A*] **by**(*simp*)
qed

lemma *is_list_rot*: *is_list*(*list_of t*)
by (*induction t rule*: *list_of.induct*) *auto*

lemma *inorder_rot*: *inorder*(*list_of t*) = *inorder t*
by (*induction t rule*: *list_of.induct*) *auto*

11.2 With positions

datatype *dir* = *L* | *R*

type_synonym *pos* = *dir list*

function (*sequential*) *rotR_poss* :: *'a tree* ⇒ *pos list* **where**
rotR_poss (*Node* (*Node A a B*) *b C*) = [] # *rotR_poss* (*Node A a* (*Node B b C*)) |
rotR_poss (*Node Leaf a A*) = *map* (*Cons R*) (*rotR_poss A*) |
rotR_poss Leaf = []
by *pat_completeness auto*

termination
proof
let *?R* = *measure*(λ*t*. *2*∗*size t* − *rlen t*)
show *wf ?R* **by** (*auto simp add*: *mlex_prod_def*)

fix *A a B b C*
show (*Node A a* (*Node B b C*), *Node* (*Node A a B*) *b C*) ∈ *?R*
using *rlen_le_size*[*of C*] **by**(*simp*)

fix *a A* **show** (*A*, *Node Leaf a A*) ∈ *?R* **using** *rlen_le_size*[*of A*] **by**(*simp*)
qed

fun *rotR* :: *'a tree* ⇒ *'a tree* **where**
rotR (*Node* (*Node A a B*) *b C*) = *Node A a* (*Node B b C*)

fun *rotL* :: *'a tree* ⇒ *'a tree* **where**
rotL (*Node A a* (*Node B b C*)) = *Node* (*Node A a B*) *b C*

fun *apply_at* :: (*'a tree* ⇒ *'a tree*) ⇒ *pos* ⇒ *'a tree* ⇒ *'a tree* **where**
apply_at f [] *t* = *f t*
| *apply_at f* (*L* # *ds*) (*Node l a r*) = *Node* (*apply_at f ds l*) *a r*
| *apply_at f* (*R* # *ds*) (*Node l a r*) = *Node l a* (*apply_at f ds r*)

fun *apply_ats* :: (*'a tree* ⇒ *'a tree*) ⇒ *pos list* ⇒ *'a tree* ⇒ *'a tree* **where**

apply_ats _ [] *t* = *t* |
apply_ats f (*p*#*ps*) *t* = *apply_ats f ps* (*apply_at f p t*)

lemma *apply_ats_append*:
  apply_ats f (*ps*$_1$ @ *ps*$_2$) *t* = *apply_ats f ps*$_2$ (*apply_ats f ps*$_1$ *t*)
by (*induction ps*$_1$ *arbitrary*: *t*) *auto*

abbreviation *rotRs* ≡ *apply_ats rotR*
abbreviation *rotLs* ≡ *apply_ats rotL*

lemma *apply_ats_map_R*: *apply_ats f* (*map* ((#) *R*) *ps*) ⟨*l*, *a*, *r*⟩ = *Node l a* (*apply_ats f ps r*)
by(*induction ps arbitrary*: *r*) *auto*

lemma *inorder_rotRs_poss*: *inorder* (*rotRs* (*rotR_poss t*) *t*) = *inorder t*
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R*)
done

lemma *is_list_rotRs*: *is_list* (*rotRs* (*rotR_poss t*) *t*)
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R*)
done

lemma *is_list* (*rotRs ps t*) ⟶ *length ps* ≤ *length*(*rotR_poss t*)
quickcheck[*expect*=*counterexample*]
oops

lemma *length_rotRs_poss*: *length* (*rotR_poss t*) = *size t* − *rlen t*
proof(*induction t rule*: *rotR_poss.induct*)
  case (*1 A a B b C*)
  then show *?case* **using** *rlen_le_size*[*of C*] **by** *simp*
qed *auto*

lemma *is_list_inorder_same*:
  is_list t1 ⟹ *is_list t2* ⟹ *inorder t1* = *inorder t2* ⟹ *t1* = *t2*
proof(*induction t1 arbitrary*: *t2*)
  case *Leaf*
  then show *?case* **by** *simp*
next
  case *Node*
  then show *?case* **by** (*cases t2*) *simp_all*
qed

lemma *rot_id*: *rotLs* (*rev* (*rotR_poss t*)) (*rotRs* (*rotR_poss t*) *t*) = *t*
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R rev_map apply_ats_append*)
done

corollary *tree_to_tree_rotations*: **assumes** *inorder t1* = *inorder t2*
shows *rotLs* (*rev* (*rotR_poss t2*)) (*rotRs* (*rotR_poss t1*) *t1*) = *t2*
proof −
have *rotRs* (*rotR_poss t1*) *t1* = *rotRs* (*rotR_poss t2*) *t2* (**is** *?L* = *?R*)
by (*simp add*: *assms inorder_rotRs_poss is_list_inorder_same is_list_rotRs*)
hence *rotLs* (*rev* (*rotR_poss t2*)) *?L* = *rotLs* (*rev* (*rotR_poss t2*)) *?R*
by *simp*
also have ... = *t2* **by**(*rule rot_id*)
finally show *?thesis* **.**
qed

lemma *size_rlen_better_ub*: *size t* − *rlen t* ≤ *size t* − *1*
by (*cases t*) *auto*

end

12 Augmented Tree (Tree2)

theory *Tree2*
imports *HOL*−*Library.Tree*
begin

This theory provides the basic infrastructure for the type (′*a* × ′*b*) *tree* of augmented trees where ′*a* is the key and ′*b* some additional information.

IMPORTANT: Inductions and cases analyses on augmented trees need to use the following two rules explicitly. They generate nodes of the form ⟨*l*, (*a*, *b*), *r*⟩ rather than ⟨*l*, *a*, *r*⟩ for trees of type ′*a tree*.

lemmas *tree2_induct* = *tree.induct*[**where** ′*a* = ′*a* ∗ ′*b*, *split_format*(*complete*)]

lemmas *tree2_cases* = *tree.exhaust*[**where** ′*a* = ′*a* ∗ ′*b*, *split_format*(*complete*)]

fun *inorder* :: (′*a*∗′*b*)*tree* ⇒ ′*a list* **where**
inorder Leaf = [] |
inorder (*Node l* (*a*,_) *r*) = *inorder l* @ *a* # *inorder r*

fun *set_tree* :: (′*a*∗′*b*) *tree* ⇒ ′*a set* **where**
set_tree Leaf = {} |
set_tree (*Node l* (*a*,_) *r*) = {*a*} ∪ *set_tree l* ∪ *set_tree r*

fun *bst* :: ('*a*::*linorder*∗'*b*) *tree* ⇒ *bool* **where**
bst Leaf = *True* |
bst (*Node l* (*a*, _) *r*) = ((∀ *x* ∈ *set_tree l*. *x* < *a*) ∧ (∀ *x* ∈ *set_tree r*. *a* < *x*) ∧ *bst l* ∧ *bst r*)

lemma *finite_set_tree*[*simp*]: *finite*(*set_tree t*)
by(*induction t*) *auto*

lemma *eq_set_tree_empty*[*simp*]: *set_tree t* = {} ⟷ *t* = *Leaf*
by (*cases t*) *auto*

lemma *set_inorder*[*simp*]: *set* (*inorder t*) = *set_tree t*
by (*induction t*) *auto*

lemma *length_inorder*[*simp*]: *length* (*inorder t*) = *size t*
by (*induction t*) *auto*

end

13 Function *isin* for Tree2

theory *Isin2*
imports
Tree2
Cmp
Set_Specs
begin

fun *isin* :: ('*a*::*linorder*∗'*b*) *tree* ⇒ '*a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node l* (*a*,_) *r*) *x* =
(*case cmp x a of*
LT ⇒ *isin l x* |
EQ ⇒ *True* |
GT ⇒ *isin r x*)

lemma *isin_set_inorder*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set*(*inorder t*))
by (*induction t rule*: *tree2_induct*) (*auto simp*: *isin_simps*)

lemma *isin_set_tree*: *bst t* ⟹ *isin t x* ⟷ *x* ∈ *set_tree t*
by(*induction t rule*: *tree2_induct*) *auto*

end

14 Interval Trees

theory *Interval_Tree*
imports
HOL−Data_Structures.Cmp
HOL−Data_Structures.List_Ins_Del
HOL−Data_Structures.Isin2
HOL−Data_Structures.Set_Specs
begin

14.1 Intervals

The following definition of intervals uses the **typedef** command to define the type of non-empty intervals as a subset of the type of pairs p where *fst* $p \leq$ *snd* p:

typedef (**overloaded**) $'a$::*linorder ivl* =
$\{p :: 'a \times 'a.\ fst\ p \leq snd\ p\}$ **by** *auto*

More precisely, $'a$ *ivl* is isomorphic with that subset via the function *Rep_ivl*. Hence the basic interval properties are not immediate but need simple proofs:

definition *low* :: $'a$::*linorder ivl* $\Rightarrow$ $'a$ **where**
low p = *fst* (*Rep_ivl p*)

definition *high* :: $'a$::*linorder ivl* $\Rightarrow$ $'a$ **where**
high p = *snd* (*Rep_ivl p*)

lemma *ivl_is_interval*: *low p* $\leq$ *high p*
by (*metis Rep_ivl high_def low_def mem_Collect_eq*)

lemma *ivl_inj*: *low p* = *low q* $\Longrightarrow$ *high p* = *high q* $\Longrightarrow$ *p* = *q*
by (*metis Rep_ivl_inverse high_def low_def prod_eqI*)

Now we can forget how exactly intervals were defined.

instantiation *ivl* :: (*linorder*) *linorder* **begin**

definition *ivl_less*: $(x < y)$ = (*low x* < *low y* | (*low x* = *low y* $\wedge$ *high x* < *high y*))
definition *ivl_less_eq*: $(x \leq y)$ = (*low x* < *low y* | (*low x* = *low y* $\wedge$ *high x* $\leq$ *high y*))

instance proof
fix $x\ y\ z :: 'a\ ivl$
show a: $(x < y) = (x \leq y \wedge \neg\ y \leq x)$
using *ivl_less ivl_less_eq* **by** *force*
show b: $x \leq x$
by (*simp add*: *ivl_less_eq*)
show c: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
using *ivl_less_eq* **by** *fastforce*
show d: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
using *ivl_less_eq a ivl_inj ivl_less* **by** *fastforce*
show e: $x \leq y \vee y \leq x$
by (*meson ivl_less_eq leI not_less_iff_gr_or_eq*)
qed end

definition *overlap* :: ('*a*::*linorder*) *ivl* ⇒ '*a ivl* ⇒ *bool* **where**
overlap x y ⟷ (*high x* ≥ *low y* ∧ *high y* ≥ *low x*)

definition *has_overlap* :: ('*a*::*linorder*) *ivl set* ⇒ '*a ivl* ⇒ *bool* **where**
has_overlap S y ⟷ (∃ *x*∈*S*. *overlap x y*)

14.2 Interval Trees

type_synonym '*a ivl_tree* = ('*a ivl* ∗ '*a*) *tree*

fun *max_hi* :: ('*a*::*order_bot*) *ivl_tree* ⇒ '*a* **where**
max_hi Leaf = *bot* |
max_hi (*Node* _ (_,*m*) _) = *m*

definition *max3* :: ('*a*::{*linorder*,*order_bot*}) *ivl* ⇒ '*a ivl_tree* ⇒ '*a ivl_tree*
⇒ '*a* **where**
max3 a l r = *max* (*high a*) (*max* (*max_hi l*) (*max_hi r*))

fun *inv_max_hi* :: ('*a*::{*linorder*,*order_bot*}) *ivl_tree* ⇒ *bool* **where**
inv_max_hi Leaf ⟷ *True* |
inv_max_hi (*Node l* (*a*, *m*) *r*) ⟷ (*m* = *max3 a l r* ∧ *inv_max_hi l* ∧
inv_max_hi r)

lemma *max_hi_is_max*:
inv_max_hi t ⟹ *a* ∈ *set_tree t* ⟹ *high a* ≤ *max_hi t*
by (*induct t*, *auto simp add*: *max3_def max_def*)

lemma *max_hi_exists*:
inv_max_hi t ⟹ *t* ≠ *Leaf* ⟹ ∃ *a*∈*set_tree t*. *high a* = *max_hi t*

proof (*induction t rule*: *tree2_induct*)
case *Leaf*
then show *?case* **by** *auto*
next
case *N*: (*Node l v m r*)
then show *?case*
proof (*cases l rule*: *tree2_cases*)
case *Leaf*
then show *?thesis*
using *N.prems*(*1*) *N.IH*(*2*) **by** (*cases r*, *auto simp add*: *max3_def max_def le_bot*)
next
case *Nl*: *Node*
then show *?thesis*
proof (*cases r rule*: *tree2_cases*)
case *Leaf*
then show *?thesis*
using *N.prems*(*1*) *N.IH*(*1*) *Nl* **by** (*auto simp add*: *max3_def max_def le_bot*)
next
case *Nr*: *Node*
obtain *p1* **where** *p1*: *p1* ∈ *set_tree l high p1* = *max_hi l*
using *N.IH*(*1*) *N.prems*(*1*) *Nl* **by** *auto*
obtain *p2* **where** *p2*: *p2* ∈ *set_tree r high p2* = *max_hi r*
using *N.IH*(*2*) *N.prems*(*1*) *Nr* **by** *auto*
then show *?thesis*
using *p1 p2 N.prems*(*1*) **by** (*auto simp add*: *max3_def max_def*)
qed
qed
qed

14.3 Insertion and Deletion

definition *node* **where**
[*simp*]: *node l a r* = *Node l* (*a*, *max3 a l r*) *r*

fun *insert* :: *'a*::{*linorder*,*order_bot*} *ivl* ⇒ *'a ivl_tree* ⇒ *'a ivl_tree* **where**
insert x Leaf = *Node Leaf* (*x*, *high x*) *Leaf* |
insert x (*Node l* (*a*, *m*) *r*) =
(*case cmp x a of*
EQ ⇒ *Node l* (*a*, *m*) *r* |
LT ⇒ *node* (*insert x l*) *a r* |
GT ⇒ *node l a* (*insert x r*))

lemma *inorder_insert*:
sorted (*inorder t*) ⟹ *inorder* (*insert x t*) = *ins_list x* (*inorder t*)
by (*induct t rule*: *tree2_induct*) (*auto simp*: *ins_list_simps*)

lemma *inv_max_hi_insert*:
inv_max_hi t ⟹ *inv_max_hi* (*insert x t*)
by (*induct t rule*: *tree2_induct*) (*auto simp add*: *max3_def*)

fun *split_min* :: *'a*::{*linorder*,*order_bot*} *ivl_tree* ⇒ *'a ivl* × *'a ivl_tree*
where
split_min (*Node l* (*a*, *m*) *r*) =
(*if l* = *Leaf then* (*a*, *r*)
else let (*x*,*l'*) = *split_min l in* (*x*, *node l' a r*))

fun *delete* :: *'a*::{*linorder*,*order_bot*} *ivl* ⇒ *'a ivl_tree* ⇒ *'a ivl_tree* **where**
delete x Leaf = *Leaf* |
delete x (*Node l* (*a*, *m*) *r*) =
(*case cmp x a of*
LT ⇒ *node* (*delete x l*) *a r* |
GT ⇒ *node l a* (*delete x r*) |
EQ ⇒ *if r* = *Leaf then l else*
let (*a'*, *r'*) = *split_min r in node l a' r'*)

lemma *split_minD*:
split_min t = (*x*,*t'*) ⟹ *t* ≠ *Leaf* ⟹ *x* # *inorder t'* = *inorder t*
by (*induct t arbitrary*: *t' rule*: *split_min.induct*)
(*auto simp*: *sorted_lems split*: *prod.splits if_splits*)

lemma *inorder_delete*:
sorted (*inorder t*) ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by (*induct t*)
(*auto simp*: *del_list_simps split_minD Let_def split*: *prod.splits*)

lemma *inv_max_hi_split_min*:
⟦ *t* ≠ *Leaf*; *inv_max_hi t* ⟧ ⟹ *inv_max_hi* (*snd* (*split_min t*))
by (*induct t*) (*auto split*: *prod.splits*)

lemma *inv_max_hi_delete*:
inv_max_hi t ⟹ *inv_max_hi* (*delete x t*)
apply (*induct t*)
apply *simp*
using *inv_max_hi_split_min* **by** (*fastforce simp add*: *Let_def split*: *prod.splits*)

14.4 Search

Does interval x overlap with any interval in the tree?

fun *search* :: *'a*::{*linorder*,*order_bot*} *ivl_tree* ⇒ *'a ivl* ⇒ *bool* **where**
search Leaf x = False |
search (*Node l* (*a*, *m*) *r*) *x* =
(*if overlap x a then True*
else if l ≠ *Leaf* ∧ *max_hi l* ≥ *low x then search l x*
else search r x)

lemma *search_correct*:
inv_max_hi t ⟹ *sorted* (*inorder t*) ⟹ *search t x* = *has_overlap* (*set_tree t*) *x*
proof (*induction t rule*: *tree2_induct*)
case *Leaf*
then show *?case* **by** (*auto simp add*: *has_overlap_def*)
next
case (*Node l a m r*)
have *search_l*: *search l x* = *has_overlap* (*set_tree l*) *x*
using *Node.IH*(*1*) *Node.prems* **by** (*auto simp*: *sorted_wrt_append*)
have *search_r*: *search r x* = *has_overlap* (*set_tree r*) *x*
using *Node.IH*(*2*) *Node.prems* **by** (*auto simp*: *sorted_wrt_append*)
show *?case*
proof (*cases overlap a x*)
case *True*
thus *?thesis* **by** (*auto simp*: *overlap_def has_overlap_def*)
next
case *a_disjoint*: *False*
then show *?thesis*
proof *cases*
assume [*simp*]: *l* = *Leaf*
have *search_eval*: *search* (*Node l* (*a*, *m*) *r*) *x* = *search r x*
using *a_disjoint overlap_def* **by** *auto*
show *?thesis*
unfolding *search_eval search_r*
by (*auto simp add*: *has_overlap_def a_disjoint*)
next
assume *l* ≠ *Leaf*
then show *?thesis*
proof (*cases max_hi l* ≥ *low x*)
case *max_hi_l_ge*: *True*
have *inv_max_hi l*
using *Node.prems*(*1*) **by** *auto*
then obtain *p* **where** *p*: *p* ∈ *set_tree l high p* = *max_hi l*

```
        using ‹l ≠ Leaf› max_hi_exists by auto
      have search_eval: search (Node l (a, m) r) x = search l x
        using a_disjoint ‹l ≠ Leaf› max_hi_l_ge by (auto simp: overlap_def)
      show ?thesis
      proof (cases low p ≤ high x)
        case True
        have overlap p x
          unfolding overlap_def using True p(2) max_hi_l_ge by auto
        then show ?thesis
          unfolding search_eval search_l
          using p(1) by(auto simp: has_overlap_def overlap_def)
      next
        case False
        have ¬overlap x rp if asm: rp ∈ set_tree r for rp
        proof −
          have low p ≤ low rp
            using asm p(1) Node(4) by(fastforce simp: sorted_wrt_append ivl_less)
          then show ?thesis
            using False by (auto simp: overlap_def)
        qed
        then show ?thesis
          unfolding search_eval search_l
          using a_disjoint by (auto simp: has_overlap_def overlap_def)
      qed
    next
      case False
      have search_eval: search (Node l (a, m) r) x = search r x
        using a_disjoint False by (auto simp: overlap_def)
      have ¬overlap x lp if asm: lp ∈ set_tree l for lp
        using asm False Node.prems(1) max_hi_is_max
        by (fastforce simp: overlap_def)
      then show ?thesis
        unfolding search_eval search_r
        using a_disjoint by (auto simp: has_overlap_def overlap_def)
    qed
  qed
qed

definition empty :: 'a ivl_tree where
empty = Leaf
```

14.5 Specification

locale *Interval_Set* = *Set* +
  fixes *has_overlap* :: *'t* ⇒ *'a*::*linorder ivl* ⇒ *bool*
  assumes *set_overlap*: *invar s* ⟹ *has_overlap s x* = *Interval_Tree.has_overlap* (*set s*) *x*

fun *invar* :: (*'a*::{*linorder*,*order_bot*}) *ivl_tree* ⇒ *bool* **where**
invar t = (*inv_max_hi t* ∧ *sorted*(*inorder t*))

interpretation *S*: *Interval_Set*
  where *empty* = *Leaf* **and** *insert* = *insert* **and** *delete* = *delete*
  and *has_overlap* = *search* **and** *isin* = *isin* **and** *set* = *set_tree*
  and *invar* = *invar*
proof (*standard*, *goal_cases*)
  case *1*
  then show *?case* **by** *auto*
next
  case *2*
  then show *?case* **by** (*simp add*: *isin_set_inorder*)
next
  case *3*
 then show *?case* **by**(*simp add*: *inorder_insert set_ins_list flip*: *set_inorder*)
next
  case *4*
 then show *?case* **by**(*simp add*: *inorder_delete set_del_list flip*: *set_inorder*)
next
  case *5*
  then show *?case* **by** *auto*
next
  case *6*
 then show *?case* **by** (*simp add*: *inorder_insert inv_max_hi_insert sorted_ins_list*)
next
  case *7*
 then show *?case* **by** (*simp add*: *inorder_delete inv_max_hi_delete sorted_del_list*)
next
  case *8*
  then show *?case* **by** (*simp add*: *search_correct*)
qed

end

15 AVL Tree Implementation of Sets

theory *AVL_Set_Code*
imports
Cmp
Isin2
begin

15.1 Code

type_synonym *'a tree_ht = ('a*nat) tree*

definition *empty :: 'a tree_ht* **where**
empty = Leaf

fun *ht :: 'a tree_ht ⇒ nat* **where**
ht Leaf = 0 |
ht (Node l (a,n) r) = n

definition *node :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht* **where**
node l a r = Node l (a, max (ht l) (ht r) + 1) r

definition *balL :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht* **where**
balL AB c C =
(if ht AB = ht C + 2 then
case AB of
Node A (a, _) B ⇒
if ht A ≥ ht B then node A a (node B c C)
else
case B of
Node B₁ (b, _) B₂ ⇒ node (node A a B₁) b (node B₂ c C)
else node AB c C)

definition *balR :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht* **where**
balR A a BC =
(if ht BC = ht A + 2 then
case BC of
Node B (c, _) C ⇒
if ht B ≤ ht C then node (node A a B) c C
else
case B of
Node B₁ (b, _) B₂ ⇒ node (node A a B₁) b (node B₂ c C)
else node A a BC)

```
fun insert :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
insert x Leaf = Node Leaf (x, 1) Leaf |
insert x (Node l (a, n) r) = (case cmp x a of
  EQ ⇒ Node l (a, n) r |
  LT ⇒ balL (insert x l) a r |
  GT ⇒ balR l a (insert x r))

fun split_max :: 'a tree_ht ⇒ 'a tree_ht * 'a where
split_max (Node l (a, _) r) =
 (if r = Leaf then (l,a) else let (r',a') = split_max r in (balL l a r', a'))

lemmas split_max_induct = split_max.induct[case_names Node Leaf]

fun delete :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
delete _ Leaf = Leaf |
delete x (Node l (a, n) r) =
 (case cmp x a of
    EQ ⇒ if l = Leaf then r
         else let (l', a') = split_max l in balR l' a' r |
    LT ⇒ balR (delete x l) a r |
    GT ⇒ balL l a (delete x r))
```

15.2 Functional Correctness Proofs

Very different from the AFP/AVL proofs

15.2.1 Proofs for insert

```
lemma inorder_balL:
  inorder (balL l a r) = inorder l @ a # inorder r
by (auto simp: node_def balL_def split:tree.splits)

lemma inorder_balR:
  inorder (balR l a r) = inorder l @ a # inorder r
by (auto simp: node_def balR_def split:tree.splits)

theorem inorder_insert:
  sorted(inorder t) ⟹ inorder(insert x t) = ins_list x (inorder t)
by (induct t)
   (auto simp: ins_list_simps inorder_balL inorder_balR)
```

15.2.2 Proofs for delete

```
lemma inorder_split_maxD:
```

⟦ *split_max t* = (*t'*,*a*); *t* ≠ *Leaf* ⟧ ⟹
inorder t' @ [*a*] = *inorder t*
by(*induction t arbitrary*: *t' rule*: *split_max.induct*)
(*auto simp*: *inorder_balL split*: *if_splits prod.splits tree.split*)

theorem *inorder_delete*:
sorted(*inorder t*) ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_balL inorder_balR inorder_split_maxD*
split: *prod.splits*)

end

15.3 Invariant

theory *AVL_Set*
imports
AVL_Set_Code
HOL−Number_Theory.Fib
begin

fun *avl* :: *'a tree_ht* ⇒ *bool* **where**
avl Leaf = *True* |
avl (*Node l* (*a*,*n*) *r*) =
(*abs*(*int*(*height l*) − *int*(*height r*)) ≤ *1* ∧
n = *max* (*height l*) (*height r*) + *1* ∧ *avl l* ∧ *avl r*)

15.3.1 Insertion maintains AVL balance

declare *Let_def* [*simp*]

lemma *ht_height*[*simp*]: *avl t* ⟹ *ht t* = *height t*
by (*cases t rule*: *tree2_cases*) *simp_all*

First, a fast but relatively manual proof with many lemmas:

lemma *height_balL*:
⟦ *avl l*; *avl r*; *height l* = *height r* + *2* ⟧ ⟹
height (*balL l a r*) ∈ {*height r* + *2*, *height r* + *3*}
by (*auto simp*:*node_def balL_def split*:*tree.split*)

lemma *height_balR*:
⟦ *avl l*; *avl r*; *height r* = *height l* + *2* ⟧ ⟹
height (*balR l a r*) : {*height l* + *2*, *height l* + *3*}
by(*auto simp add*:*node_def balR_def split*:*tree.split*)

lemma *height_node*[*simp*]: *height*(*node l a r*) = *max* (*height l*) (*height r*) + *1*
by (*simp add*: *node_def*)

lemma *height_balL2*:
⟦ *avl l*; *avl r*; *height l* ≠ *height r* + *2* ⟧ ⟹
height (*balL l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balL_def*)

lemma *height_balR2*:
⟦ *avl l*; *avl r*; *height r* ≠ *height l* + *2* ⟧ ⟹
height (*balR l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balR_def*)

lemma *avl_balL*:
⟦ *avl l*; *avl r*; *height r* − *1* ≤ *height l* ∧ *height l* ≤ *height r* + *2* ⟧ ⟹ *avl*(*balL l a r*)
by(*auto simp*: *balL_def node_def split*!: *if_split tree.split*)

lemma *avl_balR*:
⟦ *avl l*; *avl r*; *height l* − *1* ≤ *height r* ∧ *height r* ≤ *height l* + *2* ⟧ ⟹ *avl*(*balR l a r*)
by(*auto simp*: *balR_def node_def split*!: *if_split tree.split*)

Insertion maintains the AVL property. Requires simultaneous proof.

theorem *avl_insert*:
avl t ⟹ *avl*(*insert x t*)
avl t ⟹ *height* (*insert x t*) ∈ {*height t*, *height t* + *1*}
proof (*induction t rule*: *tree2_induct*)
case (*Node l a _ r*)
case *1*
show *?case*
proof(*cases x* = *a*)
case *True* **with** *1* **show** *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True* **with** *1 Node*(*1*,*2*) **show** *?thesis* **by** (*auto intro*!:*avl_balL*)
next
case *False* **with** *1 Node*(*3*,*4*) ‹*x*≠*a*› **show** *?thesis* **by** (*auto intro*!:*avl_balR*)
qed
qed

```
  case 2
  show ?case
  proof(cases x = a)
    case True with 2 show ?thesis by simp
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height (insert x l) = height r + 2)
        case False with 2 Node(1,2) ‹x < a› show ?thesis by (auto simp:
height_balL2)
      next
        case True
        hence (height (balL (insert x l) a r) = height r + 2) ∨
          (height (balL (insert x l) a r) = height r + 3) (is ?A ∨ ?B)
          using 2 Node(1,2) height_balL[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹x < a› show ?thesis by (auto)
        next
         assume ?B with 2 Node(2) True ‹x < a› show ?thesis by (simp)
arith
        qed
      qed
    next
      case False
      show ?thesis
      proof(cases height (insert x r) = height l + 2)
        case False with 2 Node(3,4) ‹¬x < a› show ?thesis by (auto simp:
height_balR2)
      next
        case True
        hence (height (balR l a (insert x r)) = height l + 2) ∨
          (height (balR l a (insert x r)) = height l + 3)  (is ?A ∨ ?B)
          using 2 Node(3) height_balR[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹¬x < a› show ?thesis by (auto)
        next
         assume ?B with 2 Node(4) True ‹¬x < a› show ?thesis by (simp)
arith
        qed
```

```
    qed
  qed
 qed
qed simp_all
```

Now an automatic proof without lemmas:

```
theorem avl_insert_auto: avl t ⟹
 avl(insert x t) ∧ height (insert x t) ∈ {height t, height t + 1}
apply (induction t rule: tree2_induct)

 apply (auto simp: balL_def balR_def node_def max_absorb2 split!: if_split
tree.split)
done
```

15.3.2 Deletion maintains AVL balance

```
lemma avl_split_max:
 ⟦ avl t; t ≠ Leaf ⟧ ⟹
 avl (fst (split_max t)) ∧
 height t ∈ {height(fst (split_max t)), height(fst (split_max t)) + 1}
by(induct t rule: split_max_induct)
  (auto simp: balL_def node_def max_absorb2 split!: prod.split if_split
tree.split)
```

Deletion maintains the AVL property:

```
theorem avl_delete:
 avl t ⟹ avl(delete x t)
 avl t ⟹ height t ∈ {height (delete x t), height (delete x t) + 1}
proof (induct t rule: tree2_induct)
 case (Node l a n r)
 case 1
 show ?case
 proof(cases x = a)
  case True thus ?thesis
   using 1 avl_split_max[of l] by (auto intro!: avl_balR split: prod.split)
 next
  case False thus ?thesis
   using Node 1 by (auto intro!: avl_balL avl_balR)
 qed
 case 2
 show ?case
 proof(cases x = a)
  case True thus ?thesis using 2 avl_split_max[of l]
  by(auto simp: balR_def max_absorb2 split!: if_splits prod.split tree.split)
```

```
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height r = height (delete x l) + 2)
        case False
        thus ?thesis using 2 Node(1,2) ‹x < a› by(auto simp: balR_def)
      next
        case True
        thus ?thesis using height_balR[OF _ _ True, of a] 2 Node(1,2) ‹x
< a› by simp linarith
      qed
    next
      case False
      show ?thesis
      proof(cases height l = height (delete x r) + 2)
        case False
         thus ?thesis using 2 Node(3,4) ‹¬x < a› ‹x ≠ a› by(auto simp:
balL_def)
      next
        case True
        thus ?thesis
         using height_balL[OF _ _ True, of a] 2 Node(3,4) ‹¬x < a› ‹x ≠
a› by simp linarith
      qed
    qed
  qed
qed simp_all
```

A more automatic proof. Complete automation as for insertion seems hard due to resource requirements.

```
theorem avl_delete_auto:
  avl t ⟹ avl(delete x t)
  avl t ⟹ height t ∈ {height (delete x t), height (delete x t) + 1}
proof (induct t rule: tree2_induct)
  case (Node l a n r)
  case 1
  thus ?case
   using Node avl_split_max[of l] by (auto intro!: avl_balL avl_balR split:
prod.split)
  case 2
  show ?case
```

using *2 Node avl_split_max*[*of l*]
by *auto*
(*auto simp*: *balL_def balR_def max_absorb1 max_absorb2 split*!:
tree.splits prod.splits if_splits)
qed *simp_all*

15.4 Overall correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by** (*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by** (*simp add*: *avl_insert*(*1*))
next
case *7* **thus** *?case* **by** (*simp add*: *avl_delete*(*1*))
qed

15.5 Height-Size Relation

Any AVL tree of height n has at least *fib* $(n+2)$ leaves:

theorem *avl_fib_bound*:
avl t $\Longrightarrow$ *fib*(*height t* + *2*) $\leq$ *size1 t*
proof (*induction rule*: *tree2_induct*)
case (*Node l a h r*)
have *1*: *height l* + *1* $\leq$ *height r* + *2* **and** *2*: *height r* + *1* $\leq$ *height l* + *2*
using *Node.prems* **by** *auto*
have *fib* (*max* (*height l*) (*height r*) + *3*) $\leq$ *size1 l* + *size1 r*
proof *cases*
assume *height l* $\geq$ *height r*
hence *fib* (*max* (*height l*) (*height r*) + *3*) = *fib* (*height l* + *3*)
by(*simp add*: *max_absorb1*)
also have ... = *fib* (*height l* + *2*) + *fib* (*height l* + *1*)
by (*simp add*: *numeral_eq_Suc*)

```
    also have ... ≤ size1 l + fib (height l + 1)
      using Node by (simp)
    also have ... ≤ size1 r + size1 l
      using Node fib_mono[OF 1] by auto
    also have ... = size1 (Node l (a,h) r)
      by simp
    finally show ?thesis
      by (simp)
  next
    assume ¬ height l ≥ height r
    hence fib (max (height l) (height r) + 3) = fib (height r + 3)
      by(simp add: max_absorb1)
    also have ... = fib (height r + 2) + fib (height r + 1)
      by (simp add: numeral_eq_Suc)
    also have ... ≤ size1 r + fib (height r + 1)
      using Node by (simp)
    also have ... ≤ size1 r + size1 l
      using Node fib_mono[OF 2] by auto
    also have ... = size1 (Node l (a,h) r)
      by simp
    finally show ?thesis
      by (simp)
  qed
  also have ... = size1 (Node l (a,h) r)
    by simp
  finally show ?case by (simp del: fib.simps add: numeral_eq_Suc)
qed auto

lemma avl_fib_bound_auto: avl t ⟹ fib (height t + 2) ≤ size1 t
proof (induction t rule: tree2_induct)
  case Leaf thus ?case by (simp)
next
  case (Node l a h r)
  have 1: height l + 1 ≤ height r + 2 and 2: height r + 1 ≤ height l + 2
    using Node.prems by auto
  have left: height l ≥ height r ⟹ ?case (is ?asm ⟹ _)
    using Node fib_mono[OF 1] by (simp add: max.absorb1)
  have right: height l ≤ height r ⟹ ?case
    using Node fib_mono[OF 2] by (simp add: max.absorb2)
  show ?case using left right using Node.prems by simp linarith
qed
```

An exponential lower bound for *fib*:

```
lemma fib_lowerbound:
```

```
  defines φ ≡ (1 + sqrt 5) / 2
  shows real (fib(n+2)) ≥ φ ^ n
proof (induction n rule: fib.induct)
  case 1
  then show ?case by simp
next
  case 2
  then show ?case by (simp add: φ_def real_le_lsqrt)
next
  case (3 n)
  have φ ^ Suc (Suc n) = φ ^ 2 * φ ^ n
    by (simp add: field_simps power2_eq_square)
  also have ... = (φ + 1) * φ ^ n
    by (simp_all add: φ_def power2_eq_square field_simps)
  also have ... = φ ^ Suc n + φ ^ n
     by (simp add: field_simps)
  also have ... ≤ real (fib (Suc n + 2)) + real (fib (n + 2))
     by (intro add_mono 3.IH)
  finally show ?case by simp
qed
```

The size of an AVL tree is (at least) exponential in its height:

```
lemma avl_size_lowerbound:
  defines φ ≡ (1 + sqrt 5) / 2
  assumes avl t
  shows   φ ^ (height t) ≤ size1 t
proof −
  have φ ^ height t ≤ fib (height t + 2)
    unfolding φ_def by(rule fib_lowerbound)
  also have ... ≤ size1 t
    using avl_fib_bound[of t] assms by simp
  finally show ?thesis .
qed
```

The height of an AVL tree is most *1 / log 2 φ ≈ 1.44* times worse than *log 2 (real (size1 t))*:

```
lemma  avl_height_upperbound:
  defines φ ≡ (1 + sqrt 5) / 2
  assumes avl t
  shows   height t ≤ (1/log 2 φ) * log 2 (size1 t)
proof −
  have φ > 0 φ > 1 by(auto simp: φ_def pos_add_strict)
  hence height t = log φ (φ ^ height t) by(simp add: log_nat_power)
  also have ... ≤ log φ (size1 t)
```

```
    using avl_size_lowerbound[OF assms(2), folded φ_def] ‹1 < φ›
    by (simp add: le_log_of_power)
  also have ... = (1/log 2 φ) * log 2 (size1 t)
    by(simp add: log_base_change[of 2 φ])
  finally show ?thesis .
qed

end
```

16 Function *lookup* for Tree2

```
theory Lookup2
imports
  Tree2
  Cmp
  Map_Specs
begin

fun lookup :: (('a::linorder * 'b) * 'c) tree ⇒ 'a ⇒ 'b option where
lookup Leaf x = None |
lookup (Node l ((a,b), _) r) x =
  (case cmp x a of LT ⇒ lookup l x | GT ⇒ lookup r x | EQ ⇒ Some b)

lemma lookup_map_of:
  sorted1(inorder t) ⟹ lookup t x = map_of (inorder t) x
by(induction t rule: tree2_induct) (auto simp: map_of_simps split: option.split)

end
```

17 AVL Tree Implementation of Maps

```
theory AVL_Map
imports
  AVL_Set
  Lookup2
begin

fun update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree_ht ⇒ ('a*'b) tree_ht where
update x y Leaf = Node Leaf ((x,y), 1) Leaf |
update x y (Node l ((a,b), h) r) = (case cmp x a of
   EQ ⇒ Node l ((x,y), h) r |
   LT ⇒ balL (update x y l) (a,b) r |
```

GT ⇒ *balR l* (*a*,*b*) (*update x y r*))

fun *delete* :: *'a*::*linorder* ⇒ (*'a*∗*'b*) *tree_ht* ⇒ (*'a*∗*'b*) *tree_ht* **where**
delete _ *Leaf* = *Leaf* |
delete x (*Node l* ((*a*,*b*), *h*) *r*) = (*case cmp x a of*
EQ ⇒ *if l* = *Leaf then r*
else let (*l'*, *ab'*) = *split_max l in balR l' ab' r* |
LT ⇒ *balR* (*delete x l*) (*a*,*b*) *r* |
GT ⇒ *balL l* (*a*,*b*) (*delete x r*))

17.1 Functional Correctness

theorem *inorder_update*:
sorted1(*inorder t*) ⟹ *inorder*(*update x y t*) = *upd_list x y* (*inorder t*)
by (*induct t*) (*auto simp*: *upd_list_simps inorder_balL inorder_balR*)

theorem *inorder_delete*:
sorted1(*inorder t*) ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_balL inorder_balR*
inorder_split_maxD split: *prod.splits*)

17.2 AVL invariants

17.2.1 Insertion maintains AVL balance

theorem *avl_update*:
assumes *avl t*
shows *avl*(*update x y t*)
(*height* (*update x y t*) = *height t* ∨ *height* (*update x y t*) = *height t*
+ *1*)
using *assms*
proof (*induction x y t rule*: *update.induct*)
case *eq2*: (*2 x y l a b h r*)
case *1*
show *?case*
proof(*cases x* = *a*)
case *True* **with** *eq2 1* **show** *?thesis* **by** *simp*
next
case *False*
with *eq2 1* **show** *?thesis*
proof(*cases x*<*a*)
case *True* **with** *eq2 1* **show** *?thesis* **by** (*auto intro*!: *avl_balL*)
next

```
    case False with eq2 1 ‹x≠a› show ?thesis by (auto intro!: avl_balR)
   qed
  qed
  case 2
  show ?case
  proof(cases x = a)
   case True with eq2 1 show ?thesis by simp
  next
   case False
   show ?thesis
   proof(cases x<a)
    case True
    show ?thesis
    proof(cases height (update x y l) = height r + 2)
        case False with eq2 2 ‹x < a› show ?thesis by (auto simp:
height_balL2)
    next
     case True
     hence (height (balL (update x y l) (a,b) r) = height r + 2) ∨
      (height (balL (update x y l) (a,b) r) = height r + 3) (is ?A ∨ ?B)
      using eq2 2 ‹x<a› height_balL[OF _ _ True] by simp
     thus ?thesis
     proof
      assume ?A with 2 ‹x < a› show ?thesis by (auto)
     next
      assume ?B with True 1 eq2(2) ‹x < a› show ?thesis by (simp)
arith
     qed
    qed
   next
    case False
    show ?thesis
    proof(cases height (update x y r) = height l + 2)
        case False with eq2 2 ‹¬x < a› show ?thesis by (auto simp:
height_balR2)
    next
     case True
     hence (height (balR l (a,b) (update x y r)) = height l + 2) ∨
      (height (balR l (a,b) (update x y r)) = height l + 3)  (is ?A ∨ ?B)
      using eq2 2 ‹¬x < a› ‹x ≠ a› height_balR[OF _ _ True] by simp
     thus ?thesis
     proof
      assume ?A with 2 ‹¬x < a› show ?thesis by (auto)
     next
```

assume *?B* **with** *True 1 eq2(4)* ‹¬*x* < *a*› **show** *?thesis* **by** (*simp*) *arith*
qed
qed
qed
qed
qed *simp_all*

17.2.2 Deletion maintains AVL balance

theorem *avl_delete*:
assumes *avl t*
shows *avl*(*delete x t*) **and** *height t* = (*height* (*delete x t*)) ∨ *height t* = *height* (*delete x t*) + *1*
using *assms*
proof (*induct t rule*: *tree2_induct*)
case (*Node l ab h r*)
obtain *a b* **where** [*simp*]: *ab* = (*a*,*b*) **by** *fastforce*
case *1*
show *?case*
proof(*cases x* = *a*)
case *True* **with** *Node 1* **show** *?thesis*
using *avl_split_max*[*of l*] **by** (*auto intro*!: *avl_balR split*: *prod.split*)
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True* **with** *Node 1* **show** *?thesis* **by** (*auto intro*!: *avl_balR*)
next
case *False* **with** *Node 1* ‹*x*≠*a*› **show** *?thesis* **by** (*auto intro*!: *avl_balL*)
qed
qed
case *2*
show *?case*
proof(*cases x* = *a*)
case *True* **then show** *?thesis* **using** *1 avl_split_max*[*of l*]
by(*auto simp*: *balR_def max_absorb2 split*!: *if_splits prod.split tree.split*)
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True*
show *?thesis*
proof(*cases height r* = *height* (*delete x l*) + *2*)

```
        case False with Node 1 ‹x < a› show ?thesis by(auto simp:
balR_def)
      next
       case True
      thus ?thesis using height_balR[OF _ _ True, of ab] 2 Node(1,2) ‹x
< a› by simp linarith
      qed
    next
      case False
      show ?thesis
      proof(cases height l = height (delete x r) + 2)
        case False with Node 1 ‹¬x < a› ‹x ≠ a› show ?thesis by(auto
simp: balL_def)
      next
       case True
       thus ?thesis
         using height_balL[OF _ _ True, of ab] 2 Node(3,4) ‹¬x < a› ‹x
≠ a› by auto
      qed
    qed
  qed
qed simp_all


interpretation M: Map_by_Ordered
where empty = empty and lookup = lookup and update = update and
delete = delete
and inorder = inorder and inv = avl
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: lookup_map_of)
next
  case 3 thus ?case by(simp add: inorder_update)
next
  case 4 thus ?case by(simp add: inorder_delete)
next
  case 5 show ?case by (simp add: empty_def)
next
  case 6 thus ?case by(simp add: avl_update(1))
next
  case 7 thus ?case by(simp add: avl_delete(1))
qed
```

end

18 AVL Tree with Balance Factors (1)

theory *AVL_Bal_Set*
imports
Cmp
Isin2
begin

This version detects height increase/decrease from above via the change in balance factors.

datatype *bal* = *Lh* | *Bal* | *Rh*

type_synonym *'a tree_bal* = (*'a* ∗ *bal*) *tree*

Invariant:

fun *avl* :: *'a tree_bal* ⇒ *bool* **where**
avl Leaf = *True* |
avl (*Node l* (*a*,*b*) *r*) =
(((*case b of*
Bal ⇒ *height r* = *height l* |
Lh ⇒ *height l* = *height r* + *1* |
Rh ⇒ *height r* = *height l* + *1*)
∧ *avl l* ∧ *avl r*)

18.1 Code

fun *is_bal* **where**
is_bal (*Node l* (*a*,*b*) *r*) = (*b* = *Bal*)

fun *incr* **where**
incr t t' = (*t* = *Leaf* ∨ *is_bal t* ∧ ¬ *is_bal t'*)

fun *rot2* **where**
rot2 A a B c C = (*case B of*
(*Node* B_1 (*b*, *bb*) B_2) ⇒
let b_1 = *if bb* = *Rh then Lh else Bal*;
b_2 = *if bb* = *Lh then Rh else Bal*
in Node (*Node A* (*a*,b_1) B_1) (*b*,*Bal*) (*Node* B_2 (*c*,b_2) *C*))

fun *balL* :: *'a tree_bal* ⇒ *'a* ⇒ *bal* ⇒ *'a tree_bal* ⇒ *'a tree_bal* **where**
balL AB c bc C = (*case bc of*
Bal ⇒ *Node AB* (*c*,*Lh*) *C* |

```
    Rh ⇒ Node AB (c,Bal) C |
    Lh ⇒ (case AB of
      Node A (a,Lh) B ⇒ Node A (a,Bal) (Node B (c,Bal) C) |
      Node A (a,Bal) B ⇒ Node A (a,Rh) (Node B (c,Lh) C) |
      Node A (a,Rh) B ⇒ rot2 A a B c C))

fun balR :: 'a tree_bal ⇒ 'a ⇒ bal ⇒ 'a tree_bal ⇒ 'a tree_bal where
balR A a ba BC = (case ba of
    Bal ⇒ Node A (a,Rh) BC |
    Lh ⇒ Node A (a,Bal) BC |
    Rh ⇒ (case BC of
      Node B (c,Rh) C ⇒ Node (Node A (a,Bal) B) (c,Bal) C |
      Node B (c,Bal) C ⇒ Node (Node A (a,Rh) B) (c,Lh) C |
      Node B (c,Lh) C ⇒ rot2 A a B c C))

fun insert :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal where
insert x Leaf = Node Leaf (x, Bal) Leaf |
insert x (Node l (a, b) r) = (case cmp x a of
  EQ ⇒ Node l (a, b) r |
  LT ⇒ let l' = insert x l in if incr l l' then balL l' a b r else Node l' (a,b)
r |
  GT ⇒ let r' = insert x r in if incr r r' then balR l a b r' else Node l (a,b)
r')

fun decr where
decr t t' = (t ≠ Leaf ∧ incr t' t)

fun split_max :: 'a tree_bal ⇒ 'a tree_bal * 'a where
split_max (Node l (a, ba) r) =
  (if r = Leaf then (l,a)
   else let (r',a') = split_max r;
            t' = if incr r' r then balL l a ba r' else Node l (a,ba) r'
        in (t', a'))

fun delete :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal where
delete _ Leaf = Leaf |
delete x (Node l (a, ba) r) =
  (case cmp x a of
    EQ ⇒ if l = Leaf then r
         else let (l', a') = split_max l in
              if incr l' l then balR l' a' ba r else Node l' (a',ba) r |
    LT ⇒ let l' = delete x l in if decr l l' then balR l' a ba r else Node l'
(a,ba) r |
    GT ⇒ let r' = delete x r in if decr r r' then balL l a ba r' else Node l
```

(*a*,*ba*) *r'*)

18.2 Proofs

lemmas *split_max_induct* = *split_max.induct*[*case_names Node Leaf*]

lemmas *splits* = *if_splits tree.splits bal.splits*

declare *Let_def* [*simp*]

18.2.1 Proofs about insertion

lemma *avl_insert*: *avl t* ⟹
avl(*insert x t*) ∧
height(*insert x t*) = *height t* + (*if incr t* (*insert x t*) *then 1 else 0*)
by (*induction x t rule*: *insert.induct*)(*auto split*!: *splits*)

The following two auxiliary lemma merely simplify the proof of *inorder_insert*.

lemma [*simp*]: [] ≠ *ins_list x xs*
by(*cases xs*) *auto*

lemma [*simp*]: *avl t* ⟹ *insert x t* ≠ ⟨*l*, (*a*, *Rh*), ⟨⟩⟩ ∧ *insert x t* ≠ ⟨⟨⟩, (*a*, *Lh*), *r*⟩
by(*drule avl_insert*[*of* _ *x*]) (*auto split*: *splits*)

theorem *inorder_insert*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by (*induction t*) (*auto simp*: *ins_list_simps split*!: *splits*)

18.2.2 Proofs about deletion

lemma *inorder_balR*:
⟦ *ba* = *Rh* ⟶ *r* ≠ *Leaf*; *avl r* ⟧
⟹ *inorder* (*balR l a ba r*) = *inorder l* @ *a* # *inorder r*
by (*auto split*: *splits*)

lemma *inorder_balL*:
⟦ *ba* = *Lh* ⟶ *l* ≠ *Leaf*; *avl l* ⟧
⟹ *inorder* (*balL l a ba r*) = *inorder l* @ *a* # *inorder r*
by (*auto split*: *splits*)

lemma *height_1_iff*: *avl t* ⟹ *height t* = *Suc 0* ⟷ (∃ *x*. *t* = *Node Leaf* (*x*,*Bal*) *Leaf*)

by(*cases t*) (*auto split*: *splits prod.splits*)

lemma *avl_split_max*:
⟦ *split_max t* = (*t'*,*a*); *avl t*; *t* ≠ *Leaf* ⟧ ⟹
avl t' ∧ *height t* = *height t'* + (*if incr t' t then 1 else 0*)
proof (*induction t arbitrary*: *t' a rule*: *split_max_induct*)
qed (*auto simp*: *max_absorb1 max_absorb2 height_1_iff split*!: *splits prod.splits*)

lemma *avl_delete*: *avl t* ⟹
avl (*delete x t*) ∧
height t = *height* (*delete x t*) + (*if decr t* (*delete x t*) *then 1 else 0*)
proof (*induction x t rule*: *delete.induct*)
qed (*auto simp*: *max_absorb1 max_absorb2 height_1_iff dest*: *avl_split_max split*!: *splits prod.splits*)

lemma *inorder_split_maxD*:
⟦ *split_max t* = (*t'*,*a*); *t* ≠ *Leaf*; *avl t* ⟧ ⟹
inorder t' @ [*a*] = *inorder t*
proof (*induction t arbitrary*: *t' rule*: *split_max.induct*)
qed (*auto split*!: *splits prod.splits*)

lemma *neq_Leaf_if_height_neq_0*: *height t* ≠ *0* ⟹ *t* ≠ *Leaf*
by *auto*

lemma *split_max_Leaf*: ⟦ *t* ≠ *Leaf*; *avl t* ⟧ ⟹ *split_max t* = (⟨⟩, *x*) ⟷ *t* = *Node Leaf* (*x*,*Bal*) *Leaf*
by(*cases t*) (*auto split*: *splits prod.splits*)

theorem *inorder_delete*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
proof (*induction t rule*: *tree2_induct*)
case *Leaf*
then show *?case* **by** *auto*
next
case (*Node x1 a b x3*)
then show *?case*
by (*auto simp*: *del_list_simps inorder_balR inorder_balL avl_delete inorder_split_maxD*
split_max_Leaf neq_Leaf_if_height_neq_0
simp del: *balL.simps balR.simps split*!: *splits prod.splits*)
qed

18.2.3 Set Implementation

interpretation *S*: *Set_by_Ordered*
where *empty* = *Leaf* **and** *isin* = *isin*
  and *insert* = *insert*
  and *delete* = *delete*
  and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp*)
next
  case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
  case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
  case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
  case *5* **thus** *?case* **by** (*simp*)
next
  case *6* **thus** *?case* **by** (*simp add*: *avl_insert*)
next
  case *7* **thus** *?case* **by** (*simp add*: *avl_delete*)
qed

end

19 AVL Tree with Balance Factors (2)

theory *AVL_Bal2_Set*
imports
  Cmp
  Isin2
begin

This version passes a flag (*Same*/*Diff*) back up to signal if the height changed.

datatype *bal* = *Lh* | *Bal* | *Rh*

type_synonym *'a tree_bal* = (*'a* ∗ *bal*) *tree*

Invariant:

fun *avl* :: *'a tree_bal* ⇒ *bool* **where**
avl Leaf = *True* |
avl (*Node l* (*a*,*b*) *r*) =
  ((*case b of*

```
    Bal ⇒ height r = height l |
    Lh ⇒ height l = height r + 1 |
    Rh ⇒ height r = height l + 1)
  ∧ avl l ∧ avl r)
```

19.1 Code

```
datatype 'a alt = Same 'a | Diff 'a

type_synonym 'a tree_bal2 = 'a tree_bal alt

fun tree :: 'a alt ⇒ 'a where
tree(Same t) = t |
tree(Diff t) = t

fun rot2 where
rot2 A a B c C = (case B of
  (Node B₁ (b, bb) B₂) ⇒
    let b₁ = if bb = Rh then Lh else Bal;
        b₂ = if bb = Lh then Rh else Bal
    in Node (Node A (a,b₁) B₁) (b,Bal) (Node B₂ (c,b₂) C))

fun balL :: 'a tree_bal2 ⇒ 'a ⇒ bal ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
balL AB' c bc C = (case AB' of
   Same AB ⇒ Same (Node AB (c,bc) C) |
   Diff AB ⇒ (case bc of
     Bal ⇒ Diff (Node AB (c,Lh) C) |
     Rh ⇒ Same (Node AB (c,Bal) C) |
     Lh ⇒ (case AB of
       Node A (a,Lh) B ⇒ Same(Node A (a,Bal) (Node B (c,Bal) C)) |
       Node A (a,Rh) B ⇒ Same(rot2 A a B c C))))

fun balR :: 'a tree_bal ⇒ 'a ⇒ bal ⇒ 'a tree_bal2 ⇒ 'a tree_bal2 where
balR A a ba BC' = (case BC' of
   Same BC ⇒ Same (Node A (a,ba) BC) |
   Diff BC ⇒ (case ba of
     Bal ⇒ Diff (Node A (a,Rh) BC) |
     Lh ⇒ Same (Node A (a,Bal) BC) |
     Rh ⇒ (case BC of
       Node B (c,Rh) C ⇒ Same(Node (Node A (a,Bal) B) (c,Bal) C) |
       Node B (c,Lh) C ⇒ Same(rot2 A a B c C))))

fun ins :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
ins x Leaf = Diff(Node Leaf (x, Bal) Leaf) |
```

```
ins x (Node l (a, b) r) = (case cmp x a of
  EQ ⇒ Same(Node l (a, b) r) |
  LT ⇒ balL (ins x l) a b r |
  GT ⇒ balR l a b (ins x r))

definition insert :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal where
insert x t = tree(ins x t)

fun baldR :: 'a tree_bal ⇒ 'a ⇒ bal ⇒ 'a tree_bal2 ⇒ 'a tree_bal2 where
baldR AB c bc C' = (case C' of
  Same C ⇒ Same (Node AB (c,bc) C) |
  Diff C ⇒ (case bc of
    Bal ⇒ Same (Node AB (c,Lh) C) |
    Rh ⇒ Diff (Node AB (c,Bal) C) |
    Lh ⇒ (case AB of
      Node A (a,Lh) B ⇒ Diff(Node A (a,Bal) (Node B (c,Bal) C)) |
      Node A (a,Bal) B ⇒ Same(Node A (a,Rh) (Node B (c,Lh) C)) |
      Node A (a,Rh) B ⇒ Diff(rot2 A a B c C))))

fun baldL :: 'a tree_bal2 ⇒ 'a ⇒ bal ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
baldL A' a ba BC = (case A' of
  Same A ⇒ Same (Node A (a,ba) BC) |
  Diff A ⇒ (case ba of
    Bal ⇒ Same (Node A (a,Rh) BC) |
    Lh ⇒ Diff (Node A (a,Bal) BC) |
    Rh ⇒ (case BC of
      Node B (c,Rh) C ⇒ Diff(Node (Node A (a,Bal) B) (c,Bal) C) |
      Node B (c,Bal) C ⇒ Same(Node (Node A (a,Rh) B) (c,Lh) C) |
      Node B (c,Lh) C ⇒ Diff(rot2 A a B c C))))

fun split_max :: 'a tree_bal ⇒ 'a tree_bal2 * 'a where
split_max (Node l (a, ba) r) =
  (if r = Leaf then (Diff l,a) else let (r',a') = split_max r in (baldR l a ba
r', a'))

fun del :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
del _ Leaf = Same Leaf |
del x (Node l (a, ba) r) =
  (case cmp x a of
    EQ ⇒ if l = Leaf then Diff r
         else let (l', a') = split_max l in baldL l' a' ba r |
    LT ⇒ baldL (del x l) a ba r |
    GT ⇒ baldR l a ba (del x r))
```

definition *delete* :: *'a::linorder* ⇒ *'a tree_bal* ⇒ *'a tree_bal* **where**
delete x t = *tree*(*del x t*)

lemmas *split_max_induct* = *split_max.induct*[*case_names Node Leaf*]

lemmas *splits* = *if_splits tree.splits alt.splits bal.splits*

19.2 Proofs

19.2.1 Proofs about insertion

lemma *avl_ins_case*: *avl t* ⟹ *case ins x t of*
Same t' ⇒ *avl t'* ∧ *height t'* = *height t* |
Diff t' ⇒ *avl t'* ∧ *height t'* = *height t* + *1* ∧
(∀ *l a r*. *t'* = *Node l* (*a*,*Bal*) *r* ⟶ *a* = *x* ∧ *l* = *Leaf* ∧ *r* = *Leaf*)
by (*induction x t rule*: *ins.induct*) (*auto simp*: *max_absorb1 split*!: *splits*)

corollary *avl_insert*: *avl t* ⟹ *avl*(*insert x t*)
using *avl_ins_case*[*of t x*] **by** (*simp add*: *insert_def split*: *splits*)

lemma *ins_Diff*[*simp*]: *avl t* ⟹
ins x t ≠ *Diff Leaf* ∧
(*ins x t* = *Diff* (*Node l* (*a*,*Bal*) *r*) ⟷ *t* = *Leaf* ∧ *a* = *x* ∧ *l*=*Leaf* ∧ *r*=*Leaf*) ∧
ins x t ≠ *Diff* (*Node l* (*a*,*Rh*) *Leaf*) ∧
ins x t ≠ *Diff* (*Node Leaf* (*a*,*Lh*) *r*)
by(*drule avl_ins_case*[*of* _ *x*]) (*auto split*: *splits*)

theorem *inorder_ins*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder*(*tree*(*ins x t*)) = *ins_list x* (*inorder t*)
by (*induction t*) (*auto simp*: *ins_list_simps split*!: *splits*)

19.2.2 Proofs about deletion

lemma *inorder_baldL*:
⟦ *ba* = *Rh* ⟶ *r* ≠ *Leaf*; *avl r* ⟧
⟹ *inorder* (*tree*(*baldL l a ba r*)) = *inorder* (*tree l*) @ *a* # *inorder r*
by (*auto split*: *splits*)

lemma *inorder_baldR*:
⟦ *ba* = *Lh* ⟶ *l* ≠ *Leaf*; *avl l* ⟧
⟹ *inorder* (*tree*(*baldR l a ba r*)) = *inorder l* @ *a* # *inorder* (*tree r*)

by (*auto split*: *splits*)

lemma *avl_split_max*:
⟦ *split_max t* = (*t'*,*a*); *avl t*; *t* ≠ *Leaf* ⟧ ⟹ *case t' of*
Same t' ⇒ *avl t'* ∧ *height t* = *height t'* |
Diff t' ⇒ *avl t'* ∧ *height t* = *height t'* + *1*
proof (*induction t arbitrary*: *t' a rule*: *split_max_induct*)
qed (*auto simp*: *max_def split*!: *splits prod.splits*)

lemma *avl_del_case*: *avl t* ⟹ *case del x t of*
Same t' ⇒ *avl t'* ∧ *height t* = *height t'* |
Diff t' ⇒ *avl t'* ∧ *height t* = *height t'* + *1*
proof (*induction x t rule*: *del.induct*)
qed (*auto simp*: *max_absorb1 max_absorb2 dest*: *avl_split_max split*!: *splits prod.splits*)

corollary *avl_delete*: *avl t* ⟹ *avl*(*delete x t*)
using *avl_del_case*[*of t x*] **by**(*simp add*: *delete_def split*: *splits*)

lemma *inorder_split_maxD*:
⟦ *split_max t* = (*t'*,*a*); *t* ≠ *Leaf*; *avl t* ⟧ ⟹
inorder (*tree t'*) @ [*a*] = *inorder t*
proof (*induction t arbitrary*: *t' rule*: *split_max.induct*)
qed (*auto split*!: *splits prod.splits*)

lemma *neq_Leaf_if_height_neq_0*[*simp*]: *height t* ≠ *0* ⟹ *t* ≠ *Leaf*
by *auto*

theorem *inorder_del*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder* (*tree*(*del x t*)) = *del_list x* (*inorder t*)
proof (*induction t rule*: *tree2_induct*)
case *Leaf*
then show *?case* **by** *simp*
next
case (*Node x1 a b x3*)
then show *?case*
by (*auto simp*: *del_list_simps inorder_baldL inorder_baldR avl_delete inorder_split_maxD*
simp del: *baldL.simps split*!: *splits prod.splits*)
qed

19.2.3 Set Implementation

interpretation *S*: *Set_by_Ordered*
where *empty* = *Leaf* **and** *isin* = *isin*
and *insert* = *insert*
and *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_ins insert_def*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_del delete_def*)
next
case *5* **thus** *?case* **by** (*simp*)
next
case *6* **thus** *?case* **by** (*simp add*: *avl_insert*)
next
case *7* **thus** *?case* **by** (*simp add*: *avl_delete*)
qed

end

20 Height-Balanced Trees

theory *Height_Balanced_Tree*
imports
Cmp
Isin2
begin

Height-balanced trees (HBTs) can be seen as a generalization of AVL trees. The code and the proofs were obtained by small modifications of the AVL theories. This is an implementation of sets via HBTs.

type_synonym *'a tree_ht* = (*'a*∗*nat*) *tree*

definition *empty* :: *'a tree_ht* **where**
empty = *Leaf*

The maximal amount by which the height of two siblings may differ:

locale *HBT* =
fixes *m* :: *nat*

assumes [*arith*]: $m > 0$
begin

Invariant:

fun *hbt* :: $'a$ *tree_ht* $\Rightarrow$ *bool* **where**
hbt Leaf = *True* |
hbt (*Node l* (*a*,*n*) *r*) =
(*abs*(*int*(*height l*) − *int*(*height r*)) ≤ *int*(*m*) ∧
n = *max* (*height l*) (*height r*) + 1 ∧ *hbt l* ∧ *hbt r*)

fun *ht* :: $'a$ *tree_ht* $\Rightarrow$ *nat* **where**
ht Leaf = 0 |
ht (*Node l* (*a*,*n*) *r*) = *n*

definition *node* :: $'a$ *tree_ht* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree_ht* $\Rightarrow$ $'a$ *tree_ht* **where**
node l a r = *Node l* (*a*, *max* (*ht l*) (*ht r*) + 1) *r*

definition *balL* :: $'a$ *tree_ht* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree_ht* $\Rightarrow$ $'a$ *tree_ht* **where**
balL AB b C =
(*if ht AB* = *ht C* + *m* + 1 *then*
case AB of
Node A (*a*, _) *B* ⇒
if ht A ≥ *ht B then node A a* (*node B b C*)
else
case B of
Node B_1 (*ab*, _) B_2 ⇒ *node* (*node A a* B_1) *ab* (*node* B_2 *b C*)
else node AB b C)

definition *balR* :: $'a$ *tree_ht* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree_ht* $\Rightarrow$ $'a$ *tree_ht* **where**
balR A a BC =
(*if ht BC* = *ht A* + *m* + 1 *then*
case BC of
Node B (*b*, _) *C* ⇒
if ht B ≤ *ht C then node* (*node A a B*) *b C*
else
case B of
Node B_1 (*ab*, _) B_2 ⇒ *node* (*node A a* B_1) *ab* (*node* B_2 *b C*)
else node A a BC)

fun *insert* :: $'a$::*linorder* $\Rightarrow$ $'a$ *tree_ht* $\Rightarrow$ $'a$ *tree_ht* **where**
insert x Leaf = *Node Leaf* (*x*, 1) *Leaf* |
insert x (*Node l* (*a*, *n*) *r*) = (*case cmp x a of*
EQ ⇒ *Node l* (*a*, *n*) *r* |
LT ⇒ *balL* (*insert x l*) *a r* |

```
  GT ⇒ balR l a (insert x r))

fun split_max :: 'a tree_ht ⇒ 'a tree_ht * 'a where
split_max (Node l (a, _) r) =
  (if r = Leaf then (l,a) else let (r',a') = split_max r in (balL l a r', a'))

lemmas split_max_induct = split_max.induct[case_names Node Leaf]

fun delete :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
delete _ Leaf = Leaf |
delete x (Node l (a, n) r) =
  (case cmp x a of
     EQ ⇒ if l = Leaf then r
          else let (l', a') = split_max l in balR l' a' r |
     LT ⇒ balR (delete x l) a r |
     GT ⇒ balL l a (delete x r))
```

20.1 Functional Correctness Proofs

20.1.1 Proofs for insert

```
lemma inorder_balL:
  inorder (balL l a r) = inorder l @ a # inorder r
by (auto simp: node_def balL_def split:tree.splits)

lemma inorder_balR:
  inorder (balR l a r) = inorder l @ a # inorder r
by (auto simp: node_def balR_def split:tree.splits)

theorem inorder_insert:
  sorted(inorder t) ⟹ inorder(insert x t) = ins_list x (inorder t)
by (induct t)
   (auto simp: ins_list_simps inorder_balL inorder_balR)
```

20.1.2 Proofs for delete

```
lemma inorder_split_maxD:
  ⟦ split_max t = (t',a); t ≠ Leaf ⟧ ⟹
   inorder t' @ [a] = inorder t
by(induction t arbitrary: t' rule: split_max.induct)
  (auto simp: inorder_balL split: if_splits prod.splits tree.split)

theorem inorder_delete:
  sorted(inorder t) ⟹ inorder (delete x t) = del_list x (inorder t)
by(induction t)
```

(*auto simp*: *del_list_simps inorder_balL inorder_balR inorder_split_maxD split*: *prod.splits*)

20.2 Invariant preservation

20.2.1 Insertion maintains balance

declare *Let_def* [*simp*]

lemma *ht_height*[*simp*]: *hbt t* $\Longrightarrow$ *ht t* = *height t*
by (*cases t rule*: *tree2_cases*) *simp_all*

First, a fast but relatively manual proof with many lemmas:

lemma *height_balL*:
$[\![$ *hbt l*; *hbt r*; *height l* = *height r* + *m* + *1* $]\!] \Longrightarrow$
height (*balL l a r*) $\in$ {*height r* + *m* + *1*, *height r* + *m* + *2*}
by (*auto simp*:*node_def balL_def split*:*tree.split*)

lemma *height_balR*:
$[\![$ *hbt l*; *hbt r*; *height r* = *height l* + *m* + *1* $]\!] \Longrightarrow$
height (*balR l a r*) $\in$ {*height l* + *m* + *1*, *height l* + *m* + *2*}
by(*auto simp add*:*node_def balR_def split*:*tree.split*)

lemma *height_node*[*simp*]: *height*(*node l a r*) = *max* (*height l*) (*height r*) + *1*
by (*simp add*: *node_def*)

lemma *height_balL2*:
$[\![$ *hbt l*; *hbt r*; *height l* $\neq$ *height r* + *m* + *1* $]\!] \Longrightarrow$
height (*balL l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balL_def*)

lemma *height_balR2*:
$[\![$ *hbt l*; *hbt r*; *height r* $\neq$ *height l* + *m* + *1* $]\!] \Longrightarrow$
height (*balR l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balR_def*)

lemma *hbt_balL*:
$[\![$ *hbt l*; *hbt r*; *height r* − *m* $\leq$ *height l* $\wedge$ *height l* $\leq$ *height r* + *m* + *1* $]\!]$
$\Longrightarrow$ *hbt*(*balL l a r*)
by(*auto simp*: *balL_def node_def max_def split*!: *if_splits tree.split*)

lemma *hbt_balR*:
$[\![$ *hbt l*; *hbt r*; *height l* − *m* $\leq$ *height r* $\wedge$ *height r* $\leq$ *height l* + *m* + *1* $]\!]$
$\Longrightarrow$ *hbt*(*balR l a r*)

by(*auto simp*: *balR_def node_def max_def split*!: *if_splits tree.split*)

Insertion maintains *hbt*. Requires simultaneous proof.

```
theorem hbt_insert:
  hbt t ⟹ hbt(insert x t)
  hbt t ⟹ height (insert x t) ∈ {height t, height t + 1}
proof (induction t rule: tree2_induct)
  case (Node l a _ r)
  case 1
  show ?case
  proof(cases x = a)
    case True with Node 1 show ?thesis by simp
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True with 1 Node(1,2) show ?thesis by (auto intro!: hbt_balL)
    next
       case False with 1 Node(3,4) ‹x≠a› show ?thesis by (auto intro!:
hbt_balR)
    qed
  qed
  case 2
  show ?case
  proof(cases x = a)
    case True with 2 show ?thesis by simp
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height (insert x l) = height r + m + 1)
        case False with 2 Node(1,2) ‹x < a› show ?thesis by (auto simp:
height_balL2)
      next
        case True
        hence (height (balL (insert x l) a r) = height r + m + 1) ∨
          (height (balL (insert x l) a r) = height r + m + 2) (is ?A ∨ ?B)
          using 2 Node(1,2) height_balL[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 Node(2) True ‹x < a› show ?thesis by (auto)
        next
```

assume *?B* **with** *2 Node(2) True* ‹$x < a$› **show** *?thesis* **by** (*simp*)
arith
qed
qed
next
case *False*
show *?thesis*
proof(*cases height* (*insert x r*) = *height l* + *m* + *1*)
case *False* **with** *2 Node(3,4)* ‹$\neg x < a$› **show** *?thesis* **by** (*auto simp*:
height_balR2)
next
case *True*
hence (*height* (*balR l a* (*insert x r*)) = *height l* + *m* + *1*) ∨
(*height* (*balR l a* (*insert x r*)) = *height l* + *m* + *2*) (**is** *?A* ∨ *?B*)
using *Node 2 height_balR*[*OF* _ _ *True*] **by** *simp*
thus *?thesis*
proof
assume *?A* **with** *2 Node(4) True* ‹$\neg x < a$› **show** *?thesis* **by** (*auto*)
next
assume *?B* **with** *2 Node(4) True* ‹$\neg x < a$› **show** *?thesis* **by** (*simp*)
arith
qed
qed
qed
qed
qed *simp_all*

Now an automatic proof without lemmas:

theorem *hbt_insert_auto*: *hbt t* ⟹
hbt(*insert x t*) ∧ *height* (*insert x t*) ∈ {*height t*, *height t* + *1*}
apply (*induction t rule*: *tree2_induct*)

apply (*auto simp*: *balL_def balR_def node_def max_absorb1 max_absorb2*
split!: *if_split tree.split*)
done

20.2.2 Deletion maintains balance

lemma *hbt_split_max*:
⟦ *hbt t*; *t* ≠ *Leaf* ⟧ ⟹
hbt (*fst* (*split_max t*)) ∧
height t ∈ {*height*(*fst* (*split_max t*)), *height*(*fst* (*split_max t*)) + *1*}
by(*induct t rule*: *split_max_induct*)
(*auto simp*: *balL_def node_def max_absorb2 split*!: *prod.split if_split*

tree.split)

Deletion maintains *hbt*:

theorem *hbt_delete*:
hbt t $\Longrightarrow$ *hbt*(*delete x t*)
hbt t $\Longrightarrow$ *height t* $\in$ {*height* (*delete x t*), *height* (*delete x t*) + *1*}
proof (*induct t rule*: *tree2_induct*)
case (*Node l a n r*)
case *1*
thus *?case*
using *Node hbt_split_max*[*of l*] **by** (*auto intro*!: *hbt_balL hbt_balR split*: *prod.split*)
case *2*
show *?case*
proof(*cases x* = *a*)
case *True* **then show** *?thesis* **using** *1 hbt_split_max*[*of l*]
by(*auto simp*: *balR_def max_absorb2 split*!: *if_splits prod.split tree.split*)
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True*
show *?thesis*
proof(*cases height r* = *height* (*delete x l*) + *m* + *1*)
case *False* **with** *Node 1* ‹*x* < *a*› **show** *?thesis* **by**(*auto simp*: *balR_def*)
next
case *True*
hence (*height* (*balR* (*delete x l*) *a r*) = *height* (*delete x l*) + *m* + *1*) ∨
height (*balR* (*delete x l*) *a r*) = *height* (*delete x l*) + *m* + *2* (**is** *?A* ∨ *?B*)
using *Node 2height_balR*[*OF* _ _ *True*] **by** *simp*
thus *?thesis*
proof
assume *?A* **with** ‹*x* < *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balR_def split*!: *if_splits*)
next
assume *?B* **with** ‹*x* < *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balR_def split*!: *if_splits*)
qed
qed
next
case *False*

```
    show ?thesis
    proof(cases height l = height (delete x r) + m + 1)
      case False with Node 1 ‹¬x < a› ‹x ≠ a› show ?thesis by(auto
simp: balL_def)
    next
      case True
      hence (height (balL l a (delete x r)) = height (delete x r) + m + 1)
∨
        height (balL l a (delete x r)) = height (delete x r) + m + 2 (is ?A
∨ ?B)
        using Node 2 height_balL[OF _ _ True] by simp
      thus ?thesis
      proof
        assume ?A with ‹¬x < a› ‹x ≠ a› Node 2 show ?thesis by(auto
simp: balL_def split: if_splits)
      next
        assume ?B with ‹¬x < a› ‹x ≠ a› Node 2 show ?thesis by(auto
simp: balL_def split: if_splits)
      qed
    qed
  qed
qed simp_all
```

A more automatic proof. Complete automation as for insertion seems hard due to resource requirements.

```
theorem hbt_delete_auto:
  hbt t ⟹ hbt(delete x t)
  hbt t ⟹ height t ∈ {height (delete x t), height (delete x t) + 1}
proof (induct t rule: tree2_induct)
  case (Node l a n r)
  case 1
  thus ?case
   using Node hbt_split_max[of l] by (auto intro!: hbt_balL hbt_balR split:
prod.split)
  case 2
  show ?case
  proof(cases x = a)
    case True thus ?thesis
      using 2 hbt_split_max[of l]
    by(auto simp: balR_def max_absorb2 split!: if_splits prod.split tree.split)
  next
    case False thus ?thesis
      using height_balL[of l delete x r a] height_balR[of delete x l r a] 2
```

Node
  by(*auto simp*: *balL_def balR_def split*!: *if_split*)
 qed
qed *simp_all*

20.3 Overall correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *hbt*
proof (*standard*, *goal_cases*)
 case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
 case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
 case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
 case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
 case *5* **thus** *?case* **by** (*simp add*: *empty_def*)
next
 case *6* **thus** *?case* **by** (*simp add*: *hbt_insert*(*1*))
next
 case *7* **thus** *?case* **by** (*simp add*: *hbt_delete*(*1*))
qed

end

end

21 Red-Black Trees

theory *RBT*
imports *Tree2*
begin

datatype *color* = *Red* | *Black*

type_synonym *'a rbt* = (*'a*∗*color*)*tree*

abbreviation *R* **where** *R l a r* ≡ *Node l* (*a*, *Red*) *r*
abbreviation *B* **where** *B l a r* ≡ *Node l* (*a*, *Black*) *r*

```
fun baliL :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baliL (R (R t1 a t2) b t3) c t4 = R (B t1 a t2) b (B t3 c t4) |
baliL (R t1 a (R t2 b t3)) c t4 = R (B t1 a t2) b (B t3 c t4) |
baliL t1 a t2 = B t1 a t2

fun baliR :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baliR t1 a (R t2 b (R t3 c t4)) = R (B t1 a t2) b (B t3 c t4) |
baliR t1 a (R (R t2 b t3) c t4) = R (B t1 a t2) b (B t3 c t4) |
baliR t1 a t2 = B t1 a t2

fun paint :: color ⇒ 'a rbt ⇒ 'a rbt where
paint c Leaf = Leaf |
paint c (Node l (a,_) r) = Node l (a,c) r

fun baldL :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baldL (R t1 a t2) b t3 = R (B t1 a t2) b t3 |
baldL t1 a (B t2 b t3) = baliR t1 a (R t2 b t3) |
baldL t1 a (R (B t2 b t3) c t4) = R (B t1 a t2) b (baliR t3 c (paint Red
t4)) |
baldL t1 a t2 = R t1 a t2

fun baldR :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baldR t1 a (R t2 b t3) = R t1 a (B t2 b t3) |
baldR (B t1 a t2) b t3 = baliL (R t1 a t2) b t3 |
baldR (R t1 a (B t2 b t3)) c t4 = R (baliL (paint Red t1) a t2) b (B t3 c
t4) |
baldR t1 a t2 = R t1 a t2

fun join :: 'a rbt ⇒ 'a rbt ⇒ 'a rbt where
join Leaf t = t |
join t Leaf = t |
join (R t1 a t2) (R t3 c t4) =
  (case join t2 t3 of
     R u2 b u3 ⇒ (R (R t1 a u2) b (R u3 c t4)) |
     t23 ⇒ R t1 a (R t23 c t4)) |
join (B t1 a t2) (B t3 c t4) =
  (case join t2 t3 of
     R u2 b u3 ⇒ R (B t1 a u2) b (B u3 c t4) |
     t23 ⇒ baldL t1 a (B t23 c t4)) |
join t1 (R t2 a t3) = R (join t1 t2) a t3 |
join (R t1 a t2) t3 = R t1 a (join t2 t3)

end
```

22 Red-Black Tree Implementation of Sets

theory *RBT_Set*
imports
Complex_Main
RBT
Cmp
Isin2
begin

definition *empty* :: *'a rbt* **where**
empty = *Leaf*

fun *ins* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
ins x Leaf = *R Leaf x Leaf* |
ins x (*B l a r*) =
(*case cmp x a of*
LT ⇒ *baliL* (*ins x l*) *a r* |
GT ⇒ *baliR l a* (*ins x r*) |
EQ ⇒ *B l a r*) |
ins x (*R l a r*) =
(*case cmp x a of*
LT ⇒ *R* (*ins x l*) *a r* |
GT ⇒ *R l a* (*ins x r*) |
EQ ⇒ *R l a r*)

definition *insert* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
insert x t = *paint Black* (*ins x t*)

fun *color* :: *'a rbt* ⇒ *color* **where**
color Leaf = *Black* |
color (*Node* _ (_, *c*) _) = *c*

fun *del* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
del x Leaf = *Leaf* |
del x (*Node l* (*a*, _) *r*) =
(*case cmp x a of*
LT ⇒ *if l* ≠ *Leaf* ∧ *color l* = *Black*
then baldL (*del x l*) *a r else R* (*del x l*) *a r* |
GT ⇒ *if r* ≠ *Leaf*∧ *color r* = *Black*
then baldR l a (*del x r*) *else R l a* (*del x r*) |
EQ ⇒ *join l r*)

definition *delete* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**

delete x t = paint Black (del x t)

22.1 Functional Correctness Proofs

lemma *inorder_paint*: *inorder(paint c t) = inorder t*
by(*cases t*) (*auto*)

lemma *inorder_baliL*:
inorder(baliL l a r) = inorder l @ a # inorder r
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto*)

lemma *inorder_baliR*:
inorder(baliR l a r) = inorder l @ a # inorder r
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto*)

lemma *inorder_ins*:
sorted(inorder t) ⟹ inorder(ins x t) = ins_list x (inorder t)
by(*induction x t rule*: *ins.induct*)
(*auto simp*: *ins_list_simps inorder_baliL inorder_baliR*)

lemma *inorder_insert*:
sorted(inorder t) ⟹ inorder(insert x t) = ins_list x (inorder t)
by (*simp add*: *insert_def inorder_ins inorder_paint*)

lemma *inorder_baldL*:
inorder(baldL l a r) = inorder l @ a # inorder r
by(*cases* (*l*,*a*,*r*) *rule*: *baldL.cases*)
(*auto simp*: *inorder_baliL inorder_baliR inorder_paint*)

lemma *inorder_baldR*:
inorder(baldR l a r) = inorder l @ a # inorder r
by(*cases* (*l*,*a*,*r*) *rule*: *baldR.cases*)
(*auto simp*: *inorder_baliL inorder_baliR inorder_paint*)

lemma *inorder_join*:
inorder(join l r) = inorder l @ inorder r
by(*induction l r rule*: *join.induct*)
(*auto simp*: *inorder_baldL inorder_baldR split*: *tree.split color.split*)

lemma *inorder_del*:
sorted(inorder t) ⟹ inorder(del x t) = del_list x (inorder t)
by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_join inorder_baldL inorder_baldR*)

lemma *inorder_delete*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by (*auto simp*: *delete_def inorder_del inorder_paint*)

22.2 Structural invariants

lemma *neq_Black*[*simp*]: (*c* $\neq$ *Black*) = (*c* = *Red*)
by (*cases c*) *auto*

The proofs are due to Markus Reiter and Alexander Krauss.

fun *bheight* :: *'a rbt* $\Rightarrow$ *nat* **where**
bheight Leaf = *0* |
bheight (*Node l* (*x*, *c*) *r*) = (*if c* = *Black then bheight l* + *1 else bheight l*)

fun *invc* :: *'a rbt* $\Rightarrow$ *bool* **where**
invc Leaf = *True* |
invc (*Node l* (*a*,*c*) *r*) =
(*c* = *Red* $\longrightarrow$ *color l* = *Black* $\wedge$ *color r* = *Black*) $\wedge$ *invc l* $\wedge$ *invc r*)

Weaker version:

abbreviation *invc2* :: *'a rbt* $\Rightarrow$ *bool* **where**
invc2 t $\equiv$ *invc*(*paint Black t*)

fun *invh* :: *'a rbt* $\Rightarrow$ *bool* **where**
invh Leaf = *True* |
invh (*Node l* (*x*, *c*) *r*) = (*bheight l* = *bheight r* $\wedge$ *invh l* $\wedge$ *invh r*)

lemma *invc2I*: *invc t* $\Longrightarrow$ *invc2 t*
by (*cases t rule*: *tree2_cases*) *simp*+

definition *rbt* :: *'a rbt* $\Rightarrow$ *bool* **where**
rbt t = (*invc t* $\wedge$ *invh t* $\wedge$ *color t* = *Black*)

lemma *color_paint_Black*: *color* (*paint Black t*) = *Black*
by (*cases t*) *auto*

lemma *paint2*: *paint c2* (*paint c1 t*) = *paint c2 t*
by (*cases t*) *auto*

lemma *invh_paint*: *invh t* $\Longrightarrow$ *invh* (*paint c t*)
by (*cases t*) *auto*

lemma *invc_baliL*:
⟦*invc2 l*; *invc r*⟧ $\Longrightarrow$ *invc* (*baliL l a r*)

by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *invc_baliR*:
⟦*invc l*; *invc2 r*⟧ ⟹ *invc* (*baliR l a r*)
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *bheight_baliL*:
bheight l = *bheight r* ⟹ *bheight* (*baliL l a r*) = *Suc* (*bheight l*)
by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *bheight_baliR*:
bheight l = *bheight r* ⟹ *bheight* (*baliR l a r*) = *Suc* (*bheight l*)
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *invh_baliL*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r* ⟧ ⟹ *invh* (*baliL l a r*)
by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *invh_baliR*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r* ⟧ ⟹ *invh* (*baliR l a r*)
by (*induct l a r rule*: *baliR.induct*) *auto*

All in one:

lemma *inv_baliR*: ⟦ *invh l*; *invh r*; *invc l*; *invc2 r*; *bheight l* = *bheight r* ⟧
⟹ *invc* (*baliR l a r*) ∧ *invh* (*baliR l a r*) ∧ *bheight* (*baliR l a r*) = *Suc* (*bheight l*)
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *inv_baliL*: ⟦ *invh l*; *invh r*; *invc2 l*; *invc r*; *bheight l* = *bheight r* ⟧
⟹ *invc* (*baliL l a r*) ∧ *invh* (*baliL l a r*) ∧ *bheight* (*baliL l a r*) = *Suc* (*bheight l*)
by (*induct l a r rule*: *baliL.induct*) *auto*

22.2.1 Insertion

lemma *invc_ins*: *invc t* ⟶ *invc2* (*ins x t*) ∧ (*color t* = *Black* ⟶ *invc* (*ins x t*))
by (*induct x t rule*: *ins.induct*) (*auto simp*: *invc_baliL invc_baliR invc2I*)

lemma *invh_ins*: *invh t* ⟹ *invh* (*ins x t*) ∧ *bheight* (*ins x t*) = *bheight t*
by(*induct x t rule*: *ins.induct*)
(*auto simp*: *invh_baliL invh_baliR bheight_baliL bheight_baliR*)

theorem *rbt_insert*: *rbt t* ⟹ *rbt* (*insert x t*)

by (*simp add*: *invc_ins invh_ins color_paint_Black invh_paint rbt_def insert_def*)

All in one:

lemma *inv_ins*: ⟦ *invc t*; *invh t* ⟧ ⟹
invc2 (*ins x t*) ∧ (*color t* = *Black* ⟶ *invc* (*ins x t*)) ∧
invh(*ins x t*) ∧ *bheight* (*ins x t*) = *bheight t*
by (*induct x t rule*: *ins.induct*) (*auto simp*: *inv_baliL inv_baliR invc2I*)

theorem *rbt_insert2*: *rbt t* ⟹ *rbt* (*insert x t*)
by (*simp add*: *inv_ins color_paint_Black invh_paint rbt_def insert_def*)

22.2.2 Deletion

lemma *bheight_paint_Red*:
color t = *Black* ⟹ *bheight* (*paint Red t*) = *bheight t* − *1*
by (*cases t*) *auto*

lemma *invh_baldL_invc*:
⟦ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *invc r* ⟧
⟹ *invh* (*baldL l a r*) ∧ *bheight* (*baldL l a r*) = *bheight r*
by (*induct l a r rule*: *baldL.induct*)
(*auto simp*: *invh_baliR invh_paint bheight_baliR bheight_paint_Red*)

lemma *invh_baldL_Black*:
⟦ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *color r* = *Black* ⟧
⟹ *invh* (*baldL l a r*) ∧ *bheight* (*baldL l a r*) = *bheight r*
by (*induct l a r rule*: *baldL.induct*) (*auto simp add*: *invh_baliR bheight_baliR*)

lemma *invc_baldL*: ⟦*invc2 l*; *invc r*; *color r* = *Black*⟧ ⟹ *invc* (*baldL l a r*)
by (*induct l a r rule*: *baldL.induct*) (*simp_all add*: *invc_baliR*)

lemma *invc2_baldL*: ⟦ *invc2 l*; *invc r* ⟧ ⟹ *invc2* (*baldL l a r*)
by (*induct l a r rule*: *baldL.induct*) (*auto simp*: *invc_baliR paint2 invc2I*)

lemma *invh_baldR_invc*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r* + *1*; *invc l* ⟧
⟹ *invh* (*baldR l a r*) ∧ *bheight* (*baldR l a r*) = *bheight l*
by(*induct l a r rule*: *baldR.induct*)
(*auto simp*: *invh_baliL bheight_baliL invh_paint bheight_paint_Red*)

lemma *invc_baldR*: ⟦*invc l*; *invc2 r*; *color l* = *Black*⟧ ⟹ *invc* (*baldR l a*

r)
by (*induct l a r rule*: *baldR.induct*) (*simp_all add*: *invc_baliL*)

lemma *invc2_baldR*: ⟦ *invc l*; *invc2 r* ⟧ ⟹*invc2* (*baldR l a r*)
by (*induct l a r rule*: *baldR.induct*) (*auto simp*: *invc_baliL paint2 invc2I*)

lemma *invh_join*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r* ⟧
⟹ *invh* (*join l r*) ∧ *bheight* (*join l r*) = *bheight l*
by (*induct l r rule*: *join.induct*)
(*auto simp*: *invh_baldL_Black split*: *tree.splits color.splits*)

lemma *invc_join*:
⟦ *invc l*; *invc r* ⟧ ⟹
(*color l* = *Black* ∧ *color r* = *Black* ⟶ *invc* (*join l r*)) ∧ *invc2* (*join l r*)
by (*induct l r rule*: *join.induct*)
(*auto simp*: *invc_baldL invc2I split*: *tree.splits color.splits*)

All in one:

lemma *inv_baldL*:
⟦ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *invc2 l*; *invc r* ⟧
⟹ *invh* (*baldL l a r*) ∧ *bheight* (*baldL l a r*) = *bheight r*
∧ *invc2* (*baldL l a r*) ∧ (*color r* = *Black* ⟶ *invc* (*baldL l a r*))
by (*induct l a r rule*: *baldL.induct*)
(*auto simp*: *inv_baliR invh_paint bheight_baliR bheight_paint_Red paint2 invc2I*)

lemma *inv_baldR*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r* + *1*; *invc l*; *invc2 r* ⟧
⟹ *invh* (*baldR l a r*) ∧ *bheight* (*baldR l a r*) = *bheight l*
∧ *invc2* (*baldR l a r*) ∧ (*color l* = *Black* ⟶ *invc* (*baldR l a r*))
by (*induct l a r rule*: *baldR.induct*)
(*auto simp*: *inv_baliL invh_paint bheight_baliL bheight_paint_Red paint2 invc2I*)

lemma *inv_join*:
⟦ *invh l*; *invh r*; *bheight l* = *bheight r*; *invc l*; *invc r* ⟧
⟹ *invh* (*join l r*) ∧ *bheight* (*join l r*) = *bheight l*
∧ *invc2* (*join l r*) ∧ (*color l* = *Black* ∧ *color r* = *Black* ⟶ *invc* (*join l r*))
by (*induct l r rule*: *join.induct*)
(*auto simp*: *invh_baldL_Black inv_baldL invc2I split*: *tree.splits color.splits*)

lemma *neq_LeafD*: *t* ≠ *Leaf* ⟹ ∃ *l x c r*. *t* = *Node l* (*x*,*c*) *r*

by(*cases t rule*: *tree2_cases*) *auto*

lemma *inv_del*: ⟦ *invh t*; *invc t* ⟧ ⟹
invh (*del x t*) ∧
(*color t* = *Red* ⟶ *bheight* (*del x t*) = *bheight t* ∧ *invc* (*del x t*)) ∧
(*color t* = *Black* ⟶ *bheight* (*del x t*) = *bheight t* − *1* ∧ *invc2* (*del x t*))
by(*induct x t rule*: *del.induct*)
(*auto simp*: *inv_baldL inv_baldR inv_join dest*!: *neq_LeafD*)

theorem *rbt_delete*: *rbt t* ⟹ *rbt* (*delete x t*)
by (*metis delete_def rbt_def color_paint_Black inv_del invh_paint*)

Overall correctness:

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *rbt*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by** (*simp add*: *rbt_def empty_def*)
next
case *6* **thus** *?case* **by** (*simp add*: *rbt_insert*)
next
case *7* **thus** *?case* **by** (*simp add*: *rbt_delete*)
qed

22.3 Height-Size Relation

lemma *rbt_height_bheight_if*: *invc t* ⟹ *invh t* ⟹
height t ≤ *2* ∗ *bheight t* + (*if color t* = *Black then 0 else 1*)
by(*induction t*) (*auto split*: *if_split_asm*)

lemma *rbt_height_bheight*: *rbt t* ⟹ *height t* / *2* ≤ *bheight t*
by(*auto simp*: *rbt_def dest*: *rbt_height_bheight_if*)

lemma *bheight_size_bound*: *invc t* ⟹ *invh t* ⟹ *2* ^ (*bheight t*) ≤ *size1 t*

by (*induction t*) *auto*

lemma *bheight_le_min_height*: *invh t* $\Longrightarrow$ *bheight t* $\leq$ *min_height t*
by (*induction t*) *auto*

lemma *rbt_height_le*: **assumes** *rbt t* **shows** *height t* $\leq$ *2* $*$ *log 2* (*size1 t*)
proof −
  have *2 powr* (*height t / 2*) $\leq$ *2 powr bheight t*
    using *rbt_height_bheight*[*OF assms*] **by** *simp*
  also have ... $\leq$ *size1 t* **using** *assms*
    by (*simp add*: *powr_realpow bheight_size_bound rbt_def*)
  finally have *2 powr* (*height t / 2*) $\leq$ *size1 t* **.**
  hence *height t / 2* $\leq$ *log 2* (*size1 t*)
    by (*simp add*: *le_log_iff size1_size del*: *divide_le_eq_numeral1*(*1*))
  thus *?thesis* **by** *simp*
qed

lemma *rbt_height_le2*: **assumes** *rbt t* **shows** *height t* $\leq$ *2* $*$ *log 2* (*size1 t*)
proof −
  have *height t* $\leq$ *2* $*$ *bheight t*
    using *rbt_height_bheight_if assms*[*simplified rbt_def*] **by** *fastforce*
  also have ... $\leq$ *2* $*$ *min_height t*
    using *bheight_le_min_height assms*[*simplified rbt_def*] **by** *auto*
  also have ... $\leq$ *2* $*$ *log 2* (*size1 t*)
    using *le_log2_of_power min_height_size1* **by** *auto*
  finally show *?thesis* **by** *simp*
qed

end

23 Alternative Deletion in Red-Black Trees

theory *RBT_Set2*
imports *RBT_Set*
begin

This is a conceptually simpler version of deletion. Instead of the tricky *join* function this version follows the standard approach of replacing the deleted element (in function *del*) by the minimal element in its right subtree.

fun *split_min* :: *'a rbt* $\Rightarrow$ *'a* $\times$ *'a rbt* **where**
split_min (*Node l* (*a*, _) *r*) =
  (*if l* = *Leaf then* (*a*,*r*)
   else let (*x*,*l'*) = *split_min l*

in (*x*, *if color l* = *Black then baldL l′ a r else R l′ a r*))

fun *del* :: *′a*::*linorder* ⇒ *′a rbt* ⇒ *′a rbt* **where**
del x Leaf = *Leaf* |
del x (*Node l* (*a*, _) *r*) =
(*case cmp x a of*
LT ⇒ *let l′* = *del x l in if l* ≠ *Leaf* ∧ *color l* = *Black*
then baldL l′ a r else R l′ a r |
GT ⇒ *let r′* = *del x r in if r* ≠ *Leaf* ∧ *color r* = *Black*
then baldR l a r′ else R l a r′ |
EQ ⇒ *if r* = *Leaf then l else let* (*a′*,*r′*) = *split_min r in*
if color r = *Black then baldR l a′ r′ else R l a′ r′*)

The first two *let*s speed up the automatic proof of *inv_del* below.

definition *delete* :: *′a*::*linorder* ⇒ *′a rbt* ⇒ *′a rbt* **where**
delete x t = *paint Black* (*del x t*)

23.1 Functional Correctness Proofs

declare *Let_def*[*simp*]

lemma *split_minD*:
split_min t = (*x*,*t′*) ⟹ *t* ≠ *Leaf* ⟹ *x* # *inorder t′* = *inorder t*
by(*induction t arbitrary*: *t′ rule*: *split_min.induct*)
(*auto simp*: *inorder_baldL sorted_lems split*: *prod.splits if_splits*)

lemma *inorder_del*:
sorted(*inorder t*) ⟹ *inorder*(*del x t*) = *del_list x* (*inorder t*)
by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_baldL inorder_baldR split_minD split*: *prod.splits*)

lemma *inorder_delete*:
sorted(*inorder t*) ⟹ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by (*auto simp*: *delete_def inorder_del inorder_paint*)

23.2 Structural invariants

lemma *neq_Red*[*simp*]: (*c* ≠ *Red*) = (*c* = *Black*)
by (*cases c*) *auto*

23.2.1 Deletion

lemma *inv_split_min*: ⟦ *split_min t* = (*x*,*t′*); *t* ≠ *Leaf*; *invh t*; *invc t* ⟧
⟹

invh $t' \wedge$
(*color* t = *Red* $\longrightarrow$ *bheight* t' = *bheight* $t \wedge$ *invc* t') $\wedge$
(*color* t = *Black* $\longrightarrow$ *bheight* t' = *bheight* $t - 1 \wedge$ *invc2* t')
apply(*induction t arbitrary*: *x t' rule*: *split_min.induct*)
apply(*auto simp*: *inv_baldR inv_baldL invc2I dest*!: *neq_LeafD*
split: *if_splits prod.splits*)
done

An automatic proof. It is quite brittle, e.g. inlining the *let*s in *RBT_Set2.del* breaks it.

lemma *inv_del*: ⟦ *invh t*; *invc t* ⟧ $\Longrightarrow$
invh (*del x t*) $\wedge$
(*color t* = *Red* $\longrightarrow$ *bheight* (*del x t*) = *bheight t* $\wedge$ *invc* (*del x t*)) $\wedge$
(*color t* = *Black* $\longrightarrow$ *bheight* (*del x t*) = *bheight t* − *1* $\wedge$ *invc2* (*del x t*))
apply(*induction x t rule*: *del.induct*)
apply(*auto simp*: *inv_baldR inv_baldL invc2I dest*!: *inv_split_min dest*: *neq_LeafD*
split!: *prod.splits if_splits*)
done

A structured proof where one can see what is used in each case.

lemma *inv_del2*: ⟦ *invh t*; *invc t* ⟧ $\Longrightarrow$
invh (*del x t*) $\wedge$
(*color t* = *Red* $\longrightarrow$ *bheight* (*del x t*) = *bheight t* $\wedge$ *invc* (*del x t*)) $\wedge$
(*color t* = *Black* $\longrightarrow$ *bheight* (*del x t*) = *bheight t* − *1* $\wedge$ *invc2* (*del x t*))
proof(*induction x t rule*: *del.induct*)
case (*1 x*)
then show *?case* **by** *simp*
next
case (*2 x l a c r*)
note *if_split*[*split del*]
show *?case*
proof *cases*
assume *x* < *a*
show *?thesis*
proof *cases*
assume *l* = *Leaf* **thus** *?thesis* **using** ‹*x* < *a*› *2.prems* **by**(*auto*)
next
assume *l*: *l* ≠ *Leaf*
show *?thesis*
proof (*cases color l*)
assume ∗: *color l* = *Black*
hence *bheight l* > *0* **using** *l neq_LeafD*[*of l*] **by** *auto*
thus *?thesis* **using** ‹*x* < *a*› *2.IH*(*1*) *2.prems inv_baldL*[*of del x l*] ∗

```
l by(auto)
      next
        assume color l = Red
        thus ?thesis using ‹x < a› 2.prems 2.IH(1) by(auto)
      qed
    qed
  next
    assume ¬ x < a
    show ?thesis
    proof cases
      assume x > a
    show ?thesis using ‹a < x› 2.IH(2) 2.prems neq_LeafD[of r] inv_baldR[of
_ del x r]
          by(auto split: if_split)

    next
      assume ¬ x > a
      show ?thesis using 2.prems ‹¬ x < a› ‹¬ x > a›
       by(auto simp: inv_baldR invc2I dest!: inv_split_min dest: neq_LeafD
split: prod.split if_split)
    qed
  qed
qed

theorem rbt_delete: rbt t ⟹ rbt (delete x t)
by (metis delete_def rbt_def color_paint_Black inv_del invh_paint)
```

Overall correctness:

```
interpretation S: Set_by_Ordered
where empty = empty and isin = isin and insert = insert and delete =
delete
and inorder = inorder and inv = rbt
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: isin_set_inorder)
next
  case 3 thus ?case by(simp add: inorder_insert)
next
  case 4 thus ?case by(simp add: inorder_delete)
next
  case 5 thus ?case by (simp add: rbt_def empty_def)
next
  case 6 thus ?case by (simp add: rbt_insert)
```

next
case *7* **thus** *?case* **by** (*simp add*: *rbt_delete*)
qed

end

24 Red-Black Tree Implementation of Maps

theory *RBT_Map*
imports
RBT_Set
Lookup2
begin

fun *upd* :: *'a::linorder* ⇒ *'b* ⇒ (*'a***'b*) *rbt* ⇒ (*'a***'b*) *rbt* **where**
upd x y Leaf = *R Leaf* (*x*,*y*) *Leaf* |
upd x y (*B l* (*a*,*b*) *r*) = (*case cmp x a of*
LT ⇒ *baliL* (*upd x y l*) (*a*,*b*) *r* |
GT ⇒ *baliR l* (*a*,*b*) (*upd x y r*) |
EQ ⇒ *B l* (*x*,*y*) *r*) |
upd x y (*R l* (*a*,*b*) *r*) = (*case cmp x a of*
LT ⇒ *R* (*upd x y l*) (*a*,*b*) *r* |
GT ⇒ *R l* (*a*,*b*) (*upd x y r*) |
EQ ⇒ *R l* (*x*,*y*) *r*)

definition *update* :: *'a::linorder* ⇒ *'b* ⇒ (*'a***'b*) *rbt* ⇒ (*'a***'b*) *rbt* **where**
update x y t = *paint Black* (*upd x y t*)

fun *del* :: *'a::linorder* ⇒ (*'a***'b*)*rbt* ⇒ (*'a***'b*)*rbt* **where**
del x Leaf = *Leaf* |
del x (*Node l* (*ab*, _) *r*) = (*case cmp x* (*fst ab*) *of*
LT ⇒ *if l* ≠ *Leaf* ∧ *color l* = *Black*
then baldL (*del x l*) *ab r else R* (*del x l*) *ab r* |
GT ⇒ *if r* ≠ *Leaf*∧ *color r* = *Black*
then baldR l ab (*del x r*) *else R l ab* (*del x r*) |
EQ ⇒ *join l r*)

definition *delete* :: *'a::linorder* ⇒ (*'a***'b*) *rbt* ⇒ (*'a***'b*) *rbt* **where**
delete x t = *paint Black* (*del x t*)

24.1 Functional Correctness Proofs

lemma *inorder_upd*:

sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*upd x y t*) = *upd_list x y* (*inorder t*)
by(*induction x y t rule*: *upd.induct*)
(*auto simp*: *upd_list_simps inorder_baliL inorder_baliR*)

lemma *inorder_update*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*update x y t*) = *upd_list x y* (*inorder t*)
by(*simp add*: *update_def inorder_upd inorder_paint*)

lemma *del_list_id*: $\forall$ *ab*$\in$*set ps*. *y* < *fst ab* $\Longrightarrow$ *x* $\leq$ *y* $\Longrightarrow$ *del_list x ps* = *ps*
by(*rule del_list_idem*) *auto*

lemma *inorder_del*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*del x t*) = *del_list x* (*inorder t*)
by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps del_list_id inorder_join inorder_baldL inorder_baldR*)

lemma *inorder_delete*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del inorder_paint*)

24.2 Structural invariants

24.2.1 Update

lemma *invc_upd*: **assumes** *invc t*
shows *color t* = *Black* $\Longrightarrow$ *invc* (*upd x y t*) *invc2* (*upd x y t*)
using *assms*
by (*induct x y t rule*: *upd.induct*) (*auto simp*: *invc_baliL invc_baliR invc2I*)

lemma *invh_upd*: **assumes** *invh t*
shows *invh* (*upd x y t*) *bheight* (*upd x y t*) = *bheight t*
using *assms*
by(*induct x y t rule*: *upd.induct*)
(*auto simp*: *invh_baliL invh_baliR bheight_baliL bheight_baliR*)

theorem *rbt_update*: *rbt t* $\Longrightarrow$ *rbt* (*update x y t*)
by (*simp add*: *invc_upd*(*2*) *invh_upd*(*1*) *color_paint_Black invh_paint rbt_def update_def*)

24.2.2 Deletion

lemma *del_invc_invh*: *invh t* $\Longrightarrow$ *invc t* $\Longrightarrow$ *invh* (*del x t*) $\wedge$

```
   (color t = Red ∧ bheight (del x t) = bheight t ∧ invc (del x t) ∨
    color t = Black ∧ bheight (del x t) = bheight t − 1 ∧ invc2 (del x t))
proof (induct x t rule: del.induct)
case (2 x _ ab c)
  have x = fst ab ∨ x < fst ab ∨ x > fst ab by auto
  thus ?case proof (elim disjE)
    assume x = fst ab
    with 2 show ?thesis
    by (cases c) (simp_all add: invh_join invc_join)
  next
    assume x < fst ab
    with 2 show ?thesis
      by(cases c)
      (auto simp: invh_baldL_invc invc_baldL invc2_baldL dest: neq_LeafD)
  next
    assume fst ab < x
    with 2 show ?thesis
      by(cases c)
      (auto simp: invh_baldR_invc invc_baldR invc2_baldR dest: neq_LeafD)
  qed
qed auto

theorem rbt_delete: rbt t ⟹ rbt (delete k t)
by (metis delete_def rbt_def color_paint_Black del_invc_invh invc2I invh_paint)

interpretation M: Map_by_Ordered
where empty = empty and lookup = lookup and update = update and
delete = delete
and inorder = inorder and inv = rbt
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: lookup_map_of)
next
  case 3 thus ?case by(simp add: inorder_update)
next
  case 4 thus ?case by(simp add: inorder_delete)
next
  case 5 thus ?case by (simp add: rbt_def empty_def)
next
  case 6 thus ?case by (simp add: rbt_update)
next
  case 7 thus ?case by (simp add: rbt_delete)
qed
```

end

25 2-3 Trees

theory *Tree23*
imports *Main*
begin

class *height* =
fixes *height* :: *'a* ⇒ *nat*

datatype *'a tree23* =
  Leaf (‹⟨⟩›) |
  Node2 'a tree23 'a 'a tree23 (‹⟨_, _, _⟩›) |
  Node3 'a tree23 'a 'a tree23 'a 'a tree23 (‹⟨_, _, _, _, _⟩›)

fun *inorder* :: *'a tree23* ⇒ *'a list* **where**
inorder Leaf = [] |
inorder(*Node2 l a r*) = *inorder l* @ *a* # *inorder r* |
inorder(*Node3 l a m b r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder r*

instantiation *tree23* :: (*type*)*height*
begin

fun *height_tree23* :: *'a tree23* ⇒ *nat* **where**
height Leaf = *0* |
height (*Node2 l* _ *r*) = *Suc*(*max* (*height l*) (*height r*)) |
height (*Node3 l* _ *m* _ *r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*height r*)))

instance ..

end

Completeness:

fun *complete* :: *'a tree23* ⇒ *bool* **where**
complete Leaf = *True* |
complete (*Node2 l* _ *r*) = (*height l* = *height r* ∧ *complete l* & *complete r*) |
complete (*Node3 l* _ *m* _ *r*) =
  (*height l* = *height m* & *height m* = *height r* & *complete l* & *complete m* & *complete r*)

lemma *ht_sz_if_complete*: *complete t* $\Longrightarrow$ *2* $\hat{}$ *height t* $\leq$ *size t + 1*
by (*induction t*) *auto*

end

26 2-3 Tree Implementation of Sets

theory *Tree23_Set*
imports
Tree23
Cmp
Set_Specs
begin

declare *sorted_wrt.simps*(*2*)[*simp del*]

definition *empty* :: *'a tree23* **where**
empty = *Leaf*

fun *isin* :: *'a*::*linorder tree23* $\Rightarrow$ *'a* $\Rightarrow$ *bool* **where**
isin Leaf x = *False* |
isin (*Node2 l a r*) *x* =
(*case cmp x a of*
LT $\Rightarrow$ *isin l x* |
EQ $\Rightarrow$ *True* |
GT $\Rightarrow$ *isin r x*) |
isin (*Node3 l a m b r*) *x* =
(*case cmp x a of*
LT $\Rightarrow$ *isin l x* |
EQ $\Rightarrow$ *True* |
GT $\Rightarrow$
(*case cmp x b of*
LT $\Rightarrow$ *isin m x* |
EQ $\Rightarrow$ *True* |
GT $\Rightarrow$ *isin r x*))

datatype *'a up*$_i$ = *Eq*$_i$ *'a tree23* | *Of 'a tree23 'a 'a tree23*

fun *tree*$_i$ :: *'a up*$_i$ $\Rightarrow$ *'a tree23* **where**
tree$_i$ (*Eq*$_i$ *t*) = *t* |
tree$_i$ (*Of l a r*) = *Node2 l a r*

fun *ins* :: *'a*::*linorder* $\Rightarrow$ *'a tree23* $\Rightarrow$ *'a up*$_i$ **where**

```
ins x Leaf = Of Leaf x Leaf |
ins x (Node2 l a r) =
  (case cmp x a of
     LT ⇒
      (case ins x l of
         Eq_i l' => Eq_i (Node2 l' a r) |
         Of l1 b l2 => Eq_i (Node3 l1 b l2 a r)) |
     EQ ⇒ Eq_i (Node2 l a r) |
     GT ⇒
      (case ins x r of
         Eq_i r' => Eq_i (Node2 l a r') |
         Of r1 b r2 => Eq_i (Node3 l a r1 b r2))) |
ins x (Node3 l a m b r) =
  (case cmp x a of
     LT ⇒
      (case ins x l of
         Eq_i l' => Eq_i (Node3 l' a m b r) |
         Of l1 c l2 => Of (Node2 l1 c l2) a (Node2 m b r)) |
     EQ ⇒ Eq_i (Node3 l a m b r) |
     GT ⇒
      (case cmp x b of
         GT ⇒
          (case ins x r of
             Eq_i r' => Eq_i (Node3 l a m b r') |
             Of r1 c r2 => Of (Node2 l a m) b (Node2 r1 c r2)) |
         EQ ⇒ Eq_i (Node3 l a m b r) |
         LT ⇒
          (case ins x m of
             Eq_i m' => Eq_i (Node3 l a m' b r) |
             Of m1 c m2 => Of (Node2 l a m1) c (Node2 m2 b r))))
```

hide_const *insert*

definition *insert* :: *'a::linorder* ⇒ *'a tree23* ⇒ *'a tree23* **where**
insert x t = $tree_i$(*ins x t*)

datatype *'a* up_d = Eq_d *'a tree23* | *Uf 'a tree23*

fun $tree_d$ :: *'a* up_d ⇒ *'a tree23* **where**
$tree_d$ (Eq_d *t*) = *t* |
$tree_d$ (*Uf t*) = *t*

fun *node21* :: *'a up$_d$* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a up$_d$* **where**
node21 (*Eq$_d$ t1*) *a t2* = *Eq$_d$*(*Node2 t1 a t2*) |
node21 (*Uf t1*) *a* (*Node2 t2 b t3*) = *Uf*(*Node3 t1 a t2 b t3*) |
node21 (*Uf t1*) *a* (*Node3 t2 b t3 c t4*) = *Eq$_d$*(*Node2* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*))

fun *node22* :: *'a tree23* ⇒ *'a* ⇒ *'a up$_d$* ⇒ *'a up$_d$* **where**
node22 t1 a (*Eq$_d$ t2*) = *Eq$_d$*(*Node2 t1 a t2*) |
node22 (*Node2 t1 b t2*) *a* (*Uf t3*) = *Uf*(*Node3 t1 b t2 a t3*) |
node22 (*Node3 t1 b t2 c t3*) *a* (*Uf t4*) = *Eq$_d$*(*Node2* (*Node2 t1 b t2*) *c* (*Node2 t3 a t4*))

fun *node31* :: *'a up$_d$* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a up$_d$* **where**
node31 (*Eq$_d$ t1*) *a t2 b t3* = *Eq$_d$*(*Node3 t1 a t2 b t3*) |
node31 (*Uf t1*) *a* (*Node2 t2 b t3*) *c t4* = *Eq$_d$*(*Node2* (*Node3 t1 a t2 b t3*) *c t4*) |
node31 (*Uf t1*) *a* (*Node3 t2 b t3 c t4*) *d t5* = *Eq$_d$*(*Node3* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*) *d t5*)

fun *node32* :: *'a tree23* ⇒ *'a* ⇒ *'a up$_d$* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a up$_d$* **where**
node32 t1 a (*Eq$_d$ t2*) *b t3* = *Eq$_d$*(*Node3 t1 a t2 b t3*) |
node32 t1 a (*Uf t2*) *b* (*Node2 t3 c t4*) = *Eq$_d$*(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node32 t1 a (*Uf t2*) *b* (*Node3 t3 c t4 d t5*) = *Eq$_d$*(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*))

fun *node33* :: *'a tree23* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a* ⇒ *'a up$_d$* ⇒ *'a up$_d$* **where**
node33 t1 a t2 b (*Eq$_d$ t3*) = *Eq$_d$*(*Node3 t1 a t2 b t3*) |
node33 t1 a (*Node2 t2 b t3*) *c* (*Uf t4*) = *Eq$_d$*(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node33 t1 a (*Node3 t2 b t3 c t4*) *d* (*Uf t5*) = *Eq$_d$*(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*))

fun *split_min* :: *'a tree23* ⇒ *'a* ∗ *'a up$_d$* **where**
split_min (*Node2 Leaf a Leaf*) = (*a*, *Uf Leaf*) |
split_min (*Node3 Leaf a Leaf b Leaf*) = (*a*, *Eq$_d$*(*Node2 Leaf b Leaf*)) |
split_min (*Node2 l a r*) = (*let* (*x*,*l'*) = *split_min l in* (*x*, *node21 l' a r*)) |
split_min (*Node3 l a m b r*) = (*let* (*x*,*l'*) = *split_min l in* (*x*, *node31 l' a m b r*))

In the base cases of *split_min* and *del* it is enough to check if one subtree is a *Leaf*, in which case completeness implies that so are the others. Exercise.

fun *del* :: *'a*::*linorder* ⇒ *'a tree23* ⇒ *'a up$_d$* **where**
del x Leaf = *Eq$_d$ Leaf* |

```
del x (Node2 Leaf a Leaf) =
  (if x = a then Uf Leaf else Eq_d(Node2 Leaf a Leaf)) |
del x (Node3 Leaf a Leaf b Leaf) =
  Eq_d(if x = a then Node2 Leaf b Leaf else
     if x = b then Node2 Leaf a Leaf
     else Node3 Leaf a Leaf b Leaf) |
del x (Node2 l a r) =
  (case cmp x a of
     LT ⇒ node21 (del x l) a r |
     GT ⇒ node22 l a (del x r) |
     EQ ⇒ let (a',r') = split_min r in node22 l a' r') |
del x (Node3 l a m b r) =
  (case cmp x a of
     LT ⇒ node31 (del x l) a m b r |
     EQ ⇒ let (a',m') = split_min m in node32 l a' m' b r |
     GT ⇒
       (case cmp x b of
          LT ⇒ node32 l a (del x m) b r |
          EQ ⇒ let (b',r') = split_min r in node33 l a m b' r' |
          GT ⇒ node33 l a m b (del x r)))
```

definition *delete* :: *'a::linorder* ⇒ *'a tree23* ⇒ *'a tree23* **where**
delete x t = *tree_d*(*del x t*)

26.1 Functional Correctness

26.1.1 Proofs for isin

lemma *isin_set*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set* (*inorder t*))
by (*induction t*) (*auto simp*: *isin_simps*)

26.1.2 Proofs for insert

lemma *inorder_ins*:
  sorted(*inorder t*) ⟹ *inorder*(*tree_i*(*ins x t*)) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps split*: *up_i.splits*)

lemma *inorder_insert*:
  sorted(*inorder t*) ⟹ *inorder*(*insert a t*) = *ins_list a* (*inorder t*)
by(*simp add*: *insert_def inorder_ins*)

26.1.3 Proofs for delete

lemma *inorder_node21*: *height r* > *0* ⟹
  inorder (*tree_d* (*node21 l' a r*)) = *inorder* (*tree_d l'*) @ *a* # *inorder r*

by(*induct* l' *a r rule*: *node21.induct*) *auto*

lemma *inorder_node22*: *height* $l > 0 \Longrightarrow$
inorder ($tree_d$ (*node22 l a* r')) = *inorder l* @ *a* # *inorder* ($tree_d$ r')
by(*induct l a* r' *rule*: *node22.induct*) *auto*

lemma *inorder_node31*: *height m* > *0* $\Longrightarrow$
inorder ($tree_d$ (*node31* l' *a m b r*)) = *inorder* ($tree_d$ l') @ *a* # *inorder m*
@ *b* # *inorder r*
by(*induct* l' *a m b r rule*: *node31.induct*) *auto*

lemma *inorder_node32*: *height r* > *0* $\Longrightarrow$
inorder ($tree_d$ (*node32 l a* m' *b r*)) = *inorder l* @ *a* # *inorder* ($tree_d$ m')
@ *b* # *inorder r*
by(*induct l a* m' *b r rule*: *node32.induct*) *auto*

lemma *inorder_node33*: *height m* > *0* $\Longrightarrow$
inorder ($tree_d$ (*node33 l a m b* r')) = *inorder l* @ *a* # *inorder m* @ *b* #
inorder ($tree_d$ r')
by(*induct l a m b* r' *rule*: *node33.induct*) *auto*

lemmas *inorder_nodes* = *inorder_node21 inorder_node22*
inorder_node31 inorder_node32 inorder_node33

lemma *split_minD*:
split_min t = (*x*,t') $\Longrightarrow$ *complete t* $\Longrightarrow$ *height t* > *0* $\Longrightarrow$
x # *inorder*($tree_d$ t') = *inorder t*
by(*induction t arbitrary*: t' *rule*: *split_min.induct*)
(*auto simp*: *inorder_nodes split*: *prod.splits*)

lemma *inorder_del*: ⟦ *complete t* ; *sorted*(*inorder t*) ⟧ $\Longrightarrow$
inorder($tree_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_split prod.splits*)

lemma *inorder_delete*: ⟦ *complete t* ; *sorted*(*inorder t*) ⟧ $\Longrightarrow$
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

26.2 Completeness

26.2.1 Proofs for insert

First a standard proof that *ins* preserves *complete*.

fun $h_i :: 'a\ up_i \Rightarrow nat$ **where**
$h_i\ (Eq_i\ t) = height\ t$ |
$h_i\ (Of\ l\ a\ r) = height\ l$

lemma *complete_ins*: $complete\ t \Longrightarrow complete\ (tree_i(ins\ a\ t)) \wedge h_i(ins\ a\ t) = height\ t$
by (*induct t*) (*auto split*!: *if_split up*$_i$*.split*)

Now an alternative proof (by Brian Huffman) that runs faster because two properties (completeness and height) are combined in one predicate.

inductive $full :: nat \Rightarrow 'a\ tree23 \Rightarrow bool$ **where**
full 0 Leaf |
$[\![full\ n\ l; full\ n\ r]\!] \Longrightarrow full\ (Suc\ n)\ (Node2\ l\ p\ r)$ |
$[\![full\ n\ l; full\ n\ m; full\ n\ r]\!] \Longrightarrow full\ (Suc\ n)\ (Node3\ l\ p\ m\ q\ r)$

inductive_cases *full_elims*:
full n Leaf
full n (Node2 l p r)
full n (Node3 l p m q r)

inductive_cases *full_0_elim*: *full 0 t*
inductive_cases *full_Suc_elim*: *full (Suc n) t*

lemma *full_0_iff* [*simp*]: $full\ 0\ t \longleftrightarrow t = Leaf$
by (*auto elim*: *full_0_elim intro*: *full.intros*)

lemma *full_Leaf_iff* [*simp*]: $full\ n\ Leaf \longleftrightarrow n = 0$
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node2_iff* [*simp*]:
$full\ (Suc\ n)\ (Node2\ l\ p\ r) \longleftrightarrow full\ n\ l \wedge full\ n\ r$
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node3_iff* [*simp*]:
$full\ (Suc\ n)\ (Node3\ l\ p\ m\ q\ r) \longleftrightarrow full\ n\ l \wedge full\ n\ m \wedge full\ n\ r$
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_imp_height*: $full\ n\ t \Longrightarrow height\ t = n$
by (*induct set*: *full*, *simp_all*)

lemma *full_imp_complete*: $full\ n\ t \Longrightarrow complete\ t$
by (*induct set*: *full*, *auto dest*: *full_imp_height*)

lemma *complete_imp_full*: $complete\ t \Longrightarrow full\ (height\ t)\ t$

by (*induct t*, *simp_all*)

lemma *complete_iff_full*: *complete t* $\longleftrightarrow$ ($\exists$ *n*. *full n t*)
by (*auto elim*!: *complete_imp_full full_imp_complete*)

The *insert* function either preserves the height of the tree, or increases it by one. The constructor returned by the *insert* function determines which: A return value of the form Eq_i *t* indicates that the height will be the same. A value of the form *Of l p r* indicates an increase in height.

fun $full_i$ :: *nat* $\Rightarrow$ *'a* up_i $\Rightarrow$ *bool* **where**
$full_i$ *n* (Eq_i *t*) $\longleftrightarrow$ *full n t* |
$full_i$ *n* (*Of l p r*) $\longleftrightarrow$ *full n l* $\wedge$ *full n r*

lemma $full_i$*_ins*: *full n t* $\Longrightarrow$ $full_i$ *n* (*ins a t*)
by (*induct rule*: *full.induct*) (*auto split*: up_i*.split*)

The *insert* operation preserves completeance.

lemma *complete_insert*: *complete t* $\Longrightarrow$ *complete* (*insert a t*)
unfolding *complete_iff_full insert_def*
apply (*erule exE*)
apply (*drule* $full_i$*_ins* [*of* _ _ *a*])
apply (*cases ins a t*)
apply (*auto intro*: *full.intros*)
done

26.3 Proofs for delete

fun h_d :: *'a* up_d $\Rightarrow$ *nat* **where**
h_d (Eq_d *t*) = *height t* |
h_d (*Uf t*) = *height t* + *1*

lemma *complete_tree*$_d$*_node21*:
⟦*complete r*; *complete* ($tree_d$ *l'*); *height r* = h_d *l'* ⟧ $\Longrightarrow$ *complete* ($tree_d$ (*node21 l' a r*))
by(*induct l' a r rule*: *node21.induct*) *auto*

lemma *complete_tree*$_d$*_node22*:
⟦*complete*($tree_d$ *r'*); *complete l*; h_d *r'* = *height l* ⟧ $\Longrightarrow$ *complete* ($tree_d$ (*node22 l a r'*))
by(*induct l a r' rule*: *node22.induct*) *auto*

lemma *complete_tree*$_d$*_node31*:
⟦ *complete* ($tree_d$ *l'*); *complete m*; *complete r*; h_d *l'* = *height r*; *height m* = *height r* ⟧

$\Longrightarrow$ *complete* ($tree_d$ (*node31* l' *a m b r*))
by(*induct* l' *a m b r rule*: *node31.induct*) *auto*

lemma *complete_tree*$_d$*_node32*:
$[\![$ *complete l*; *complete* ($tree_d$ m'); *complete r*; *height l* = *height r*; h_d m' = *height r* $]\!]$
$\Longrightarrow$ *complete* ($tree_d$ (*node32 l a* m' *b r*))
by(*induct l a* m' *b r rule*: *node32.induct*) *auto*

lemma *complete_tree*$_d$*_node33*:
$[\![$ *complete l*; *complete m*; *complete*($tree_d$ r'); *height l* = h_d r'; *height m* = h_d r' $]\!]$
$\Longrightarrow$ *complete* ($tree_d$ (*node33 l a m b* r'))
by(*induct l a m b* r' *rule*: *node33.induct*) *auto*

lemmas *completes* = *complete_tree*$_d$*_node21 complete_tree*$_d$*_node22*
complete_tree$_d$*_node31 complete_tree*$_d$*_node32 complete_tree*$_d$*_node33*

lemma *height'_node21*:
height r > *0* $\Longrightarrow$ h_d(*node21* l' *a r*) = *max* (h_d l') (*height r*) + *1*
by(*induct* l' *a r rule*: *node21.induct*)(*simp_all*)

lemma *height'_node22*:
height l > *0* $\Longrightarrow$ h_d(*node22 l a* r') = *max* (*height l*) (h_d r') + *1*
by(*induct l a* r' *rule*: *node22.induct*)(*simp_all*)

lemma *height'_node31*:
height m > *0* $\Longrightarrow$ h_d(*node31 l a m b r*) =
max (h_d *l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node31.induct*)(*simp_all add*: *max_def*)

lemma *height'_node32*:
height r > *0* $\Longrightarrow$ h_d(*node32 l a m b r*) =
max (*height l*) (*max* (h_d *m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node32.induct*)(*simp_all add*: *max_def*)

lemma *height'_node33*:
height m > *0* $\Longrightarrow$ h_d(*node33 l a m b r*) =
max (*height l*) (*max* (*height m*) (h_d *r*)) + *1*
by(*induct l a m b r rule*: *node33.induct*)(*simp_all add*: *max_def*)

lemmas *heights* = *height'_node21 height'_node22*
height'_node31 height'_node32 height'_node33

lemma *height_split_min*:
split_min t = (*x*, *t'*) ⟹ *height t* > *0* ⟹ *complete t* ⟹ $h_d\ t' = height\ t$
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
(*auto simp*: *heights split*: *prod.splits*)

lemma *height_del*: *complete t* ⟹ $h_d(del\ x\ t) = height\ t$
by(*induction x t rule*: *del.induct*)
(*auto simp*: *heights max_def height_split_min split*: *prod.splits*)

lemma *complete_split_min*:
⟦ *split_min t* = (*x*, *t'*); *complete t*; *height t* > *0* ⟧ ⟹ $complete\ (tree_d\ t')$
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
(*auto simp*: *heights height_split_min completes split*: *prod.splits*)

lemma $complete\_tree_d\_del$: *complete t* ⟹ $complete(tree_d(del\ x\ t))$
by(*induction x t rule*: *del.induct*)
(*auto simp*: *completes complete_split_min height_del height_split_min split*: *prod.splits*)

corollary *complete_delete*: *complete t* ⟹ *complete*(*delete x t*)
by(*simp add*: *delete_def* $complete\_tree_d\_del$)

26.4 Overall Correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *complete*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *6* **thus** *?case* **by**(*simp add*: *complete_insert*)
next
case *7* **thus** *?case* **by**(*simp add*: *complete_delete*)
qed (*simp add*: *empty_def*)+

end

27 2-3 Tree Implementation of Maps

theory *Tree23_Map*
imports
Tree23_Set
Map_Specs
begin

fun *lookup* :: (*'a::linorder* ∗ *'b*) *tree23* ⇒ *'a* ⇒ *'b option* **where**
lookup Leaf x = *None* |
lookup (*Node2 l* (*a*,*b*) *r*) *x* = (*case cmp x a of*
LT ⇒ *lookup l x* |
GT ⇒ *lookup r x* |
EQ ⇒ *Some b*) |
lookup (*Node3 l* (*a1*,*b1*) *m* (*a2*,*b2*) *r*) *x* = (*case cmp x a1 of*
LT ⇒ *lookup l x* |
EQ ⇒ *Some b1* |
GT ⇒ (*case cmp x a2 of*
LT ⇒ *lookup m x* |
EQ ⇒ *Some b2* |
GT ⇒ *lookup r x*))

fun *upd* :: *'a::linorder* ⇒ *'b* ⇒ (*'a*∗*'b*) *tree23* ⇒ (*'a*∗*'b*) *up*$_i$ **where**
upd x y Leaf = *Of Leaf* (*x*,*y*) *Leaf* |
upd x y (*Node2 l ab r*) = (*case cmp x* (*fst ab*) *of*
LT ⇒ (*case upd x y l of*
Eq$_i$ *l'* => *Eq*$_i$ (*Node2 l' ab r*)
| *Of l1 ab' l2* => *Eq*$_i$ (*Node3 l1 ab' l2 ab r*)) |
EQ ⇒ *Eq*$_i$ (*Node2 l* (*x*,*y*) *r*) |
GT ⇒ (*case upd x y r of*
Eq$_i$ *r'* => *Eq*$_i$ (*Node2 l ab r'*)
| *Of r1 ab' r2* => *Eq*$_i$ (*Node3 l ab r1 ab' r2*))) |
upd x y (*Node3 l ab1 m ab2 r*) = (*case cmp x* (*fst ab1*) *of*
LT ⇒ (*case upd x y l of*
Eq$_i$ *l'* => *Eq*$_i$ (*Node3 l' ab1 m ab2 r*)
| *Of l1 ab' l2* => *Of* (*Node2 l1 ab' l2*) *ab1* (*Node2 m ab2 r*)) |
EQ ⇒ *Eq*$_i$ (*Node3 l* (*x*,*y*) *m ab2 r*) |
GT ⇒ (*case cmp x* (*fst ab2*) *of*
LT ⇒ (*case upd x y m of*
Eq$_i$ *m'* => *Eq*$_i$ (*Node3 l ab1 m' ab2 r*)
| *Of m1 ab' m2* => *Of* (*Node2 l ab1 m1*) *ab'* (*Node2 m2 ab2*
r)) |
EQ ⇒ *Eq*$_i$ (*Node3 l ab1 m* (*x*,*y*) *r*) |
GT ⇒ (*case upd x y r of*

```
                Eq_i r' => Eq_i (Node3 l ab1 m ab2 r')
                | Of r1 ab' r2 => Of (Node2 l ab1 m) ab2 (Node2 r1 ab' r2))))

definition update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree23 ⇒ ('a*'b) tree23
where
update a b t = tree_i(upd a b t)

fun del :: 'a::linorder ⇒ ('a*'b) tree23 ⇒ ('a*'b) up_d where
del x Leaf = Eq_d Leaf |
del x (Node2 Leaf ab1 Leaf) = (if x=fst ab1 then Uf Leaf else Eq_d(Node2 Leaf ab1 Leaf)) |
del x (Node3 Leaf ab1 Leaf ab2 Leaf) = Eq_d(if x=fst ab1 then Node2 Leaf ab2 Leaf
  else if x=fst ab2 then Node2 Leaf ab1 Leaf else Node3 Leaf ab1 Leaf ab2 Leaf) |
del x (Node2 l ab1 r) = (case cmp x (fst ab1) of
  LT ⇒ node21 (del x l) ab1 r |
  GT ⇒ node22 l ab1 (del x r) |
  EQ ⇒ let (ab1',t) = split_min r in node22 l ab1' t) |
del x (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of
  LT ⇒ node31 (del x l) ab1 m ab2 r |
  EQ ⇒ let (ab1',m') = split_min m in node32 l ab1' m' ab2 r |
  GT ⇒ (case cmp x (fst ab2) of
        LT ⇒ node32 l ab1 (del x m) ab2 r |
        EQ ⇒ let (ab2',r') = split_min r in node33 l ab1 m ab2' r' |
        GT ⇒ node33 l ab1 m ab2 (del x r)))

definition delete :: 'a::linorder ⇒ ('a*'b) tree23 ⇒ ('a*'b) tree23 where
delete x t = tree_d(del x t)
```

27.1 Functional Correctness

```
lemma lookup_map_of:
  sorted1(inorder t) ⟹ lookup t x = map_of (inorder t) x
by (induction t) (auto simp: map_of_simps split: option.split)


lemma inorder_upd:
  sorted1(inorder t) ⟹ inorder(tree_i(upd x y t)) = upd_list x y (inorder t)
by(induction t) (auto simp: upd_list_simps split: up_i.splits)

corollary inorder_update:
  sorted1(inorder t) ⟹ inorder(update x y t) = upd_list x y (inorder t)
```

by(*simp add*: *update_def inorder_upd*)

lemma *inorder_del*: ⟦ *complete t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder(*tree*$_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_split prod.splits*)

corollary *inorder_delete*: ⟦ *complete t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

27.2 Balancedness

lemma *complete_upd*: *complete t* ⟹ *complete* (*tree*$_i$(*upd x y t*)) ∧ *h*$_i$(*upd x y t*) = *height t*
by (*induct t*) (*auto split*!: *if_split up*$_i$.*split*)

corollary *complete_update*: *complete t* ⟹ *complete* (*update x y t*)
by (*simp add*: *update_def complete_upd*)

lemma *height_del*: *complete t* ⟹ *h*$_d$(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
(*auto simp add*: *heights max_def height_split_min split*: *prod.split*)

lemma *complete_tree*$_d$*_del*: *complete t* ⟹ *complete*(*tree*$_d$(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *completes complete_split_min height_del height_split_min split*: *prod.split*)

corollary *complete_delete*: *complete t* ⟹ *complete*(*delete x t*)
by(*simp add*: *delete_def complete_tree*$_d$*_del*)

27.3 Overall Correctness

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *complete*
proof (*standard*, *goal_cases*)
case *1* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)

next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *complete_update*)
next
case *7* **thus** *?case* **by**(*simp add*: *complete_delete*)
qed

end

28 2-3 Tree from List

theory *Tree23_of_List*
imports
Tree23
HOL−Library.Time_Commands
begin

Linear-time bottom up conversion of a list of items into a complete 2-3 tree whose inorder traversal yields the list of items.

28.1 Code

Nonempty lists of 2-3 trees alternating with items, starting and ending with a 2-3 tree:

datatype *'a tree23s* = *T 'a tree23* | *TTs 'a tree23 'a 'a tree23s*

abbreviation *not_T ts* == ¬(∃ *t*. *ts* = *T t*)

fun *len* :: *'a tree23s* ⇒ *nat* **where**
len (*T* _) = *1* |
len (*TTs* _ _ *ts*) = *len ts* + *1*

fun *trees* :: *'a tree23s* ⇒ *'a tree23 set* **where**
trees (*T t*) = {*t*} |
trees (*TTs t a ts*) = {*t*} ∪ *trees ts*

Join pairs of adjacent trees:

fun *join_adj* :: *'a tree23s* ⇒ *'a tree23s* **where**
join_adj (*TTs t1 a* (*T t2*)) = *T*(*Node2 t1 a t2*) |

join_adj (TTs t1 a (TTs t2 b (T t3))) = T(Node3 t1 a t2 b t3) |
join_adj (TTs t1 a (TTs t2 b ts)) = TTs (Node2 t1 a t2) b (join_adj ts)

Towards termination of *join_all*:

lemma *len_ge2*:
not_T ts ⟹ len ts ≥ 2
by(*cases ts rule*: *join_adj.cases*) *auto*

lemma [*measure_function*]: *is_measure len*
by(*rule is_measure_trivial*)

lemma *len_join_adj_div2*:
not_T ts ⟹ len(*join_adj ts*) ≤ *len ts div 2*
by(*induction ts rule*: *join_adj.induct*) *auto*

lemma *len_join_adj1*: *not_T ts ⟹ len*(*join_adj ts*) < *len ts*
using *len_join_adj_div2*[*of ts*] *len_ge2*[*of ts*] **by** *simp*

corollary *len_join_adj2*[*termination_simp*]: *len*(*join_adj (TTs t a ts*)) ≤ *len ts*
using *len_join_adj1*[*of TTs t a ts*] **by** *simp*

fun *join_all* :: *'a tree23s ⇒ 'a tree23* **where**
join_all (T t) = t |
join_all ts = join_all (*join_adj ts*)

fun *leaves* :: *'a list ⇒ 'a tree23s* **where**
leaves [] = *T Leaf |*
leaves (*a # as*) = *TTs Leaf a* (*leaves as*)

definition *tree23_of_list* :: *'a list ⇒ 'a tree23* **where**
tree23_of_list as = join_all(*leaves as*)

28.2 Functional correctness

28.2.1 *inorder*:

fun *inorder2* :: *'a tree23s ⇒ 'a list* **where**
inorder2 (*T t*) = *inorder t |*
inorder2 (*TTs t a ts*) = *inorder t* @ *a # inorder2 ts*

lemma *inorder2_join_adj*: *not_T ts ⟹ inorder2*(*join_adj ts*) = *inorder2 ts*
by (*induction ts rule*: *join_adj.induct*) *auto*

lemma *inorder_join_all*: *inorder* (*join_all ts*) = *inorder2 ts*
proof (*induction ts rule*: *join_all.induct*)
case *1* **thus** *?case* **by** *simp*
next
case (*2 t a ts*)
thus *?case* **using** *inorder2_join_adj*[*of TTs t a ts*]
by (*simp add*: *le_imp_less_Suc*)
qed

lemma *inorder2_leaves*: *inorder2*(*leaves as*) = *as*
by(*induction as*) *auto*

lemma *inorder*: *inorder*(*tree23_of_list as*) = *as*
by(*simp add*: *tree23_of_list_def inorder_join_all inorder2_leaves*)

28.2.2 Completeness:

lemma *complete_join_adj*:
∀ *t* ∈ *trees ts*. *complete t* ∧ *height t* = *n* ⟹ *not_T ts* ⟹
∀ *t* ∈ *trees* (*join_adj ts*). *complete t* ∧ *height t* = *Suc n*
by (*induction ts rule*: *join_adj.induct*) *auto*

lemma *complete_join_all*:
∀ *t* ∈ *trees ts*. *complete t* ∧ *height t* = *n* ⟹ *complete* (*join_all ts*)
proof (*induction ts arbitrary*: *n rule*: *join_all.induct*)
case *1* **thus** *?case* **by** *simp*
next
case (*2 t a ts*)
thus *?case*
apply *simp* **using** *complete_join_adj*[*of TTs t a ts n*, *simplified*] **by** *blast*
qed

lemma *complete_leaves*: *t* ∈ *trees* (*leaves as*) ⟹ *complete t* ∧ *height t* = *0*
by (*induction as*) *auto*

corollary *complete*: *complete*(*tree23_of_list as*)
by(*simp add*: *tree23_of_list_def complete_leaves complete_join_all*[*of _ 0*])

28.3 Linear running time

time_fun *join_adj*

time_fun *join_all*
time_fun *leaves*
time_fun *tree23_of_list*

lemma *T_join_adj*: *not_T ts* $\Longrightarrow$ *T_join_adj ts* $\leq$ *len ts div 2*
by(*induction ts rule*: *T_join_adj.induct*) *auto*

lemma *len_ge_1*: *len ts* $\geq$ *1*
by(*cases ts*) *auto*

lemma *T_join_all*: *T_join_all ts* $\leq$ *2* $*$ *len ts*
proof(*induction ts rule*: *join_all.induct*)
case *1* **thus** *?case* **by** *simp*
next
case (*2 t a ts*)
let *?ts* = *TTs t a ts*
have *T_join_all ?ts* = *T_join_adj ?ts* + *T_join_all* (*join_adj ?ts*) + *1*
by *simp*
also have ... $\leq$ *len ?ts div 2* + *T_join_all* (*join_adj ?ts*) + *1*
using *T_join_adj*[*of ?ts*] **by** *simp*
also have ... $\leq$ *len ?ts div 2* + *2* $*$ *len* (*join_adj ?ts*) + *1*
using *2.IH* **by** *simp*
also have ... $\leq$ *len ?ts div 2* + *2* $*$ (*len ?ts div 2*) + *1*
using *len_join_adj_div2*[*of ?ts*] **by** *simp*
also have ... $\leq$ *2* $*$ *len ?ts* **using** *len_ge_1*[*of ?ts*] **by** *linarith*
finally show *?case* **.**
qed

lemma *T_leaves*: *T_leaves as* = *length as* + *1*
by(*induction as*) *auto*

lemma *len_leaves*: *len*(*leaves as*) = *length as* + *1*
by(*induction as*) *auto*

lemma *T_tree23_of_list*: *T_tree23_of_list as* $\leq$ *3*$*$(*length as*) + *3*
using *T_join_all*[*of leaves as*] **by**(*simp add*: *T_leaves len_leaves*)

end

29 2-3-4 Trees

theory *Tree234*

imports *Main*
begin

class *height* =
fixes *height* :: *'a* ⇒ *nat*

datatype *'a tree234* =
Leaf (‹⟨⟩›) |
Node2 'a tree234 'a 'a tree234 (‹⟨_, _, _⟩›) |
Node3 'a tree234 'a 'a tree234 'a 'a tree234 (‹⟨_, _, _, _, _⟩›) |
Node4 'a tree234 'a 'a tree234 'a 'a tree234 'a 'a tree234
(‹⟨_, _, _, _, _, _, _⟩›)

fun *inorder* :: *'a tree234* ⇒ *'a list* **where**
inorder Leaf = [] |
inorder(*Node2 l a r*) = *inorder l* @ *a* # *inorder r* |
inorder(*Node3 l a m b r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder r* |
inorder(*Node4 l a m b n c r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder r*

instantiation *tree234* :: (*type*)*height*
begin

fun *height_tree234* :: *'a tree234* ⇒ *nat* **where**
height Leaf = *0* |
height (*Node2 l* _ *r*) = *Suc*(*max* (*height l*) (*height r*)) |
height (*Node3 l* _ *m* _ *r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*height r*))) |
height (*Node4 l* _ *m* _ *n* _ *r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))))

instance ..

end

Balanced:

fun *bal* :: *'a tree234* ⇒ *bool* **where**
bal Leaf = *True* |
bal (*Node2 l* _ *r*) = (*bal l* & *bal r* & *height l* = *height r*) |
bal (*Node3 l* _ *m* _ *r*) = (*bal l* & *bal m* & *bal r* & *height l* = *height m* & *height m* = *height r*) |
bal (*Node4 l* _ *m* _ *n* _ *r*) = (*bal l* & *bal m* & *bal n* & *bal r* & *height l* = *height m* & *height m* = *height n* & *height n* = *height r*)

end

30 2-3-4 Tree Implementation of Sets

theory *Tree234_Set*
imports
Tree234
Cmp
Set_Specs
begin

declare *sorted_wrt.simps*(*2*)[*simp del*]

30.1 Set operations on 2-3-4 trees

definition *empty* :: *'a tree234* **where**
empty = *Leaf*

fun *isin* :: *'a*::*linorder tree234* ⇒ *'a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node2 l a r*) *x* =
(*case cmp x a of LT* ⇒ *isin l x* | *EQ* ⇒ *True* | *GT* ⇒ *isin r x*) |
isin (*Node3 l a m b r*) *x* =
(*case cmp x a of LT* ⇒ *isin l x* | *EQ* ⇒ *True* | *GT* ⇒ (*case cmp x b of*
LT ⇒ *isin m x* | *EQ* ⇒ *True* | *GT* ⇒ *isin r x*)) |
isin (*Node4 t1 a t2 b t3 c t4*) *x* =
(*case cmp x b of*
LT ⇒
(*case cmp x a of*
LT ⇒ *isin t1 x* |
EQ ⇒ *True* |
GT ⇒ *isin t2 x*) |
EQ ⇒ *True* |
GT ⇒
(*case cmp x c of*
LT ⇒ *isin t3 x* |
EQ ⇒ *True* |
GT ⇒ *isin t4 x*))

datatype *'a up*$_i$ = *T*$_i$ *'a tree234* | *Up*$_i$ *'a tree234 'a 'a tree234*

fun *tree*$_i$ :: *'a up*$_i$ ⇒ *'a tree234* **where**
tree$_i$ (*T*$_i$ *t*) = *t* |

```
tree_i (Up_i l a r) = Node2 l a r

fun ins :: 'a::linorder ⇒ 'a tree234 ⇒ 'a up_i where
ins x Leaf = Up_i Leaf x Leaf |
ins x (Node2 l a r) =
  (case cmp x a of
    LT ⇒ (case ins x l of
           T_i l' => T_i (Node2 l' a r)
         | Up_i l1 b l2 => T_i (Node3 l1 b l2 a r)) |
    EQ ⇒ T_i (Node2 l x r) |
    GT ⇒ (case ins x r of
           T_i r' => T_i (Node2 l a r')
         | Up_i r1 b r2 => T_i (Node3 l a r1 b r2))) |
ins x (Node3 l a m b r) =
  (case cmp x a of
    LT ⇒ (case ins x l of
           T_i l' => T_i (Node3 l' a m b r)
         | Up_i l1 c l2 => Up_i (Node2 l1 c l2) a (Node2 m b r)) |
    EQ ⇒ T_i (Node3 l a m b r) |
    GT ⇒ (case cmp x b of
           GT ⇒ (case ins x r of
                  T_i r' => T_i (Node3 l a m b r')
                | Up_i r1 c r2 => Up_i (Node2 l a m) b (Node2 r1 c r2)) |
           EQ ⇒ T_i (Node3 l a m b r) |
           LT ⇒ (case ins x m of
                  T_i m' => T_i (Node3 l a m' b r)
                | Up_i m1 c m2 => Up_i (Node2 l a m1) c (Node2 m2 b
r)))) |
ins x (Node4 t1 a t2 b t3 c t4) =
 (case cmp x b of
   LT ⇒
    (case cmp x a of
      LT ⇒
       (case ins x t1 of
          T_i t => T_i (Node4 t a t2 b t3 c t4) |
          Up_i l y r => Up_i (Node2 l y r) a (Node3 t2 b t3 c t4)) |
      EQ ⇒ T_i (Node4 t1 a t2 b t3 c t4) |
      GT ⇒
       (case ins x t2 of
          T_i t => T_i (Node4 t1 a t b t3 c t4) |
          Up_i l y r => Up_i (Node2 t1 a l) y (Node3 r b t3 c t4))) |
   EQ ⇒ T_i (Node4 t1 a t2 b t3 c t4) |
   GT ⇒
    (case cmp x c of
```

```
    LT ⇒
      (case ins x t3 of
        T_i t => T_i (Node4 t1 a t2 b t c t4) |
        Up_i l y r => Up_i (Node2 t1 a t2) b (Node3 l y r c t4)) |
    EQ ⇒ T_i (Node4 t1 a t2 b t3 c t4) |
    GT ⇒
      (case ins x t4 of
        T_i t => T_i (Node4 t1 a t2 b t3 c t) |
        Up_i l y r => Up_i (Node2 t1 a t2) b (Node3 t3 c l y r))))
```

hide_const *insert*

definition *insert* :: *'a::linorder ⇒ 'a tree234 ⇒ 'a tree234* **where**
insert x t = tree_i(ins x t)

datatype *'a up_d = T_d 'a tree234 | Up_d 'a tree234*

fun *tree_d* :: *'a up_d ⇒ 'a tree234* **where**
tree_d (T_d t) = t |
tree_d (Up_d t) = t

fun *node21* :: *'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d* **where**
node21 (T_d l) a r = T_d(Node2 l a r) |
node21 (Up_d l) a (Node2 lr b rr) = Up_d(Node3 l a lr b rr) |
node21 (Up_d l) a (Node3 lr b mr c rr) = T_d(Node2 (Node2 l a lr) b (Node2 mr c rr)) |
node21 (Up_d t1) a (Node4 t2 b t3 c t4 d t5) = T_d(Node2 (Node2 t1 a t2) b (Node3 t3 c t4 d t5))

fun *node22* :: *'a tree234 ⇒ 'a ⇒ 'a up_d ⇒ 'a up_d* **where**
node22 l a (T_d r) = T_d(Node2 l a r) |
node22 (Node2 ll b rl) a (Up_d r) = Up_d(Node3 ll b rl a r) |
node22 (Node3 ll b ml c rl) a (Up_d r) = T_d(Node2 (Node2 ll b ml) c (Node2 rl a r)) |
node22 (Node4 t1 a t2 b t3 c t4) d (Up_d t5) = T_d(Node2 (Node2 t1 a t2) b (Node3 t3 c t4 d t5))

fun *node31* :: *'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d* **where**
node31 (T_d t1) a t2 b t3 = T_d(Node3 t1 a t2 b t3) |
node31 (Up_d t1) a (Node2 t2 b t3) c t4 = T_d(Node2 (Node3 t1 a t2 b t3) c t4) |
node31 (Up_d t1) a (Node3 t2 b t3 c t4) d t5 = T_d(Node3 (Node2 t1 a t2) b (Node2 t3 c t4) d t5) |
node31 (Up_d t1) a (Node4 t2 b t3 c t4 d t5) e t6 = T_d(Node3 (Node2 t1 a

t2) *b* (*Node3 t3 c t4 d t5*) *e t6*)

fun *node32* :: *'a tree234* ⇒ *'a* ⇒ *'a up*$_d$ ⇒ *'a* ⇒ *'a tree234* ⇒ *'a up*$_d$ **where**
node32 t1 a (*T*$_d$ *t2*) *b t3* = *T*$_d$(*Node3 t1 a t2 b t3*) |
node32 t1 a (*Up*$_d$ *t2*) *b* (*Node2 t3 c t4*) = *T*$_d$(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node32 t1 a (*Up*$_d$ *t2*) *b* (*Node3 t3 c t4 d t5*) = *T*$_d$(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*)) |
node32 t1 a (*Up*$_d$ *t2*) *b* (*Node4 t3 c t4 d t5 e t6*) = *T*$_d$(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node3 t4 d t5 e t6*))

fun *node33* :: *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a up*$_d$ ⇒ *'a up*$_d$ **where**
node33 l a m b (*T*$_d$ *r*) = *T*$_d$(*Node3 l a m b r*) |
node33 t1 a (*Node2 t2 b t3*) *c* (*Up*$_d$ *t4*) = *T*$_d$(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node33 t1 a (*Node3 t2 b t3 c t4*) *d* (*Up*$_d$ *t5*) = *T*$_d$(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*)) |
node33 t1 a (*Node4 t2 b t3 c t4 d t5*) *e* (*Up*$_d$ *t6*) = *T*$_d$(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node3 t4 d t5 e t6*))

fun *node41* :: *'a up*$_d$ ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a up*$_d$ **where**
node41 (*T*$_d$ *t1*) *a t2 b t3 c t4* = *T*$_d$(*Node4 t1 a t2 b t3 c t4*) |
node41 (*Up*$_d$ *t1*) *a* (*Node2 t2 b t3*) *c t4 d t5* = *T*$_d$(*Node3* (*Node3 t1 a t2 b t3*) *c t4 d t5*) |
node41 (*Up*$_d$ *t1*) *a* (*Node3 t2 b t3 c t4*) *d t5 e t6* = *T*$_d$(*Node4* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*) *d t5 e t6*) |
node41 (*Up*$_d$ *t1*) *a* (*Node4 t2 b t3 c t4 d t5*) *e t6 f t7* = *T*$_d$(*Node4* (*Node2 t1 a t2*) *b* (*Node3 t3 c t4 d t5*) *e t6 f t7*)

fun *node42* :: *'a tree234* ⇒ *'a* ⇒ *'a up*$_d$ ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a up*$_d$ **where**
node42 t1 a (*T*$_d$ *t2*) *b t3 c t4* = *T*$_d$(*Node4 t1 a t2 b t3 c t4*) |
node42 (*Node2 t1 a t2*) *b* (*Up*$_d$ *t3*) *c t4 d t5* = *T*$_d$(*Node3* (*Node3 t1 a t2 b t3*) *c t4 d t5*) |
node42 (*Node3 t1 a t2 b t3*) *c* (*Up*$_d$ *t4*) *d t5 e t6* = *T*$_d$(*Node4* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*) *d t5 e t6*) |
node42 (*Node4 t1 a t2 b t3 c t4*) *d* (*Up*$_d$ *t5*) *e t6 f t7* = *T*$_d$(*Node4* (*Node2 t1 a t2*) *b* (*Node3 t3 c t4 d t5*) *e t6 f t7*)

fun *node43* :: *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a up*$_d$ ⇒ *'a* ⇒ *'a tree234* ⇒ *'a up*$_d$ **where**
node43 t1 a t2 b (*T*$_d$ *t3*) *c t4* = *T*$_d$(*Node4 t1 a t2 b t3 c t4*) |
node43 t1 a (*Node2 t2 b t3*) *c* (*Up*$_d$ *t4*) *d t5* = *T*$_d$(*Node3 t1 a* (*Node3 t2 b*

```
t3 c t4) d t5) |
node43 t1 a (Node3 t2 b t3 c t4) d (Up_d t5) e t6 = T_d(Node4 t1 a (Node2
t2 b t3) c (Node2 t4 d t5) e t6) |
node43 t1 a (Node4 t2 b t3 c t4 d t5) e (Up_d t6) f t7 = T_d(Node4 t1 a
(Node2 t2 b t3) c (Node3 t4 d t5 e t6) f t7)

fun node44 :: 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a
up_d ⇒ 'a up_d where
node44 t1 a t2 b t3 c (T_d t4) = T_d(Node4 t1 a t2 b t3 c t4) |
node44 t1 a t2 b (Node2 t3 c t4) d (Up_d t5) = T_d(Node3 t1 a t2 b (Node3
t3 c t4 d t5)) |
node44 t1 a t2 b (Node3 t3 c t4 d t5) e (Up_d t6) = T_d(Node4 t1 a t2 b
(Node2 t3 c t4) d (Node2 t5 e t6)) |
node44 t1 a t2 b (Node4 t3 c t4 d t5 e t6) f (Up_d t7) = T_d(Node4 t1 a t2
b (Node2 t3 c t4) d (Node3 t5 e t6 f t7))

fun split_min :: 'a tree234 ⇒ 'a * 'a up_d where
split_min (Node2 Leaf a Leaf) = (a, Up_d Leaf) |
split_min (Node3 Leaf a Leaf b Leaf) = (a, T_d(Node2 Leaf b Leaf)) |
split_min (Node4 Leaf a Leaf b Leaf c Leaf) = (a, T_d(Node3 Leaf b Leaf c
Leaf)) |
split_min (Node2 l a r) = (let (x,l') = split_min l in (x, node21 l' a r)) |
split_min (Node3 l a m b r) = (let (x,l') = split_min l in (x, node31 l' a
m b r)) |
split_min (Node4 l a m b n c r) = (let (x,l') = split_min l in (x, node41 l'
a m b n c r))

fun del :: 'a::linorder ⇒ 'a tree234 ⇒ 'a up_d where
del k Leaf = T_d Leaf |
del k (Node2 Leaf p Leaf) = (if k=p then Up_d Leaf else T_d(Node2 Leaf p
Leaf)) |
del k (Node3 Leaf p Leaf q Leaf) = T_d(if k=p then Node2 Leaf q Leaf
  else if k=q then Node2 Leaf p Leaf else Node3 Leaf p Leaf q Leaf) |
del k (Node4 Leaf a Leaf b Leaf c Leaf) =
  T_d(if k=a then Node3 Leaf b Leaf c Leaf else
    if k=b then Node3 Leaf a Leaf c Leaf else
    if k=c then Node3 Leaf a Leaf b Leaf
    else Node4 Leaf a Leaf b Leaf c Leaf) |
del k (Node2 l a r) = (case cmp k a of
  LT ⇒ node21 (del k l) a r |
  GT ⇒ node22 l a (del k r) |
  EQ ⇒ let (a',t) = split_min r in node22 l a' t) |
del k (Node3 l a m b r) = (case cmp k a of
  LT ⇒ node31 (del k l) a m b r |
```

```
  EQ ⇒ let (a',m') = split_min m in node32 l a' m' b r |
  GT ⇒ (case cmp k b of
         LT ⇒ node32 l a (del k m) b r |
         EQ ⇒ let (b',r') = split_min r in node33 l a m b' r' |
         GT ⇒ node33 l a m b (del k r))) |
del k (Node4 l a m b n c r) = (case cmp k b of
  LT ⇒ (case cmp k a of
         LT ⇒ node41 (del k l) a m b n c r |
         EQ ⇒ let (a',m') = split_min m in node42 l a' m' b n c r |
         GT ⇒ node42 l a (del k m) b n c r) |
  EQ ⇒ let (b',n') = split_min n in node43 l a m b' n' c r |
  GT ⇒ (case cmp k c of
         LT ⇒ node43 l a m b (del k n) c r |
         EQ ⇒ let (c',r') = split_min r in node44 l a m b n c' r' |
         GT ⇒ node44 l a m b n c (del k r)))
```

definition *delete* :: *'a::linorder ⇒ 'a tree234 ⇒ 'a tree234* **where**
delete x t = $tree_d$(*del x t*)

30.2 Functional correctness

30.2.1 Functional correctness of isin:

lemma *isin_set*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set* (*inorder t*))
by (*induction t*) (*auto simp*: *isin_simps*)

30.2.2 Functional correctness of insert:

lemma *inorder_ins*:
sorted(*inorder t*) ⟹ *inorder*($tree_i$(*ins x t*)) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto*, *auto simp*: *ins_list_simps split*!: *if_splits up*$_i$.*splits*)

lemma *inorder_insert*:
sorted(*inorder t*) ⟹ *inorder*(*insert a t*) = *ins_list a* (*inorder t*)
by(*simp add*: *insert_def inorder_ins*)

30.2.3 Functional correctness of delete

lemma *inorder_node21*: *height r* > *0* ⟹
inorder ($tree_d$ (*node21 l' a r*)) = *inorder* ($tree_d$ *l'*) @ *a* # *inorder r*
by(*induct l' a r rule*: *node21.induct*) *auto*

lemma *inorder_node22*: *height l* > *0* ⟹
inorder ($tree_d$ (*node22 l a r'*)) = *inorder l* @ *a* # *inorder* ($tree_d$ *r'*)
by(*induct l a r' rule*: *node22.induct*) *auto*

lemma *inorder_node31*: *height m > 0 ⟹*
inorder (*tree*$_d$ (*node31 l' a m b r*)) = *inorder* (*tree*$_d$ *l'*) @ *a* # *inorder m* @ *b* # *inorder r*
by(*induct l' a m b r rule*: *node31.induct*) *auto*

lemma *inorder_node32*: *height r > 0 ⟹*
inorder (*tree*$_d$ (*node32 l a m' b r*)) = *inorder l* @ *a* # *inorder* (*tree*$_d$ *m'*) @ *b* # *inorder r*
by(*induct l a m' b r rule*: *node32.induct*) *auto*

lemma *inorder_node33*: *height m > 0 ⟹*
inorder (*tree*$_d$ (*node33 l a m b r'*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder* (*tree*$_d$ *r'*)
by(*induct l a m b r' rule*: *node33.induct*) *auto*

lemma *inorder_node41*: *height m > 0 ⟹*
inorder (*tree*$_d$ (*node41 l' a m b n c r*)) = *inorder* (*tree*$_d$ *l'*) @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder r*
by(*induct l' a m b n c r rule*: *node41.induct*) *auto*

lemma *inorder_node42*: *height l > 0 ⟹*
inorder (*tree*$_d$ (*node42 l a m b n c r*)) = *inorder l* @ *a* # *inorder* (*tree*$_d$ *m*) @ *b* # *inorder n* @ *c* # *inorder r*
by(*induct l a m b n c r rule*: *node42.induct*) *auto*

lemma *inorder_node43*: *height m > 0 ⟹*
inorder (*tree*$_d$ (*node43 l a m b n c r*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder*(*tree*$_d$ *n*) @ *c* # *inorder r*
by(*induct l a m b n c r rule*: *node43.induct*) *auto*

lemma *inorder_node44*: *height n > 0 ⟹*
inorder (*tree*$_d$ (*node44 l a m b n c r*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder* (*tree*$_d$ *r*)
by(*induct l a m b n c r rule*: *node44.induct*) *auto*

lemmas *inorder_nodes* = *inorder_node21 inorder_node22*
inorder_node31 inorder_node32 inorder_node33
inorder_node41 inorder_node42 inorder_node43 inorder_node44

lemma *split_minD*:
split_min t = (*x*,*t'*) ⟹ *bal t* ⟹ *height t > 0 ⟹*
x # *inorder*(*tree*$_d$ *t'*) = *inorder t*
by(*induction t arbitrary*: *t' rule*: *split_min.induct*)

(*auto simp*: *inorder_nodes split*: *prod.splits*)

lemma *inorder_del*: ⟦ *bal t* ; *sorted*(*inorder t*) ⟧ ⟹
inorder(*tree*$_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *inorder_nodes del_list_simps split_minD split*!: *if_split prod.splits*)

lemma *inorder_delete*: ⟦ *bal t* ; *sorted*(*inorder t*) ⟧ ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

30.3 Balancedness

30.3.1 Proofs for insert

First a standard proof that *ins* preserves *bal*.

instantiation *up*$_i$ :: (*type*)*height*
begin

fun *height_up*$_i$ :: '*a up*$_i$ ⇒ *nat* **where**
height (*T*$_i$ *t*) = *height t* |
height (*Up*$_i$ *l a r*) = *height l*

instance ..

end

lemma *bal_ins*: *bal t* ⟹ *bal* (*tree*$_i$(*ins a t*)) ∧ *height*(*ins a t*) = *height t*
by (*induct t*) (*auto split*!: *if_split up*$_i$.*split*)

Now an alternative proof (by Brian Huffman) that runs faster because two properties (balance and height) are combined in one predicate.

inductive *full* :: *nat* ⇒ '*a tree234* ⇒ *bool* **where**
full 0 Leaf |
⟦*full n l*; *full n r*⟧ ⟹ *full* (*Suc n*) (*Node2 l p r*) |
⟦*full n l*; *full n m*; *full n r*⟧ ⟹ *full* (*Suc n*) (*Node3 l p m q r*) |
⟦*full n l*; *full n m*; *full n m'*; *full n r*⟧ ⟹ *full* (*Suc n*) (*Node4 l p m q m' q' r*)

inductive_cases *full_elims*:
full n Leaf
full n (*Node2 l p r*)
full n (*Node3 l p m q r*)

full n (Node4 l p m q m′ q′ r)

inductive_cases *full_0_elim*: *full 0 t*
inductive_cases *full_Suc_elim*: *full (Suc n) t*

lemma *full_0_iff* [*simp*]: *full 0 t* ⟷ *t = Leaf*
by (*auto elim*: *full_0_elim intro*: *full.intros*)

lemma *full_Leaf_iff* [*simp*]: *full n Leaf* ⟷ *n = 0*
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node2_iff* [*simp*]:
full (Suc n) (Node2 l p r) ⟷ *full n l* ∧ *full n r*
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node3_iff* [*simp*]:
full (Suc n) (Node3 l p m q r) ⟷ *full n l* ∧ *full n m* ∧ *full n r*
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node4_iff* [*simp*]:
full (Suc n) (Node4 l p m q m′ q′ r) ⟷ *full n l* ∧ *full n m* ∧ *full n m′* ∧ *full n r*
by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_imp_height*: *full n t* ⟹ *height t = n*
by (*induct set*: *full*, *simp_all*)

lemma *full_imp_bal*: *full n t* ⟹ *bal t*
by (*induct set*: *full*, *auto dest*: *full_imp_height*)

lemma *bal_imp_full*: *bal t* ⟹ *full* (*height t*) *t*
by (*induct t*, *simp_all*)

lemma *bal_iff_full*: *bal t* ⟷ (∃ *n*. *full n t*)
by (*auto elim*!: *bal_imp_full full_imp_bal*)

The *insert* function either preserves the height of the tree, or increases it by one. The constructor returned by the *insert* function determines which: A return value of the form T_i *t* indicates that the height will be the same. A value of the form Up_i *l p r* indicates an increase in height.

primrec $full_i$ :: *nat* ⇒ *′a* up_i ⇒ *bool* **where**
$full_i$ *n* (T_i *t*) ⟷ *full n t* |
$full_i$ *n* (Up_i *l p r*) ⟷ *full n l* ∧ *full n r*

lemma $full_i\_ins$: $full\ n\ t \Longrightarrow full_i\ n\ (ins\ a\ t)$
by (*induct rule*: *full.induct*) (*auto*, *auto split*: $up_i.split$)

The *insert* operation preserves balance.

lemma *bal_insert*: $bal\ t \Longrightarrow bal\ (insert\ a\ t)$
unfolding *bal_iff_full insert_def*
apply (*erule exE*)
apply (*drule* $full_i\_ins$ [*of* _ _ *a*])
apply (*cases ins a t*)
apply (*auto intro*: *full.intros*)
done

30.3.2 Proofs for delete

instantiation $up_d :: (type)height$
begin

fun $height\_up_d :: 'a\ up_d \Rightarrow nat$ **where**
$height\ (T_d\ t) = height\ t\ |$
$height\ (Up_d\ t) = height\ t + 1$

instance ..

end

lemma $bal\_tree_d\_node21$:
$[\![bal\ r;\ bal\ (tree_d\ l);\ height\ r = height\ l\ ]\!] \Longrightarrow bal\ (tree_d\ (node21\ l\ a\ r))$
by(*induct l a r rule*: *node21.induct*) *auto*

lemma $bal\_tree_d\_node22$:
$[\![bal(tree_d\ r);\ bal\ l;\ height\ r = height\ l\ ]\!] \Longrightarrow bal\ (tree_d\ (node22\ l\ a\ r))$
by(*induct l a r rule*: *node22.induct*) *auto*

lemma $bal\_tree_d\_node31$:
$[\![\ bal\ (tree_d\ l);\ bal\ m;\ bal\ r;\ height\ l = height\ r;\ height\ m = height\ r\ ]\!]$
$\Longrightarrow bal\ (tree_d\ (node31\ l\ a\ m\ b\ r))$
by(*induct l a m b r rule*: *node31.induct*) *auto*

lemma $bal\_tree_d\_node32$:
$[\![\ bal\ l;\ bal\ (tree_d\ m);\ bal\ r;\ height\ l = height\ r;\ height\ m = height\ r\ ]\!]$
$\Longrightarrow bal\ (tree_d\ (node32\ l\ a\ m\ b\ r))$
by(*induct l a m b r rule*: *node32.induct*) *auto*

lemma $bal\_tree_d\_node33$:

⟦ *bal l*; *bal m*; *bal*($tree_d$ *r*); *height l* = *height r*; *height m* = *height r* ⟧
⟹ *bal* ($tree_d$ (*node33 l a m b r*))
by(*induct l a m b r rule*: *node33.induct*) *auto*

lemma *bal_tree*$_d$*_node41*:
⟦ *bal* ($tree_d$ *l*); *bal m*; *bal n*; *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
⟹ *bal* ($tree_d$ (*node41 l a m b n c r*))
by(*induct l a m b n c r rule*: *node41.induct*) *auto*

lemma *bal_tree*$_d$*_node42*:
⟦ *bal l*; *bal* ($tree_d$ *m*); *bal n*; *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
⟹ *bal* ($tree_d$ (*node42 l a m b n c r*))
by(*induct l a m b n c r rule*: *node42.induct*) *auto*

lemma *bal_tree*$_d$*_node43*:
⟦ *bal l*; *bal m*; *bal* ($tree_d$ *n*); *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
⟹ *bal* ($tree_d$ (*node43 l a m b n c r*))
by(*induct l a m b n c r rule*: *node43.induct*) *auto*

lemma *bal_tree*$_d$*_node44*:
⟦ *bal l*; *bal m*; *bal n*; *bal* ($tree_d$ *r*); *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
⟹ *bal* ($tree_d$ (*node44 l a m b n c r*))
by(*induct l a m b n c r rule*: *node44.induct*) *auto*

lemmas *bals* = *bal_tree*$_d$*_node21 bal_tree*$_d$*_node22*
bal_tree$_d$*_node31 bal_tree*$_d$*_node32 bal_tree*$_d$*_node33*
bal_tree$_d$*_node41 bal_tree*$_d$*_node42 bal_tree*$_d$*_node43 bal_tree*$_d$*_node44*

lemma *height_node21*:
height r > *0* ⟹ *height*(*node21 l a r*) = *max* (*height l*) (*height r*) + *1*
by(*induct l a r rule*: *node21.induct*)(*simp_all add*: *max.assoc*)

lemma *height_node22*:
height l > *0* ⟹ *height*(*node22 l a r*) = *max* (*height l*) (*height r*) + *1*
by(*induct l a r rule*: *node22.induct*)(*simp_all add*: *max.assoc*)

lemma *height_node31*:
height m > *0* ⟹ *height*(*node31 l a m b r*) =
max (*height l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node31.induct*)(*simp_all add*: *max_def*)

lemma *height_node32*:
height r > 0 ⟹ height(node32 l a m b r) =
max (height l) (max (height m) (height r)) + 1
by(*induct l a m b r rule*: *node32.induct*)(*simp_all add*: *max_def*)

lemma *height_node33*:
height m > 0 ⟹ height(node33 l a m b r) =
max (height l) (max (height m) (height r)) + 1
by(*induct l a m b r rule*: *node33.induct*)(*simp_all add*: *max_def*)

lemma *height_node41*:
height m > 0 ⟹ height(node41 l a m b n c r) =
max (height l) (max (height m) (max (height n) (height r))) + 1
by(*induct l a m b n c r rule*: *node41.induct*)(*simp_all add*: *max_def*)

lemma *height_node42*:
height l > 0 ⟹ height(node42 l a m b n c r) =
max (height l) (max (height m) (max (height n) (height r))) + 1
by(*induct l a m b n c r rule*: *node42.induct*)(*simp_all add*: *max_def*)

lemma *height_node43*:
height m > 0 ⟹ height(node43 l a m b n c r) =
max (height l) (max (height m) (max (height n) (height r))) + 1
by(*induct l a m b n c r rule*: *node43.induct*)(*simp_all add*: *max_def*)

lemma *height_node44*:
height n > 0 ⟹ height(node44 l a m b n c r) =
max (height l) (max (height m) (max (height n) (height r))) + 1
by(*induct l a m b n c r rule*: *node44.induct*)(*simp_all add*: *max_def*)

lemmas *heights = height_node21 height_node22*
height_node31 height_node32 height_node33
height_node41 height_node42 height_node43 height_node44

lemma *height_split_min*:
split_min t = (x, t′) ⟹ height t > 0 ⟹ bal t ⟹ height t′ = height t
by(*induct t arbitrary*: *x t′ rule*: *split_min.induct*)
(*auto simp*: *heights split*: *prod.splits*)

lemma *height_del*: *bal t ⟹ height(del x t) = height t*
by(*induction x t rule*: *del.induct*)
(*auto simp add*: *heights height_split_min split*!: *if_split prod.split*)

lemma *bal_split_min*:
⟦ *split_min t* = (*x*, *t'*); *bal t*; *height t* > *0* ⟧ ⟹ *bal* (*tree*$_d$ *t'*)
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
(*auto simp*: *heights height_split_min bals split*: *prod.splits*)

lemma *bal_tree*$_d$*_del*: *bal t* ⟹ *bal*(*tree*$_d$(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *bals bal_split_min height_del height_split_min split*!: *if_split prod.split*)

corollary *bal_delete*: *bal t* ⟹ *bal*(*delete x t*)
by(*simp add*: *delete_def bal_tree*$_d$*_del*)

30.4 Overall Correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *bal*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *6* **thus** *?case* **by**(*simp add*: *bal_insert*)
next
case *7* **thus** *?case* **by**(*simp add*: *bal_delete*)
qed (*simp add*: *empty_def*)+

end

31 2-3-4 Tree Implementation of Maps

theory *Tree234_Map*
imports
Tree234_Set
Map_Specs
begin

31.1 Map operations on 2-3-4 trees

fun *lookup* :: ('*a*::*linorder* ∗ '*b*) *tree234* ⇒ '*a* ⇒ '*b option* **where**

```
lookup Leaf x = None |
lookup (Node2 l (a,b) r) x = (case cmp x a of
  LT ⇒ lookup l x |
  GT ⇒ lookup r x |
  EQ ⇒ Some b) |
lookup (Node3 l (a1,b1) m (a2,b2) r) x = (case cmp x a1 of
  LT ⇒ lookup l x |
  EQ ⇒ Some b1 |
  GT ⇒ (case cmp x a2 of
        LT ⇒ lookup m x |
        EQ ⇒ Some b2 |
        GT ⇒ lookup r x)) |
lookup (Node4 t1 (a1,b1) t2 (a2,b2) t3 (a3,b3) t4) x = (case cmp x a2 of
  LT ⇒ (case cmp x a1 of
         LT ⇒ lookup t1 x | EQ ⇒ Some b1 | GT ⇒ lookup t2 x) |
  EQ ⇒ Some b2 |
  GT ⇒ (case cmp x a3 of
         LT ⇒ lookup t3 x | EQ ⇒ Some b3 | GT ⇒ lookup t4 x))

fun upd :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree234 ⇒ ('a*'b) up_i where
upd x y Leaf = Up_i Leaf (x,y) Leaf |
upd x y (Node2 l ab r) = (case cmp x (fst ab) of
   LT ⇒ (case upd x y l of
          T_i l' => T_i (Node2 l' ab r)
        | Up_i l1 ab' l2 => T_i (Node3 l1 ab' l2 ab r)) |
   EQ ⇒ T_i (Node2 l (x,y) r) |
   GT ⇒ (case upd x y r of
          T_i r' => T_i (Node2 l ab r')
        | Up_i r1 ab' r2 => T_i (Node3 l ab r1 ab' r2))) |
upd x y (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of
   LT ⇒ (case upd x y l of
          T_i l' => T_i (Node3 l' ab1 m ab2 r)
        | Up_i l1 ab' l2 => Up_i (Node2 l1 ab' l2) ab1 (Node2 m ab2 r)) |
   EQ ⇒ T_i (Node3 l (x,y) m ab2 r) |
   GT ⇒ (case cmp x (fst ab2) of
          LT ⇒ (case upd x y m of
                  T_i m' => T_i (Node3 l ab1 m' ab2 r)
                | Up_i m1 ab' m2 => Up_i (Node2 l ab1 m1) ab' (Node2 m2
ab2 r)) |
          EQ ⇒ T_i (Node3 l ab1 m (x,y) r) |
          GT ⇒ (case upd x y r of
                  T_i r' => T_i (Node3 l ab1 m ab2 r')
                | Up_i r1 ab' r2 => Up_i (Node2 l ab1 m) ab2 (Node2 r1 ab'
r2)))) |
```

```
upd x y (Node4 t1 ab1 t2 ab2 t3 ab3 t4) = (case cmp x (fst ab2) of
  LT ⇒ (case cmp x (fst ab1) of
        LT ⇒ (case upd x y t1 of
                T_i t1' => T_i (Node4 t1' ab1 t2 ab2 t3 ab3 t4)
              | Up_i t11 q t12 => Up_i (Node2 t11 q t12) ab1 (Node3 t2 ab2
t3 ab3 t4)) |
        EQ ⇒ T_i (Node4 t1 (x,y) t2 ab2 t3 ab3 t4) |
        GT ⇒ (case upd x y t2 of
                T_i t2' => T_i (Node4 t1 ab1 t2' ab2 t3 ab3 t4)
              | Up_i t21 q t22 => Up_i (Node2 t1 ab1 t21) q (Node3 t22 ab2
t3 ab3 t4))) |
  EQ ⇒ T_i (Node4 t1 ab1 t2 (x,y) t3 ab3 t4) |
  GT ⇒ (case cmp x (fst ab3) of
        LT ⇒ (case upd x y t3 of
                T_i t3' ⇒ T_i (Node4 t1 ab1 t2 ab2 t3' ab3 t4)
              | Up_i t31 q t32 => Up_i (Node2 t1 ab1 t2) ab2 (Node3 t31
q t32 ab3 t4)) |
        EQ ⇒ T_i (Node4 t1 ab1 t2 ab2 t3 (x,y) t4) |
        GT ⇒ (case upd x y t4 of
                T_i t4' => T_i (Node4 t1 ab1 t2 ab2 t3 ab3 t4')
              | Up_i t41 q t42 => Up_i (Node2 t1 ab1 t2) ab2 (Node3 t3 ab3
t41 q t42))))

definition update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree234 ⇒ ('a*'b) tree234
where
update x y t = tree_i(upd x y t)

fun del :: 'a::linorder ⇒ ('a*'b) tree234 ⇒ ('a*'b) up_d where
del x Leaf = T_d Leaf |
del x (Node2 Leaf ab1 Leaf) = (if x=fst ab1 then Up_d Leaf else T_d(Node2
Leaf ab1 Leaf)) |
del x (Node3 Leaf ab1 Leaf ab2 Leaf) = T_d(if x=fst ab1 then Node2 Leaf
ab2 Leaf
  else if x=fst ab2 then Node2 Leaf ab1 Leaf else Node3 Leaf ab1 Leaf ab2
Leaf) |
del x (Node4 Leaf ab1 Leaf ab2 Leaf ab3 Leaf) =
  T_d(if x = fst ab1 then Node3 Leaf ab2 Leaf ab3 Leaf else
    if x = fst ab2 then Node3 Leaf ab1 Leaf ab3 Leaf else
    if x = fst ab3 then Node3 Leaf ab1 Leaf ab2 Leaf
    else Node4 Leaf ab1 Leaf ab2 Leaf ab3 Leaf) |
del x (Node2 l ab1 r) = (case cmp x (fst ab1) of
  LT ⇒ node21 (del x l) ab1 r |
  GT ⇒ node22 l ab1 (del x r) |
  EQ ⇒ let (ab1',t) = split_min r in node22 l ab1' t) |
```

```
del x (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of
  LT ⇒ node31 (del x l) ab1 m ab2 r |
  EQ ⇒ let (ab1',m') = split_min m in node32 l ab1' m' ab2 r |
  GT ⇒ (case cmp x (fst ab2) of
         LT ⇒ node32 l ab1 (del x m) ab2 r |
         EQ ⇒ let (ab2',r') = split_min r in node33 l ab1 m ab2' r' |
         GT ⇒ node33 l ab1 m ab2 (del x r))) |
del x (Node4 t1 ab1 t2 ab2 t3 ab3 t4) = (case cmp x (fst ab2) of
  LT ⇒ (case cmp x (fst ab1) of
         LT ⇒ node41 (del x t1) ab1 t2 ab2 t3 ab3 t4 |
         EQ ⇒ let (ab',t2') = split_min t2 in node42 t1 ab' t2' ab2 t3 ab3
t4 |
         GT ⇒ node42 t1 ab1 (del x t2) ab2 t3 ab3 t4) |
  EQ ⇒ let (ab',t3') = split_min t3 in node43 t1 ab1 t2 ab' t3' ab3 t4 |
  GT ⇒ (case cmp x (fst ab3) of
        LT ⇒ node43 t1 ab1 t2 ab2 (del x t3) ab3 t4 |
         EQ ⇒ let (ab',t4') = split_min t4 in node44 t1 ab1 t2 ab2 t3 ab'
t4' |
         GT ⇒ node44 t1 ab1 t2 ab2 t3 ab3 (del x t4)))
```

definition *delete* :: *'a::linorder* ⇒ *('a*'b) tree234* ⇒ *('a*'b) tree234* **where**
delete x t = $tree_d$(*del x t*)

31.2 Functional correctness

lemma *lookup_map_of*:
sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by (*induction t*) (*auto simp*: *map_of_simps split*: *option.split*)

lemma *inorder_upd*:
sorted1(*inorder t*) ⟹ *inorder*($tree_i$(*upd a b t*)) = *upd_list a b* (*inorder t*)
by(*induction t*)
(*auto simp*: *upd_list_simps*, *auto simp*: *upd_list_simps split*: up_i.*splits*)

lemma *inorder_update*:
sorted1(*inorder t*) ⟹ *inorder*(*update a b t*) = *upd_list a b* (*inorder t*)
by(*simp add*: *update_def inorder_upd*)

lemma *inorder_del*: ⟦ *bal t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder($tree_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_splits prod.splits*)

lemma *inorder_delete*: ⟦ *bal t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

31.3 Balancedness

lemma *bal_upd*: *bal t* ⟹ *bal* (*tree*$_i$(*upd x y t*)) ∧ *height*(*upd x y t*) = *height t*
by (*induct t*) (*auto*, *auto split*!: *if_split up*$_i$.*split*)

lemma *bal_update*: *bal t* ⟹ *bal* (*update x y t*)
by (*simp add*: *update_def bal_upd*)

lemma *height_del*: *bal t* ⟹ *height*(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
(*auto simp add*: *heights height_split_min split*!: *if_split prod.split*)

lemma *bal_tree*$_d$*_del*: *bal t* ⟹ *bal*(*tree*$_d$(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *bals bal_split_min height_del height_split_min split*!: *if_split prod.split*)

corollary *bal_delete*: *bal t* ⟹ *bal*(*delete x t*)
by(*simp add*: *delete_def bal_tree*$_d$*_del*)

31.4 Overall Correctness

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *bal*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *6* **thus** *?case* **by**(*simp add*: *bal_update*)
next
case *7* **thus** *?case* **by**(*simp add*: *bal_delete*)
qed (*simp add*: *empty_def*)+

end

32 1-2 Brother Tree Implementation of Sets

theory *Brother12_Set*
imports
Cmp
Set_Specs
HOL−Number_Theory.Fib
begin

32.1 Data Type and Operations

datatype *'a bro* =
N0 |
N1 'a bro |
N2 'a bro 'a 'a bro |

L2 'a |
N3 'a bro 'a 'a bro 'a 'a bro

definition *empty* :: *'a bro* **where**
empty = *N0*

fun *inorder* :: *'a bro* ⇒ *'a list* **where**
inorder N0 = [] |
inorder (*N1 t*) = *inorder t* |
inorder (*N2 l a r*) = *inorder l* @ *a* # *inorder r* |
inorder (*L2 a*) = [*a*] |
inorder (*N3 t1 a1 t2 a2 t3*) = *inorder t1* @ *a1* # *inorder t2* @ *a2* # *inorder t3*

fun *isin* :: *'a bro* ⇒ *'a*::*linorder* ⇒ *bool* **where**
isin N0 x = *False* |
isin (*N1 t*) *x* = *isin t x* |
isin (*N2 l a r*) *x* =
(*case cmp x a of*
LT ⇒ *isin l x* |
EQ ⇒ *True* |
GT ⇒ *isin r x*)

fun *n1* :: *'a bro* ⇒ *'a bro* **where**
n1 (*L2 a*) = *N2 N0 a N0* |
n1 (*N3 t1 a1 t2 a2 t3*) = *N2* (*N2 t1 a1 t2*) *a2* (*N1 t3*) |

```
  n1 t = N1 t

hide_const (open) insert

locale insert
begin

fun n2 :: 'a bro ⇒ 'a ⇒ 'a bro ⇒ 'a bro where
  n2 (L2 a1) a2 t = N3 N0 a1 N0 a2 t |
  n2 (N3 t1 a1 t2 a2 t3) a3 (N1 t4) = N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4) |
  n2 (N3 t1 a1 t2 a2 t3) a3 t4 = N3 (N2 t1 a1 t2) a2 (N1 t3) a3 t4 |
  n2 t1 a1 (L2 a2) = N3 t1 a1 N0 a2 N0 |
  n2 (N1 t1) a1 (N3 t2 a2 t3 a3 t4) = N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4) |
  n2 t1 a1 (N3 t2 a2 t3 a3 t4) = N3 t1 a1 (N1 t2) a2 (N2 t3 a3 t4) |
  n2 t1 a t2 = N2 t1 a t2

fun ins :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
  ins x N0 = L2 x |
  ins x (N1 t) = n1 (ins x t) |
  ins x (N2 l a r) =
  (case cmp x a of
     LT ⇒ n2 (ins x l) a r |
     EQ ⇒ N2 l a r |
     GT ⇒ n2 l a (ins x r))

fun tree :: 'a bro ⇒ 'a bro where
  tree (L2 a) = N2 N0 a N0 |
  tree (N3 t1 a1 t2 a2 t3) = N2 (N2 t1 a1 t2) a2 (N1 t3) |
  tree t = t

definition insert :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
  insert x t = tree(ins x t)

end

locale delete
begin

fun n2 :: 'a bro ⇒ 'a ⇒ 'a bro ⇒ 'a bro where
  n2 (N1 t1) a1 (N1 t2) = N1 (N2 t1 a1 t2) |
  n2 (N1 (N1 t1)) a1 (N2 (N1 t2) a2 (N2 t3 a3 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
  n2 (N1 (N1 t1)) a1 (N2 (N2 t2 a2 t3) a3 (N1 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
```

```
  n2 (N1 (N1 t1)) a1 (N2 (N2 t2 a2 t3) a3 (N2 t4 a4 t5)) =
  N2 (N2 (N1 t1) a1 (N2 t2 a2 t3)) a3 (N1 (N2 t4 a4 t5)) |
  n2 (N2 (N1 t1) a1 (N2 t2 a2 t3)) a3 (N1 (N1 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
  n2 (N2 (N2 t1 a1 t2) a2 (N1 t3)) a3 (N1 (N1 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
  n2 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) a5 (N1 (N1 t5)) =
  N2 (N1 (N2 t1 a1 t2)) a2 (N2 (N2 t3 a3 t4) a5 (N1 t5)) |
  n2 t1 a1 t2 = N2 t1 a1 t2

fun split_min :: 'a bro ⇒ ('a × 'a bro) option where
  split_min N0 = None |
  split_min (N1 t) =
  (case split_min t of
     None ⇒ None |
     Some (a, t') ⇒ Some (a, N1 t')) |
  split_min (N2 t1 a t2) =
  (case split_min t1 of
     None ⇒ Some (a, N1 t2) |
     Some (b, t1') ⇒ Some (b, n2 t1' a t2))

fun del :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
  del _ N0          = N0 |
  del x (N1 t)      = N1 (del x t) |
  del x (N2 l a r) =
  (case cmp x a of
     LT ⇒ n2 (del x l) a r |
     GT ⇒ n2 l a (del x r) |
     EQ ⇒ (case split_min r of
             None ⇒ N1 l |
             Some (b, r') ⇒ n2 l b r'))

fun tree :: 'a bro ⇒ 'a bro where
  tree (N1 t) = t |
  tree t = t

definition delete :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
  delete a t = tree (del a t)

end
```

32.2 Invariants

```
fun B :: nat ⇒ 'a bro set
```

and $U :: nat \Rightarrow 'a\ bro\ set$ **where**
$B\ 0 = \{N0\}$ |
$B\ (Suc\ h) = \{\ N2\ t1\ a\ t2 \mid t1\ a\ t2.$
$t1 \in B\ h \cup U\ h \wedge t2 \in B\ h \vee t1 \in B\ h \wedge t2 \in B\ h \cup U\ h\}$ |
$U\ 0 = \{\}$ |
$U\ (Suc\ h) = N1\ `\ B\ h$

abbreviation $T\ h \equiv B\ h \cup U\ h$

fun $Bp :: nat \Rightarrow 'a\ bro\ set$ **where**
$Bp\ 0 = B\ 0 \cup L2\ `\ UNIV$ |
$Bp\ (Suc\ 0) = B\ (Suc\ 0) \cup \{N3\ N0\ a\ N0\ b\ N0 | a\ b.\ True\}$ |
$Bp\ (Suc(Suc\ h)) = B\ (Suc(Suc\ h)) \cup$
$\{N3\ t1\ a\ t2\ b\ t3 \mid t1\ a\ t2\ b\ t3.\ t1 \in B\ (Suc\ h) \wedge t2 \in U\ (Suc\ h) \wedge t3 \in B\ (Suc\ h)\}$

fun $Um :: nat \Rightarrow 'a\ bro\ set$ **where**
$Um\ 0 = \{\}$ |
$Um\ (Suc\ h) = N1\ `\ T\ h$

32.3 Functional Correctness Proofs

32.3.1 Proofs for isin

lemma *isin_set*:
$t \in T\ h \Longrightarrow sorted(inorder\ t) \Longrightarrow isin\ t\ x = (x \in set(inorder\ t))$
by(*induction h arbitrary*: *t*) (*fastforce simp*: *isin_simps split*: *if_splits*)+

32.3.2 Proofs for insertion

lemma *inorder_n1*: $inorder(n1\ t) = inorder\ t$
by(*cases t rule*: *n1.cases*) (*auto simp*: *sorted_lems*)

context *insert*
begin

lemma *inorder_n2*: $inorder(n2\ l\ a\ r) = inorder\ l\ @\ a\ \#\ inorder\ r$
by(*cases* (*l*,*a*,*r*) *rule*: *n2.cases*) (*auto simp*: *sorted_lems*)

lemma *inorder_tree*: $inorder(tree\ t) = inorder\ t$
by(*cases t*) *auto*

lemma *inorder_ins*: $t \in T\ h \Longrightarrow$
$sorted(inorder\ t) \Longrightarrow inorder(ins\ a\ t) = ins\_list\ a\ (inorder\ t)$

by(*induction h arbitrary*: *t*) (*auto simp*: *ins_list_simps inorder_n1 inorder_n2*)

lemma *inorder_insert*: $t \in T\ h \Longrightarrow$
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*insert a t*) = *ins_list a* (*inorder t*)
by(*simp add*: *insert_def inorder_ins inorder_tree*)

end

32.3.3 Proofs for deletion

context *delete*
begin

lemma *inorder_tree*: *inorder*(*tree t*) = *inorder t*
by(*cases t*) *auto*

lemma *inorder_n2*: *inorder*(*n2 l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *n2.cases*) (*auto*)

lemma *inorder_split_min*:
$t \in T\ h \Longrightarrow$ (*split_min t* = *None* $\longleftrightarrow$ *inorder t* = []) $\wedge$
(*split_min t* = *Some*(*a*,*t'*) $\longrightarrow$ *inorder t* = *a* # *inorder t'*)
by(*induction h arbitrary*: *t a t'*) (*auto simp*: *inorder_n2 split*: *option.splits*)

lemma *inorder_del*:
$t \in T\ h \Longrightarrow$ *sorted*(*inorder t*) $\Longrightarrow$ *inorder*(*del x t*) = *del_list x* (*inorder t*)
apply (*induction h arbitrary*: *t*)
apply (*auto simp*: *del_list_simps inorder_n2 split*: *option.splits*)
apply (*auto simp*: *del_list_simps inorder_n2*
inorder_split_min[*OF UnI1*] *inorder_split_min*[*OF UnI2*] *split*: *option.splits*)
done

lemma *inorder_delete*:
$t \in T\ h \Longrightarrow$ *sorted*(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del inorder_tree*)

end

32.4 Invariant Proofs

32.4.1 Proofs for insertion

lemma *n1_type*: $t \in Bp\ h \Longrightarrow n1\ t \in T\ (Suc\ h)$
by(*cases h rule*: *Bp.cases*) *auto*

context *insert*
begin

lemma *tree_type*: $t \in Bp\ h \Longrightarrow tree\ t \in B\ h \cup B\ (Suc\ h)$
by(*cases h rule*: *Bp.cases*) *auto*

lemma *n2_type*:
$(t1 \in Bp\ h \wedge t2 \in T\ h \longrightarrow n2\ t1\ a\ t2 \in Bp\ (Suc\ h)) \wedge$
$(t1 \in T\ h \wedge t2 \in Bp\ h \longrightarrow n2\ t1\ a\ t2 \in Bp\ (Suc\ h))$
apply(*cases h rule*: *Bp.cases*)
apply (*auto*)[*2*]
apply(*rule conjI impI | erule conjE exE imageE | simp | erule disjE*)+
done

lemma *Bp_if_B*: $t \in B\ h \Longrightarrow t \in Bp\ h$
by (*cases h rule*: *Bp.cases*) *simp_all*

An automatic proof:

lemma
$(t \in B\ h \longrightarrow ins\ x\ t \in Bp\ h) \wedge (t \in U\ h \longrightarrow ins\ x\ t \in T\ h)$
proof (*induction h arbitrary*: *t*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc h*)
then show *?case* **by** (*fastforce simp*: *Bp_if_B n2_type dest*: *n1_type*)
qed

A detailed proof:

lemma *ins_type*:
shows $t \in B\ h \Longrightarrow ins\ x\ t \in Bp\ h$ **and** $t \in U\ h \Longrightarrow ins\ x\ t \in T\ h$
proof(*induction h arbitrary*: *t*)
case *0*
{ case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp* **}**
next
case (*Suc h*)

```
  { case 1
    then obtain t1 a t2 where [simp]: t = N2 t1 a t2 and
      t1: t1 ∈ T h and t2: t2 ∈ T h and t12: t1 ∈ B h ∨ t2 ∈ B h
      by auto
    have ?case if x < a
    proof −
      have n2 (ins x t1) a t2 ∈ Bp (Suc h)
      proof cases
        assume t1 ∈ B h
        with t2 show ?thesis by (simp add: Suc.IH(1) n2_type)
      next
        assume t1 ∉ B h
        hence 1: t1 ∈ U h and 2: t2 ∈ B h using t1 t12 by auto
        show ?thesis by (metis Suc.IH(2)[OF 1] Bp_if_B[OF 2] n2_type)
      qed
      with ‹x < a› show ?case by simp
    qed
    moreover
    have ?case if a < x
    proof −
      have n2 t1 a (ins x t2) ∈ Bp (Suc h)
      proof cases
        assume t2 ∈ B h
        with t1 show ?thesis by (simp add: Suc.IH(1) n2_type)
      next
        assume t2 ∉ B h
        hence 1: t1 ∈ B h and 2: t2 ∈ U h using t2 t12 by auto
        show ?thesis by (metis Bp_if_B[OF 1] Suc.IH(2)[OF 2] n2_type)
      qed
      with ‹a < x› show ?case by simp
    qed
    moreover
    have ?case if x = a
    proof −
      from 1 have t ∈ Bp (Suc h) by(rule Bp_if_B)
      thus ?case using ‹x = a› by simp
    qed
    ultimately show ?case by auto
  next
    case 2 thus ?case using Suc(1) n1_type by fastforce }
qed

lemma insert_type:
  t ∈ B h ⟹ insert x t ∈ B h ∪ B (Suc h)
```

unfolding *insert_def* **by** (*metis ins_type*(*1*) *tree_type*)

end

32.4.2 Proofs for deletion

lemma *B_simps*[*simp*]:
N1 t ∈ *B h* = *False*
L2 y ∈ *B h* = *False*
(*N3 t1 a1 t2 a2 t3*) ∈ *B h* = *False*
N0 ∈ *B h* ⟷ *h* = *0*
by (*cases h*, *auto*)+

context *delete*
begin

lemma *n2_type1*:
⟦*t1* ∈ *Um h*; *t2* ∈ *B h*⟧ ⟹ *n2 t1 a t2* ∈ *T* (*Suc h*)
apply(*cases h rule*: *Bp.cases*)
apply *auto*[*2*]
apply(*erule exE bexE conjE imageE* | *simp* | *erule disjE*)+
done

lemma *n2_type2*:
⟦*t1* ∈ *B h* ; *t2* ∈ *Um h* ⟧ ⟹ *n2 t1 a t2* ∈ *T* (*Suc h*)
apply(*cases h rule*: *Bp.cases*)
using *Um.simps*(*1*) **apply** *blast*
apply *force*
apply(*erule exE bexE conjE imageE* | *simp* | *erule disjE*)+
done

lemma *n2_type3*:
⟦*t1* ∈ *T h* ; *t2* ∈ *T h* ⟧ ⟹ *n2 t1 a t2* ∈ *T* (*Suc h*)
apply(*cases h rule*: *Bp.cases*)
apply *auto*[*2*]
apply(*erule exE bexE conjE imageE* | *simp* | *erule disjE*)+
done

lemma *split_minNoneN0*: ⟦*t* ∈ *B h*; *split_min t* = *None*⟧ ⟹ *t* = *N0*
by (*cases t*) (*auto split*: *option.splits*)

lemma *split_minNoneN1* : ⟦*t* ∈ *U h*; *split_min t* = *None*⟧ ⟹ *t* = *N1 N0*
by (*cases h*) (*auto simp*: *split_minNoneN0* *split*: *option.splits*)

```
lemma split_min_type:
  t ∈ B h ⟹ split_min t = Some (a, t') ⟹ t' ∈ T h
  t ∈ U h ⟹ split_min t = Some (a, t') ⟹ t' ∈ Um h
proof (induction h arbitrary: t a t')
  case (Suc h)
  { case 1
    then obtain t1 a t2 where [simp]: t = N2 t1 a t2 and
      t12: t1 ∈ T h t2 ∈ T h t1 ∈ B h ∨ t2 ∈ B h
      by auto
    show ?case
    proof (cases split_min t1)
      case None
      show ?thesis
      proof cases
        assume t1 ∈ B h
        with split_minNoneN0[OF this None] 1 show ?thesis by(auto)
      next
        assume t1 ∉ B h
        thus ?thesis using 1 None by (auto)
      qed
    next
      case [simp]: (Some bt')
      obtain b t1' where [simp]: bt' = (b,t1') by fastforce
      show ?thesis
      proof cases
        assume t1 ∈ B h
        from Suc.IH(1)[OF this] 1 have t1' ∈ T h by simp
        from n2_type3[OF this t12(2)] 1 show ?thesis by auto
      next
        assume t1 ∉ B h
        hence t1: t1 ∈ U h and t2: t2 ∈ B h using t12 by auto
        from Suc.IH(2)[OF t1] have t1' ∈ Um h by simp
        from n2_type1[OF this t2] 1 show ?thesis by auto
      qed
    qed
  }
  { case 2
    then obtain t1 where [simp]: t = N1 t1 and t1: t1 ∈ B h by auto
    show ?case
    proof (cases split_min t1)
      case None
      with split_minNoneN0[OF t1 None] 2 show ?thesis by(auto)
    next
      case [simp]: (Some bt')
```

```
    obtain b t1' where [simp]: bt' = (b,t1') by fastforce
    from Suc.IH(1)[OF t1] have t1' ∈ T h by simp
    thus ?thesis using 2 by auto
   qed
  }
qed auto

lemma del_type:
  t ∈ B h ⟹ del x t ∈ T h
  t ∈ U h ⟹ del x t ∈ Um h
proof (induction h arbitrary: x t)
  case (Suc h)
  { case 1
   then obtain l a r where [simp]: t = N2 l a r and
     lr: l ∈ T h r ∈ T h l ∈ B h ∨ r ∈ B h by auto
   have ?case if x < a
   proof cases
     assume l ∈ B h
     from n2_type3[OF Suc.IH(1)[OF this] lr(2)]
     show ?thesis using ‹x<a› by(simp)
   next
     assume l ∉ B h
     hence l ∈ U h r ∈ B h using lr by auto
     from n2_type1[OF Suc.IH(2)[OF this(1)] this(2)]
     show ?thesis using ‹x<a› by(simp)
   qed
   moreover
   have ?case if x > a
   proof cases
     assume r ∈ B h
     from n2_type3[OF lr(1) Suc.IH(1)[OF this]]
     show ?thesis using ‹x>a› by(simp)
   next
     assume r ∉ B h
     hence l ∈ B h r ∈ U h using lr by auto
     from n2_type2[OF this(1) Suc.IH(2)[OF this(2)]]
     show ?thesis using ‹x>a› by(simp)
   qed
   moreover
   have ?case if [simp]: x=a
   proof (cases split_min r)
     case None
     show ?thesis
     proof cases
```

```
      assume r ∈ B h
      with split_minNoneN0[OF this None] lr show ?thesis by(simp)
    next
      assume r ∉ B h
      hence r ∈ U h using lr by auto
     with split_minNoneN1[OF this None] lr(3) show ?thesis by (simp)
    qed
  next
    case [simp]: (Some br')
    obtain b r' where [simp]: br' = (b,r') by fastforce
    show ?thesis
    proof cases
      assume r ∈ B h
      from split_min_type(1)[OF this] n2_type3[OF lr(1)]
      show ?thesis by simp
    next
      assume r ∉ B h
      hence l ∈ B h and r ∈ U h using lr by auto
      from split_min_type(2)[OF this(2)] n2_type2[OF this(1)]
      show ?thesis by simp
    qed
  qed
  ultimately show ?case by auto
 }
 { case 2 with Suc.IH(1) show ?case by auto }
qed auto

lemma tree_type: t ∈ T (h+1) ⟹ tree t ∈ B (h+1) ∪ B h
 by(auto)

lemma delete_type: t ∈ B h ⟹ delete x t ∈ B h ∪ B(h−1)
 unfolding delete_def
 by (cases h) (simp, metis del_type(1) tree_type Suc_eq_plus1 diff_Suc_1)

end
```

32.5 Overall correctness

```
interpretation Set_by_Ordered
 where empty = empty and isin = isin and insert = insert.insert
   and delete = delete.delete and inorder = inorder and inv = λt. ∃h. t
∈ B h
proof (standard, goal_cases)
 case 2 thus ?case by(auto intro!: isin_set)
```

next
case *3* **thus** *?case* **by**(*auto intro*!: *insert.inorder_insert*)
next
case *4* **thus** *?case* **by**(*auto intro*!: *delete.inorder_delete*)
next
case *6* **thus** *?case* **using** *insert.insert_type* **by** *blast*
next
case *7* **thus** *?case* **using** *delete.delete_type* **by** *blast*
qed (*auto simp*: *empty_def*)

32.6 Height-Size Relation

By Daniel Stüwe

fun *fib_tree* :: *nat* ⇒ *unit bro* **where**
fib_tree 0 = *N0*
| *fib_tree* (*Suc 0*) = *N2 N0* () *N0*
| *fib_tree* (*Suc*(*Suc h*)) = *N2* (*fib_tree* (*h*+*1*)) () (*N1* (*fib_tree h*))

fun *fib′* :: *nat* ⇒ *nat* **where**
fib′ 0 = *0*
| *fib′* (*Suc 0*) = *1*
| *fib′* (*Suc*(*Suc h*)) = *1* + *fib′* (*Suc h*) + *fib′ h*

fun *size* :: *′a bro* ⇒ *nat* **where**
size N0 = *0*
| *size* (*N1 t*) = *size t*
| *size* (*N2 t1 _ t2*) = *1* + *size t1* + *size t2*

lemma *fib_tree_B*: *fib_tree h* ∈ *B h*
by (*induction h rule*: *fib_tree.induct*) *auto*

declare [[*names_short*]]

lemma *size_fib′*: *size* (*fib_tree h*) = *fib′ h*
by (*induction h rule*: *fib_tree.induct*) *auto*

lemma *fibfib*: *fib′ h* + *1* = *fib* (*Suc*(*Suc h*))
by (*induction h rule*: *fib_tree.induct*) *auto*

lemma *B_N2_cases*[*consumes 1*]:
assumes *N2 t1 a t2* ∈ *B* (*Suc n*)
obtains
(*BB*) *t1* ∈ *B n* **and** *t2* ∈ *B n* |
(*UB*) *t1* ∈ *U n* **and** *t2* ∈ *B n* |

```
    (BU) t1 ∈ B n and t2 ∈ U n
  using assms by auto

lemma size_bounded: t ∈ B h ⟹ size t ≥ size (fib_tree h)
  unfolding size_fib' proof (induction h arbitrary: t rule: fib'.induct)
  case (3 h t')
  note main = 3
  then obtain t1 a t2 where t': t' = N2 t1 a t2 by auto
  with main have N2 t1 a t2 ∈ B (Suc (Suc h)) by auto
  thus ?case proof (cases rule: B_N2_cases)
    case BB
    then obtain x y z where t2: t2 = N2 x y z ∨ t2 = N2 z y x x ∈ B h
by auto
     show ?thesis unfolding t' using main(1)[OF BB(1)] main(2)[OF
t2(2)] t2(1) by auto
  next
    case UB
    then obtain t11 where t1: t1 = N1 t11 t11 ∈ B h by auto
   show ?thesis unfolding t' t1(1) using main(2)[OF t1(2)] main(1)[OF
UB(2)] by simp
  next
    case BU
    then obtain t22 where t2: t2 = N1 t22 t22 ∈ B h by auto
   show ?thesis unfolding t' t2(1) using main(2)[OF t2(2)] main(1)[OF
BU(1)] by simp
  qed
qed auto

theorem t ∈ B h ⟹ fib (h + 2) ≤ size t + 1
  using size_bounded
  by (simp add: size_fib' fibfib[symmetric] del: fib.simps)

end
```

33 1-2 Brother Tree Implementation of Maps

```
theory Brother12_Map
imports
  Brother12_Set
  Map_Specs
begin

fun lookup :: ('a × 'b) bro ⇒ 'a::linorder ⇒ 'b option where
```

```
lookup N0 x = None |
lookup (N1 t) x = lookup t x |
lookup (N2 l (a,b) r) x =
  (case cmp x a of
     LT ⇒ lookup l x |
     EQ ⇒ Some b |
     GT ⇒ lookup r x)

locale update = insert
begin

fun upd :: 'a::linorder ⇒ 'b ⇒ ('a×'b) bro ⇒ ('a×'b) bro where
upd x y N0 = L2 (x,y) |
upd x y (N1 t) = n1 (upd x y t) |
upd x y (N2 l (a,b) r) =
  (case cmp x a of
     LT ⇒ n2 (upd x y l) (a,b) r |
     EQ ⇒ N2 l (a,y) r |
     GT ⇒ n2 l (a,b) (upd x y r))

definition update :: 'a::linorder ⇒ 'b ⇒ ('a×'b) bro ⇒ ('a×'b) bro where
update x y t = tree(upd x y t)

end

context delete
begin

fun del :: 'a::linorder ⇒ ('a×'b) bro ⇒ ('a×'b) bro where
del _ N0          = N0 |
del x (N1 t)      = N1 (del x t) |
del x (N2 l (a,b) r) =
  (case cmp x a of
     LT ⇒ n2 (del x l) (a,b) r |
     GT ⇒ n2 l (a,b) (del x r) |
     EQ ⇒ (case split_min r of
             None ⇒ N1 l |
             Some (ab, r') ⇒ n2 l ab r'))

definition delete :: 'a::linorder ⇒ ('a×'b) bro ⇒ ('a×'b) bro where
delete a t = tree (del a t)

end
```

33.1 Functional Correctness Proofs

33.1.1 Proofs for lookup

lemma *lookup_map_of*: $t \in T\ h \Longrightarrow$
$sorted1(inorder\ t) \Longrightarrow lookup\ t\ x = map\_of\ (inorder\ t)\ x$
by(*induction h arbitrary*: *t*) (*auto simp*: *map_of_simps split*: *option.splits*)

33.1.2 Proofs for update

context *update*
begin

lemma *inorder_upd*: $t \in T\ h \Longrightarrow$
$sorted1(inorder\ t) \Longrightarrow inorder(upd\ x\ y\ t) = upd\_list\ x\ y\ (inorder\ t)$
by(*induction h arbitrary*: *t*) (*auto simp*: *upd_list_simps inorder_n1 inorder_n2*)

lemma *inorder_update*: $t \in T\ h \Longrightarrow$
$sorted1(inorder\ t) \Longrightarrow inorder(update\ x\ y\ t) = upd\_list\ x\ y\ (inorder\ t)$
by(*simp add*: *update_def inorder_upd inorder_tree*)

end

33.1.3 Proofs for deletion

context *delete*
begin

lemma *inorder_del*:
$t \in T\ h \Longrightarrow sorted1(inorder\ t) \Longrightarrow inorder(del\ x\ t) = del\_list\ x\ (inorder\ t)$
apply (*induction h arbitrary*: *t*)
apply (*auto simp*: *del_list_simps inorder_n2*)
apply (*auto simp*: *del_list_simps inorder_n2*
inorder_split_min[*OF UnI1*] *inorder_split_min*[*OF UnI2*] *split*: *option.splits*)
done

lemma *inorder_delete*:
$t \in T\ h \Longrightarrow sorted1(inorder\ t) \Longrightarrow inorder(delete\ x\ t) = del\_list\ x\ (inorder\ t)$
by(*simp add*: *delete_def inorder_del inorder_tree*)

end

33.2 Invariant Proofs

33.2.1 Proofs for update

context *update*
begin

lemma *upd_type*:
  $(t \in B\ h \longrightarrow upd\ x\ y\ t \in Bp\ h) \wedge (t \in U\ h \longrightarrow upd\ x\ y\ t \in T\ h)$
apply(*induction h arbitrary*: *t*)
 apply (*simp*)
apply (*fastforce simp*: *Bp_if_B n2_type dest*: *n1_type*)
done

lemma *update_type*:
  $t \in B\ h \Longrightarrow update\ x\ y\ t \in B\ h \cup B\ (Suc\ h)$
unfolding *update_def* **by** (*metis upd_type tree_type*)

end

33.2.2 Proofs for deletion

context *delete*
begin

lemma *del_type*:
  $t \in B\ h \Longrightarrow del\ x\ t \in T\ h$
  $t \in U\ h \Longrightarrow del\ x\ t \in Um\ h$
proof (*induction h arbitrary*: *x t*)
  case (*Suc h*)
  { case *1*
    then obtain *l a b r* **where** [*simp*]: $t = N2\ l\ (a,b)\ r$ **and**
      lr: $l \in T\ h\ r \in T\ h\ l \in B\ h \vee r \in B\ h$ **by** *auto*
    have *?case* **if** $x < a$
    proof *cases*
      assume $l \in B\ h$
      from *n2_type3*[*OF Suc.IH*(*1*)[*OF this*] *lr*(*2*)]
      show *?thesis* **using** ‹*x*<*a*› **by**(*simp*)
    next
      assume $l \notin B\ h$
      hence $l \in U\ h\ r \in B\ h$ **using** *lr* **by** *auto*
      from *n2_type1*[*OF Suc.IH*(*2*)[*OF this*(*1*)] *this*(*2*)]
      show *?thesis* **using** ‹*x*<*a*› **by**(*simp*)
    qed
    moreover

```
    have ?case if x > a
    proof cases
      assume r ∈ B h
      from n2_type3[OF lr(1) Suc.IH(1)[OF this]]
      show ?thesis using ‹x>a› by(simp)
    next
      assume r ∉ B h
      hence l ∈ B h r ∈ U h using lr by auto
      from n2_type2[OF this(1) Suc.IH(2)[OF this(2)]]
      show ?thesis using ‹x>a› by(simp)
    qed
    moreover
    have ?case if [simp]: x=a
    proof (cases split_min r)
      case None
      show ?thesis
      proof cases
        assume r ∈ B h
        with split_minNoneN0[OF this None] lr show ?thesis by(simp)
      next
        assume r ∉ B h
        hence r ∈ U h using lr by auto
       with split_minNoneN1[OF this None] lr(3) show ?thesis by (simp)
      qed
    next
      case [simp]: (Some br')
      obtain b r' where [simp]: br' = (b,r') by fastforce
      show ?thesis
      proof cases
        assume r ∈ B h
        from split_min_type(1)[OF this] n2_type3[OF lr(1)]
        show ?thesis by simp
      next
        assume r ∉ B h
        hence l ∈ B h and r ∈ U h using lr by auto
        from split_min_type(2)[OF this(2)] n2_type2[OF this(1)]
        show ?thesis by simp
      qed
    qed
    ultimately show ?case by auto
  }
  { case 2 with Suc.IH(1) show ?case by auto }
qed auto
```

lemma *delete_type*:
t ∈ *B h* ⟹ *delete x t* ∈ *B h* ∪ *B*(*h*−*1*)
unfolding *delete_def*
by (*cases h*) (*simp*, *metis del_type*(*1*) *tree_type Suc_eq_plus1 diff_Suc_1*)

end

33.3 Overall correctness

interpretation *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update.update* **and** *delete* = *delete.delete* **and** *inorder* = *inorder* **and** *inv* = λ*t*. ∃ *h*. *t* ∈ *B h*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*auto intro*!: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*auto intro*!: *update.inorder_update*)
next
case *4* **thus** *?case* **by**(*auto intro*!: *delete.inorder_delete*)
next
case *6* **thus** *?case* **using** *update.update_type* **by** (*metis Un_iff*)
next
case *7* **thus** *?case* **using** *delete.delete_type* **by** *blast*
qed (*auto simp*: *empty_def*)

end

34 AA Tree Implementation of Sets

theory *AA_Set*
imports
Isin2
Cmp
begin

type_synonym *'a aa_tree* = (*'a*∗*nat*) *tree*

definition *empty* :: *'a aa_tree* **where**
empty = *Leaf*

fun *lvl* :: *'a aa_tree* ⇒ *nat* **where**
lvl Leaf = *0* |
lvl (*Node* _ (_, *lv*) _) = *lv*

```
fun invar :: 'a aa_tree ⇒ bool where
invar Leaf = True |
invar (Node l (a, h) r) =
 (invar l ∧ invar r ∧
  h = lvl l + 1 ∧ (h = lvl r + 1 ∨ (∃ lr b rr. r = Node lr (b,h) rr ∧ h =
lvl rr + 1)))

fun skew :: 'a aa_tree ⇒ 'a aa_tree where
skew (Node (Node t1 (b, lvb) t2) (a, lva) t3) =
  (if lva = lvb then Node t1 (b, lvb) (Node t2 (a, lva) t3) else Node (Node
t1 (b, lvb) t2) (a, lva) t3) |
skew t = t

fun split :: 'a aa_tree ⇒ 'a aa_tree where
split (Node t1 (a, lva) (Node t2 (b, lvb) (Node t3 (c, lvc) t4))) =
   (if lva = lvb ∧ lvb = lvc — lva = lvc suffices
    then Node (Node t1 (a,lva) t2) (b,lva+1) (Node t3 (c, lva) t4)
    else Node t1 (a,lva) (Node t2 (b,lvb) (Node t3 (c,lvc) t4))) |
split t = t

hide_const (open) insert

fun insert :: 'a::linorder ⇒ 'a aa_tree ⇒ 'a aa_tree where
insert x Leaf = Node Leaf (x, 1) Leaf |
insert x (Node t1 (a,lv) t2) =
  (case cmp x a of
     LT ⇒ split (skew (Node (insert x t1) (a,lv) t2)) |
     GT ⇒ split (skew (Node t1 (a,lv) (insert x t2))) |
     EQ ⇒ Node t1 (x, lv) t2)

fun sngl :: 'a aa_tree ⇒ bool where
sngl Leaf = False |
sngl (Node _ _ Leaf) = True |
sngl (Node _ (_, lva) (Node _ (_, lvb) _)) = (lva > lvb)

definition adjust :: 'a aa_tree ⇒ 'a aa_tree where
adjust t =
 (case t of
  Node l (x,lv) r ⇒
   (if lvl l >= lv−1 ∧ lvl r >= lv−1 then t else
    if lvl r < lv−1 ∧ sngl l then skew (Node l (x,lv−1) r) else
    if lvl r < lv−1
    then case l of
           Node t1 (a,lva) (Node t2 (b,lvb) t3)
```

```
        ⇒ Node (Node t1 (a,lva) t2) (b,lvb+1) (Node t3 (x,lv−1) r)
  else
  if lvl r < lv then split (Node l (x,lv−1) r)
  else
    case r of
      Node t1 (b,lvb) t4 ⇒
        (case t1 of
           Node t2 (a,lva) t3
             ⇒ Node (Node l (x,lv−1) t2) (a,lva+1)
                  (split (Node t3 (b, if sngl t1 then lva else lva+1) t4)))))
```

In the paper, the last case of *adjust* is expressed with the help of an incorrect auxiliary function `nlvl`.

Function *split_max* below is called `dellrg` in the paper. The latter is incorrect for two reasons: `dellrg` is meant to delete the largest element but recurses on the left instead of the right subtree; the invariant is not restored.

```
fun split_max :: 'a aa_tree ⇒ 'a aa_tree * 'a where
split_max (Node l (a,lv) Leaf) = (l,a) |
split_max (Node l (a,lv) r) = (let (r',b) = split_max r in (adjust(Node l
(a,lv) r'), b))

fun delete :: 'a::linorder ⇒ 'a aa_tree ⇒ 'a aa_tree where
delete _ Leaf = Leaf |
delete x (Node l (a,lv) r) =
  (case cmp x a of
     LT ⇒ adjust (Node (delete x l) (a,lv) r) |
     GT ⇒ adjust (Node l (a,lv) (delete x r)) |
     EQ ⇒ (if l = Leaf then r
           else let (l',b) = split_max l in adjust (Node l' (b,lv) r)))

fun pre_adjust where
pre_adjust (Node l (a,lv) r) = (invar l ∧ invar r ∧
    ((lv = lvl l + 1 ∧ (lv = lvl r + 1 ∨ lv = lvl r + 2 ∨ lv = lvl r ∧ sngl
r)) ∨
     (lv = lvl l + 2 ∧ (lv = lvl r + 1 ∨ lv = lvl r ∧ sngl r))))

declare pre_adjust.simps [simp del]
```

34.1 Auxiliary Proofs

```
lemma split_case: split t = (case t of
  Node t1 (x,lvx) (Node t2 (y,lvy) (Node t3 (z,lvz) t4)) ⇒
   (if lvx = lvy ∧ lvy = lvz
    then Node (Node t1 (x,lvx) t2) (y,lvx+1) (Node t3 (z,lvx) t4)
```

else t)
| *t* ⇒ *t*)
by(*auto split*: *tree.split*)

lemma *skew_case*: *skew t* = (*case t of*
Node (*Node t1* (*y*,*lvy*) *t2*) (*x*,*lvx*) *t3* ⇒
(*if lvx* = *lvy then Node t1* (*y*, *lvx*) (*Node t2* (*x*,*lvx*) *t3*) *else t*)
| *t* ⇒ *t*)
by(*auto split*: *tree.split*)

lemma *lvl_0_iff*: *invar t* ⟹ *lvl t* = *0* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *lvl_Suc_iff*: *lvl t* = *Suc n* ⟷ (∃ *l a r*. *t* = *Node l* (*a*,*Suc n*) *r*)
by(*cases t*) *auto*

lemma *lvl_skew*: *lvl* (*skew t*) = *lvl t*
by(*cases t rule*: *skew.cases*) *auto*

lemma *lvl_split*: *lvl* (*split t*) = *lvl t* ∨ *lvl* (*split t*) = *lvl t* + *1* ∧ *sngl* (*split t*)
by(*cases t rule*: *split.cases*) *auto*

lemma *invar_2Nodes*:*invar* (*Node l* (*x*,*lv*) (*Node rl* (*rx*, *rlv*) *rr*)) =
(*invar l* ∧ *invar* ⟨*rl*, (*rx*, *rlv*), *rr*⟩ ∧ *lv* = *Suc* (*lvl l*) ∧
(*lv* = *Suc rlv* ∨ *rlv* = *lv* ∧ *lv* = *Suc* (*lvl rr*)))
by *simp*

lemma *invar_NodeLeaf*[*simp*]:
invar (*Node l* (*x*,*lv*) *Leaf*) = (*invar l* ∧ *lv* = *Suc* (*lvl l*) ∧ *lv* = *Suc 0*)
by *simp*

lemma *sngl_if_invar*: *invar* (*Node l* (*a*, *n*) *r*) ⟹ *n* = *lvl r* ⟹ *sngl r*
by(*cases r rule*: *sngl.cases*) *clarsimp*+

34.2 Invariance

34.2.1 Proofs for insert

lemma *lvl_insert_aux*:
lvl (*insert x t*) = *lvl t* ∨ *lvl* (*insert x t*) = *lvl t* + *1* ∧ *sngl* (*insert x t*)
apply(*induction t*)
apply (*auto simp*: *lvl_skew*)
apply (*metis Suc_eq_plus1 lvl.simps*(*2*) *lvl_split lvl_skew*)+

done

lemma *lvl_insert*: **obtains**
  (*Same*) *lvl* (*insert x t*) = *lvl t* |
  (*Incr*) *lvl* (*insert x t*) = *lvl t* + *1 sngl* (*insert x t*)
using *lvl_insert_aux* **by** *blast*

lemma *lvl_insert_sngl*: *invar t* ⟹ *sngl t* ⟹ *lvl*(*insert x t*) = *lvl t*
proof (*induction t rule*: *insert.induct*)
  case (*2 x t1 a lv t2*)
  consider (*LT*) *x* < *a* | (*GT*) *x* > *a* | (*EQ*) *x* = *a*
    using *less_linear* **by** *blast*
  thus *?case* **proof** *cases*
    case *LT*
     thus *?thesis* **using** *2* **by** (*auto simp add*: *skew_case split_case split*: *tree.splits*)
  next
    case *GT*
    thus *?thesis* **using** *2*
    proof (*cases t1 rule*: *tree2_cases*)
      case *Node*
      thus *?thesis* **using** *2 GT*
        apply (*auto simp add*: *skew_case split_case split*: *tree.splits*)
        by (*metis less_not_refl2 lvl.simps*(*2*) *lvl_insert_aux n_not_Suc_n sngl.simps*(*3*))+
    qed (*auto simp add*: *lvl_0_iff*)
  qed *simp*
qed *simp*

lemma *skew_invar*: *invar t* ⟹ *skew t* = *t*
by(*cases t rule*: *skew.cases*) *auto*

lemma *split_invar*: *invar t* ⟹ *split t* = *t*
by(*cases t rule*: *split.cases*) *clarsimp*+

lemma *invar_NodeL*:
  ⟦ *invar*(*Node l* (*x*, *n*) *r*); *invar l*′; *lvl l*′ = *lvl l* ⟧ ⟹ *invar*(*Node l*′ (*x*, *n*) *r*)
by(*auto*)

lemma *invar_NodeR*:
  ⟦ *invar*(*Node l* (*x*, *n*) *r*); *n* = *lvl r* + *1*; *invar r*′; *lvl r*′ = *lvl r* ⟧ ⟹ *invar*(*Node l* (*x*, *n*) *r*′)
by(*auto*)

lemma *invar_NodeR2*:
⟦ *invar*(*Node l* (*x*, *n*) *r*); *sngl r′*; *n* = *lvl r* + *1*; *invar r′*; *lvl r′* = *n* ⟧ ⟹
invar(*Node l* (*x*, *n*) *r′*)
by(*cases r′ rule*: *sngl.cases*) *clarsimp*+

lemma *lvl_insert_incr_iff*: (*lvl*(*insert a t*) = *lvl t* + *1*) ⟷
(∃ *l x r*. *insert a t* = *Node l* (*x*, *lvl t* + *1*) *r* ∧ *lvl l* = *lvl r*)
apply(*cases t rule*: *tree2_cases*)
apply(*auto simp add*: *skew_case split_case split*: *if_splits*)
apply(*auto split*: *tree.splits if_splits*)
done

lemma *invar_insert*: *invar t* ⟹ *invar*(*insert a t*)
proof(*induction t rule*: *tree2_induct*)
case *N*: (*Node l x n r*)
hence *il*: *invar l* **and** *ir*: *invar r* **by** *auto*
note *iil* = *N.IH*(*1*)[*OF il*]
note *iir* = *N.IH*(*2*)[*OF ir*]
let *?t* = *Node l* (*x*, *n*) *r*
have *a* < *x* ∨ *a* = *x* ∨ *x* < *a* **by** *auto*
moreover
have *?case* **if** *a* < *x*
proof (*cases rule*: *lvl_insert*[*of a l*])
case (*Same*) **thus** *?thesis*
using ‹*a*<*x*› *invar_NodeL*[*OF N.prems iil Same*]
by (*simp add*: *skew_invar split_invar del*: *invar.simps*)
next
case (*Incr*)
then obtain *t1 w t2* **where** *ial*[*simp*]: *insert a l* = *Node t1* (*w*, *n*) *t2*
using *N.prems* **by** (*auto simp*: *lvl_Suc_iff*)
have *l12*: *lvl t1* = *lvl t2*
by (*metis Incr*(*1*) *ial lvl_insert_incr_iff tree.inject*)
have *insert a ?t* = *split*(*skew*(*Node* (*insert a l*) (*x*,*n*) *r*))
by(*simp add*: ‹*a*<*x*›)
also have *skew*(*Node* (*insert a l*) (*x*,*n*) *r*) = *Node t1* (*w*,*n*) (*Node t2* (*x*,*n*) *r*)
by(*simp*)
also have *invar*(*split* . . .)
proof (*cases r rule*: *tree2_cases*)
case *Leaf*
hence *l* = *Leaf* **using** *N.prems* **by**(*auto simp*: *lvl_0_iff*)
thus *?thesis* **using** *Leaf ial* **by** *simp*

```
  next
    case [simp]: (Node t3 y m t4)
    show ?thesis
    proof cases
      assume m = n thus ?thesis using N(3) iil by(auto)
    next
      assume m ≠ n thus ?thesis using N(3) iil l12 by(auto)
    qed
  qed
  finally show ?thesis .
qed
moreover
have ?case if x < a
proof −
  from ‹invar ?t› have n = lvl r ∨ n = lvl r + 1 by auto
  thus ?case
  proof
    assume 0: n = lvl r
    have insert a ?t = split(skew(Node l (x, n) (insert a r)))
      using ‹a>x› by(auto)
    also have skew(Node l (x,n) (insert a r)) = Node l (x,n) (insert a r)
      using N.prems by(simp add: skew_case split: tree.split)
    also have invar(split ...)
    proof −
      from lvl_insert_sngl[OF ir sngl_if_invar[OF ‹invar ?t› 0], of a]
      obtain t1 y t2 where iar: insert a r = Node t1 (y,n) t2
        using N.prems 0 by (auto simp: lvl_Suc_iff)
      from N.prems iar 0 iir
      show ?thesis by (auto simp: split_case split: tree.splits)
    qed
    finally show ?thesis .
  next
    assume 1: n = lvl r + 1
    hence sngl ?t by(cases r) auto
    show ?thesis
    proof (cases rule: lvl_insert[of a r])
      case (Same)
     show ?thesis using ‹x<a› il ir invar_NodeR[OF N.prems 1 iir Same]
        by (auto simp add: skew_invar split_invar)
    next
      case (Incr)
      thus ?thesis using invar_NodeR2[OF ‹invar ?t› Incr(2) 1 iir] 1 ‹x
< a›
        by (auto simp add: skew_invar split_invar split: if_splits)
```

qed
qed
qed
moreover
have $a = x \Longrightarrow$ *?case* **using** *N.prems* **by** *auto*
ultimately show *?case* **by** *blast*
qed *simp*

34.2.2 Proofs for delete

lemma *invarL*: *ASSUMPTION*(*invar* $\langle l, (a, lv), r\rangle$) $\Longrightarrow$ *invar l*
by(*simp add*: *ASSUMPTION_def*)

lemma *invarR*: *ASSUMPTION*(*invar* $\langle l, (a,lv), r\rangle$) $\Longrightarrow$ *invar r*
by(*simp add*: *ASSUMPTION_def*)

lemma *sngl_NodeI*:
sngl (*Node l* (*a*,*lv*) *r*) $\Longrightarrow$ *sngl* (*Node l'* (*a'*, *lv*) *r*)
by(*cases r rule*: *tree2_cases*) (*simp_all*)

declare *invarL*[*simp*] *invarR*[*simp*]

lemma *pre_cases*:
assumes *pre_adjust* (*Node l* (*x*,*lv*) *r*)
obtains
(*tSngl*) *invar l* $\wedge$ *invar r* $\wedge$
lv = *Suc* (*lvl r*) $\wedge$ *lvl l* = *lvl r* |
(*tDouble*) *invar l* $\wedge$ *invar r* $\wedge$
lv = *lvl r* $\wedge$ *Suc* (*lvl l*) = *lvl r* $\wedge$ *sngl r* |
(*rDown*) *invar l* $\wedge$ *invar r* $\wedge$
lv = *Suc* (*Suc* (*lvl r*)) $\wedge$ *lv* = *Suc* (*lvl l*) |
(*lDown_tSngl*) *invar l* $\wedge$ *invar r* $\wedge$
lv = *Suc* (*lvl r*) $\wedge$ *lv* = *Suc* (*Suc* (*lvl l*)) |
(*lDown_tDouble*) *invar l* $\wedge$ *invar r* $\wedge$
lv = *lvl r* $\wedge$ *lv* = *Suc* (*Suc* (*lvl l*)) $\wedge$ *sngl r*
using *assms* **unfolding** *pre_adjust.simps*
by *auto*

declare *invar.simps*(*2*)[*simp del*] *invar_2Nodes*[*simp add*]

lemma *invar_adjust*:
assumes *pre*: *pre_adjust* (*Node l* (*a*,*lv*) *r*)
shows *invar*(*adjust* (*Node l* (*a*,*lv*) *r*))

using *pre* **proof** (*cases rule*: *pre_cases*)
  case (*tDouble*) **thus** *?thesis* **unfolding** *adjust_def* **by** (*cases r*) (*auto simp*: *invar.simps*(*2*))
next
  case (*rDown*)
  from *rDown* **obtain** *llv ll la lr* **where** *l*: *l* = *Node ll* (*la*, *llv*) *lr* **by** (*cases l*) *auto*
  from *rDown* **show** *?thesis* **unfolding** *adjust_def* **by** (*auto simp*: *l invar.simps*(*2*) *split*: *tree.splits*)
next
  case (*lDown_tDouble*)
  from *lDown_tDouble* **obtain** *rlv rr ra rl* **where** *r*: *r* = *Node rl* (*ra*, *rlv*) *rr* **by** (*cases r*) *auto*
  from *lDown_tDouble* **and** *r* **obtain** *rrlv rrr rra rrl* **where**
    rr :*rr* = *Node rrr* (*rra*, *rrlv*) *rrl* **by** (*cases rr*) *auto*
  from *lDown_tDouble* **show** *?thesis* **unfolding** *adjust_def r rr*
   apply (*cases rl rule*: *tree2_cases*) **apply** (*auto simp add*: *invar.simps*(*2*) *split*!: *if_split*)
   using *lDown_tDouble* **by** (*auto simp*: *split_case lvl_0_iff elim*:*lvl.elims split*: *tree.split*)
qed (*auto simp*: *split_case invar.simps*(*2*) *adjust_def split*: *tree.splits*)

lemma *lvl_adjust*:
  assumes *pre_adjust* (*Node l* (*a*,*lv*) *r*)
  shows *lv* = *lvl* (*adjust*(*Node l* (*a*,*lv*) *r*)) ∨ *lv* = *lvl* (*adjust*(*Node l* (*a*,*lv*) *r*)) + *1*
using *assms*(*1*)
proof(*cases rule*: *pre_cases*)
  case *lDown_tSngl* **thus** *?thesis*
    using *lvl_split*[*of* ⟨*l*, (*a*, *lvl r*), *r*⟩] **by** (*auto simp*: *adjust_def*)
next
  case *lDown_tDouble* **thus** *?thesis*
    by (*auto simp*: *adjust_def invar.simps*(*2*) *split*: *tree.split*)
qed (*auto simp*: *adjust_def split*: *tree.splits*)

lemma *sngl_adjust*: **assumes** *pre_adjust* (*Node l* (*a*,*lv*) *r*)
  sngl ⟨*l*, (*a*, *lv*), *r*⟩ *lv* = *lvl* (*adjust* ⟨*l*, (*a*, *lv*), *r*⟩)
  shows *sngl* (*adjust* ⟨*l*, (*a*, *lv*), *r*⟩)
using *assms* **proof** (*cases rule*: *pre_cases*)
  case *rDown*
  thus *?thesis* **using** *assms*(*2*,*3*) **unfolding** *adjust_def*
    by (*auto simp add*: *skew_case*) (*auto split*: *tree.split*)
qed (*auto simp*: *adjust_def skew_case split_case split*: *tree.split*)

definition *post_del t t′ ==*
invar t′ ∧
(lvl t′ = lvl t ∨ lvl t′ + 1 = lvl t) ∧
(lvl t′ = lvl t ∧ sngl t ⟶ sngl t′)

lemma *pre_adj_if_postR*:
invar⟨lv, (l, a), r⟩ ⟹ post_del r r′ ⟹ pre_adjust ⟨lv, (l, a), r′⟩
by(*cases sngl r*)
(*auto simp*: *pre_adjust.simps post_del_def invar.simps*(*2*) *elim*: *sngl.elims*)

lemma *pre_adj_if_postL*:
invar⟨l, (a, lv), r⟩ ⟹ post_del l l′ ⟹ pre_adjust ⟨l′, (b, lv), r⟩
by(*cases sngl r*)
(*auto simp*: *pre_adjust.simps post_del_def invar.simps*(*2*) *elim*: *sngl.elims*)

lemma *post_del_adjL*:
⟦ invar⟨l, (a, lv), r⟩; pre_adjust ⟨l′, (b, lv), r⟩ ⟧
⟹ post_del ⟨l, (a, lv), r⟩ (adjust ⟨l′, (b, lv), r⟩)
unfolding *post_del_def*
by (*metis invar_adjust lvl_adjust sngl_NodeI sngl_adjust lvl.simps*(*2*))

lemma *post_del_adjR*:
assumes *invar⟨l, (a,lv), r⟩ pre_adjust ⟨l, (a,lv), r′⟩ post_del r r′*
shows *post_del ⟨l, (a,lv), r⟩ (adjust ⟨l, (a,lv), r′⟩)*
proof(*unfold post_del_def*, *safe del*: *disjCI*)
let *?t = ⟨l, (a,lv), r⟩*
let *?t′ = adjust ⟨l, (a,lv), r′⟩*
show *invar ?t′* **by**(*rule invar_adjust*[*OF assms*(*2*)])
show *lvl ?t′ = lvl ?t ∨ lvl ?t′ + 1 = lvl ?t*
using *lvl_adjust*[*OF assms*(*2*)] **by** *auto*
show *sngl ?t′* **if** *as*: *lvl ?t′ = lvl ?t sngl ?t*
proof −
have *s*: *sngl ⟨l, (a,lv), r′⟩*
proof(*cases r′ rule*: *tree2_cases*)
case *Leaf* **thus** *?thesis* **by** *simp*
next
case *Node* **thus** *?thesis* **using** *as*(*2*) *assms*(*1,3*)
by (*cases r rule*: *tree2_cases*) (*auto simp*: *post_del_def*)
qed
show *?thesis* **using** *as*(*1*) *sngl_adjust*[*OF assms*(*2*) *s*] **by** *simp*
qed
qed

declare *prod.splits*[*split*]

```
theorem post_split_max:
 ⟦ invar t; (t', x) = split_max t; t ≠ Leaf ⟧ ⟹ post_del t t'
proof (induction t arbitrary: t' rule: split_max.induct)
  case (2 l a lv rl bl rr)
  let ?r = ⟨rl, bl, rr⟩
  let ?t = ⟨l, (a, lv), ?r⟩
  from 2.prems(2) obtain r' where r': (r', x) = split_max ?r
    and [simp]: t' = adjust ⟨l, (a, lv), r'⟩ by auto
  from  2.IH[OF _ r'] ‹invar ?t› have post: post_del ?r r' by simp
  note preR = pre_adj_if_postR[OF ‹invar ?t› post]
  show ?case by (simp add: post_del_adjR[OF 2.prems(1) preR post])
qed (auto simp: post_del_def)

theorem post_delete: invar t ⟹ post_del t (delete x t)
proof (induction t rule: tree2_induct)
  case (Node l a lv r)

  let ?l' = delete x l and ?r' = delete x r
  let ?t = Node l (a,lv) r let ?t' = delete x ?t

  from Node.prems have inv_l: invar l and inv_r: invar r by (auto)

  note post_l' = Node.IH(1)[OF inv_l]
  note preL = pre_adj_if_postL[OF Node.prems post_l']

  note post_r' = Node.IH(2)[OF inv_r]
  note preR = pre_adj_if_postR[OF Node.prems post_r']

  show ?case
  proof (cases rule: linorder_cases[of x a])
    case less
    thus ?thesis using Node.prems by (simp add: post_del_adjL preL)
  next
    case greater
     thus ?thesis using Node.prems by (simp add: post_del_adjR preR
post_r')
  next
    case equal
    show ?thesis
    proof cases
      assume l = Leaf thus ?thesis using equal Node.prems
        by(auto simp: post_del_def invar.simps(2))
    next
```

assume $l \neq Leaf$ **thus** *?thesis* **using** *equal*
by *simp* (*metis Node.prems inv_l post_del_adjL post_split_max pre_adj_if_postL*)
qed
qed
qed (*simp add*: *post_del_def*)

declare *invar_2Nodes*[*simp del*]

34.3 Functional Correctness

34.3.1 Proofs for insert

lemma *inorder_split*: *inorder*(*split t*) = *inorder t*
by(*cases t rule*: *split.cases*) (*auto*)

lemma *inorder_skew*: *inorder*(*skew t*) = *inorder t*
by(*cases t rule*: *skew.cases*) (*auto*)

lemma *inorder_insert*:
sorted(*inorder t*) ⟹ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps inorder_split inorder_skew*)

34.3.2 Proofs for delete

lemma *inorder_adjust*: *t* ≠ *Leaf* ⟹ *pre_adjust t* ⟹ *inorder*(*adjust t*) = *inorder t*
by(*cases t*)
(*auto simp*: *adjust_def inorder_skew inorder_split invar.simps*(*2*) *pre_adjust.simps split*: *tree.splits*)

lemma *split_maxD*:
⟦ *split_max t* = (*t'*,*x*); *t* ≠ *Leaf*; *invar t* ⟧ ⟹ *inorder t'* @ [*x*] = *inorder t*
by(*induction t arbitrary*: *t'* *rule*: *split_max.induct*)
(*auto simp*: *sorted_lems inorder_adjust pre_adj_if_postR post_split_max split*: *prod.splits*)

lemma *inorder_delete*:
invar t ⟹ *sorted*(*inorder t*) ⟹ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_adjust pre_adj_if_postL pre_adj_if_postR post_split_max post_delete split_maxD split*: *prod.splits*)

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *invar*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *invar_insert*)
next
case *7* **thus** *?case* **using** *post_delete* **by**(*auto simp*: *post_del_def*)
qed

end

35 AA Tree Implementation of Maps

theory *AA_Map*
imports
AA_Set
Lookup2
begin

fun *update* :: *'a*::*linorder* ⇒ *'b* ⇒ (*'a*∗*'b*) *aa_tree* ⇒ (*'a*∗*'b*) *aa_tree* **where**
update x y Leaf = *Node Leaf* ((*x*,*y*), *1*) *Leaf* |
update x y (*Node t1* ((*a*,*b*), *lv*) *t2*) =
(*case cmp x a of*
LT ⇒ *split* (*skew* (*Node* (*update x y t1*) ((*a*,*b*), *lv*) *t2*)) |
GT ⇒ *split* (*skew* (*Node t1* ((*a*,*b*), *lv*) (*update x y t2*))) |
EQ ⇒ *Node t1* ((*x*,*y*), *lv*) *t2*)

fun *delete* :: *'a*::*linorder* ⇒ (*'a*∗*'b*) *aa_tree* ⇒ (*'a*∗*'b*) *aa_tree* **where**
delete _ *Leaf* = *Leaf* |
delete x (*Node l* ((*a*,*b*), *lv*) *r*) =
(*case cmp x a of*

LT ⇒ *adjust* (*Node* (*delete x l*) ((*a*,*b*), *lv*) *r*) |
GT ⇒ *adjust* (*Node l* ((*a*,*b*), *lv*) (*delete x r*)) |
EQ ⇒ (*if l* = *Leaf then r*
else let (*l*′,*ab*′) = *split_max l in adjust* (*Node l*′ (*ab*′, *lv*) *r*)))

35.1 Invariance

35.1.1 Proofs for insert

lemma *lvl_update_aux*:
lvl (*update x y t*) = *lvl t* ∨ *lvl* (*update x y t*) = *lvl t* + *1* ∧ *sngl* (*update x y t*)
apply(*induction t*)
apply (*auto simp*: *lvl_skew*)
apply (*metis Suc_eq_plus1 lvl.simps*(*2*) *lvl_split lvl_skew*)+
done

lemma *lvl_update*: **obtains**
(*Same*) *lvl* (*update x y t*) = *lvl t* |
(*Incr*) *lvl* (*update x y t*) = *lvl t* + *1 sngl* (*update x y t*)
using *lvl_update_aux* **by** *fastforce*

declare *invar.simps*(*2*)[*simp*]

lemma *lvl_update_sngl*: *invar t* ⟹ *sngl t* ⟹ *lvl*(*update x y t*) = *lvl t*
proof (*induction t rule*: *update.induct*)
case (*2 x y t1 a b lv t2*)
consider (*LT*) *x* < *a* | (*GT*) *x* > *a* | (*EQ*) *x* = *a*
using *less_linear* **by** *blast*
thus *?case* **proof** *cases*
case *LT*
thus *?thesis* **using** *2* **by** (*auto simp add*: *skew_case split_case split*: *tree.splits*)
next
case *GT*
thus *?thesis* **using** *2* **proof** (*cases t1*)
case *Node*
thus *?thesis* **using** *2 GT*
apply (*auto simp add*: *skew_case split_case split*: *tree.splits*)
by (*metis less_not_refl2 lvl.simps*(*2*) *lvl_update_aux n_not_Suc_n sngl.simps*(*3*))+
qed (*auto simp add*: *lvl_0_iff*)
qed *simp*
qed *simp*

lemma *lvl_update_incr_iff*: (*lvl*(*update a b t*) = *lvl t* + *1*) ⟷
(∃ *l x r*. *update a b t* = *Node l* (*x*,*lvl t* + *1*) *r* ∧ *lvl l* = *lvl r*)
apply(*cases t*)
apply(*auto simp add*: *skew_case split_case split*: *if_splits*)
apply(*auto split*: *tree.splits if_splits*)
done

lemma *invar_update*: *invar t* ⟹ *invar*(*update a b t*)
proof(*induction t rule*: *tree2_induct*)
case *N*: (*Node l xy n r*)
hence *il*: *invar l* **and** *ir*: *invar r* **by** *auto*
note *iil* = *N.IH*(*1*)[*OF il*]
note *iir* = *N.IH*(*2*)[*OF ir*]
obtain *x y* **where** [*simp*]: *xy* = (*x*,*y*) **by** *fastforce*
let *?t* = *Node l* (*xy*, *n*) *r*
have *a* < *x* ∨ *a* = *x* ∨ *x* < *a* **by** *auto*
moreover
have *?case* **if** *a* < *x*
proof (*cases rule*: *lvl_update*[*of a b l*])
case (*Same*) **thus** *?thesis*
using ‹*a*<*x*› *invar_NodeL*[*OF N.prems iil Same*]
by (*simp add*: *skew_invar split_invar del*: *invar.simps*)
next
case (*Incr*)
then obtain *t1 w t2* **where** *ial*[*simp*]: *update a b l* = *Node t1* (*w*, *n*) *t2*
using *N.prems* **by** (*auto simp*: *lvl_Suc_iff*)
have *l12*: *lvl t1* = *lvl t2*
by (*metis Incr*(*1*) *ial lvl_update_incr_iff tree.inject*)
have *update a b ?t* = *split*(*skew*(*Node* (*update a b l*) (*xy*, *n*) *r*))
by(*simp add*: ‹*a*<*x*›)
also have *skew*(*Node* (*update a b l*) (*xy*, *n*) *r*) = *Node t1* (*w*, *n*) (*Node t2* (*xy*, *n*) *r*)
by(*simp*)
also have *invar*(*split* . . .)
proof (*cases r rule*: *tree2_cases*)
case *Leaf*
hence *l* = *Leaf* **using** *N.prems* **by**(*auto simp*: *lvl_0_iff*)
thus *?thesis* **using** *Leaf ial* **by** *simp*
next
case [*simp*]: (*Node t3 y m t4*)
show *?thesis*
proof *cases*
assume *m* = *n* **thus** *?thesis* **using** *N*(*3*) *iil* **by**(*auto*)

```
    next
      assume m ≠ n thus ?thesis using N(3) iil l12 by(auto)
    qed
  qed
  finally show ?thesis .
qed
moreover
have ?case if x < a
proof −
  from ‹invar ?t› have n = lvl r ∨ n = lvl r + 1 by auto
  thus ?case
  proof
    assume 0: n = lvl r
    have update a b ?t = split(skew(Node l (xy, n) (update a b r)))
      using ‹a>x› by(auto)
   also have skew(Node l (xy, n) (update a b r)) = Node l (xy, n) (update
a b r)
      using N.prems by(simp add: skew_case split: tree.split)
    also have invar(split ...)
    proof −
      from lvl_update_sngl[OF ir sngl_if_invar[OF ‹invar ?t› 0], of a b]
      obtain t1 p t2 where iar: update a b r = Node t1 (p, n) t2
        using N.prems 0 by (auto simp: lvl_Suc_iff)
      from N.prems iar 0 iir
      show ?thesis by (auto simp: split_case split: tree.splits)
    qed
    finally show ?thesis .
  next
    assume 1: n = lvl r + 1
    hence sngl ?t by(cases r) auto
    show ?thesis
    proof (cases rule: lvl_update[of a b r])
      case (Same)
     show ?thesis using ‹x<a› il ir invar_NodeR[OF N.prems 1 iir Same]
        by (auto simp add: skew_invar split_invar)
    next
      case (Incr)
     thus ?thesis using invar_NodeR2[OF ‹invar ?t› Incr(2) 1 iir] 1 ‹x
< a›
        by (auto simp add: skew_invar split_invar split: if_splits)
    qed
  qed
qed
moreover
```

have $a = x \Longrightarrow$ *?case* **using** *N.prems* **by** *auto*
ultimately show *?case* **by** *blast*
qed *simp*

35.1.2 Proofs for delete

declare *invar.simps*(*2*)[*simp del*]

theorem *post_delete*: *invar t* $\Longrightarrow$ *post_del t* (*delete x t*)
proof (*induction t rule*: *tree2_induct*)
case (*Node l ab lv r*)

obtain *a b* **where** [*simp*]: *ab* = (*a*,*b*) **by** *fastforce*

let *?l'* = *delete x l* **and** *?r'* = *delete x r*
let *?t* = *Node l* (*ab*, *lv*) *r* **let** *?t'* = *delete x ?t*

from *Node.prems* **have** *inv_l*: *invar l* **and** *inv_r*: *invar r* **by** (*auto*)

note *post_l'* = *Node.IH*(*1*)[*OF inv_l*]
note *preL* = *pre_adj_if_postL*[*OF Node.prems post_l'*]

note *post_r'* = *Node.IH*(*2*)[*OF inv_r*]
note *preR* = *pre_adj_if_postR*[*OF Node.prems post_r'*]

show *?case*
proof (*cases rule*: *linorder_cases*[*of x a*])
case *less*
thus *?thesis* **using** *Node.prems* **by** (*simp add*: *post_del_adjL preL*)
next
case *greater*
thus *?thesis* **using** *Node.prems preR* **by** (*simp add*: *post_del_adjR post_r'*)
next
case *equal*
show *?thesis*
proof *cases*
assume *l* = *Leaf* **thus** *?thesis* **using** *equal Node.prems*
by(*auto simp*: *post_del_def invar.simps*(*2*))
next
assume *l* ≠ *Leaf* **thus** *?thesis* **using** *equal Node.prems*
by *simp* (*metis inv_l post_del_adjL post_split_max pre_adj_if_postL*)
qed
qed

qed (*simp add*: *post_del_def*)

35.2 Functional Correctness Proofs

theorem *inorder_update*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*update x y t*) = *upd_list x y* (*inorder t*)
by (*induct t*) (*auto simp*: *upd_list_simps inorder_split inorder_skew*)

theorem *inorder_delete*:
⟦*invar t*; *sorted1*(*inorder t*)⟧ $\Longrightarrow$
inorder (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_adjust pre_adj_if_postL pre_adj_if_postR*

post_split_max post_delete split_maxD split: *prod.splits*)

interpretation *I*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and**
delete = *delete*
and *inorder* = *inorder* **and** *inv* = *invar*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *invar_update*)
next
case *7* **thus** *?case* **using** *post_delete* **by**(*auto simp*: *post_del_def*)
qed

end

36 Join-Based Implementation of Sets

theory *Set2_Join*
imports
Isin2
begin

This theory implements the set operations *insert*, *delete*, *union*, *inter*section and *diff*erence. The implementation is based on binary search trees. All operations are reduced to a single operation *join* l x r that joins two BSTs l and r and an element x such that $l < x < r$.

The theory is based on theory *HOL−Data_Structures.Tree2* where nodes have an additional field. This field is ignored here but it means that this theory can be instantiated with red-black trees (see theory `Set2_Join_RBT.thy`) and other balanced trees. This approach is very concrete and fixes the type of trees. Alternatively, one could assume some abstract type $'t$ of trees with suitable decomposition and recursion operators on it.

locale *Set2_Join* =
fixes *join* :: ($'a$::*linorder*∗$'b$) *tree* ⇒ $'a$ ⇒ ($'a$∗$'b$) *tree* ⇒ ($'a$∗$'b$) *tree*
fixes *inv* :: ($'a$∗$'b$) *tree* ⇒ *bool*
assumes *set_join*: *set_tree* (*join* *l* *a* *r*) = *set_tree* *l* ∪ {*a*} ∪ *set_tree* *r*
assumes *bst_join*: *bst* (*Node* *l* (*a*, *b*) *r*) ⟹ *bst* (*join* *l* *a* *r*)
assumes *inv_Leaf*: *inv* ⟨⟩
assumes *inv_join*: ⟦ *inv* *l*; *inv* *r* ⟧ ⟹ *inv* (*join* *l* *a* *r*)
assumes *inv_Node*: ⟦ *inv* (*Node* *l* (*a*,*b*) *r*) ⟧ ⟹ *inv* *l* ∧ *inv* *r*
begin

declare *set_join* [*simp*] *Let_def*[*simp*]

36.1 *split_min*

fun *split_min* :: ($'a$∗$'b$) *tree* ⇒ $'a$ × ($'a$∗$'b$) *tree* **where**
split_min (*Node* *l* (*a*, _) *r*) =
(*if* *l* = *Leaf* *then* (*a*,*r*) *else let* (*m*,*l'*) = *split_min* *l* *in* (*m*, *join* *l'* *a* *r*))

lemma *split_min_set*:
⟦ *split_min* *t* = (*m*,*t'*); *t* ≠ *Leaf* ⟧ ⟹ *m* ∈ *set_tree* *t* ∧ *set_tree* *t* = {*m*} ∪ *set_tree* *t'*
proof(*induction* *t* *arbitrary*: *t'* *rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*auto* *split*: *prod.splits* *if_splits* *dest*: *inv_Node*)
next
case *Leaf* **thus** *?case* **by** *simp*
qed

lemma *split_min_bst*:
⟦ *split_min* *t* = (*m*,*t'*); *bst* *t*; *t* ≠ *Leaf* ⟧ ⟹ *bst* *t'* ∧ (∀ *x* ∈ *set_tree* *t'*. *m* < *x*)
proof(*induction* *t* *arbitrary*: *t'* *rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*fastforce* *simp*: *split_min_set* *bst_join* *split*: *prod.splits* *if_splits*)
next

case *Leaf* **thus** *?case* **by** *simp*
qed

lemma *split_min_inv*:
⟦ *split_min t* = (*m*,*t'*); *inv t*; *t* ≠ *Leaf* ⟧ ⟹ *inv t'*
proof(*induction t arbitrary*: *t' rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*auto simp*: *inv_join split*: *prod.splits if_splits dest*: *inv_Node*)
next
case *Leaf* **thus** *?case* **by** *simp*
qed

36.2 *join2*

definition *join2* :: ('*a*∗'*b*) *tree* ⇒ ('*a*∗'*b*) *tree* ⇒ ('*a*∗'*b*) *tree* **where**
join2 l r = (*if r* = *Leaf then l else let* (*m*,*r'*) = *split_min r in join l m r'*)

lemma *set_join2*[*simp*]: *set_tree* (*join2 l r*) = *set_tree l* ∪ *set_tree r*
by(*cases r*)(*simp_all add*: *split_min_set join2_def split*: *prod.split*)

lemma *bst_join2*: ⟦ *bst l*; *bst r*; ∀ *x* ∈ *set_tree l*. ∀ *y* ∈ *set_tree r*. *x* < *y* ⟧
⟹ *bst* (*join2 l r*)
by(*cases r*)(*simp_all add*: *bst_join split_min_set split_min_bst join2_def split*: *prod.split*)

lemma *inv_join2*: ⟦ *inv l*; *inv r* ⟧ ⟹ *inv* (*join2 l r*)
by(*cases r*)(*simp_all add*: *inv_join split_min_set split_min_inv join2_def split*: *prod.split*)

36.3 *split*

fun *split* :: '*a* ⇒ ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* × *bool* × ('*a*∗'*b*)*tree* **where**
split x Leaf = (*Leaf*, *False*, *Leaf*) |
split x (*Node l* (*a*, _) *r*) =
(*case cmp x a of*
LT ⇒ *let* (*l1*,*b*,*l2*) = *split x l in* (*l1*, *b*, *join l2 a r*) |
GT ⇒ *let* (*r1*,*b*,*r2*) = *split x r in* (*join l a r1*, *b*, *r2*) |
EQ ⇒ (*l*, *True*, *r*))

lemma *split*: *split x t* = (*l*,*b*,*r*) ⟹ *bst t* ⟹
set_tree l = {*a* ∈ *set_tree t*. *a* < *x*} ∧ *set_tree r* = {*a* ∈ *set_tree t*. *x* < *a*}
∧ (*b* = (*x* ∈ *set_tree t*)) ∧ *bst l* ∧ *bst r*
proof(*induction t arbitrary*: *l b r rule*: *tree2_induct*)

```
  case Leaf thus ?case by simp
next
  case (Node y a b z l c r)
  consider (LT) l1 xin l2 where (l1,xin,l2) = split x y
    and split x ⟨y, (a, b), z⟩ = (l1, xin, join l2 a z) and cmp x a = LT
  | (GT) r1 xin r2 where (r1,xin,r2) = split x z
    and split x ⟨y, (a, b), z⟩ = (join y a r1, xin, r2) and cmp x a = GT
  | (EQ) split x ⟨y, (a, b), z⟩ = (y, True, z) and cmp x a = EQ
    by (force split: cmp_val.splits prod.splits if_splits)

  thus ?case
  proof cases
    case (LT l1 xin l2)
    with Node.IH(1)[OF ‹(l1,xin,l2) = split x y›[symmetric]] Node.prems
    show ?thesis by (force intro!: bst_join)
  next
    case (GT r1 xin r2)
    with Node.IH(2)[OF ‹(r1,xin,r2) = split x z›[symmetric]] Node.prems
    show ?thesis by (force intro!: bst_join)
  next
    case EQ
    with Node.prems show ?thesis by auto
  qed
qed

lemma split_inv: split x t = (l,b,r) ⟹ inv t ⟹ inv l ∧ inv r
proof(induction t arbitrary: l b r rule: tree2_induct)
  case Leaf thus ?case by simp
next
  case Node
 thus ?case by(force simp: inv_join split!: prod.splits if_splits dest!: inv_Node)
qed

declare split.simps[simp del]
```

36.4 *insert*

```
definition insert :: 'a ⇒ ('a*'b) tree ⇒ ('a*'b) tree where
insert x t = (let (l,_,r) = split x t in join l x r)

lemma set_tree_insert: bst t ⟹ set_tree (insert x t) = {x} ∪ set_tree t
by(auto simp add: insert_def split split: prod.split)

lemma bst_insert: bst t ⟹ bst (insert x t)
```

by(*auto simp add*: *insert_def bst_join dest*: *split split*: *prod.split*)

lemma *inv_insert*: *inv t* $\Longrightarrow$ *inv* (*insert x t*)
by(*force simp*: *insert_def inv_join dest*: *split_inv split*: *prod.split*)

36.5 *delete*

definition *delete* :: *'a* $\Rightarrow$ (*'a*$*$*'b*) *tree* $\Rightarrow$ (*'a*$*$*'b*) *tree* **where**
delete x t = (*let* (*l*,_,*r*) = *split x t in join2 l r*)

lemma *set_tree_delete*: *bst t* $\Longrightarrow$ *set_tree* (*delete x t*) = *set_tree t* − {*x*}
by(*auto simp*: *delete_def split split*: *prod.split*)

lemma *bst_delete*: *bst t* $\Longrightarrow$ *bst* (*delete x t*)
by(*force simp add*: *delete_def intro*: *bst_join2 dest*: *split split*: *prod.split*)

lemma *inv_delete*: *inv t* $\Longrightarrow$ *inv* (*delete x t*)
by(*force simp*: *delete_def inv_join2 dest*: *split_inv split*: *prod.split*)

36.6 *union*

fun *union* :: (*'a*$*$*'b*)*tree* $\Rightarrow$ (*'a*$*$*'b*)*tree* $\Rightarrow$ (*'a*$*$*'b*)*tree* **where**
union t1 t2 =
(*if t1* = *Leaf then t2 else*
if t2 = *Leaf then t1 else*
case t1 of Node l1 (*a*, _) *r1* $\Rightarrow$
let (*l2*,_ ,*r2*) = *split a t2*;
l′ = *union l1 l2*; *r′* = *union r1 r2*
in join l′ a r′)

declare *union.simps* [*simp del*]

lemma *set_tree_union*: *bst t2* $\Longrightarrow$ *set_tree* (*union t1 t2*) = *set_tree t1* $\cup$ *set_tree t2*
proof(*induction t1 t2 rule*: *union.induct*)
case (*1 t1 t2*)
then show *?case*
by (*auto simp*: *union.simps*[*of t1 t2*] *split split*: *tree.split prod.split*)
qed

lemma *bst_union*: ⟦ *bst t1*; *bst t2* ⟧ $\Longrightarrow$ *bst* (*union t1 t2*)
proof(*induction t1 t2 rule*: *union.induct*)
case (*1 t1 t2*)
thus *?case*

by(*fastforce simp*: *union.simps*[*of t1 t2*] *set_tree_union split intro*!: *bst_join*
split: *tree.split prod.split*)
qed

lemma *inv_union*: ⟦ *inv t1*; *inv t2* ⟧ ⟹ *inv* (*union t1 t2*)
proof(*induction t1 t2 rule*: *union.induct*)
case (*1 t1 t2*)
thus *?case*
by(*auto simp*:*union.simps*[*of t1 t2*] *inv_join split_inv*
split!: *tree.split prod.split dest*: *inv_Node*)
qed

36.7 *inter*

fun *inter* :: ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* **where**
inter t1 t2 =
(*if t1* = *Leaf then Leaf else*
if t2 = *Leaf then Leaf else*
case t1 of Node l1 (*a*, _) *r1* ⇒
let (*l2*,*b*,*r2*) = *split a t2*;
l' = *inter l1 l2*; *r'* = *inter r1 r2*
in if b then join l' a r' else join2 l' r')

declare *inter.simps* [*simp del*]

lemma *set_tree_inter*:
⟦ *bst t1*; *bst t2* ⟧ ⟹ *set_tree* (*inter t1 t2*) = *set_tree t1* ∩ *set_tree t2*
proof(*induction t1 t2 rule*: *inter.induct*)
case (*1 t1 t2*)
show *?case*
proof (*cases t1 rule*: *tree2_cases*)
case *Leaf* **thus** *?thesis* **by** (*simp add*: *inter.simps*)
next
case [*simp*]: (*Node l1 a _ r1*)
show *?thesis*
proof (*cases t2* = *Leaf*)
case *True* **thus** *?thesis* **by** (*simp add*: *inter.simps*)
next
case *False*
let *?L1* = *set_tree l1* **let** *?R1* = *set_tree r1*
have ∗: *a* ∉ *?L1* ∪ *?R1* **using** ‹*bst t1*› **by** (*fastforce*)
obtain *l2 b r2* **where** *sp*: *split a t2* = (*l2*,*b*,*r2*) **using** *prod_cases3*
by *blast*

let *?L2* = *set_tree l2* **let** *?R2* = *set_tree r2* **let** *?A* = *if b then* {*a*} *else* {}
have *t2*: *set_tree t2* = *?L2* ∪ *?R2* ∪ *?A* **and**
∗∗: *?L2* ∩ *?R2* = {} *a* ∉ *?L2* ∪ *?R2 ?L1* ∩ *?R2* = {} *?L2* ∩ *?R1* = {}
using *split*[*OF sp*] ‹*bst t1*› ‹*bst t2*› **by** (*force*, *force*, *force*, *force*, *force*)
have *IHl*: *set_tree* (*inter l1 l2*) = *set_tree l1* ∩ *set_tree l2*
using *1.IH*(*1*)[*OF* _ *False* _ _ *sp*[*symmetric*]] *1.prems*(*1*,*2*) *split*[*OF sp*] **by** *simp*
have *IHr*: *set_tree* (*inter r1 r2*) = *set_tree r1* ∩ *set_tree r2*
using *1.IH*(*2*)[*OF* _ *False* _ _ *sp*[*symmetric*]] *1.prems*(*1*,*2*) *split*[*OF sp*] **by** *simp*
have *set_tree t1* ∩ *set_tree t2* = (*?L1* ∪ *?R1* ∪ {*a*}) ∩ (*?L2* ∪ *?R2* ∪ *?A*)
by(*simp add*: *t2*)
also have ... = (*?L1* ∩ *?L2*) ∪ (*?R1* ∩ *?R2*) ∪ *?A*
using ∗ ∗∗ **by** *auto*
also have ... = *set_tree* (*inter t1 t2*)
using *IHl IHr sp inter.simps*[*of t1 t2*] *False* **by**(*simp*)
finally show *?thesis* **by** *simp*
qed
qed
qed

lemma *bst_inter*: ⟦ *bst t1*; *bst t2* ⟧ ⟹ *bst* (*inter t1 t2*)
proof(*induction t1 t2 rule*: *inter.induct*)
case (*1 t1 t2*)
thus *?case*
by(*fastforce simp*: *inter.simps*[*of t1 t2*] *set_tree_inter split*
intro!: *bst_join bst_join2 split*: *tree.split prod.split*)
qed

lemma *inv_inter*: ⟦ *inv t1*; *inv t2* ⟧ ⟹ *inv* (*inter t1 t2*)
proof(*induction t1 t2 rule*: *inter.induct*)
case (*1 t1 t2*)
thus *?case*
by(*auto simp*: *inter.simps*[*of t1 t2*] *inv_join inv_join2 split_inv*
split!: *tree.split prod.split dest*: *inv_Node*)
qed

36.8 *diff*

fun *diff* :: ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* **where**

```
diff t1 t2 =
  (if t1 = Leaf then Leaf else
   if t2 = Leaf then t1 else
   case t2 of Node l2 (a, _) r2 ⇒
   let (l1,_,r1) = split a t1;
       l' = diff l1 l2; r' = diff r1 r2
   in join2 l' r')

declare diff.simps [simp del]

lemma set_tree_diff:
  ⟦ bst t1; bst t2 ⟧ ⟹ set_tree (diff t1 t2) = set_tree t1 − set_tree t2
proof(induction t1 t2 rule: diff.induct)
  case (1 t1 t2)
  show ?case
  proof (cases t2 rule: tree2_cases)
    case Leaf thus ?thesis by (simp add: diff.simps)
  next
    case [simp]: (Node l2 a _ r2)
    show ?thesis
    proof (cases t1 = Leaf)
      case True thus ?thesis by (simp add: diff.simps)
    next
      case False
      let ?L2 = set_tree l2 let ?R2 = set_tree r2
       obtain l1 b r1 where sp: split a t1 = (l1,b,r1) using prod_cases3
by blast
      let ?L1 = set_tree l1 let ?R1 = set_tree r1 let ?A = if b then {a}
else {}
      have t1: set_tree t1 = ?L1 ∪ ?R1 ∪ ?A and
           **: a ∉ ?L1 ∪ ?R1 ?L1 ∩ ?R2 = {} ?L2 ∩ ?R1 = {}
        using split[OF sp] ‹bst t1› ‹bst t2› by (force, force, force, force)
      have IHl: set_tree (diff l1 l2) = set_tree l1 − set_tree l2
       using 1.IH(1)[OF False _ _ _ sp[symmetric]] 1.prems(1,2) split[OF
sp] by simp
      have IHr: set_tree (diff r1 r2) = set_tree r1 − set_tree r2
       using 1.IH(2)[OF False _ _ _ sp[symmetric]] 1.prems(1,2) split[OF
sp] by simp
     have set_tree t1 − set_tree t2 = (?L1 ∪ ?R1) − (?L2 ∪ ?R2 ∪ {a})
       by(simp add: t1)
     also have ... = (?L1 − ?L2) ∪ (?R1 − ?R2)
       using ** by auto
     also have ... = set_tree (diff t1 t2)
     using IHl IHr sp diff.simps[of t1 t2] False by(simp)
```

```
    finally show ?thesis by simp
   qed
  qed
qed

lemma bst_diff: ⟦ bst t1; bst t2 ⟧ ⟹ bst (diff t1 t2)
proof(induction t1 t2 rule: diff.induct)
  case (1 t1 t2)
  thus ?case
   by(fastforce simp: diff.simps[of t1 t2] set_tree_diff split
       intro!: bst_join bst_join2 split: tree.split prod.split)
qed

lemma inv_diff: ⟦ inv t1; inv t2 ⟧ ⟹ inv (diff t1 t2)
proof(induction t1 t2 rule: diff.induct)
  case (1 t1 t2)
  thus ?case
   by(auto simp: diff.simps[of t1 t2] inv_join inv_join2 split_inv
       split!: tree.split prod.split dest: inv_Node)
qed
```

Locale *Set2_Join* implements locale *Set2*:

```
sublocale Set2
where empty = Leaf and insert = insert and delete = delete and isin =
isin
and union = union and inter = inter and diff = diff
and set = set_tree and invar = λt. inv t ∧ bst t
proof (standard, goal_cases)
  case 1 show ?case by (simp)
next
  case 2 thus ?case by(simp add: isin_set_tree)
next
  case 3 thus ?case by (simp add: set_tree_insert)
next
  case 4 thus ?case by (simp add: set_tree_delete)
next
  case 5 thus ?case by (simp add: inv_Leaf)
next
  case 6 thus ?case by (simp add: bst_insert inv_insert)
next
  case 7 thus ?case by (simp add: bst_delete inv_delete)
next
  case 8 thus ?case by(simp add: set_tree_union)
next
```

case *9* **thus** *?case* **by**(*simp add*: *set_tree_inter*)
next
case *10* **thus** *?case* **by**(*simp add*: *set_tree_diff*)
next
case *11* **thus** *?case* **by** (*simp add*: *bst_union inv_union*)
next
case *12* **thus** *?case* **by** (*simp add*: *bst_inter inv_inter*)
next
case *13* **thus** *?case* **by** (*simp add*: *bst_diff inv_diff*)
qed

end

interpretation *unbal*: *Set2_Join*
where *join* = $\lambda l\ x\ r$. *Node* l $(x, ())$ r **and** *inv* = λt. *True*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp*
next
case *3* **thus** *?case* **by** *simp*
next
case *4* **thus** *?case* **by** *simp*
next
case *5* **thus** *?case* **by** *simp*
qed

end

37 Join-Based Implementation of Sets via RBTs

theory *Set2_Join_RBT*
imports
Set2_Join
RBT_Set
begin

37.1 Code

Function *joinL* joins two trees (and an element). Precondition: *bheight* $l \leq$ *bheight* r. Method: Descend along the left spine of r until you find a subtree with the same *bheight* as l, then combine them into a new red node.

fun *joinL* :: $'a\ rbt \Rightarrow 'a \Rightarrow 'a\ rbt \Rightarrow 'a\ rbt$ **where**

joinL l x r =
(if bheight l ≥ bheight r then R l x r
else case r of
B l′ x′ r′ ⇒ baliL (joinL l x l′) x′ r′ |
R l′ x′ r′ ⇒ R (joinL l x l′) x′ r′)

fun *joinR :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt* **where**
joinR l x r =
(if bheight l ≤ bheight r then R l x r
else case l of
B l′ x′ r′ ⇒ baliR l′ x′ (joinR r′ x r) |
R l′ x′ r′ ⇒ R l′ x′ (joinR r′ x r))

definition *join :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt* **where**
join l x r =
(if bheight l > bheight r
then paint Black (joinR l x r)
else if bheight l < bheight r
then paint Black (joinL l x r)
else B l x r)

declare *joinL.simps*[*simp del*]
declare *joinR.simps*[*simp del*]

37.2 Properties

37.2.1 Color and height invariants

lemma *invc2_joinL*:
⟦ *invc l*; *invc r*; *bheight l ≤ bheight r* ⟧ ⟹
invc2 (joinL l x r)
∧ (*bheight l ≠ bheight r* ∧ *color r = Black* ⟶ *invc*(*joinL l x r*))
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *invc_baliL invc2I joinL.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *invc2_joinR*:
⟦ *invc l*; *invh l*; *invc r*; *invh r*; *bheight l ≥ bheight r* ⟧ ⟹
invc2 (joinR l x r)
∧ (*bheight l ≠ bheight r* ∧ *color l = Black* ⟶ *invc*(*joinR l x r*))
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*

by(*fastforce simp*: *invc_baliR invc2I joinR.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *bheight_joinL*:
⟦ *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧ ⟹ *bheight* (*joinL l x r*) = *bheight r*
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *bheight_baliL joinL.simps*[*of l x r*] *split*!: *tree.split*)
qed

lemma *invh_joinL*:
⟦ *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧ ⟹ *invh* (*joinL l x r*)
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *invh_baliL bheight_joinL joinL.simps*[*of l x r*] *split*!: *tree.split color.split*)
qed

lemma *bheight_joinR*:
⟦ *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧ ⟹ *bheight* (*joinR l x r*) = *bheight l*
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*fastforce simp*: *bheight_baliR joinR.simps*[*of l x r*] *split*!: *tree.split*)
qed

lemma *invh_joinR*:
⟦ *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧ ⟹ *invh* (*joinR l x r*)
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*fastforce simp*: *invh_baliR bheight_joinR joinR.simps*[*of l x r*]
split!: *tree.split color.split*)
qed

All invariants in one:

lemma *inv_joinL*: ⟦ *invc l*; *invc r*; *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧
⟹ *invc2* (*joinL l x r*) ∧ (*bheight l* ≠ *bheight r* ∧ *color r* = *Black* ⟶ *invc* (*joinL l x r*))
∧ *invh* (*joinL l x r*) ∧ *bheight* (*joinL l x r*) = *bheight r*
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *inv_baliL invc2I joinL.simps*[*of l x r*] *split*!: *tree.splits*

if_splits)
qed

lemma *inv_joinR*: ⟦ *invc l*; *invc r*; *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧
⟹ *invc2* (*joinR l x r*) ∧ (*bheight l* ≠ *bheight r* ∧ *color l* = *Black* ⟶ *invc* (*joinR l x r*))
∧ *invh* (*joinR l x r*) ∧ *bheight* (*joinR l x r*) = *bheight l*
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *inv_baliR invc2I joinR.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *rbt_join*: ⟦ *invc l*; *invh l*; *invc r*; *invh r* ⟧ ⟹ *rbt*(*join l x r*)
by(*simp add*: *inv_joinL inv_joinR invh_paint rbt_def color_paint_Black join_def*)

To make sure the the black height is not increased unnecessarily:

lemma *bheight_paint_Black*: *bheight*(*paint Black t*) ≤ *bheight t* + *1*
by(*cases t*) *auto*

lemma ⟦ *rbt l*; *rbt r* ⟧ ⟹ *bheight*(*join l x r*) ≤ *max* (*bheight l*) (*bheight r*) + *1*
using *bheight_paint_Black*[*of joinL l x r*] *bheight_paint_Black*[*of joinR l x r*]
bheight_joinL[*of l r x*] *bheight_joinR*[*of l r x*]
by(*auto simp*: *max_def rbt_def join_def*)

37.2.2 Inorder properties

Currently unused. Instead *Tree2.set_tree* and *Tree2.bst* properties are proved directly.

lemma *inorder_joinL*: *bheight l* ≤ *bheight r* ⟹ *inorder*(*joinL l x r*) = *inorder l* @ *x* # *inorder r*
proof(*induction l x r rule*: *joinL.induct*)
case (*1 l x r*)
thus *?case* **by**(*auto simp*: *inorder_baliL joinL.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *inorder_joinR*:
inorder(*joinR l x r*) = *inorder l* @ *x* # *inorder r*
proof(*induction l x r rule*: *joinR.induct*)

case (*1 l x r*)
thus *?case* **by** (*force simp*: *inorder_baliR joinR.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *inorder*(*join l x r*) = *inorder l* @ *x* # *inorder r*
by(*auto simp*: *inorder_joinL inorder_joinR inorder_paint join_def*
split!: *tree.splits color.splits if_splits*
dest!: *arg_cong*[**where** *f* = *inorder*])

37.2.3 Set and bst properties

lemma *set_baliL*:
set_tree(*baliL l a r*) = *set_tree l* ∪ {*a*} ∪ *set_tree r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto*)

lemma *set_joinL*:
bheight l ≤ *bheight r* ⟹ *set_tree* (*joinL l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
proof(*induction l x r rule*: *joinL.induct*)
case (*1 l x r*)
thus *?case* **by**(*auto simp*: *set_baliL joinL.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *set_baliR*:
set_tree(*baliR l a r*) = *set_tree l* ∪ {*a*} ∪ *set_tree r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto*)

lemma *set_joinR*:
set_tree (*joinR l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
proof(*induction l x r rule*: *joinR.induct*)
case (*1 l x r*)
thus *?case* **by**(*force simp*: *set_baliR joinR.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *set_paint*: *set_tree* (*paint c t*) = *set_tree t*
by (*cases t*) *auto*

lemma *set_join*: *set_tree* (*join l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
by(*simp add*: *set_joinL set_joinR set_paint join_def*)

lemma *bst_baliL*:

⟦*bst l*; *bst r*; ∀ *x*∈*set_tree l*. *x* < *a*; ∀ *x*∈*set_tree r*. *a* < *x*⟧
⟹ *bst* (*baliL l a r*)
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto simp*: *ball_Un*)

lemma *bst_baliR*:
⟦*bst l*; *bst r*; ∀ *x*∈*set_tree l*. *x* < *a*; ∀ *x*∈*set_tree r*. *a* < *x*⟧
⟹ *bst* (*baliR l a r*)
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto simp*: *ball_Un*)

lemma *bst_joinL*:
⟦*bst* (*Node l* (*a*, *n*) *r*); *bheight l* ≤ *bheight r*⟧
⟹ *bst* (*joinL l a r*)
proof(*induction l a r rule*: *joinL.induct*)
case (*1 l a r*)
thus *?case*
by(*auto simp*: *set_baliL joinL.simps*[*of l a r*] *set_joinL ball_Un intro*!: *bst_baliL*
split!: *tree.splits color.splits*)
qed

lemma *bst_joinR*:
⟦*bst l*; *bst r*; ∀ *x*∈*set_tree l*. *x* < *a*; ∀ *y*∈*set_tree r*. *a* < *y* ⟧
⟹ *bst* (*joinR l a r*)
proof(*induction l a r rule*: *joinR.induct*)
case (*1 l a r*)
thus *?case*
by(*auto simp*: *set_baliR joinR.simps*[*of l a r*] *set_joinR ball_Un intro*!: *bst_baliR*
split!: *tree.splits color.splits*)
qed

lemma *bst_paint*: *bst* (*paint c t*) = *bst t*
by(*cases t*) *auto*

lemma *bst_join*:
bst (*Node l* (*a*, *n*) *r*) ⟹ *bst* (*join l a r*)
by(*auto simp*: *bst_paint bst_joinL bst_joinR join_def*)

lemma *inv_join*: ⟦ *invc l*; *invh l*; *invc r*; *invh r* ⟧ ⟹ *invc*(*join l x r*) ∧ *invh*(*join l x r*)
by (*simp add*: *inv_joinL inv_joinR invh_paint join_def*)

37.2.4 Interpretation of *Set2_Join* with Red-Black Tree

global_interpretation *RBT*: *Set2_Join*
where *join* = *join* **and** *inv* = $\lambda t.$ *invc t* $\wedge$ *invh t*
defines *insert_rbt* = *RBT.insert* **and** *delete_rbt* = *RBT.delete* **and** *split_rbt* = *RBT.split*
and *join2_rbt* = *RBT.join2* **and** *split_min_rbt* = *RBT.split_min*
and *inter_rbt* = *RBT.inter* **and** *union_rbt* = *RBT.union* **and** *diff_rbt* = *RBT.diff*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*rule set_join*)
next
  case *2* **thus** *?case* **by** (*simp add*: *bst_join*)
next
  case *3* **show** *?case* **by** *simp*
next
  case *4* **thus** *?case* **by** (*simp add*: *inv_join*)
next
  case *5* **thus** *?case* **by** *simp*
qed

The invariant does not guarantee that the root node is black. This is not required to guarantee that the height is logarithmic in the size — Exercise.

end
theory *Array_Specs*
imports *Main*
begin

Array Specifications

locale *Array* =
fixes *lookup* :: $'ar \Rightarrow nat \Rightarrow 'a$
fixes *update* :: $nat \Rightarrow 'a \Rightarrow 'ar \Rightarrow 'ar$
fixes *len* :: $'ar \Rightarrow nat$
fixes *array* :: $'a\ list \Rightarrow 'ar$

fixes *list* :: $'ar \Rightarrow 'a\ list$
fixes *invar* :: $'ar \Rightarrow bool$

assumes *lookup*: *invar ar* $\Longrightarrow$ *n* $<$ *len ar* $\Longrightarrow$ *lookup ar n* = *list ar* ! *n*
assumes *update*: *invar ar* $\Longrightarrow$ *n* $<$ *len ar* $\Longrightarrow$ *list*(*update n x ar*) = (*list ar*)[*n*:=*x*]
assumes *len_array*: *invar ar* $\Longrightarrow$ *len ar* = *length* (*list ar*)
assumes *array*: *list* (*array xs*) = *xs*

assumes *invar_update*: *invar ar* $\Longrightarrow$ *n* $<$ *len ar* $\Longrightarrow$ *invar*(*update n x ar*)

assumes *invar_array*: *invar*(*array xs*)

locale *Array_Flex* = *Array* +
fixes *add_lo* :: *'a* ⇒ *'ar* ⇒ *'ar*
fixes *del_lo* :: *'ar* ⇒ *'ar*
fixes *add_hi* :: *'a* ⇒ *'ar* ⇒ *'ar*
fixes *del_hi* :: *'ar* ⇒ *'ar*

assumes *add_lo*: *invar ar* ⟹ *list*(*add_lo a ar*) = *a* # *list ar*
assumes *del_lo*: *invar ar* ⟹ *list*(*del_lo ar*) = *tl* (*list ar*)
assumes *add_hi*: *invar ar* ⟹ *list*(*add_hi a ar*) = *list ar* @ [*a*]
assumes *del_hi*: *invar ar* ⟹ *list*(*del_hi ar*) = *butlast* (*list ar*)

assumes *invar_add_lo*: *invar ar* ⟹ *invar* (*add_lo a ar*)
assumes *invar_del_lo*: *invar ar* ⟹ *invar* (*del_lo ar*)
assumes *invar_add_hi*: *invar ar* ⟹ *invar* (*add_hi a ar*)
assumes *invar_del_hi*: *invar ar* ⟹ *invar* (*del_hi ar*)

end

38 Braun Trees

theory *Braun_Tree*
imports *HOL−Library.Tree_Real*
begin

Braun Trees were studied by Braun and Rem [5] and later Hoogerwoord [10].

fun *braun* :: *'a tree* ⇒ *bool* **where**
braun Leaf = *True* |
braun (*Node l x r*) = ((*size l* = *size r* ∨ *size l* = *size r* + *1*) ∧ *braun l* ∧ *braun r*)

lemma *braun_Node'*:
braun (*Node l x r*) = (*size r* ≤ *size l* ∧ *size l* ≤ *size r* + *1* ∧ *braun l* ∧ *braun r*)
by *auto*

The shape of a Braun-tree is uniquely determined by its size:

lemma *braun_unique*: ⟦ *braun* (*t1*::*unit tree*); *braun t2*; *size t1* = *size t2* ⟧ ⟹ *t1* = *t2*
proof (*induction t1 arbitrary*: *t2*)
case *Leaf* **thus** *?case* **by** *simp*
next

case (*Node l1 _ r1*)
from *Node.prems*(*3*) **have** *t2* ≠ *Leaf* **by** *auto*
then obtain *l2 x2 r2* **where** [*simp*]: *t2* = *Node l2 x2 r2* **by** (*meson neq_Leaf_iff*)
with *Node.prems* **have** *size l1* = *size l2* ∧ *size r1* = *size r2* **by** *auto*
thus *?case* **using** *Node.prems*(*1*,*2*) *Node.IH* **by** *auto*
qed

Braun trees are almost complete:

lemma *acomplete_if_braun*: *braun t* ⟹ *acomplete t*
proof(*induction t*)
case *Leaf* **show** *?case* **by** (*simp add*: *acomplete_def*)
next
case (*Node l x r*) **thus** *?case* **using** *acomplete_Node_if_wbal2* **by** *force*
qed

38.1 Numbering Nodes

We show that a tree is a Braun tree iff a parity-based numbering (*braun_indices*) of nodes yields an interval of numbers.

fun *braun_indices* :: *'a tree* ⇒ *nat set* **where**
braun_indices Leaf = {} |
braun_indices (*Node l _ r*) = {*1*} ∪ (∗) *2* ‘ *braun_indices l* ∪ *Suc* ‘ (∗) *2* ‘ *braun_indices r*

lemma *braun_indices1*: *0* ∉ *braun_indices t*
by (*induction t*) *auto*

lemma *finite_braun_indices*: *finite*(*braun_indices t*)
by (*induction t*) *auto*

One direction:

lemma *braun_indices_if_braun*: *braun t* ⟹ *braun_indices t* = {*1*..*size t*}
proof(*induction t*)
case *Leaf* **thus** *?case* **by** *simp*
next
have ∗: (∗) *2* ‘ {*a*..*b*} ∪ *Suc* ‘ (∗) *2* ‘ {*a*..*b*} = {*2*∗*a*..*2*∗*b*+*1*} (**is** *?l* = *?r*)
for *a b*
proof
show *?l* ⊆ *?r* **by** *auto*
next
have ∃ *x2*∈{*a*..*b*}. *x* ∈ {*Suc* (*2*∗*x2*), *2*∗*x2*} **if** ∗: *x* ∈ {*2*∗*a* .. *2*∗*b*+*1*}
for *x*

```
    proof −
      have x div 2 ∈ {a..b} using ∗ by auto
      moreover have x ∈ {2 ∗ (x div 2), Suc(2 ∗ (x div 2))} by auto
      ultimately show ?thesis by blast
    qed
    thus ?r ⊆ ?l by fastforce
  qed
  case (Node l x r)
  hence size l = size r ∨ size l = size r + 1 (is ?A ∨ ?B) by auto
  thus ?case
  proof
    assume ?A
    with Node show ?thesis by (auto simp: ∗)
  next
    assume ?B
    with Node show ?thesis by (auto simp: ∗ atLeastAtMostSuc_conv)
  qed
qed
```

The other direction is more complicated. The following proof is due to Thomas Sewell.

```
lemma disj_evens_odds: (∗) 2 ‘ A ∩ Suc ‘ (∗) 2 ‘ B = {}
  using double_not_eq_Suc_double by auto

lemma card_braun_indices: card (braun_indices t) = size t
proof (induction t)
  case Leaf thus ?case by simp
next
  case Node
  thus ?case
   by(auto simp: UNION_singleton_eq_range finite_braun_indices card_Un_disjoint
      card_insert_if disj_evens_odds card_image inj_on_def braun_indices1)
qed

lemma braun_indices_intvl_base_1:
  assumes bi: braun_indices t = {m..n}
  shows {m..n} = {1..size t}
proof (cases t = Leaf)
  case True then show ?thesis using bi by simp
next
  case False
  note eqs = eqset_imp_iff[OF bi]
  from eqs[of 0] have 0: 0 < m
    by (simp add: braun_indices1)
```

```
  from eqs[of 1] have 1: m ≤ 1
    by (cases t; simp add: False)
  from 0 1 have eq1: m = 1 by simp
  from card_braun_indices[of t] show ?thesis
    by (simp add: bi eq1)
qed

lemma even_of_intvl_intvl:
  fixes S :: nat set
  assumes S = {m..n} ∩ {i. even i}
  shows ∃ m' n'. S = (λi. i * 2) ` {m'..n'}
proof −
  have S = (λi. i * 2) ` {Suc m div 2..n div 2}
    by (fastforce simp add: assms mult.commute)
  then show ?thesis
    by blast
qed

lemma odd_of_intvl_intvl:
  fixes S :: nat set
  assumes S = {m..n} ∩ {i. odd i}
  shows ∃ m' n'. S = Suc ` (λi. i * 2) ` {m'..n'}
proof −
  have S = Suc ` ({if n = 0 then 1 else m − 1..n − 1} ∩ Collect even)
    by (auto simp: assms image_def elim!: oddE)
  thus ?thesis
    by (metis even_of_intvl_intvl)
qed

lemma image_int_eq_image:
  (∀ i ∈ S. f i ∈ T) ⟹ (f ` S) ∩ T = f ` S
  (∀ i ∈ S. f i ∉ T) ⟹ (f ` S) ∩ T = {}
  by auto

lemma braun_indices1_le:
  i ∈ braun_indices t ⟹ Suc 0 ≤ i
  using braun_indices1 not_less_eq_eq by blast

lemma braun_if_braun_indices: braun_indices t = {1..size t} ⟹ braun
t
proof(induction t)
  case Leaf
  then show ?case by simp
next
```

case (*Node l x r*)
obtain *t* **where** *t*: *t* = *Node l x r* **by** *simp*
then have *insert* (*Suc 0*) ((∗) *2* ‘ *braun_indices l* ∪ *Suc* ‘ (∗) *2* ‘ *braun_indices r*) ∩ {*2*..} = {*Suc 0*..*Suc* (*size l* + *size r*)} ∩ {*2*..}
by (*metis Node.prems One_nat_def Suc_eq_plus1 Un_insert_left braun_indices.simps*(*2*) *sup_bot_left tree.size*(*4*))
then have *eq*: {*2* .. *size t*} = (λ*i*. *i* ∗ *2*) ‘ *braun_indices l* ∪ *Suc* ‘ (λ*i*. *i* ∗ *2*) ‘ *braun_indices r*
(**is** *?R* = *?S* ∪ *?T*)
by (*simp add*: *t mult.commute Int_Un_distrib2 image_int_eq_image braun_indices1_le*)
then have *ST*: *?S* = *?R* ∩ {*i*. *even i*} *?T* = *?R* ∩ {*i*. *odd i*}
by (*simp_all add*: *Int_Un_distrib2 image_int_eq_image*)
from *ST* **have** *l*: *braun_indices l* = {*1* .. *size l*}
by (*fastforce dest*: *braun_indices_intvl_base_1 dest*!: *even_of_intvl_intvl simp*: *mult.commute inj_image_eq_iff*[*OF inj_onI*])
from *ST* **have** *r*: *braun_indices r* = {*1* .. *size r*}
by (*fastforce dest*: *braun_indices_intvl_base_1 dest*!: *odd_of_intvl_intvl simp*: *mult.commute inj_image_eq_iff*[*OF inj_onI*])
note *STa* = *ST*[*THEN eqset_imp_iff*, *THEN iffD2*]
note *STb* = *STa*[*of size t*] *STa*[*of size t* − *1*]
then have *size l* = *size r* ∨ *size l* = *size r* + *1*
using *t l r* **by** *atomize auto*
with *l r* **show** *?case*
by (*clarsimp simp*: *Node.IH*)
qed

lemma *braun_iff_braun_indices*: *braun t* ⟷ *braun_indices t* = {*1*..*size t*}
using *braun_if_braun_indices braun_indices_if_braun* **by** *blast*

end

39 Arrays via Braun Trees

theory *Array_Braun*
imports
HOL−Library.Time_Functions
Array_Specs
Braun_Tree
begin

39.1 Array

fun *lookup1* :: *'a tree* ⇒ *nat* ⇒ *'a* **where**
lookup1 (*Node l x r*) *n* = (*if n=1 then x else lookup1* (*if even n then l else r*) (*n div 2*))

fun *update1* :: *nat* ⇒ *'a* ⇒ *'a tree* ⇒ *'a tree* **where**
update1 n x Leaf = *Node Leaf x Leaf* |
update1 n x (*Node l a r*) =
(*if n=1 then Node l x r else*
if even n then Node (*update1* (*n div 2*) *x l*) *a r*
else Node l a (*update1* (*n div 2*) *x r*))

fun *adds* :: *'a list* ⇒ *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
adds [] *n t* = *t* |
adds (*x*#*xs*) *n t* = *adds xs* (*n*+*1*) (*update1* (*n*+*1*) *x t*)

fun *list* :: *'a tree* ⇒ *'a list* **where**
list Leaf = [] |
list (*Node l x r*) = *x* # *splice* (*list l*) (*list r*)

39.1.1 Functional Correctness

lemma *size_list*: *size*(*list t*) = *size t*
by(*induction t*)(*auto*)

lemma *minus1_div2*: (*n* − *Suc 0*) *div 2* = (*if odd n then n div 2 else n div 2* − *1*)
by *auto arith*

lemma *nth_splice*: ⟦ *n* < *size xs* + *size ys*; *size ys* ≤ *size xs*; *size xs* ≤ *size ys* + *1* ⟧
⟹ *splice xs ys* ! *n* = (*if even n then xs else ys*) ! (*n div 2*)
proof(*induction xs ys arbitrary*: *n rule*: *splice.induct*)
qed (*auto simp*: *nth_Cons' minus1_div2*)

lemma *div2_in_bounds*:
⟦ *braun* (*Node l x r*); *n* ∈ {*1..size*(*Node l x r*)}; *n* > *1* ⟧ ⟹
(*odd n* ⟶ *n div 2* ∈ {*1..size r*}) ∧ (*even n* ⟶ *n div 2* ∈ {*1..size l*})
by *auto arith*

declare *upt_Suc*[*simp del*]

lookup1 **lemma** *nth_list_lookup1*: ⟦*braun t*; *i* < *size t*⟧ ⟹ *list t* ! *i* = *lookup1 t* (*i*+*1*)
proof(*induction t arbitrary*: *i*)
case *Leaf* **thus** *?case* **by** *simp*
next
case *Node*
thus *?case* **using** *div2_in_bounds*[*OF Node.prems*(*1*), *of i*+*1*]
by (*auto simp*: *nth_splice minus1_div2 size_list*)
qed

lemma *list_eq_map_lookup1*: *braun t* ⟹ *list t* = *map* (*lookup1 t*) [*1*..<*size t* + *1*]
by(*auto simp add*: *list_eq_iff_nth_eq size_list nth_list_lookup1*)

update1 **lemma** *size_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *size*(*update1 n x t*) = *size t*
proof(*induction t arbitrary*: *n*)
case *Leaf* **thus** *?case* **by** *simp*
next
case *Node* **thus** *?case* **using** *div2_in_bounds*[*OF Node.prems*] **by** *simp*
qed

lemma *braun_update1*: ⟦*braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *braun*(*update1 n x t*)
proof(*induction t arbitrary*: *n*)
case *Leaf* **thus** *?case* **by** *simp*
next
case *Node* **thus** *?case*
using *div2_in_bounds*[*OF Node.prems*] **by** (*simp add*: *size_update1*)
qed

lemma *lookup1_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹
lookup1 (*update1 n x t*) *m* = (*if n*=*m then x else lookup1 t m*)
proof(*induction t arbitrary*: *m n*)
case *Leaf*
then show *?case* **by** *simp*
next
have *aux*: ⟦ *odd n*; *odd m* ⟧ ⟹ *n div 2* = (*m*::*nat*) *div 2* ⟷ *m*=*n* **for** *m n*
using *odd_two_times_div_two_succ* **by** *fastforce*
case *Node*
thus *?case* **using** *div2_in_bounds*[*OF Node.prems*] **by** (*auto simp*: *aux*)
qed

lemma *list_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *list*(*update1 n x t*) = (*list t*)[*n*−*1* := *x*]
by(*auto simp add*: *list_eq_map_lookup1 list_eq_iff_nth_eq lookup1_update1 size_update1 braun_update1*)

A second proof of ⟦*braun ?t*; *?n* ∈ {*1*..*size ?t*}⟧ ⟹ *list* (*update1 ?n ?x ?t*) = (*list ?t*)[*?n* − *1* := *?x*]:

lemma *diff1_eq_iff*: *n* > *0* ⟹ *n* − *Suc 0* = *m* ⟷ *n* = *m*+*1*
by *arith*

lemma *list_update_splice*:
⟦ *n* < *size xs* + *size ys*; *size ys* ≤ *size xs*; *size xs* ≤ *size ys* + *1* ⟧ ⟹
(*splice xs ys*) [*n* := *x*] =
(*if even n then splice* (*xs*[*n div 2* := *x*]) *ys else splice xs* (*ys*[*n div 2* := *x*]))
by(*induction xs ys arbitrary*: *n rule*: *splice.induct*) (*auto split*: *nat.split*)

lemma *list_update2*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *list*(*update1 n x t*) = (*list t*)[*n*−*1* := *x*]
proof(*induction t arbitrary*: *n*)
case *Leaf* **thus** *?case* **by** *simp*
next
case (*Node l a r*) **thus** *?case* **using** *div2_in_bounds*[*OF Node.prems*]
by(*auto simp*: *list_update_splice diff1_eq_iff size_list split*: *nat.split*)
qed

adds **lemma** *splice_last*: **shows**
size ys ≤ *size xs* ⟹ *splice* (*xs* @ [*x*]) *ys* = *splice xs ys* @ [*x*]
and *size ys*+*1* ≥ *size xs* ⟹ *splice xs* (*ys* @ [*y*]) = *splice xs ys* @ [*y*]
by(*induction xs ys arbitrary*: *x y rule*: *splice.induct*) (*auto*)

lemma *list_add_hi*: *braun t* ⟹ *list*(*update1* (*Suc*(*size t*)) *x t*) = *list t* @ [*x*]
by(*induction t*)(*auto simp*: *splice_last size_list*)

lemma *size_add_hi*: *braun t* ⟹ *m* = *size t* ⟹ *size*(*update1* (*Suc m*) *x t*) = *size t* + *1*
by(*induction t arbitrary*: *m*)(*auto*)

lemma *braun_add_hi*: *braun t* ⟹ *braun*(*update1* (*Suc*(*size t*)) *x t*)
by(*induction t*)(*auto simp*: *size_add_hi*)

lemma *size_braun_adds*:
⟦ *braun t*; *size t* = *n* ⟧ ⟹ *size*(*adds xs n t*) = *size t* + *length xs* ∧ *braun*

(*adds xs n t*)
by(*induction xs arbitrary*: *t n*)(*auto simp*: *braun_add_hi size_add_hi*)

lemma *list_adds*: ⟦ *braun t*; *size t* = *n* ⟧ ⟹ *list*(*adds xs n t*) = *list t* @ *xs*
by(*induction xs arbitrary*: *t n*)(*auto simp*: *size_braun_adds list_add_hi size_add_hi braun_add_hi*)

39.1.2 Array Implementation

interpretation *A*: *Array*
where *lookup* = λ(*t*,*l*) *n*. *lookup1 t* (*n*+*1*)
and *update* = λ*n x* (*t*,*l*). (*update1* (*n*+*1*) *x t*, *l*)
and *len* = λ(*t*,*l*). *l*
and *array* = λ*xs*. (*adds xs 0 Leaf*, *length xs*)
and *invar* = λ(*t*,*l*). *braun t* ∧ *l* = *size t*
and *list* = λ(*t*,*l*). *list t*
proof (*standard*, *goal_cases*)
case *1* **thus** *?case* **by** (*simp add*: *nth_list_lookup1 split*: *prod.splits*)
next
case *2* **thus** *?case* **by** (*simp add*: *list_update1 split*: *prod.splits*)
next
case *3* **thus** *?case* **by** (*simp add*: *size_list split*: *prod.splits*)
next
case *4* **thus** *?case* **by** (*simp add*: *list_adds*)
next
case *5* **thus** *?case* **by** (*simp add*: *braun_update1 size_update1 split*: *prod.splits*)
next
case *6* **thus** *?case* **by** (*simp add*: *size_braun_adds split*: *prod.splits*)
qed

39.2 Flexible Array

fun *add_lo* **where**
add_lo x Leaf = *Node Leaf x Leaf* |
add_lo x (*Node l a r*) = *Node* (*add_lo a r*) *x l*

fun *merge* **where**
merge Leaf r = *r* |
merge (*Node l a r*) *rr* = *Node rr a* (*merge l r*)

fun *del_lo* **where**
del_lo Leaf = *Leaf* |
del_lo (*Node l a r*) = *merge l r*

fun *del_hi* :: *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
del_hi n Leaf = *Leaf* |
del_hi n (*Node l x r*) =
(*if n* = *1 then Leaf*
else if even n
then Node (*del_hi* (*n div 2*) *l*) *x r*
else Node l x (*del_hi* (*n div 2*) *r*))

39.2.1 Functional Correctness

add_lo **lemma** *list_add_lo*: *braun t* ⟹ *list* (*add_lo a t*) = *a* # *list t*
by(*induction t arbitrary*: *a*) *auto*

lemma *braun_add_lo*: *braun t* ⟹ *braun*(*add_lo x t*)
by(*induction t arbitrary*: *x*) (*auto simp add*: *list_add_lo simp flip*: *size_list*)

del_lo **lemma** *list_merge*: *braun* (*Node l x r*) ⟹ *list*(*merge l r*) = *splice* (*list l*) (*list r*)
by (*induction l r rule*: *merge.induct*) *auto*

lemma *braun_merge*: *braun* (*Node l x r*) ⟹ *braun*(*merge l r*)
by (*induction l r rule*: *merge.induct*)(*auto simp add*: *list_merge simp flip*: *size_list*)

lemma *list_del_lo*: *braun t* ⟹ *list*(*del_lo t*) = *tl* (*list t*)
by (*cases t*) (*simp_all add*: *list_merge*)

lemma *braun_del_lo*: *braun t* ⟹ *braun*(*del_lo t*)
by (*cases t*) (*simp_all add*: *braun_merge*)

del_hi **lemma** *list_Nil_iff*: *list t* = [] ⟷ *t* = *Leaf*
by(*cases t*) *simp_all*

lemma *butlast_splice*: *butlast* (*splice xs ys*) =
(*if size xs* > *size ys then splice* (*butlast xs*) *ys else splice xs* (*butlast ys*))
by(*induction xs ys rule*: *splice.induct*) (*auto*)

lemma *list_del_hi*: *braun t* ⟹ *size t* = *st* ⟹ *list*(*del_hi st t*) = *butlast*(*list t*)
by (*induction t arbitrary*: *st*) (*auto simp*: *list_Nil_iff size_list butlast_splice*)

lemma *braun_del_hi*: *braun t* ⟹ *size t* = *st* ⟹ *braun*(*del_hi st t*)
by (*induction t arbitrary*: *st*) (*auto simp*: *list_del_hi simp flip*: *size_list*)

39.2.2 Flexible Array Implementation

interpretation *AF*: *Array_Flex*
  where *lookup* = λ(*t*,*l*) *n*. *lookup1 t* (*n*+*1*)
    and *update* = λ*n x* (*t*,*l*). (*update1* (*n*+*1*) *x t*, *l*)
    and *len* = λ(*t*,*l*). *l*
    and *array* = λ*xs*. (*adds xs 0 Leaf*, *length xs*)
    and *invar* = λ(*t*,*l*). *braun t* ∧ *l* = *size t*
    and *list* = λ(*t*,*l*). *list t*
    and *add_lo* = λ*x* (*t*,*l*). (*add_lo x t*, *l*+*1*)
    and *del_lo* = λ(*t*,*l*). (*del_lo t*, *l*−*1*)
    and *add_hi* = λ*x* (*t*,*l*). (*update1* (*Suc l*) *x t*, *l*+*1*)
    and *del_hi* = λ(*t*,*l*). (*del_hi l t*, *l*−*1*)
proof (*standard*, *goal_cases*)
  case *1* **thus** *?case* **by** (*simp add*: *list_add_lo split*: *prod.splits*)
next
  case *2* **thus** *?case* **by** (*simp add*: *list_del_lo split*: *prod.splits*)
next
  case *3* **thus** *?case* **by** (*simp add*: *list_add_hi braun_add_hi split*: *prod.splits*)
next
  case *4* **thus** *?case* **by** (*simp add*: *list_del_hi split*: *prod.splits*)
next
  case *5* **thus** *?case* **by** (*simp add*: *braun_add_lo list_add_lo flip*: *size_list split*: *prod.splits*)
next
  case *6* **thus** *?case* **by** (*simp add*: *braun_del_lo list_del_lo flip*: *size_list split*: *prod.splits*)
next
  case *7* **thus** *?case* **by** (*simp add*: *size_add_hi braun_add_hi split*: *prod.splits*)
next
  case *8* **thus** *?case* **by** (*simp add*: *braun_del_hi list_del_hi flip*: *size_list split*: *prod.splits*)
qed

39.3 Faster

39.3.1 Size

fun *diff* :: *'a tree* ⇒ *nat* ⇒ *nat* **where**
  diff Leaf _ = *0* |
  diff (*Node l x r*) *n* = (*if n=0 then 1 else if even n then diff r* (*n div 2* − *1*) *else diff l* (*n div 2*))

fun *size_fast* :: *'a tree* ⇒ *nat* **where**
  size_fast Leaf = *0* |

size_fast (*Node l x r*) = (*let n* = *size_fast r in 1* + *2*∗*n* + *diff l n*)

declare *Let_def*[*simp*]

lemma *diff*: *braun t* ⟹ *size t* : {*n*, *n* + *1*} ⟹ *diff t n* = *size t* − *n*
by (*induction t arbitrary*: *n*) *auto*

lemma *size_fast*: *braun t* ⟹ *size_fast t* = *size t*
by (*induction t*) (*auto simp add*: *diff*)

39.3.2 Initialization with 1 element

fun *braun_of_naive* :: *'a* ⇒ *nat* ⇒ *'a tree* **where**
braun_of_naive x n = (*if n=0 then Leaf*
else let m = (*n−1*) *div 2*
in if odd n then Node (*braun_of_naive x m*) *x* (*braun_of_naive x m*)
else Node (*braun_of_naive x* (*m* + *1*)) *x* (*braun_of_naive x m*))

fun *braun2_of* :: *'a* ⇒ *nat* ⇒ *'a tree* ∗ *'a tree* **where**
braun2_of x n = (*if n* = *0 then* (*Leaf*, *Node Leaf x Leaf*)
else let (*s*,*t*) = *braun2_of x* ((*n−1*) *div 2*)
in if odd n then (*Node s x s*, *Node t x s*) *else* (*Node t x s*, *Node t x t*))

definition *braun_of* :: *'a* ⇒ *nat* ⇒ *'a tree* **where**
braun_of x n = *fst* (*braun2_of x n*)

declare *braun2_of.simps* [*simp del*]

lemma *braun2_of_size_braun*: *braun2_of x n* = (*s*,*t*) ⟹ *size s* = *n* ∧ *size t* = *n+1* ∧ *braun s* ∧ *braun t*
proof(*induction x n arbitrary*: *s t rule*: *braun2_of.induct*)
case (*1 x n*)
then show *?case*
by (*auto simp*: *braun2_of.simps*[*of x n*] *split*: *prod.splits if_splits*) *presburger+*
qed

lemma *braun2_of_replicate*:
braun2_of x n = (*s*,*t*) ⟹ *list s* = *replicate n x* ∧ *list t* = *replicate* (*n+1*) *x*
proof(*induction x n arbitrary*: *s t rule*: *braun2_of.induct*)
case (*1 x n*)
have *x* # *replicate m x* = *replicate* (*m+1*) *x* **for** *m* **by** *simp*
with *1* **show** *?case*

apply (*auto simp*: *braun2_of.simps*[*of x n*] *replicate.simps*(*2*)[*of 0 x*]
simp del: *replicate.simps*(*2*) *split*: *prod.splits if_splits*)
by *presburger*+
qed

corollary *braun_braun_of*: *braun*(*braun_of x n*)
unfolding *braun_of_def* **by** (*metis eq_fst_iff braun2_of_size_braun*)

corollary *list_braun_of*: *list*(*braun_of x n*) = *replicate n x*
unfolding *braun_of_def* **by** (*metis eq_fst_iff braun2_of_replicate*)

39.3.3 Proof Infrastructure

Originally due to Thomas Sewell.

take_nths **fun** *take_nths* :: *nat* ⇒ *nat* ⇒ *'a list* ⇒ *'a list* **where**
take_nths i k [] = [] |
take_nths i k (*x* # *xs*) = (*if i* = *0 then x* # *take_nths* (*2^k* − *1*) *k xs*
else take_nths (*i* − *1*) *k xs*)

This is the more concise definition but seems to complicate the proofs:

lemma *take_nths_eq_nths*: *take_nths i k xs* = *nths xs* (⋃ *n*. {*n*∗*2^k* + *i*})
proof(*induction xs arbitrary*: *i*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons x xs*)
show *?case*
proof *cases*
assume [*simp*]: *i* = *0*
have ⋀*x n*. *Suc x* = *n* ∗ *2* ^ *k* ⟹ ∃ *xa*. *x* = *Suc xa* ∗ *2* ^ *k* − *Suc 0*
by (*metis diff_Suc_Suc diff_zero mult_eq_0_iff not0_implies_Suc*)
then have (⋃ *n*. {(*n*+*1*) ∗ *2* ^ *k* − *1*}) = {*m*. ∃ *n*. *Suc m* = *n* ∗ *2* ^ *k*}
by (*auto simp del*: *mult_Suc*)
thus *?thesis* **by** (*simp add*: *Cons.IH ac_simps nths_Cons*)
next
assume [*arith*]: *i* ≠ *0*
have ⋀*x n*. *Suc x* = *n* ∗ *2* ^ *k* + *i* ⟹ ∃ *xa*. *x* = *xa* ∗ *2* ^ *k* + *i* − *Suc 0*
by (*metis diff_Suc_Suc diff_zero*)
then have (⋃ *n*. {*n* ∗ *2* ^ *k* + *i* − *1*}) = {*m*. ∃ *n*. *Suc m* = *n* ∗ *2* ^ *k* + *i*}
by *auto*
thus *?thesis* **by** (*simp add*: *Cons.IH nths_Cons*)

qed
qed

lemma *take_nths_drop*:
take_nths i k (drop j xs) = take_nths (i + j) k xs
by (*induct xs arbitrary*: *i j*; *simp add*: *drop_Cons split*: *nat.split*)

lemma *take_nths_00*:
take_nths 0 0 xs = xs
by (*induct xs*; *simp*)

lemma *splice_take_nths*:
splice (take_nths 0 (Suc 0) xs) (take_nths (Suc 0) (Suc 0) xs) = xs
by (*induct xs*; *simp*)

lemma *take_nths_take_nths*:
*take_nths i m (take_nths j n xs) = take_nths ((i * 2^n) + j) (m + n) xs*
by (*induct xs arbitrary*: *i j*; *simp add*: *algebra_simps power_add*)

lemma *take_nths_empty*:
(take_nths i k xs = []) = (length xs ≤ i)
by (*induction xs arbitrary*: *i k*) *auto*

lemma *hd_take_nths*:
i < length xs ⟹ hd(take_nths i k xs) = xs ! i
by (*induction xs arbitrary*: *i k*) *auto*

lemma *take_nths_01_splice*:
⟦ length xs = length ys ∨ length xs = length ys + 1 ⟧ ⟹
take_nths 0 (Suc 0) (splice xs ys) = xs ∧
take_nths (Suc 0) (Suc 0) (splice xs ys) = ys
by (*induct xs arbitrary*: *ys*; *case_tac ys*; *simp*)

lemma *length_take_nths_00*:
length (take_nths 0 (Suc 0) xs) = length (take_nths (Suc 0) (Suc 0) xs)
∨
length (take_nths 0 (Suc 0) xs) = length (take_nths (Suc 0) (Suc 0) xs)
+ 1
by (*induct xs*) *auto*

braun_list **fun** *braun_list* :: *'a tree ⇒ 'a list ⇒ bool* **where**
braun_list Leaf xs = (xs = []) |
braun_list (Node l x r) xs = (xs ≠ [] ∧ x = hd xs ∧

braun_list l (take_nths 1 1 xs) ∧
braun_list r (take_nths 2 1 xs))

lemma *braun_list_eq*:
braun_list t xs = (braun t ∧ xs = list t)
proof (*induct t arbitrary*: *xs*)
case *Leaf*
show *?case* **by** *simp*
next
case *Node*
show *?case*
using *length_take_nths_00*[*of xs*] *splice_take_nths*[*of xs*]
by (*auto simp*: *neq_Nil_conv Node.hyps size_list*[*symmetric*] *take_nths_01_splice*)
qed

39.3.4 Converting a list of elements into a Braun tree

fun *nodes* :: *'a tree list ⇒ 'a list ⇒ 'a tree list ⇒ 'a tree list* **where**
nodes (l#ls) (x#xs) (r#rs) = Node l x r # nodes ls xs rs |
nodes (l#ls) (x#xs) [] = Node l x Leaf # nodes ls xs [] |
nodes [] (x#xs) (r#rs) = Node Leaf x r # nodes [] xs rs |
nodes [] (x#xs) [] = Node Leaf x Leaf # nodes [] xs [] |
nodes ls [] rs = []

fun *brauns* :: *nat ⇒ 'a list ⇒ 'a tree list* **where**
brauns k xs = (if xs = [] then [] else
let ys = take (2^k) xs;
zs = drop (2^k) xs;
ts = brauns (k+1) zs
in nodes ts ys (drop (2^k) ts))

declare *brauns.simps*[*simp del*]

definition *brauns1* :: *'a list ⇒ 'a tree* **where**
brauns1 xs = (if xs = [] then Leaf else brauns 0 xs ! 0)

Functional correctness The proof is originally due to Thomas Sewell.

lemma *length_nodes*:
length (nodes ls xs rs) = length xs
by (*induct ls xs rs rule*: *nodes.induct*; *simp*)

lemma *nth_nodes*:
i < length xs ⟹ nodes ls xs rs ! i =
Node (if i < length ls then ls ! i else Leaf) (xs ! i)

```
    (if i < length rs then rs ! i else Leaf)
  by (induct ls xs rs arbitrary: i rule: nodes.induct;
      simp add: nth_Cons split: nat.split)

theorem length_brauns:
  length (brauns k xs) = min (length xs) (2 ^ k)
proof (induct xs arbitrary: k rule: measure_induct_rule[where f=length])
  case (less xs) thus ?case by (simp add: brauns.simps[of k xs] length_nodes)
qed

theorem brauns_correct:
  i < min (length xs) (2 ^ k) ⟹ braun_list (brauns k xs ! i) (take_nths i
k xs)
proof (induct xs arbitrary: i k rule: measure_induct_rule[where f=length])
  case (less xs)
  have xs ≠ [] using less.prems by auto
  let ?zs = drop (2^k) xs
  let ?ts = brauns (Suc k) ?zs
  from less.hyps[of ?zs _ Suc k]
  have IH: ⟦ j = i + 2 ^ k;  i < min (length ?zs) (2 ^ (k+1)) ⟧ ⟹
    braun_list (?ts ! i) (take_nths j (Suc k) xs) for i j
    using ‹xs ≠ []› by (simp add: take_nths_drop)
  show ?case
    using less.prems
    by (auto simp: brauns.simps[of k xs] nth_nodes take_nths_take_nths
        IH take_nths_empty hd_take_nths length_brauns)
qed

corollary brauns1_correct:
  braun (brauns1 xs) ∧ list (brauns1 xs) = xs
  using brauns_correct[of 0 xs 0]
  by (simp add: brauns1_def braun_list_eq take_nths_00)
```

Running Time Analysis time_fun_0 (^)

```
time_fun nodes

lemma T_nodes: T_nodes ls xs rs = length xs + 1
by(induction ls xs rs rule: T_nodes.induct) auto

time_fun brauns

lemma T_brauns_simple: T_brauns k xs = (if xs = [] then 0 else
```

```
  3 * (min (2^k) (length xs) + 1) + (min (2^k) (length xs - 2^k) + 1)
+ T_brauns (k+1) (drop (2^k) xs)) + 1
by(simp add: T_nodes T_take T_drop length_brauns min_def)

theorem T_brauns_ub:
  T_brauns k xs ≤ 9 * (length xs + 1)
proof (induction xs arbitrary: k rule: measure_induct_rule[where f =
length])
  case (less xs)
  show ?case
  proof cases
    assume xs = []
    thus ?thesis by(simp)
  next
    assume xs ≠ []
    let ?n = length xs let ?zs = drop (2^k) xs
    have *: ?n - 2^k + 1 ≤ ?n
      using ‹xs ≠ []› less_eq_Suc_le by fastforce
    have T_brauns k xs =
      3 * (min (2^k) ?n + 1) + (min (2^k) (?n - 2^k) + 1) + T_brauns
(k+1) ?zs + 1
       unfolding T_brauns_simple[of k xs] using ‹xs ≠ []› by(simp del:
T_brauns.simps)
    also have ... ≤ 4 * min (2^k) ?n + T_brauns (k+1) ?zs + 5
      by(simp add: min_def)
    also have ... ≤ 4 * min (2^k) ?n + 9 * (length ?zs + 1) + 5
      using less[of ?zs k+1] ‹xs ≠ []›
      by (simp del: T_brauns.simps)
    also have ... = 4 * min (2^k) ?n + 9 * (?n - 2^k + 1) + 5
      by(simp)
    also have ... = 4 * min (2^k) ?n + 4 * (?n - 2^k) + 5 * (?n - 2^k
+ 1) + 9
      by(simp)
    also have ... = 4 * ?n + 5 * (?n - 2^k + 1) + 9
      by(simp)
    also have ... ≤ 4 * ?n + 5 * ?n + 9
      using * by(simp)
    also have ... = 9 * (?n + 1)
      by (simp add: Suc_leI)
    finally show ?thesis .
  qed
qed
```

39.3.5 Converting a Braun Tree into a List of Elements

The code and the proof are originally due to Thomas Sewell (except running time).

function *list_fast_rec* :: *'a tree list* ⇒ *'a list* **where**
list_fast_rec ts = (*let us* = *filter* (λ*t*. *t* ≠ *Leaf*) *ts in*
if us = [] *then* [] *else*
map value us @ *list_fast_rec* (*map left us* @ *map right us*))
by (*pat_completeness*, *auto*)

lemma *list_fast_rec_term1*: *ts* ≠ [] ⟹ *Leaf* ∉ *set ts* ⟹
sum_list (*map* (*size o left*) *ts*) + *sum_list* (*map* (*size o right*) *ts*) < *sum_list* (*map size ts*)
apply (*clarsimp simp*: *sum_list_addf*[*symmetric*] *sum_list_map_filter'*)
apply (*rule sum_list_strict_mono*; *clarsimp?*)
apply (*case_tac x*; *simp*)
done

lemma *list_fast_rec_term*: *us* ≠ [] ⟹ *us* = *filter* (λ*t*. *t* ≠ ⟨⟩) *ts* ⟹
sum_list (*map* (*size o left*) *us*) + *sum_list* (*map* (*size o right*) *us*) < *sum_list* (*map size ts*)
apply (*rule order_less_le_trans*, *rule list_fast_rec_term1*, *simp_all*)
apply (*rule sum_list_filter_le_nat*)
done

termination
by (*relation measure* (*sum_list o map size*); *simp add*: *list_fast_rec_term*)

declare *list_fast_rec.simps*[*simp del*]

definition *list_fast* :: *'a tree* ⇒ *'a list* **where**
list_fast t = *list_fast_rec* [*t*]

definition *filter_not_Leaf* = *filter* (λ*t*. *t* ≠ *Leaf*)

definition *map_left* = *map left*
definition *map_right* = *map right*
definition *map_value* = *map value*

definition *T_filter_not_Leaf ts* = *length ts* + *1*
definition *T_map_left ts* = *length ts* + *1*
definition *T_map_right ts* = *length ts* + *1*

definition *T_map_value ts = length ts + 1*

lemmas *defs = filter_not_Leaf_def map_left_def map_right_def map_value_def*
T_filter_not_Leaf_def T_map_value_def T_map_left_def T_map_right_def

lemma *list_fast_rec_simp*:
list_fast_rec ts = (let us = filter_not_Leaf ts in
if us = [] then [] else
map_value us @ list_fast_rec (map_left us @ map_right us))
unfolding *defs list_fast_rec.simps*[*of ts*] **by**(*rule refl*)

time_function *list_fast_rec* **equations** *list_fast_rec_simp*
termination
by (*relation measure* (*sum_list o map size*); *simp add*: *list_fast_rec_term defs*)

declare *T_list_fast_rec.simps*[*simp del*]

Functional Correctness **lemma** *list_fast_rec_all_Leaf*:
$\forall$ *t* $\in$ *set ts. t = Leaf* $\Longrightarrow$ *list_fast_rec ts = []*
by (*simp add*: *filter_empty_conv list_fast_rec.simps*)

lemma *take_nths_eq_single*:
length xs − i < 2^n $\Longrightarrow$ *take_nths i n xs = take 1* (*drop i xs*)
by (*induction xs arbitrary*: *i n*; *simp add*: *drop_Cons'*)

lemma *braun_list_Nil*:
braun_list t [] = (t = Leaf)
by (*cases t*; *simp*)

lemma *braun_list_not_Nil*: *xs* $\neq$ *[]* $\Longrightarrow$
braun_list t xs =
(∃ *l x r. t = Node l x r* ∧ *x = hd xs* ∧
braun_list l (*take_nths 1 1 xs*) ∧
braun_list r (*take_nths 2 1 xs*))
by(*cases t*; *simp*)

theorem *list_fast_rec_correct*:
⟦ *length ts = 2 ^ k*; $\forall$ *i < 2 ^ k. braun_list* (*ts ! i*) (*take_nths i k xs*) ⟧
$\Longrightarrow$ *list_fast_rec ts = xs*
proof (*induct xs arbitrary*: *k ts rule*: *measure_induct_rule*[**where** *f*=*length*])

case (*less xs*)
show *?case*
proof (*cases length xs < 2 ^ k*)
case *True*
from *less.prems True* **have** *filter*:
∃ *n*. *ts* = *map* (λ*x*. *Node Leaf x Leaf*) *xs* @ *replicate n Leaf*
apply (*rule_tac x*=*length ts* − *length xs* **in** *exI*)
apply (*clarsimp simp*: *list_eq_iff_nth_eq*)
apply(*auto simp*: *nth_append braun_list_not_Nil take_nths_eq_single braun_list_Nil hd_drop_conv_nth*)
done
thus *?thesis*
by (*clarsimp simp*: *list_fast_rec.simps*[*of ts*] *o_def list_fast_rec_all_Leaf*)
next
case *False*
with *less.prems*(*2*) **have** ∗:
∀ *i* < *2* ^ *k*. *ts* ! *i* ≠ *Leaf*
∧ *value* (*ts* ! *i*) = *xs* ! *i*
∧ *braun_list* (*left* (*ts* ! *i*)) (*take_nths* (*i* + *2* ^ *k*) (*Suc k*) *xs*)
∧ (∀ *ys*. *ys* = *take_nths* (*i* + *2* ∗ *2* ^ *k*) (*Suc k*) *xs*
⟶ *braun_list* (*right* (*ts* ! *i*)) *ys*)
by (*auto simp*: *take_nths_empty hd_take_nths braun_list_not_Nil take_nths_take_nths*
algebra_simps)
have *1*: *map value ts* = *take* (*2* ^ *k*) *xs*
using *less.prems*(*1*) *False* **by** (*simp add*: *list_eq_iff_nth_eq* ∗)
have *2*: *list_fast_rec* (*map left ts* @ *map right ts*) = *drop* (*2* ^ *k*) *xs*
using *less.prems*(*1*) *False*
by (*auto intro*!: *Nat.diff_less less.hyps*[**where** *k*= *Suc k*]
simp: *nth_append* ∗ *take_nths_drop algebra_simps*)
from *less.prems*(*1*) *False* **show** *?thesis*
by (*auto simp*: *list_fast_rec.simps*[*of ts*] *1 2* ∗ *all_set_conv_all_nth*)
qed
qed

corollary *list_fast_correct*:
braun t ⟹ *list_fast t* = *list t*
by (*simp add*: *list_fast_def take_nths_00 braun_list_eq list_fast_rec_correct*[**where** *k*=*0*])

Running Time Analysis **lemma** *sum_tree_list_children*: ∀ *t* ∈ *set ts*. *t* ≠ *Leaf* ⟹
(∑ *t*←*ts*. *k* ∗ *size t*) = (∑ *t* ← *map left ts* @ *map right ts*. *k* ∗ *size t*) +

```
k * length ts
  by(induction ts)(auto simp add: neq_Leaf_iff algebra_simps)

theorem T_list_fast_rec_ub:
  T_list_fast_rec ts ≤ sum_list (map (λt. 14*size t + 1) ts) + 2
proof (induction ts rule: measure_induct_rule[where f=sum_list o map size])
  case (less ts)
  let ?us = filter (λt. t ≠ Leaf) ts
  show ?case
  proof cases
    assume ?us = []
    thus ?thesis using T_list_fast_rec.simps[of ts]
      by(simp add: defs sum_list_Suc)
  next
    assume ?us ≠ []
    let ?children = map left ?us @ map right ?us
    have 1: 1 ≤ length ?us
      using ‹?us ≠ []› linorder_not_less by auto
    have T_list_fast_rec ts = T_list_fast_rec ?children + 5 * length ?us
+ length ts + 7
        using ‹?us ≠ []› T_list_fast_rec.simps[of ts] by(simp add: defs
T_append)
    also have ... ≤ (∑ t←?children. 14 * size t + 1) + 5 * length ?us +
length ts + 9
      using less[of ?children] list_fast_rec_term[of ?us] ‹?us ≠ []›
      by (simp)
   also have ... = (∑ t←?children. 14 * size t) + 7 * length ?us + length
ts + 9
      by(simp add: sum_list_Suc o_def)
     also have ... ≤ (∑ t←?children. 14 * size t) + 14 * length ?us +
length ts + 2
      using 1 by(simp add: sum_list_Suc o_def)
    also have ... = (∑ t←?us. 14 * size t) + length ts + 2
      by(simp add: sum_tree_list_children)
    also have ... ≤ (∑ t←ts. 14 * size t) + length ts + 2
      by(simp add: sum_list_filter_le_nat)
    also have ... = (∑ t←ts. 14 * size t + 1) + 2
      by(simp add: sum_list_Suc)
    finally show ?case .
  qed
qed

end
```

40 Tries via Functions

theory *Trie_Fun*
imports
Set_Specs
begin

A trie where each node maps a key to sub-tries via a function. Nice abstract model. Not efficient because of the function space.

datatype *'a trie = Nd bool 'a ⇒ 'a trie option*

definition *empty* :: *'a trie* **where**
[*simp*]: *empty = Nd False* (*λ_. None*)

fun *isin* :: *'a trie ⇒ 'a list ⇒ bool* **where**
isin (*Nd b m*) [] = *b* |
isin (*Nd b m*) (*k # xs*) = (*case m k of None ⇒ False | Some t ⇒ isin t xs*)

fun *insert* :: *'a list ⇒ 'a trie ⇒ 'a trie* **where**
insert [] (*Nd b m*) = *Nd True m* |
insert (*x#xs*) (*Nd b m*) =
(*let s* = (*case m x of None ⇒ empty | Some t ⇒ t*) *in Nd b* (*m*(*x* := *Some*(*insert xs s*))))

fun *delete* :: *'a list ⇒ 'a trie ⇒ 'a trie* **where**
delete [] (*Nd b m*) = *Nd False m* |
delete (*x#xs*) (*Nd b m*) = *Nd b*
(*case m x of*
None ⇒ m |
Some t ⇒ m(*x* := *Some*(*delete xs t*)))

Use (a tuned version of) *isin* as an abstraction function:

lemma *isin_case*: *isin* (*Nd b m*) *xs* =
(*case xs of*
[] ⇒ *b* |
x # ys ⇒ (*case m x of None ⇒ False | Some t ⇒ isin t ys*))
by(*cases xs*)*auto*

definition *set_trie* :: *'a trie ⇒ 'a list set* **where**
[*simp*]: *set_trie t* = {*xs. isin t xs*}

lemma *isin_set_trie*: *isin t xs* = (*xs* ∈ *set_trie t*)
by *simp*

lemma *set_trie_insert*: *set_trie* (*insert xs t*) = *set_trie t* ∪ {*xs*}
by (*induction xs t rule*: *insert.induct*)
(*auto simp*: *isin_case split*!: *if_splits option.splits list.splits*)

lemma *set_trie_delete*: *set_trie* (*delete xs t*) = *set_trie t* − {*xs*}
by (*induction xs t rule*: *delete.induct*)
(*auto simp*: *isin_case split*!: *if_splits option.splits list.splits*)

interpretation *S*: *Set*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *set* = *set_trie* **and** *invar* = λ_. *True*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *isin_case split*: *list.split*)
next
case *2* **show** *?case* **by**(*rule isin_set_trie*)
next
case *3* **show** *?case* **by**(*rule set_trie_insert*)
next
case *4* **show** *?case* **by**(*rule set_trie_delete*)
qed (*rule TrueI*)+

end

41 Binary Tries and Patricia Tries

theory *Tries_Binary*
imports *Set_Specs*
begin

hide_const (**open**) *insert*

declare *Let_def*[*simp*]

fun *sel2* :: *bool* ⇒ *'a* ∗ *'a* ⇒ *'a* **where**
sel2 b (*a1*,*a2*) = (*if b then a2 else a1*)

fun *mod2* :: (*'a* ⇒ *'a*) ⇒ *bool* ⇒ *'a* ∗ *'a* ⇒ *'a* ∗ *'a* **where**
mod2 f b (*a1*,*a2*) = (*if b then* (*a1*,*f a2*) *else* (*f a1*,*a2*))

41.1 Trie

datatype *trie* = *Lf* | *Nd bool trie* ∗ *trie*

definition *empty* :: *trie* **where**
  [*simp*]: *empty* = *Lf*

fun *isin* :: *trie* ⇒ *bool list* ⇒ *bool* **where**
  isin Lf ks = *False* |
  isin (*Nd b lr*) *ks* =
  (*case ks of*
    [] ⇒ *b* |
    k#*ks* ⇒ *isin* (*sel2 k lr*) *ks*)

fun *insert* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
  insert [] *Lf* = *Nd True* (*Lf*,*Lf*) |
  insert [] (*Nd b lr*) = *Nd True lr* |
  insert (*k*#*ks*) *Lf* = *Nd False* (*mod2* (*insert ks*) *k* (*Lf*,*Lf*)) |
  insert (*k*#*ks*) (*Nd b lr*) = *Nd b* (*mod2* (*insert ks*) *k lr*)

lemma *isin_insert*: *isin* (*insert xs t*) *ys* = (*xs* = *ys* ∨ *isin t ys*)
proof (*induction xs t arbitrary*: *ys rule*: *insert.induct*)
qed (*auto split*: *list.splits if_splits*)

A simple implementation of delete; does not shrink the trie!

fun *delete0* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
  delete0 ks Lf = *Lf* |
  delete0 ks (*Nd b lr*) =
  (*case ks of*
    [] ⇒ *Nd False lr* |
    k#*ks*′ ⇒ *Nd b* (*mod2* (*delete0 ks*′) *k lr*))

lemma *isin_delete0*: *isin* (*delete0 as t*) *bs* = (*as* ≠ *bs* ∧ *isin t bs*)
proof (*induction as t arbitrary*: *bs rule*: *delete0.induct*)
qed (*auto split*: *list.splits if_splits*)

Now deletion with shrinking:

fun *node* :: *bool* ⇒ *trie* ∗ *trie* ⇒ *trie* **where**
  node b lr = (*if* ¬ *b* ∧ *lr* = (*Lf*,*Lf*) *then Lf else Nd b lr*)

fun *delete* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
  delete ks Lf = *Lf* |
  delete ks (*Nd b lr*) =
  (*case ks of*
    [] ⇒ *node False lr* |
    k#*ks*′ ⇒ *node b* (*mod2* (*delete ks*′) *k lr*))

lemma *isin_delete*: *isin* (*delete xs t*) *ys* = (*xs* ≠ *ys* ∧ *isin t ys*)

apply(*induction xs t arbitrary*: *ys rule*: *delete.induct*)
apply (*auto split*: *list.splits if_splits*)
apply (*metis isin.simps*(*1*))+
done

definition *set_trie* :: *trie* ⇒ *bool list set* **where**
set_trie t = {*xs*. *isin t xs*}

lemma *set_trie_empty*: *set_trie empty* = {}
by(*simp add*: *set_trie_def*)

lemma *set_trie_isin*: *isin t xs* = (*xs* ∈ *set_trie t*)
by(*simp add*: *set_trie_def*)

lemma *set_trie_insert*: *set_trie*(*insert xs t*) = *set_trie t* ∪ {*xs*}
by(*auto simp add*: *isin_insert set_trie_def*)

lemma *set_trie_delete*: *set_trie*(*delete xs t*) = *set_trie t* − {*xs*}
by(*auto simp add*: *isin_delete set_trie_def*)

Invariant: tries are fully shrunk:

fun *invar* **where**
invar Lf = *True* |
invar (*Nd b* (*l*,*r*)) = (*invar l* ∧ *invar r* ∧ (*l* = *Lf* ∧ *r* = *Lf* ⟶ *b*))

lemma *insert_Lf*: *insert xs t* ≠ *Lf*
using *insert.elims* **by** *blast*

lemma *invar_insert*: *invar t* ⟹ *invar*(*insert xs t*)
proof(*induction xs t rule*: *insert.induct*)
case *1* **thus** *?case* **by** *simp*
next
case (*2 b lr*)
thus *?case* **by**(*cases lr*; *simp*)
next
case (*3 k ks*)
thus *?case* **by**(*simp*; *cases ks*; *auto*)
next
case (*4 k ks b lr*)
then show *?case* **by**(*cases lr*; *auto simp*: *insert_Lf*)
qed

lemma *invar_delete*: *invar t* ⟹ *invar*(*delete xs t*)
proof(*induction t arbitrary*: *xs*)

case *Lf* **thus** *?case* **by** *simp*
next
case (*Nd b lr*)
thus *?case* **by**(*cases lr*)(*auto split*: *list.split*)
qed

interpretation *S*: *Set*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *set* = *set_trie* **and** *invar* = *invar*
unfolding *Set_def*
by (*smt* (*verit*, *best*) *Tries_Binary.empty_def invar.simps*(*1*) *invar_delete invar_insert set_trie_delete set_trie_empty set_trie_insert set_trie_isin*)

41.2 Patricia Trie

datatype *trieP* = *LfP* | *NdP bool list bool trieP* ∗ *trieP*

Fully shrunk:

fun *invarP* **where**
invarP LfP = *True* |
invarP (*NdP ps b* (*l*,*r*)) = (*invarP l* ∧ *invarP r* ∧ (*l* = *LfP* ∨ *r* = *LfP* ⟶ *b*))

fun *isinP* :: *trieP* ⇒ *bool list* ⇒ *bool* **where**
isinP LfP ks = *False* |
isinP (*NdP ps b lr*) *ks* =
(*let n* = *length ps in*
if ps = *take n ks*
then case drop n ks of [] ⇒ *b* | *k*#*ks*′ ⇒ *isinP* (*sel2 k lr*) *ks*′
else False)

definition *emptyP* :: *trieP* **where**
[*simp*]: *emptyP* = *LfP*

fun *lcp* :: ′*a list* ⇒ ′*a list* ⇒ ′*a list* × ′*a list* × ′*a list* **where**
lcp [] *ys* = ([],[],*ys*) |
lcp xs [] = (*xs*,[],[]) |
lcp (*x*#*xs*) (*y*#*ys*) =
(*if x*≠*y then* ([],*x*#*xs*,*y*#*ys*)
else let (*ps*,*xs*′,*ys*′) = *lcp xs ys in* (*x*#*ps*,*xs*′,*ys*′))

lemma *mod2_cong*[*fundef_cong*]:

```
  ⟦ lr = lr'; k = k'; ⋀a b. lr'=(a,b) ⟹ f (a) = f' (a) ; ⋀a b. lr'=(a,b) ⟹
f (b) = f' (b) ⟧
  ⟹ mod2 f k lr= mod2 f' k' lr'
  by(cases lr, cases lr', auto)


fun insertP :: bool list ⇒ trieP ⇒ trieP where
  insertP ks LfP  = NdP ks True (LfP,LfP) |
  insertP ks (NdP ps b lr) =
  (case lcp ks ps of
     (qs, k#ks', p#ps') ⇒
       let tp = NdP ps' b lr; tk = NdP ks' True (LfP,LfP) in
       NdP qs False (if k then (tp,tk) else (tk,tp)) |
     (qs, k#ks', []) ⇒
       NdP ps b (mod2 (insertP ks') k lr) |
     (qs, [], p#ps') ⇒
       let t = NdP ps' b lr in
       NdP qs True (if p then (LfP,t) else (t,LfP)) |
     (qs,[],[]) ⇒ NdP ps True lr)
```

Smart constructor that shrinks:

```
definition nodeP :: bool list ⇒ bool ⇒ trieP ∗ trieP ⇒ trieP where
  nodeP ps b lr =
 (if b then  NdP ps b lr
  else case lr of
   (LfP,LfP) ⇒ LfP |
   (LfP, NdP ks b lr) ⇒ NdP (ps @ True # ks) b lr |
   (NdP ks b lr, LfP) ⇒ NdP (ps @ False # ks) b lr |
   _ ⇒ NdP ps b lr)

fun deleteP :: bool list ⇒ trieP ⇒ trieP where
  deleteP ks LfP  = LfP |
  deleteP ks (NdP ps b lr) =
  (case lcp ks ps of
     (_, _, _#_) ⇒ NdP ps b lr |
     (_, k#ks', []) ⇒ nodeP ps b (mod2 (deleteP ks') k lr) |
     (_, [], []) ⇒ nodeP ps False lr)
```

41.2.1 Functional Correctness

First step: *trieP* implements *trie* via the abstraction function *abs_trieP*:

```
fun prefix_trie :: bool list ⇒ trie ⇒ trie where
  prefix_trie [] t = t |
  prefix_trie (k#ks) t =
```

```
  (let t' = prefix_trie ks t in Nd False (if k then (Lf,t') else (t',Lf)))

fun abs_trieP :: trieP ⇒ trie where
  abs_trieP LfP = Lf |
  abs_trieP (NdP ps b (l,r)) = prefix_trie ps (Nd b (abs_trieP l, abs_trieP
r))
```

Correctness of *isinP*:

```
lemma isin_prefix_trie:
  isin (prefix_trie ps t) ks
   = (ps = take (length ps) ks ∧ isin t (drop (length ps) ks))
  by (induction ps arbitrary: ks) (auto split: list.split)

lemma abs_trieP_isinP:
  isinP t ks = isin (abs_trieP t) ks
proof (induction t arbitrary: ks rule: abs_trieP.induct)
qed (auto simp: isin_prefix_trie split: list.split)
```

Correctness of *insertP*:

```
lemma prefix_trie_Lfs: prefix_trie ks (Nd True (Lf,Lf)) = insert ks Lf
  by (induction ks) auto

lemma insert_prefix_trie_same:
  insert ps (prefix_trie ps (Nd b lr)) = prefix_trie ps (Nd True lr)
  by (induction ps) auto

lemma insert_append: insert (ks @ ks') (prefix_trie ks t) = prefix_trie ks
(insert ks' t)
  by (induction ks) auto

lemma prefix_trie_append: prefix_trie (ps @ qs) t = prefix_trie ps (prefix_trie
qs t)
  by (induction ps) auto

lemma lcp_if: lcp ks ps = (qs, ks', ps') ⟹
  ks = qs @ ks' ∧ ps = qs @ ps' ∧ (ks' ≠ [] ∧ ps' ≠ [] ⟶ hd ks' ≠ hd ps')
proof (induction ks ps arbitrary: qs ks' ps' rule: lcp.induct)
qed (auto split: prod.splits if_splits)

lemma abs_trieP_insertP:
  abs_trieP (insertP ks t) = insert ks (abs_trieP t)
proof (induction t arbitrary: ks)
qed (auto simp: prefix_trie_Lfs insert_prefix_trie_same insert_append
prefix_trie_append
```

dest!: *lcp_if split*: *list.split prod.split if_splits*)

Correctness of *deleteP*:

lemma *prefix_trie_Lf*: *prefix_trie xs t* = *Lf* ⟷ *xs* = [] ∧ *t* = *Lf*
by(*cases xs*)(*auto*)

lemma *abs_trieP_Lf*: *abs_trieP t* = *Lf* ⟷ *t* = *LfP*
by(*cases t*) (*auto simp*: *prefix_trie_Lf*)

lemma *delete_prefix_trie*:
delete xs (*prefix_trie xs* (*Nd b* (*l*,*r*)))
= (*if* (*l*,*r*) = (*Lf*,*Lf*) *then Lf else prefix_trie xs* (*Nd False* (*l*,*r*)))
by(*induction xs*)(*auto simp*: *prefix_trie_Lf*)

lemma *delete_append_prefix_trie*:
delete (*xs* @ *ys*) (*prefix_trie xs t*)
= (*if delete ys t* = *Lf then Lf else prefix_trie xs* (*delete ys t*))
by(*induction xs*)(*auto simp*: *prefix_trie_Lf*)

lemma *nodeP_LfP2*: *nodeP xs False* (*LfP*, *LfP*) = *LfP*
by(*simp add*: *nodeP_def*)

Some non-inductive aux. lemmas:

lemma *abs_trieP_nodeP*: *a*≠*LfP* ∨ *b* ≠ *LfP* ⟹
abs_trieP (*nodeP xs f* (*a*, *b*)) = *prefix_trie xs* (*Nd f* (*abs_trieP a*, *abs_trieP b*))
by(*auto simp add*: *nodeP_def prefix_trie_append split*: *trieP.split*)

lemma *nodeP_True*: *nodeP ps True lr* = *NdP ps True lr*
by(*simp add*: *nodeP_def*)

lemma *delete_abs_trieP*:
delete ks (*abs_trieP t*) = *abs_trieP* (*deleteP ks t*)
proof (*induction t arbitrary*: *ks*)
qed (*auto simp*: *delete_prefix_trie delete_append_prefix_trie*
prefix_trie_append prefix_trie_Lf abs_trieP_Lf nodeP_LfP2 abs_trieP_nodeP nodeP_True
dest!: *lcp_if split*: *if_splits list.split prod.split*)

Invariant preservation:

lemma *insertP_LfP*: *insertP xs t* ≠ *LfP*
by(*cases t*)(*auto split*: *prod.split list.split*)

lemma *invarP_insertP*: *invarP t* ⟹ *invarP*(*insertP xs t*)

proof(*induction t arbitrary*: *xs*)
case *LfP* **thus** *?case* **by** *simp*
next
case (*NdP bs b lr*)
then show *?case*
by(*cases lr*)(*auto simp*: *insertP_LfP split*: *prod.split list.split*)
qed

lemma *invarP_nodeP*: ⟦ *invarP t1*; *invarP t2*⟧ ⟹ *invarP* (*nodeP xs b* (*t1*, *t2*))
by (*auto simp add*: *nodeP_def split*: *trieP.split*)

lemma *invarP_deleteP*: *invarP t* ⟹ *invarP*(*deleteP xs t*)
proof(*induction t arbitrary*: *xs*)
case *LfP* **thus** *?case* **by** *simp*
next
case (*NdP ks b lr*)
thus *?case* **by**(*cases lr*)(*auto simp*: *invarP_nodeP split*: *prod.split list.split*)
qed

The overall correctness proof. Simply composes correctness lemmas.

definition *set_trieP* :: *trieP* ⇒ *bool list set* **where**
set_trieP = *set_trie o abs_trieP*

lemma *isinP_set_trieP*: *isinP t xs* = (*xs* ∈ *set_trieP t*)
by(*simp add*: *abs_trieP_isinP set_trie_isin set_trieP_def*)

lemma *set_trieP_insertP*: *set_trieP* (*insertP xs t*) = *set_trieP t* ∪ {*xs*}
by(*simp add*: *abs_trieP_insertP set_trie_insert set_trieP_def*)

lemma *set_trieP_deleteP*: *set_trieP* (*deleteP xs t*) = *set_trieP t* − {*xs*}
by(*auto simp*: *set_trie_delete set_trieP_def simp flip*: *delete_abs_trieP*)

interpretation *SP*: *Set*
where *empty* = *emptyP* **and** *isin* = *isinP* **and** *insert* = *insertP* **and** *delete* = *deleteP*
and *set* = *set_trieP* **and** *invar* = *invarP*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *set_trieP_def set_trie_def*)
next
case *2* **show** *?case* **by**(*rule isinP_set_trieP*)
next
case *3* **thus** *?case* **by** (*auto simp*: *set_trieP_insertP*)

next
case *4* **thus** *?case* **by**(*auto simp*: *set_trieP_deleteP*)
next
case *5* **thus** *?case* **by**(*simp*)
next
case *6* **thus** *?case* **by**(*rule invarP_insertP*)
next
case *7* **thus** *?case* **by**(*rule invarP_deleteP*)
qed

end

42 Ternary Tries

theory *Trie_Ternary*
imports
Tree_Map
Trie_Fun
begin

An implementation of tries for an arbitrary alphabet *'a* where the mapping from an element of type *'a* to the sub-trie is implemented by an (unbalanced) binary search tree. In principle, other search trees (e.g. red-black trees) work just as well, with some small adjustments (Exercise!).

This is an implementation of the "ternary search trees" by Bentley and Sedgewick [SODA 1997, Dr. Dobbs 1998]. The name derives from the fact that a node in the BST can now be drawn to have 3 children, where the middle child is the sub-trie that the node maps its key to. Hence the name *trie3*.

Example from https://en.wikipedia.org/wiki/Ternary_search_tree#Description:
c / | a u h | | | t. t e. u / / | / | s. p. e. i. s.

Characters with a dot are final. Thus the tree represents the set of strings "cute","cup","at","as","he","us" and "i".

datatype *'a trie3 = Nd3 bool ('a * 'a trie3) tree*

The development below works almost verbatim for any search tree implementation, eg *RBT_Map*, and not just *Tree_Map*, except for the termination lemma *lookup_size*.

term *size_tree*
lemma *lookup_size*[*termination_simp*]:
fixes *t* :: (*'a*::*linorder* * *'a trie3*) *tree*
shows *lookup t a = Some b* $\Longrightarrow$ *size b < Suc* (*size_tree* (λ*ab*. *Suc* (*size* (*snd*(*ab*)))) *t*)
by (*induction t a rule*: *lookup.induct*)(*auto split*: *if_splits*)

definition *empty3* :: *'a trie3* **where**
[*simp*]: *empty3 = Nd3 False Leaf*

fun *isin3* :: (*'a*::*linorder*) *trie3* ⇒ *'a list* ⇒ *bool* **where**
isin3 (*Nd3 b m*) [] = *b* |
isin3 (*Nd3 b m*) (*x # xs*) = (*case lookup m x of None* ⇒ *False* | *Some t* ⇒ *isin3 t xs*)

fun *insert3* :: (*'a*::*linorder*) *list* ⇒ *'a trie3* ⇒ *'a trie3* **where**
insert3 [] (*Nd3 b m*) = *Nd3 True m* |
insert3 (*x*#*xs*) (*Nd3 b m*) =
Nd3 b (*update x* (*insert3 xs* (*case lookup m x of None* ⇒ *empty3* | *Some t* ⇒ *t*)) *m*)

fun *delete3* :: (*'a*::*linorder*) *list* ⇒ *'a trie3* ⇒ *'a trie3* **where**
delete3 [] (*Nd3 b m*) = *Nd3 False m* |
delete3 (*x*#*xs*) (*Nd3 b m*) = *Nd3 b*
(*case lookup m x of*
None ⇒ *m* |
Some t ⇒ *update x* (*delete3 xs t*) *m*)

42.1 Correctness

Proof by stepwise refinement. First abs3tract to type *'a trie*.

fun *abs3* :: *'a*::*linorder trie3* ⇒ *'a trie* **where**
abs3 (*Nd3 b t*) = *Nd b* (λ*a*. *map_option abs3* (*lookup t a*))

fun *invar3* :: (*'a*::*linorder*)*trie3* ⇒ *bool* **where**
invar3 (*Nd3 b m*) = (*M.invar m* ∧ (∀ *a t*. *lookup m a* = *Some t* ⟶ *invar3 t*))

lemma *isin_abs3*: *isin3 t xs* = *isin* (*abs3 t*) *xs*
by (*induction t xs rule*: *isin3.induct*)(*auto split*: *option.split*)

lemma *abs3_insert3*: *invar3 t* ⟹ *abs3*(*insert3 xs t*) = *insert xs* (*abs3 t*)
proof (*induction xs t rule*: *insert3.induct*)
qed (*auto simp*: *M.map_specs Tree_Set.empty_def*[*symmetric*] *split*: *option.split*)

lemma *abs3_delete3*: *invar3 t* ⟹ *abs3*(*delete3 xs t*) = *delete xs* (*abs3 t*)
by (*induction xs t rule*: *delete3.induct*)(*auto simp*: *M.map_specs split*: *option.split*)

lemma *invar3_insert3*: *invar3 t* $\Longrightarrow$ *invar3* (*insert3 xs t*)
proof (*induction xs t rule*: *insert3.induct*)
qed (*auto simp*: *M.map_specs simp flip*: *Tree_Set.empty_def split*: *option.split*)

lemma *invar3_delete3*: *invar3 t* $\Longrightarrow$ *invar3* (*delete3 xs t*)
by (*induction xs t rule*: *delete3.induct*)(*auto simp*: *M.map_specs split*: *option.split*)

Overall correctness w.r.t. the *Set* ADT:

interpretation *S2*: *Set*
where *empty* = *empty3* **and** *isin* = *isin3* **and** *insert* = *insert3* **and** *delete* = *delete3*
and *set* = *set_trie o abs3* **and** *invar* = *invar3*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *isin_case split*: *list.split*)
next
case *2* **thus** *?case* **by** (*simp add*: *isin_abs3*)
next
case *3* **thus** *?case* **by** (*simp add*: *set_trie_insert abs3_insert3 del*: *set_trie_def*)
next
case *4* **thus** *?case* **by** (*simp add*: *set_trie_delete abs3_delete3 del*: *set_trie_def*)
next
case *5* **thus** *?case* **by** (*simp add*: *M.map_specs Tree_Set.empty_def*[*symmetric*])
next
case *6* **thus** *?case* **by** (*simp add*: *invar3_insert3*)
next
case *7* **thus** *?case* **by** (*simp add*: *invar3_delete3*)
qed

end

43 Queue Specification

theory *Queue_Spec*
imports *Main*
begin

The basic queue interface with *list*-based specification:

locale *Queue* =
fixes *empty* :: $'q$
fixes *enq* :: $'a \Rightarrow 'q \Rightarrow 'q$
fixes *first* :: $'q \Rightarrow 'a$

fixes *deq* :: $'q \Rightarrow 'q$
fixes *is_empty* :: $'q \Rightarrow$ *bool*
fixes *list* :: $'q \Rightarrow 'a$ *list*
fixes *invar* :: $'q \Rightarrow$ *bool*
assumes *list_empty*: *list empty* = []
assumes *list_enq*: *invar q* $\Longrightarrow$ *list*(*enq x q*) = *list q* @ [*x*]
assumes *list_deq*: *invar q* $\Longrightarrow$ *list*(*deq q*) = *tl*(*list q*)
assumes *list_first*: *invar q* $\Longrightarrow \neg$ *list q* = [] $\Longrightarrow$ *first q* = *hd*(*list q*)
assumes *list_is_empty*: *invar q* $\Longrightarrow$ *is_empty q* = (*list q* = [])
assumes *invar_empty*: *invar empty*
assumes *invar_enq*: *invar q* $\Longrightarrow$ *invar*(*enq x q*)
assumes *invar_deq*: *invar q* $\Longrightarrow$ *invar*(*deq q*)

end

44 Queue Implementation via 2 Lists

theory *Queue_2Lists*
imports
Queue_Spec
HOL−Library.Time_Functions
begin

Definitions:

type_synonym $'a$ *queue* = $'a$ *list* × $'a$ *list*

fun *norm* :: $'a$ *queue* ⇒ $'a$ *queue* **where**
norm (*fs*,*rs*) = (*if fs* = [] *then* (*itrev rs* [], []) *else* (*fs*,*rs*))

fun *enq* :: $'a$ ⇒ $'a$ *queue* ⇒ $'a$ *queue* **where**
enq a (*fs*,*rs*) = *norm*(*fs*, *a* # *rs*)

fun *deq* :: $'a$ *queue* ⇒ $'a$ *queue* **where**
deq (*fs*,*rs*) = (*if fs* = [] *then* (*fs*,*rs*) *else norm*(*tl fs*,*rs*))

fun *first* :: $'a$ *queue* ⇒ $'a$ **where**
first (*a* # *fs*,*rs*) = *a*

fun *is_empty* :: $'a$ *queue* ⇒ *bool* **where**
is_empty (*fs*,*rs*) = (*fs* = [])

fun *list* :: $'a$ *queue* ⇒ $'a$ *list* **where**
list (*fs*,*rs*) = *fs* @ *rev rs*

fun *invar* :: *'a queue* ⇒ *bool* **where**
invar (*fs*,*rs*) = (*fs* = [] ⟶ *rs* = [])

Implementation correctness:

interpretation *Queue*
where *empty* = ([],[]) **and** *enq* = *enq* **and** *deq* = *deq* **and** *first* = *first*
and *is_empty* = *is_empty* **and** *list* = *list* **and** *invar* = *invar*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp*)
next
case (*2 q*) **thus** *?case* **by**(*cases q*) (*simp*)
next
case (*3 q*) **thus** *?case* **by**(*cases q*) (*simp add*: *itrev_Nil*)
next
case (*4 q*) **thus** *?case* **by**(*cases q*) (*auto simp*: *neq_Nil_conv*)
next
case (*5 q*) **thus** *?case* **by**(*cases q*) (*auto*)
next
case *6* **show** *?case* **by**(*simp*)
next
case (*7 q*) **thus** *?case* **by**(*cases q*) (*simp*)
next
case (*8 q*) **thus** *?case* **by**(*cases q*) (*simp*)
qed

Running times:

time_fun *norm*
time_fun *enq*
time_fun *deq*

Amortized running times:

fun Φ :: *'a queue* ⇒ *nat* **where**
Φ(*fs*,*rs*) = *length rs*

lemma *a_enq*: *T_enq a* (*fs*,*rs*) + Φ(*enq a* (*fs*,*rs*)) − Φ(*fs*,*rs*) ≤ *2*
by(*auto simp*: *T_itrev*)

lemma *a_deq*: *T_deq* (*fs*,*rs*) + Φ(*deq* (*fs*,*rs*)) − Φ(*fs*,*rs*) ≤ *1*
by(*auto simp*: *T_itrev T_tl*)

end

45 Priority Queue Specifications

theory *Priority_Queue_Specs*
imports *HOL−Library.Multiset*
begin

Priority queue interface + specification:

locale *Priority_Queue* =
fixes *empty* :: $'q$
and *is_empty* :: $'q \Rightarrow$ *bool*
and *insert* :: $'a$::*linorder* $\Rightarrow$ $'q \Rightarrow$ $'q$
and *get_min* :: $'q \Rightarrow$ $'a$
and *del_min* :: $'q \Rightarrow$ $'q$
and *invar* :: $'q \Rightarrow$ *bool*
and *mset* :: $'q \Rightarrow$ $'a$ *multiset*
assumes *mset_empty*: *mset empty* = {#}
and *is_empty*: *invar q* $\Longrightarrow$ *is_empty q* = (*mset q* = {#})
and *mset_insert*: *invar q* $\Longrightarrow$ *mset* (*insert x q*) = *mset q* + {#*x*#}
and *mset_del_min*: *invar q* $\Longrightarrow$ *mset q* $\neq$ {#} $\Longrightarrow$
mset (*del_min q*) = *mset q* − {# *get_min q* #}
and *mset_get_min*: *invar q* $\Longrightarrow$ *mset q* $\neq$ {#} $\Longrightarrow$ *get_min q* = *Min_mset* (*mset q*)
and *invar_empty*: *invar empty*
and *invar_insert*: *invar q* $\Longrightarrow$ *invar* (*insert x q*)
and *invar_del_min*: *invar q* $\Longrightarrow$ *mset q* $\neq$ {#} $\Longrightarrow$ *invar* (*del_min q*)

Extend locale with *merge*. Need to enforce that $'q$ is the same in both locales.

locale *Priority_Queue_Merge* = *Priority_Queue* **where** *empty* = *empty*
for *empty* :: $'q$ +
fixes *merge* :: $'q \Rightarrow$ $'q \Rightarrow$ $'q$
assumes *mset_merge*: ⟦ *invar q1*; *invar q2* ⟧ $\Longrightarrow$ *mset* (*merge q1 q2*) = *mset q1* + *mset q2*
and *invar_merge*: ⟦ *invar q1*; *invar q2* ⟧ $\Longrightarrow$ *invar* (*merge q1 q2*)

end

46 Heaps

theory *Heaps*
imports
HOL−Library.Tree_Multiset
Priority_Queue_Specs
begin

Heap = priority queue on trees:

locale *Heap* =
fixes *insert* :: ('*a*::*linorder*) ⇒ '*a tree* ⇒ '*a tree*
and *del_min* :: '*a tree* ⇒ '*a tree*
assumes *mset_insert*: *heap q* ⟹ *mset_tree* (*insert x q*) = {#*x*#} + *mset_tree q*
and *mset_del_min*: ⟦ *heap q*; *q* ≠ *Leaf* ⟧ ⟹ *mset_tree* (*del_min q*) = *mset_tree q* − {#*value q*#}
and *heap_insert*: *heap q* ⟹ *heap*(*insert x q*)
and *heap_del_min*: *heap q* ⟹ *heap*(*del_min q*)
begin

definition *empty* :: '*a tree* **where**
empty = *Leaf*

fun *is_empty* :: '*a tree* ⇒ *bool* **where**
is_empty t = (*t* = *Leaf*)

fun *get_min* :: '*a tree* ⇒ '*a* **where**
get_min (*Node l a r*) = *a*

sublocale *Priority_Queue* **where** *empty* = *empty* **and** *is_empty* = *is_empty*
and *insert* = *insert*
and *get_min* = *get_min* **and** *del_min* = *del_min* **and** *invar* = *heap* **and** *mset* = *mset_tree*
proof (*standard*, *goal_cases*)
case *1* **thus** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*auto*)
next
case *3* **thus** *?case* **by**(*simp add*: *mset_insert*)
next
case *4* **thus** *?case* **by**(*auto simp add*: *mset_del_min neq_Leaf_iff*)
next
case *5* **thus** *?case* **by**(*auto simp*: *neq_Leaf_iff Min_insert2 simp del*: *Un_iff*)
next
case *6* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *7* **thus** *?case* **by**(*simp add*: *heap_insert*)
next
case *8* **thus** *?case* **by**(*simp add*: *heap_del_min*)
qed

end

Once you have *merge*, *insert* and *del_min* are easy:

locale *Heap_Merge* =
fixes *merge* :: *'a*::*linorder tree* ⇒ *'a tree* ⇒ *'a tree*
assumes *mset_merge*: ⟦ *heap q1*; *heap q2* ⟧ ⟹ *mset_tree* (*merge q1 q2*)
= *mset_tree q1* + *mset_tree q2*
and *invar_merge*: ⟦ *heap q1*; *heap q2* ⟧ ⟹ *heap* (*merge q1 q2*)
begin

fun *insert* :: *'a* ⇒ *'a tree* ⇒ *'a tree* **where**
insert x t = *merge* (*Node Leaf x Leaf*) *t*

fun *del_min* :: *'a tree* ⇒ *'a tree* **where**
del_min Leaf = *Leaf* |
del_min (*Node l x r*) = *merge l r*

interpretation *Heap insert del_min*
proof(*standard*, *goal_cases*)
case *1* **thus** *?case* **by**(*simp add*:*mset_merge*)
next
case (*2 q*) **thus** *?case* **by**(*cases q*)(*auto simp*: *mset_merge*)
next
case *3* **thus** *?case* **by** (*simp add*: *invar_merge*)
next
case (*4 q*) **thus** *?case* **by** (*cases q*)(*auto simp*: *invar_merge*)
qed

lemmas *local_empty_def* = *local.empty_def*
lemmas *local_get_min_def* = *local.get_min.simps*

sublocale *PQM*: *Priority_Queue_Merge* **where** *empty* = *empty* **and** *is_empty*
= *is_empty* **and** *insert* = *insert*
and *get_min* = *get_min* **and** *del_min* = *del_min* **and** *invar* = *heap* **and**
mset = *mset_tree* **and** *merge* = *merge*
proof(*standard*, *goal_cases*)
case *1* **thus** *?case* **by** (*simp add*: *mset_merge*)
next
case *2* **thus** *?case* **by** (*simp add*: *invar_merge*)
qed

end

end

47 Leftist Heap

theory *Leftist_Heap*
imports
HOL−Library.Pattern_Aliases
Tree2
Priority_Queue_Specs
Complex_Main
HOL−Library.Time_Commands
begin

fun *mset_tree* :: (*'a*∗*'b*) *tree* ⇒ *'a multiset* **where**
mset_tree Leaf = {#} |
mset_tree (*Node l* (*a*, _) *r*) = {#*a*#} + *mset_tree l* + *mset_tree r*

type_synonym *'a lheap* = (*'a*∗*nat*)*tree*

fun *mht* :: *'a lheap* ⇒ *nat* **where**
mht Leaf = *0* |
mht (*Node* _ (_, *n*) _) = *n*

The invariants:

fun (**in** *linorder*) *heap* :: (*'a*∗*'b*) *tree* ⇒ *bool* **where**
heap Leaf = *True* |
heap (*Node l* (*m*, _) *r*) =
((∀ *x* ∈ *set_tree l* ∪ *set_tree r*. *m* ≤ *x*) ∧ *heap l* ∧ *heap r*)

fun *ltree* :: *'a lheap* ⇒ *bool* **where**
ltree Leaf = *True* |
ltree (*Node l* (*a*, *n*) *r*) =
(*min_height l* ≥ *min_height r* ∧ *n* = *min_height r* + *1* ∧ *ltree l* & *ltree r*)

definition *empty* :: *'a lheap* **where**
empty = *Leaf*

definition *node* :: *'a lheap* ⇒ *'a* ⇒ *'a lheap* ⇒ *'a lheap* **where**
node l a r =
(*let mhl* = *mht l*; *mhr* = *mht r*
in if mhl ≥ *mhr then Node l* (*a*,*mhr*+*1*) *r else Node r* (*a*,*mhl*+*1*) *l*)

fun *get_min* :: *'a lheap* ⇒ *'a* **where**

get_min(*Node l* (*a*, *n*) *r*) = *a*

For function *merge*:

unbundle *pattern_aliases*

fun *merge* :: *'a*::*ord lheap* ⇒ *'a lheap* ⇒ *'a lheap* **where**
merge Leaf t = *t* |
merge t Leaf = *t* |
merge (*Node l1* (*a1*, *n1*) *r1* =: *t1*) (*Node l2* (*a2*, *n2*) *r2* =: *t2*) =
(*if a1* ≤ *a2 then node l1 a1* (*merge r1 t2*)
else node l2 a2 (*merge t1 r2*))

Termination of *merge*: by sum or lexicographic product of the sizes of the two arguments. Isabelle uses a lexicographic product.

lemma *merge_code*: *merge t1 t2* = (*case* (*t1*,*t2*) *of*
(*Leaf*, _) ⇒ *t2* |
(_, *Leaf*) ⇒ *t1* |
(*Node l1* (*a1*, *n1*) *r1*, *Node l2* (*a2*, *n2*) *r2*) ⇒
if a1 ≤ *a2 then node l1 a1* (*merge r1 t2*) *else node l2 a2* (*merge t1 r2*))
by(*induction t1 t2 rule*: *merge.induct*) (*simp_all split*: *tree.split*)

hide_const (**open**) *insert*

definition *insert* :: *'a*::*ord* ⇒ *'a lheap* ⇒ *'a lheap* **where**
insert x t = *merge* (*Node Leaf* (*x*,*1*) *Leaf*) *t*

fun *del_min* :: *'a*::*ord lheap* ⇒ *'a lheap* **where**
del_min Leaf = *Leaf* |
del_min (*Node l* _ *r*) = *merge l r*

47.1 Lemmas

lemma *mset_tree_empty*: *mset_tree t* = {#} ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *mht_eq_min_height*: *ltree t* ⟹ *mht t* = *min_height t*
by(*cases t*) *auto*

lemma *ltree_node*: *ltree* (*node l a r*) ⟷ *ltree l* ∧ *ltree r*
by(*auto simp add*: *node_def mht_eq_min_height*)

lemma *heap_node*: *heap* (*node l a r*) ⟷
heap l ∧ *heap r* ∧ (∀ *x* ∈ *set_tree l* ∪ *set_tree r*. *a* ≤ *x*)
by(*auto simp add*: *node_def*)

lemma *set_tree_mset*: *set_tree t* = *set_mset*(*mset_tree t*)
by(*induction t*) *auto*

47.2 Functional Correctness

lemma *mset_merge*: *mset_tree* (*merge t1 t2*) = *mset_tree t1* + *mset_tree t2*
by (*induction t1 t2 rule*: *merge.induct*) (*auto simp add*: *node_def ac_simps*)

lemma *mset_insert*: *mset_tree* (*insert x t*) = *mset_tree t* + {#*x*#}
by (*auto simp add*: *insert_def mset_merge*)

lemma *get_min*: ⟦ *heap t*; *t* ≠ *Leaf* ⟧ ⟹ *get_min t* = *Min*(*set_tree t*)
by (*cases t*) (*auto simp add*: *eq_Min_iff*)

lemma *mset_del_min*: *mset_tree* (*del_min t*) = *mset_tree t* − {# *get_min t* #}
by (*cases t*) (*auto simp*: *mset_merge*)

lemma *ltree_merge*: ⟦ *ltree l*; *ltree r* ⟧ ⟹ *ltree* (*merge l r*)
by(*induction l r rule*: *merge.induct*)(*auto simp*: *ltree_node*)

lemma *heap_merge*: ⟦ *heap l*; *heap r* ⟧ ⟹ *heap* (*merge l r*)
proof(*induction l r rule*: *merge.induct*)
case *3* **thus** *?case* **by**(*auto simp*: *heap_node mset_merge ball_Un set_tree_mset*)
qed *simp_all*

lemma *ltree_insert*: *ltree t* ⟹ *ltree*(*insert x t*)
by(*simp add*: *insert_def ltree_merge del*: *merge.simps split*: *tree.split*)

lemma *heap_insert*: *heap t* ⟹ *heap*(*insert x t*)
by(*simp add*: *insert_def heap_merge del*: *merge.simps split*: *tree.split*)

lemma *ltree_del_min*: *ltree t* ⟹ *ltree*(*del_min t*)
by(*cases t*)(*auto simp add*: *ltree_merge simp del*: *merge.simps*)

lemma *heap_del_min*: *heap t* ⟹ *heap*(*del_min t*)
by(*cases t*)(*auto simp add*: *heap_merge simp del*: *merge.simps*)

Last step of functional correctness proof: combine all the above lemmas to show that leftist heaps satisfy the specification of priority queues with merge.

interpretation *lheap*: *Priority_Queue_Merge*

where *empty* = *empty* **and** *is_empty* = $\lambda t.\ t$ = *Leaf*
and *insert* = *insert* **and** *del_min* = *del_min*
and *get_min* = *get_min* **and** *merge* = *merge*
and *invar* = $\lambda t.$ *heap t* $\wedge$ *ltree t* **and** *mset* = *mset_tree*
proof(*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case (*2 q*) **show** *?case* **by** (*cases q*) *auto*
next
case *3* **show** *?case* **by**(*rule mset_insert*)
next
case *4* **show** *?case* **by**(*rule mset_del_min*)
next
case *5* **thus** *?case* **by**(*simp add*: *get_min mset_tree_empty set_tree_mset*)
next
case *6* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *7* **thus** *?case* **by**(*simp add*: *heap_insert ltree_insert*)
next
case *8* **thus** *?case* **by**(*simp add*: *heap_del_min ltree_del_min*)
next
case *9* **thus** *?case* **by** (*simp add*: *mset_merge*)
next
case *10* **thus** *?case* **by** (*simp add*: *heap_merge ltree_merge*)
qed

47.3 Complexity

Auxiliary time functions (which are both 0):

time_fun *mht*
time_fun *node*

lemma *T_mht_0*[*simp*]: *T_mht t* = *0*
by(*cases t*)*auto*

Define timing function

time_fun *merge*
time_fun *insert*
time_fun *del_min*

lemma *T_merge_min_height*: *ltree l* $\Longrightarrow$ *ltree r* $\Longrightarrow$ *T_merge l r* $\leq$ *min_height l* + *min_height r* + *1*
proof(*induction l r rule*: *merge.induct*)
case *3* **thus** *?case* **by**(*auto*)

qed *simp_all*

corollary *T_merge_log*: **assumes** *ltree l ltree r*
shows *T_merge l r ≤ log 2 (size1 l) + log 2 (size1 r) + 1*
using *le_log2_of_power*[*OF min_height_size1*[*of l*]]
le_log2_of_power[*OF min_height_size1*[*of r*]] *T_merge_min_height*[*of l r*] *assms*
by *linarith*

corollary *T_insert_log*: *ltree t* ⟹ *T_insert x t ≤ log 2 (size1 t) + 2*
using *T_merge_log*[*of Node Leaf* (*x*, *1*) *Leaf t*]
by(*simp split*: *tree.split*)

corollary *T_del_min_log*: **assumes** *ltree t*
shows *T_del_min t ≤ 2 ∗ log 2 (size1 t) + 1*
proof(*cases t rule*: *tree2_cases*)
case *Leaf* **thus** *?thesis* **using** *assms* **by** *simp*
next
case [*simp*]: (*Node l _ _ r*)
show *?thesis*
using ‹*ltree t*› *T_merge_log*[*of l r*]
log_mono[*of 2 size1 l size1 t*] *log_mono*[*of 2 size1 r size1 t*]
by (*simp del*: *T_merge.simps*)
qed

end

theory *Leftist_Heap_List*
imports
Leftist_Heap
Complex_Main
begin

47.4 Converting a list into a leftist heap

fun *merge_adj* :: ('*a*::*ord*) *lheap list* ⇒ '*a lheap list* **where**
merge_adj [] = [] |
merge_adj [*t*] = [*t*] |
merge_adj (*t1* # *t2* # *ts*) = *merge t1 t2* # *merge_adj ts*

For the termination proof of *merge_all* below.

lemma *length_merge_adjacent*[*termination_simp*]: *length* (*merge_adj ts*) = (*length ts* + *1*) *div 2*

by (*induction ts rule*: *merge_adj.induct*) *auto*

fun *merge_all* :: ('*a*::*ord*) *lheap list* ⇒ '*a lheap* **where**
merge_all [] = *Leaf* |
merge_all [*t*] = *t* |
merge_all ts = *merge_all* (*merge_adj ts*)

47.4.1 Functional correctness

lemma *heap_merge_adj*: ∀ *t* ∈ *set ts*. *heap t* ⟹ ∀ *t* ∈ *set* (*merge_adj ts*). *heap t*
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *heap_merge*)

lemma *ltree_merge_adj*: ∀ *t* ∈ *set ts*. *ltree t* ⟹ ∀ *t* ∈ *set* (*merge_adj ts*). *ltree t*
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *ltree_merge*)

lemma *heap_merge_all*: ∀ *t* ∈ *set ts*. *heap t* ⟹ *heap* (*merge_all ts*)
apply(*induction ts rule*: *merge_all.induct*)
using [[*simp_depth_limit*=*3*]] **by** (*auto simp add*: *heap_merge_adj*)

lemma *ltree_merge_all*: ∀ *t* ∈ *set ts*. *ltree t* ⟹ *ltree* (*merge_all ts*)
apply(*induction ts rule*: *merge_all.induct*)
using [[*simp_depth_limit*=*3*]] **by** (*auto simp add*: *ltree_merge_adj*)

lemma *mset_merge_adj*:
  $\sum_{\#}$ (*image_mset mset_tree* (*mset* (*merge_adj ts*))) =
  $\sum_{\#}$ (*image_mset mset_tree* (*mset ts*))
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *mset_merge*)

lemma *mset_merge_all*:
  mset_tree (*merge_all ts*) = $\sum_{\#}$ (*mset* (*map mset_tree ts*))
by(*induction ts rule*: *merge_all.induct*) (*auto simp*: *mset_merge mset_merge_adj*)

fun *lheap_list* :: '*a*::*ord list* ⇒ '*a lheap* **where**
lheap_list xs = *merge_all* (*map* (λ*x*. *Node Leaf* (*x*,*1*) *Leaf*) *xs*)

lemma *mset_lheap_list*: *mset_tree* (*lheap_list xs*) = *mset xs*
by (*simp add*: *mset_merge_all o_def*)

lemma *ltree_lheap_list*: *ltree* (*lheap_list ts*)
by(*simp add*: *ltree_merge_all*)

lemma *heap_lheap_list*: *heap* (*lheap_list ts*)

by(*simp add*: *heap_merge_all*)

lemma *size_merge*: *size*(*merge t1 t2*) = *size t1* + *size t2*
by(*induction t1 t2 rule*: *merge.induct*) (*auto simp*: *node_def*)

47.4.2 Running time

Not defined automatically because we only count the time for *merge*.

fun *T_merge_adj* :: ('*a*::*ord*) *lheap list* ⇒ *nat* **where**
T_merge_adj [] = *0* |
T_merge_adj [*t*] = *0* |
T_merge_adj (*t1* # *t2* # *ts*) = *T_merge t1 t2* + *T_merge_adj ts*

fun *T_merge_all* :: ('*a*::*ord*) *lheap list* ⇒ *nat* **where**
T_merge_all [] = *0* |
T_merge_all [*t*] = *0* |
T_merge_all ts = *T_merge_adj ts* + *T_merge_all* (*merge_adj ts*)

fun *T_lheap_list* :: '*a*::*ord list* ⇒ *nat* **where**
T_lheap_list xs = *T_merge_all* (*map* (λ*x*. *Node Leaf* (*x*,*1*) *Leaf*) *xs*)

abbreviation *Tm* **where**
Tm n == *2* ∗ *log 2* (*n*+*1*) + *1*

lemma *T_merge_adj*: ⟦ ∀ *t* ∈ *set ts*. *ltree t*; ∀ *t* ∈ *set ts*. *size t* = *n* ⟧
⟹ *T_merge_adj ts* ≤ (*length ts div 2*) ∗ *Tm n*
proof(*induction ts rule*: *T_merge_adj.induct*)
case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp*
next
case (*3 t1 t2*) **thus** *?case* **using** *T_merge_log*[*of t1 t2*] **by** (*simp add*: *algebra_simps size1_size*)
qed

lemma *size_merge_adj*:
⟦ *even*(*length ts*); ∀ *t* ∈ *set ts*. *ltree t*; ∀ *t* ∈ *set ts*. *size t* = *n* ⟧
⟹ ∀ *t* ∈ *set* (*merge_adj ts*). *size t* = *2*∗*n*
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *size_merge*)

lemma *T_merge_all*:
⟦ ∀ *t* ∈ *set ts*. *ltree t*; ∀ *t* ∈ *set ts*. *size t* = *n*; *length ts* = *2*^*k* ⟧
⟹ *T_merge_all ts* ≤ (∑ *i*=*1*..*k*. *2*^(*k*−*i*) ∗ *Tm*(*2* ^ (*i*−*1*) ∗ *n*))
proof (*induction ts arbitrary*: *k n rule*: *merge_all.induct*)

```
  case 1 thus ?case by simp
next
  case 2 thus ?case by simp
next
  case (3 t1 t2 ts)
  let ?ts = t1 # t2 # ts
  let ?ts2 = merge_adj ?ts
  obtain k' where k': k = Suc k' using 3.prems(3)
   by (metis length_Cons nat.inject nat_power_eq_Suc_0_iff nat.exhaust)
  have 1: ∀ x ∈ set(merge_adj ?ts). ltree x
    by(rule ltree_merge_adj[OF 3.prems(1)])
  have even (length ts) using 3.prems(3) even_Suc_Suc_iff by fastforce
  from 3.prems(2) size_merge_adj[OF this] 3.prems(1)
 have 2: ∀ x ∈ set(merge_adj ?ts). size x = 2*n by(auto simp: size_merge)
 have 3: length ?ts2 = 2 ^ k' using 3.prems(3) k' by (simp add: length_merge_adjacent)
  have 4: length ?ts div 2 = 2 ^ k'
    using 3.prems(3) k' by(simp add: power_eq_if[of 2 k] split: if_splits)
 have T_merge_all ?ts = T_merge_adj ?ts + T_merge_all ?ts2 by simp
  also have ... ≤ 2^k' * Tm n + T_merge_all ?ts2
    using 4 T_merge_adj[OF 3.prems(1,2)] by auto
  also have ... ≤ 2^k' * Tm n + (∑ i=1..k'. 2^(k'-i) * Tm(2 ^ (i-1) *
(2*n)))
    using 3.IH[OF 1 2 3] by simp
 also have ... = 2^k' * Tm n + (∑ i=1..k'. 2^(k'-i) * Tm(2 ^ (Suc(i-1))
* n))
    by (simp add: mult_ac cong del: sum.cong)
  also have ... = 2^k' * Tm n + (∑ i=1..k'. 2^(k'-i) * Tm(2 ^ i * n))
     by (simp)
  also have ... = (∑ i=1..k. 2^(k-i) * Tm(2 ^ (i-1) * real n))
   by(simp add: sum.atLeast_Suc_atMost[of Suc 0 Suc k'] sum.atLeast_Suc_atMost_Suc_shift[of
_ Suc 0] k'
        del: sum.cl_ivl_Suc)
  finally show ?case .
qed

lemma summation: (∑ i=1..k. 2^(k-i) * ((2::real)*i+1)) = 5*2^k -
(2::real)*k - 5
proof (induction k)
  case 0 thus ?case by simp
next
  case (Suc k)
  have (∑ i=1..Suc k. 2^(Suc k - i) * ((2::real)*i+1))
    = (∑ i=1..k. 2^(k+1-i) * ((2::real)*i+1)) + 2*k+3
    by(simp)
```

also have ... = ($\sum$ *i=1..k. (2::real)*(2^(k−i) * ((2::real)*i+1))) + 2*k+3*
by (*simp add*: *Suc_diff_le mult.assoc*)
also have ... = *2*(*$\sum$ *i=1..k. 2^(k−i) * ((2::real)*i+1)) + 2*k+3*
by(*simp add*: *sum_distrib_left*)
also have ... = *(2::real)*(5*2^k − (2::real)*k − 5) + 2*k+3*
using *Suc.IH* **by** *simp*
also have ... = *5*2^(Suc k) − (2::real)*(Suc k) − 5*
by *simp*
finally show *?case* **.**
qed

lemma *T_lheap_list*: **assumes** *length xs = 2 ^ k*
shows *T_lheap_list xs ≤ 5 * length xs − 2 * log 2 (length xs)*
proof −
let *?ts = map* (*λx. Node Leaf* (*x*,*1*) *Leaf*) *xs*
have *T_lheap_list xs = T_merge_all ?ts* **by** *simp*
also have ... ≤ ($\sum$ *i = 1..k. 2^(k−i) * (2 * log 2 (2^(i−1) + 1) + 1))*
using *T_merge_all*[*of ?ts 1 k*] *assms* **by** (*simp*)
also have ... ≤ ($\sum$ *i = 1..k. 2^(k−i) * (2 * log 2 (2*2^(i−1)) + 1))*
apply(*rule sum_mono*)
using *zero_le_power*[*of 2::real*] **by** (*simp add*: *add_pos_nonneg*)
also have ... = ($\sum$ *i = 1..k. 2^(k−i) * (2 * log 2 (2^(1+(i−1))) + 1))*
by (*simp del*: *Suc_pred*)
also have ... = ($\sum$ *i = 1..k. 2^(k−i) * (2 * log 2 (2^i) + 1))*
by (*simp*)
also have ... = ($\sum$ *i = 1..k. 2^(k−i) * ((2::real)*i+1))*
by (*simp add*:*log_nat_power algebra_simps*)
also have ... = *5*(2::real)^k − (2::real)*k − 5*
using *summation* **by** (*simp*)
finally show *?thesis*
using *assms of_nat_le_iff* **by** *simp*
qed

end

48 Binomial Priority Queue

theory *Binomial_Heap*
imports
HOL−Library.Pattern_Aliases
Complex_Main
Priority_Queue_Specs

HOL−Library.Time_Functions
begin

We formalize the presentation from Okasaki's book. We show the functional correctness and complexity of all operations.

The presentation is engineered for simplicity, and most proofs are straightforward and automatic.

48.1 Binomial Tree and Forest Types

datatype *'a tree* = *Node* (*rank*: *nat*) (*root*: *'a*) (*children*: *'a tree list*)

type_synonym *'a forest* = *'a tree list*

48.1.1 Multiset of elements

fun *mset_tree* :: *'a::linorder tree* ⇒ *'a multiset* **where**
mset_tree (*Node* _ *a ts*) = {#*a*#} + (∑ *t*∈#*mset ts*. *mset_tree t*)

definition *mset_forest* :: *'a::linorder forest* ⇒ *'a multiset* **where**
mset_forest ts = (∑ *t*∈#*mset ts*. *mset_tree t*)

lemma *mset_tree_simp_alt*[*simp*]:
mset_tree (*Node r a ts*) = {#*a*#} + *mset_forest ts*
unfolding *mset_forest_def* **by** *auto*
declare *mset_tree.simps*[*simp del*]

lemma *mset_tree_nonempty*[*simp*]: *mset_tree t* ≠ {#}
by (*cases t*) *auto*

lemma *mset_forest_Nil*[*simp*]:
mset_forest [] = {#}
by (*auto simp*: *mset_forest_def*)

lemma *mset_forest_Cons*[*simp*]: *mset_forest* (*t*#*ts*) = *mset_tree t* + *mset_forest ts*
by (*auto simp*: *mset_forest_def*)

lemma *mset_forest_empty_iff*[*simp*]: *mset_forest ts* = {#} ⟷ *ts*=[]
by (*auto simp*: *mset_forest_def*)

lemma *root_in_mset*[*simp*]: *root t* ∈# *mset_tree t*
by (*cases t*) *auto*

lemma *mset_forest_rev_eq*[*simp*]: *mset_forest* (*rev ts*) = *mset_forest ts*

by (*auto simp*: *mset_forest_def*)

48.1.2 Invariants

Binomial tree

fun *btree* :: *'a*::*linorder tree* ⇒ *bool* **where**
btree (*Node r x ts*) ⟷
  (∀ *t*∈*set ts*. *btree t*) ∧ *map rank ts* = *rev* [*0*..<*r*]

Heap invariant

fun *heap* :: *'a*::*linorder tree* ⇒ *bool* **where**
heap (*Node* _ *x ts*) ⟷ (∀ *t*∈*set ts*. *heap t* ∧ *x* ≤ *root t*)

definition *bheap t* ⟷ *btree t* ∧ *heap t*

Binomial Forest invariant:

definition *invar ts* ⟷ (∀ *t*∈*set ts*. *bheap t*) ∧ (*sorted_wrt* (<) (*map rank ts*))

A binomial forest is often called a binomial heap, but this overloads the latter term.

The children of a binomial heap node are a valid forest:

lemma *invar_children*:
  bheap (*Node r v ts*) ⟹ *invar* (*rev ts*)
  by (*auto simp*: *bheap_def invar_def rev_map*[*symmetric*])

48.2 Operations and Their Functional Correctness

48.2.1 *link*

context
includes *pattern_aliases*
begin

fun *link* :: (*'a*::*linorder*) *tree* ⇒ *'a tree* ⇒ *'a tree* **where**
  link (*Node r* x_1 ts_1 =: t_1) (*Node r'* x_2 ts_2 =: t_2) =
    (*if* x_1≤x_2 *then Node* (*r*+*1*) x_1 (t_2#ts_1) *else Node* (*r*+*1*) x_2 (t_1#ts_2))

end

lemma *invar_link*:
  assumes *bheap* t_1
  assumes *bheap* t_2
  assumes *rank* t_1 = *rank* t_2
  shows *bheap* (*link* t_1 t_2)

using *assms* **unfolding** *bheap_def*
by (*cases* (t_1, t_2) *rule*: *link.cases*) *auto*

lemma *rank_link*[*simp*]: *rank* (*link* t_1 t_2) = *rank* t_1 + *1*
by (*cases* (t_1, t_2) *rule*: *link.cases*) *simp*

lemma *mset_link*[*simp*]: *mset_tree* (*link* t_1 t_2) = *mset_tree* t_1 + *mset_tree* t_2
by (*cases* (t_1, t_2) *rule*: *link.cases*) *simp*

48.2.2 *ins_tree*

fun *ins_tree* :: *'a*::*linorder tree* ⇒ *'a forest* ⇒ *'a forest* **where**
  ins_tree t [] = [*t*]
| *ins_tree* t_1 (t_2#*ts*) =
  (*if rank* t_1 < *rank* t_2 *then* t_1#t_2#*ts else ins_tree* (*link* t_1 t_2) *ts*)

lemma *bheap0*[*simp*]: *bheap* (*Node 0 x* [])
unfolding *bheap_def* **by** *auto*

lemma *invar_Cons*[*simp*]:
  invar (*t*#*ts*)
  ⟷ *bheap t* ∧ *invar ts* ∧ (∀ *t'*∈*set ts*. *rank t* < *rank t'*)
by (*auto simp*: *invar_def*)

lemma *invar_ins_tree*:
  assumes *bheap t*
  assumes *invar ts*
  assumes ∀ *t'*∈*set ts*. *rank t* ≤ *rank t'*
  shows *invar* (*ins_tree t ts*)
using *assms*
by (*induction t ts rule*: *ins_tree.induct*) (*auto simp*: *invar_link less_eq_Suc_le*[*symmetric*])

lemma *mset_forest_ins_tree*[*simp*]:
  mset_forest (*ins_tree t ts*) = *mset_tree t* + *mset_forest ts*
by (*induction t ts rule*: *ins_tree.induct*) *auto*

lemma *ins_tree_rank_bound*:
  assumes *t'* ∈ *set* (*ins_tree t ts*)
  assumes ∀ *t'*∈*set ts*. *rank* t_0 < *rank t'*
  assumes *rank* t_0 < *rank t*
  shows *rank* t_0 < *rank t'*
using *assms*
by (*induction t ts rule*: *ins_tree.induct*) (*auto split*: *if_splits*)

48.2.3 *insert*

hide_const (**open**) *insert*

definition *insert* :: *'a::linorder* ⇒ *'a forest* ⇒ *'a forest* **where**
insert x ts = *ins_tree* (*Node 0 x* []) *ts*

lemma *invar_insert*[*simp*]: *invar t* ⟹ *invar* (*insert x t*)
by (*auto intro*!: *invar_ins_tree simp*: *insert_def*)

lemma *mset_forest_insert*[*simp*]: *mset_forest* (*insert x t*) = {#*x*#} + *mset_forest t*
by(*auto simp*: *insert_def*)

48.2.4 *merge*

context
includes *pattern_aliases*
begin

fun *merge* :: *'a::linorder forest* ⇒ *'a forest* ⇒ *'a forest* **where**
  merge ts$_1$ [] = *ts*$_1$
| *merge* [] *ts*$_2$ = *ts*$_2$
| *merge* (*t*$_1$#*ts*$_1$ =: *f*$_1$) (*t*$_2$#*ts*$_2$ =: *f*$_2$) = (
    if rank t$_1$ < *rank t*$_2$ *then t*$_1$ # *merge ts*$_1$ *f*$_2$ *else*
    if rank t$_2$ < *rank t*$_1$ *then t*$_2$ # *merge f*$_1$ *ts*$_2$
    else ins_tree (*link t*$_1$ *t*$_2$) (*merge ts*$_1$ *ts*$_2$)
)

end

lemma *merge_simp2*[*simp*]: *merge* [] *ts*$_2$ = *ts*$_2$
by (*cases ts*$_2$) *auto*

lemma *merge_rank_bound*:
  assumes *t'* ∈ *set* (*merge ts*$_1$ *ts*$_2$)
  assumes ∀ *t*$_{12}$∈*set ts*$_1$ ∪ *set ts*$_2$. *rank t* < *rank t*$_{12}$
  shows *rank t* < *rank t'*
using *assms*
by (*induction ts*$_1$ *ts*$_2$ *arbitrary*: *t'* *rule*: *merge.induct*)
   (*auto split*: *if_splits simp*: *ins_tree_rank_bound*)

lemma *invar_merge*[*simp*]:
  assumes *invar ts*$_1$

assumes *invar* ts_2
shows *invar* (*merge* ts_1 ts_2)
using *assms*
by (*induction* ts_1 ts_2 *rule*: *merge.induct*)
(*auto 0 3 simp*: *Suc_le_eq intro*!: *invar_ins_tree invar_link elim*!: *merge_rank_bound*)

Longer, more explicit proof of *invar_merge*, to illustrate the application of the *merge_rank_bound* lemma.

lemma
assumes *invar* ts_1
assumes *invar* ts_2
shows *invar* (*merge* ts_1 ts_2)
using *assms*
proof (*induction* ts_1 ts_2 *rule*: *merge.induct*)
case (*3* t_1 ts_1 t_2 ts_2)
— Invariants of the parts can be shown automatically
from *3.prems* **have** [*simp*]:
bheap t_1 *bheap* t_2

by *auto*

— These are the three cases of the *merge* function
consider (*LT*) *rank* t_1 < *rank* t_2
| (*GT*) *rank* t_1 > *rank* t_2
| (*EQ*) *rank* t_1 = *rank* t_2
using *antisym_conv3* **by** *blast*
then show *?case* **proof** *cases*
case *LT*
— *merge* takes the first tree from the left heap
then have *merge* (t_1 # ts_1) (t_2 # ts_2) = t_1 # *merge* ts_1 (t_2 # ts_2) **by** *simp*
also have *invar* ... **proof** (*simp*, *intro conjI*)
— Invariant follows from induction hypothesis
show *invar* (*merge* ts_1 (t_2 # ts_2))
using *LT 3.IH 3.prems* **by** *simp*

— It remains to show that t_1 has smallest rank.
show ∀ t'∈*set* (*merge* ts_1 (t_2 # ts_2)). *rank* t_1 < *rank* t'
— Which is done by auxiliary lemma *merge_rank_bound*
using *LT 3.prems* **by** (*force elim*!: *merge_rank_bound*)
qed
finally show *?thesis* **.**
next
— *merge* takes the first tree from the right heap

case *GT*
— The proof is anaologous to the *LT* case
then show *?thesis* **using** *3.prems 3.IH* **by** (*force elim*!: *merge_rank_bound*)
next
case [*simp*]: *EQ*
— *merge* links both first forest, and inserts them into the merged remaining heaps
have *merge* (t_1 # ts_1) (t_2 # ts_2) = *ins_tree* (*link* t_1 t_2) (*merge* ts_1 ts_2) **by** *simp*
also have *invar* ... **proof** (*intro invar_ins_tree invar_link*)
— Invariant of merged remaining heaps follows by IH
show *invar* (*merge* ts_1 ts_2)
using *EQ 3.prems 3.IH* **by** *auto*

— For insertion, we have to show that the rank of the linked tree is $\leq$ the ranks in the merged remaining heaps
show $\forall t' \in$ *set* (*merge* ts_1 ts_2). *rank* (*link* t_1 t_2) $\leq$ *rank* t'
proof −
— Which is, again, done with the help of *merge_rank_bound*
have *rank* (*link* t_1 t_2) = *Suc* (*rank* t_2) **by** *simp*
thus *?thesis* **using** *3.prems* **by** (*auto simp*: *Suc_le_eq elim*!: *merge_rank_bound*)
qed
qed *simp_all*
finally show *?thesis* **.**
qed
qed *auto*

lemma *mset_forest_merge*[*simp*]:
mset_forest (*merge* ts_1 ts_2) = *mset_forest* ts_1 + *mset_forest* ts_2
by (*induction* ts_1 ts_2 *rule*: *merge.induct*) *auto*

48.2.5 *get_min*

fun *get_min* :: $'a$::*linorder forest* ⇒ $'a$ **where**
get_min [*t*] = *root t*
| *get_min* (*t*#*ts*) = *min* (*root t*) (*get_min ts*)

lemma *bheap_root_min*:
assumes *bheap t*
assumes *x* ∈# *mset_tree t*
shows *root t* ≤ *x*
using *assms* **unfolding** *bheap_def*

by (*induction t arbitrary*: *x rule*: *mset_tree.induct*) (*fastforce simp*: *mset_forest_def*)

lemma *get_min_mset*:
  assumes *ts*≠[]
  assumes *invar ts*
  assumes *x* ∈# *mset_forest ts*
  shows *get_min ts* ≤ *x*
  using *assms*
apply (*induction ts arbitrary*: *x rule*: *get_min.induct*)
apply (*auto*
      simp: *bheap_root_min min_def intro*: *order_trans*;
      meson linear order_trans bheap_root_min
)+
done

lemma *get_min_member*:
  ts≠[] ⟹ *get_min ts* ∈# *mset_forest ts*
by (*induction ts rule*: *get_min.induct*) (*auto simp*: *min_def*)

lemma *get_min*:
  assumes *mset_forest ts* ≠ {#}
  assumes *invar ts*
  shows *get_min ts* = *Min_mset* (*mset_forest ts*)
using *assms get_min_member get_min_mset*
by (*auto simp*: *eq_Min_iff*)

48.2.6 *get_min_rest*

fun *get_min_rest* :: *'a*::*linorder forest* ⇒ *'a tree* × *'a forest* **where**
  get_min_rest [*t*] = (*t*,[])
| *get_min_rest* (*t*#*ts*) = (*let* (*t'*,*ts'*) = *get_min_rest ts*
                  in if root t ≤ *root t' then* (*t*,*ts*) *else* (*t'*,*t*#*ts'*))

lemma *get_min_rest_get_min_same_root*:
  assumes *ts*≠[]
  assumes *get_min_rest ts* = (*t'*,*ts'*)
  shows *root t'* = *get_min ts*
using *assms*
by (*induction ts arbitrary*: *t' ts' rule*: *get_min.induct*) (*auto simp*: *min_def*
split: *prod.splits*)

lemma *mset_get_min_rest*:
  assumes *get_min_rest ts* = (*t'*,*ts'*)
  assumes *ts*≠[]

shows *mset ts* = {#*t'*#} + *mset ts'*
using *assms*
by (*induction ts arbitrary*: *t' ts' rule*: *get_min.induct*) (*auto split*: *prod.splits if_splits*)

lemma *set_get_min_rest*:
assumes *get_min_rest ts* = (*t'*, *ts'*)
assumes *ts*≠[]
shows *set ts* = *Set.insert t'* (*set ts'*)
using *mset_get_min_rest*[*OF assms*, *THEN arg_cong*[**where** *f*=*set_mset*]]
by *auto*

lemma *invar_get_min_rest*:
assumes *get_min_rest ts* = (*t'*,*ts'*)
assumes *ts*≠[]
assumes *invar ts*
shows *bheap t'* **and** *invar ts'*
proof −
have *bheap t'* ∧ *invar ts'*
using *assms*
proof (*induction ts arbitrary*: *t' ts' rule*: *get_min.induct*)
case (*2 t v va*)
then show *?case*
apply (*clarsimp split*: *prod.splits if_splits*)
apply (*drule set_get_min_rest*; *fastforce*)
done
qed *auto*
thus *bheap t'* **and** *invar ts'* **by** *auto*
qed

48.2.7 *del_min*

definition *del_min* :: *'a*::*linorder forest* ⇒ *'a*::*linorder forest* **where**
del_min ts = (*case get_min_rest ts of*
(*Node r x ts*$_1$, *ts*$_2$) ⇒ *merge* (*itrev ts*$_1$ []) *ts*$_2$)

lemma *invar_del_min*[*simp*]:
assumes *ts* ≠ []
assumes *invar ts*
shows *invar* (*del_min ts*)
using *assms*
unfolding *del_min_def itrev_Nil*
by (*auto*
split: *prod.split tree.split*

intro!: *invar_merge invar_children*
dest: *invar_get_min_rest*
)

lemma *mset_forest_del_min*:
assumes *ts* ≠ []
shows *mset_forest ts* = *mset_forest* (*del_min ts*) + {# *get_min ts* #}
using *assms*
unfolding *del_min_def itrev_Nil*
apply (*clarsimp split*: *tree.split prod.split*)
apply (*frule* (*1*) *get_min_rest_get_min_same_root*)
apply (*frule* (*1*) *mset_get_min_rest*)
apply (*auto simp*: *mset_forest_def*)
done

48.2.8 Instantiating the Priority Queue Locale

Last step of functional correctness proof: combine all the above lemmas to show that binomial heaps satisfy the specification of priority queues with merge.

interpretation *bheaps*: *Priority_Queue_Merge*
where *empty* = [] **and** *is_empty* = (=) [] **and** *insert* = *insert*
and *get_min* = *get_min* **and** *del_min* = *del_min* **and** *merge* = *merge*
and *invar* = *invar* **and** *mset* = *mset_forest*
proof (*unfold_locales*, *goal_cases*)
case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *auto*
next
case *3* **thus** *?case* **by** *auto*
next
case (*4 q*)
thus *?case* **using** *mset_forest_del_min*[*of q*] *get_min*[*OF* _ ‹*invar q*›]
by (*auto simp*: *union_single_eq_diff*)
next
case (*5 q*) **thus** *?case* **using** *get_min*[*of q*] **by** *auto*
next
case *6* **thus** *?case* **by** (*auto simp add*: *invar_def*)
next
case *7* **thus** *?case* **by** *simp*
next
case *8* **thus** *?case* **by** *simp*
next
case *9* **thus** *?case* **by** *simp*

next
  case *10* **thus** *?case* **by** *simp*
qed

48.3 Complexity

The size of a binomial tree is determined by its rank

lemma *size_mset_btree*:
  assumes *btree t*
  shows *size* (*mset_tree t*) = *2^rank t*
  using *assms*
proof (*induction t*)
  case (*Node r v ts*)
  hence *IH*: *size* (*mset_tree t*) = *2^rank t* **if** *t* ∈ *set ts* **for** *t*
    using *that* **by** *auto*

  from *Node* **have** *COMPL*: *map rank ts* = *rev* [*0..<r*] **by** *auto*

  have *size* (*mset_forest ts*) = (∑ *t*←*ts*. *size* (*mset_tree t*))
    by (*induction ts*) *auto*
  also have ... = (∑ *t*←*ts*. *2^rank t*) **using** *IH*
    by (*auto cong*: *map_cong*)
  also have ... = (∑ *r*←*map rank ts*. *2^r*)
    by (*induction ts*) *auto*
  also have ... = (∑ *i*∈{*0..<r*}. *2^i*)
    unfolding *COMPL*
   by (*auto simp*: *rev_map*[*symmetric*] *interv_sum_list_conv_sum_set_nat*)
  also have ... = *2^r* − *1*
    by (*induction r*) *auto*
  finally show *?case*
    by (*simp*)
qed

lemma *size_mset_tree*:
  assumes *bheap t*
  shows *size* (*mset_tree t*) = *2^rank t*
using *assms* **unfolding** *bheap_def*
by (*simp add*: *size_mset_btree*)

The length of a binomial heap is bounded by the number of its elements

lemma *size_mset_forest*:
  assumes *invar ts*
  shows *length ts* ≤ *log 2* (*size* (*mset_forest ts*) + *1*)
proof −

from ‹*invar ts*› **have**
ASC: *sorted_wrt* (<) (*map rank ts*) **and**
TINV: ∀ *t*∈*set ts*. *bheap t*
unfolding *invar_def* **by** *auto*

have (*2*::*nat*)^*length ts* = (∑ *i*∈{*0*..<*length ts*}. *2*^*i*) + *1*
by (*simp add*: *sum_power2*)
also have ... = (∑ *i*←[*0*..<*length ts*]. *2*^*i*) + *1* (**is** _ = *?S* + *1*)
by (*simp add*: *interv_sum_list_conv_sum_set_nat*)
also have *?S* ≤ (∑ *t*←*ts*. *2*^*rank t*) (**is** _ ≤ *?T*)
using *sorted_wrt_less_idx*[*OF ASC*] **by**(*simp add*: *sum_list_mono2*)
also have *?T* + *1* ≤ (∑ *t*←*ts*. *size* (*mset_tree t*)) + *1* **using** *TINV*
by (*auto cong*: *map_cong simp*: *size_mset_tree*)
also have ... = *size* (*mset_forest ts*) + *1*
unfolding *mset_forest_def* **by** (*induction ts*) *auto*
finally have *2*^*length ts* ≤ *size* (*mset_forest ts*) + *1* **by** *simp*
then show *?thesis* **using** *le_log2_of_power* **by** *blast*
qed

48.3.1 Timing Functions

time_fun *link*

lemma *T_link*[*simp*]: *T_link t*$_1$ *t*$_2$ = *0*
by(*cases t*$_1$; *cases t*$_2$, *auto*)

time_fun *rank*

lemma *T_rank*[*simp*]: *T_rank t* = *0*
by(*cases t*, *auto*)

time_fun *ins_tree*

time_fun *insert*

lemma *T_ins_tree_simple_bound*: *T_ins_tree t ts* ≤ *length ts* + *1*
by (*induction t ts rule*: *T_ins_tree.induct*) *auto*

48.3.2 *T_insert*

lemma *T_insert_bound*:
assumes *invar ts*
shows *T_insert x ts* ≤ *log 2* (*size* (*mset_forest ts*) + *1*) + *1*
proof −

have *real* (*T_insert x ts*) ≤ *real* (*length ts*) + *1*
unfolding *T_insert.simps* **using** *T_ins_tree_simple_bound*
by (*metis of_nat_1 of_nat_add of_nat_mono*)
also note *size_mset_forest*[*OF* ‹*invar ts*›]
finally show *?thesis* **by** *simp*
qed

48.3.3 *T_merge*

time_fun *merge*

A crucial idea is to estimate the time in correlation with the result length, as each carry reduces the length of the result.

lemma *T_ins_tree_length*:
T_ins_tree t ts + *length* (*ins_tree t ts*) = *2* + *length ts*
by (*induction t ts rule*: *ins_tree.induct*) *auto*

lemma *T_merge_length*:
T_merge ts_1 ts_2 + *length* (*merge* ts_1 ts_2) ≤ *2* ∗ (*length* ts_1 + *length* ts_2) + *1*
by (*induction* ts_1 ts_2 *rule*: *merge.induct*)
(*auto simp*: *T_ins_tree_length algebra_simps*)

Finally, we get the desired logarithmic bound

lemma *T_merge_bound*:
fixes ts_1 ts_2
defines n_1 ≡ *size* (*mset_forest* ts_1)
defines n_2 ≡ *size* (*mset_forest* ts_2)
assumes *invar* ts_1 *invar* ts_2
shows *T_merge* ts_1 ts_2 ≤ *4*∗*log 2* (n_1 + n_2 + *1*) + *1*
proof −
note *n_defs* = *assms*(*1*,*2*)

have *T_merge* ts_1 ts_2 ≤ *2* ∗ *real* (*length* ts_1) + *2* ∗ *real* (*length* ts_2) + *1*
using *T_merge_length*[*of* ts_1 ts_2] **by** *simp*
also note *size_mset_forest*[*OF* ‹*invar* ts_1›]
also note *size_mset_forest*[*OF* ‹*invar* ts_2›]
finally have *T_merge* ts_1 ts_2 ≤ *2* ∗ *log 2* (n_1 + *1*) + *2* ∗ *log 2* (n_2 + *1*) + *1*
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)
also have *log 2* (n_1 + *1*) ≤ *log 2* (n_1 + n_2 + *1*)
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)
also have *log 2* (n_2 + *1*) ≤ *log 2* (n_1 + n_2 + *1*)
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)

finally show *?thesis* **by** (*simp add*: *algebra_simps*)
qed

48.3.4 *T_get_min*

time_fun *root*

lemma *T_root*[*simp*]: *T_root t* = *0*
by(*cases t*)(*simp_all*)

time_fun *min*

time_fun *get_min*

lemma *T_get_min_estimate*: *ts*≠[] ⟹ *T_get_min ts* = *length ts*
by (*induction ts rule*: *T_get_min.induct*) *auto*

lemma *T_get_min_bound*:
assumes *invar ts*
assumes *ts*≠[]
shows *T_get_min ts* ≤ *log 2* (*size* (*mset_forest ts*) + *1*)
proof −
have *1*: *T_get_min ts* = *length ts* **using** *assms T_get_min_estimate* **by** *auto*
also note *size_mset_forest*[*OF* ‹*invar ts*›]
finally show *?thesis* **.**
qed

48.3.5 *T_del_min*

time_fun *get_min_rest*

lemma *T_get_min_rest_estimate*: *ts*≠[] ⟹ *T_get_min_rest ts* = *length ts*
by (*induction ts rule*: *T_get_min_rest.induct*) *auto*

lemma *T_get_min_rest_bound*:
assumes *invar ts*
assumes *ts*≠[]
shows *T_get_min_rest ts* ≤ *log 2* (*size* (*mset_forest ts*) + *1*)
proof −
have *1*: *T_get_min_rest ts* = *length ts* **using** *assms T_get_min_rest_estimate* **by** *auto*
also note *size_mset_forest*[*OF* ‹*invar ts*›]

```
  finally show ?thesis .
qed

time_fun del_min

lemma T_del_min_bound:
  fixes ts
  defines n ≡ size (mset_forest ts)
  assumes invar ts and ts≠[]
  shows T_del_min ts ≤ 6 * log 2 (n+1) + 2
proof −
  obtain r x ts1 ts2 where GM: get_min_rest ts = (Node r x ts1, ts2)
    by (metis surj_pair tree.exhaust_sel)

  have I1: invar (rev ts1) and I2: invar ts2
    using invar_get_min_rest[OF GM ‹ts≠[]› ‹invar ts›] invar_children
    by auto

  define n1 where n1 = size (mset_forest ts1)
  define n2 where n2 = size (mset_forest ts2)

  have n1 ≤ n n1 + n2 ≤ n unfolding n_def n1_def n2_def
    using mset_get_min_rest[OF GM ‹ts≠[]›]
    by (auto simp: mset_forest_def)

  have T_del_min ts = real (T_get_min_rest ts) + real (T_itrev ts1 [])
+ real (T_merge (rev ts1) ts2)
    unfolding T_del_min.simps GM T_itrev itrev_Nil
    by simp
  also have T_get_min_rest ts ≤ log 2 (n+1)
    using T_get_min_rest_bound[OF ‹invar ts› ‹ts≠[]›] unfolding n_def
by simp
  also have T_itrev ts1 [] ≤ 1 + log 2 (n1 + 1)
    unfolding T_itrev n1_def using size_mset_forest[OF I1] by simp
  also have T_merge (rev ts1) ts2 ≤ 4*log 2 (n1 + n2 + 1) + 1
    unfolding n1_def n2_def using T_merge_bound[OF I1 I2] by (simp
add: algebra_simps)
  finally have T_del_min ts ≤ log 2 (n+1) + log 2 (n1 + 1) + 4*log 2
(real (n1 + n2) + 1) + 2
    by (simp add: algebra_simps)
  also note ‹n1 + n2 ≤ n›
  also note ‹n1 ≤ n›
  finally show ?thesis by (simp add: algebra_simps)
qed
```

end

49 The Median-of-Medians Selection Algorithm

theory *Selection*
imports *Complex_Main HOL−Library.Time_Functions Sorting*
begin

Note that there is significant overlap between this theory (which is intended mostly for the Functional Data Structures book) and the Median-of-Medians AFP entry.

49.1 Auxiliary material

lemma *replicate_numeral*: *replicate* (*numeral n*) *x* = *x* # *replicate* (*pred_numeral n*) *x*
by (*simp add*: *numeral_eq_Suc*)

lemma *insort_correct*: *insort xs* = *sort xs*
using *sorted_insort mset_insort* **by** (*metis properties_for_sort*)

lemma *sum_list_replicate* [*simp*]: *sum_list* (*replicate n x*) = *n* ∗ *x*
by (*induction n*) *auto*

lemma *mset_concat*: *mset* (*concat xss*) = *sum_list* (*map mset xss*)
by (*induction xss*) *simp_all*

lemma *set_mset_sum_list* [*simp*]: *set_mset* (*sum_list xs*) = (⋃ *x*∈*set xs*. *set_mset x*)
by (*induction xs*) *auto*

lemma *filter_mset_image_mset*:
filter_mset P (*image_mset f A*) = *image_mset f* (*filter_mset* (λ*x*. *P* (*f x*)) *A*)
by (*induction A*) *auto*

lemma *filter_mset_sum_list*: *filter_mset P* (*sum_list xs*) = *sum_list* (*map* (*filter_mset P*) *xs*)
by (*induction xs*) *simp_all*

lemma *sum_mset_mset_mono*:
assumes (⋀*x*. *x* ∈# *A* ⟹ *f x* ⊆# *g x*)
shows (∑ *x*∈#*A*. *f x*) ⊆# (∑ *x*∈#*A*. *g x*)

using *assms* **by** (*induction A*) (*auto intro*!: *subset_mset.add_mono*)

lemma *mset_filter_mono*:
assumes $A \subseteq\# B$ $\bigwedge x.\ x \in\# A \Longrightarrow P\ x \Longrightarrow Q\ x$
shows *filter_mset P A* $\subseteq\#$ *filter_mset Q B*
by (*rule mset_subset_eqI*) (*insert assms*, *auto simp*: *mset_subset_eq_count count_eq_zero_iff*)

lemma *size_mset_sum_mset_distrib*: *size* (*sum_mset A* :: *'a multiset*) = *sum_mset* (*image_mset size A*)
by (*induction A*) *auto*

lemma *sum_mset_mono*:
assumes $\bigwedge x.\ x \in\# A \Longrightarrow f\ x \leq (g\ x :: 'a :: \{ordered\_ab\_semigroup\_add, comm\_monoid\_add\})$
shows $(\sum x \in\# A.\ f\ x) \leq (\sum x \in\# A.\ g\ x)$
using *assms* **by** (*induction A*) (*auto intro*!: *add_mono*)

lemma *filter_mset_is_empty_iff*: *filter_mset P A* = $\{\#\} \longleftrightarrow (\forall x.\ x \in\# A \longrightarrow \neg P\ x)$
by (*auto simp*: *multiset_eq_iff count_eq_zero_iff*)

lemma *sort_eq_Nil_iff* [*simp*]: *sort xs* = $[] \longleftrightarrow xs = []$
by (*metis set_empty set_sort*)

lemma *sort_mset_cong*: *mset xs* = *mset ys* $\Longrightarrow$ *sort xs* = *sort ys*
by (*metis sorted_list_of_multiset_mset*)

lemma *Min_set_sorted*: *sorted xs* $\Longrightarrow$ $xs \neq [] \Longrightarrow$ *Min* (*set xs*) = *hd xs*
by (*cases xs*; *force intro*: *Min_insert2*)

lemma *hd_sort*:
fixes *xs* :: *'a* :: *linorder list*
shows $xs \neq [] \Longrightarrow$ *hd* (*sort xs*) = *Min* (*set xs*)
by (*subst Min_set_sorted* [*symmetric*]) *auto*

lemma *length_filter_conv_size_filter_mset*: *length* (*filter P xs*) = *size* (*filter_mset P* (*mset xs*))
by (*induction xs*) *auto*

lemma *sorted_filter_less_subset_take*:
assumes *sorted xs* **and** *i* < *length xs*
shows $\{\#x \in\# mset\ xs.\ x < xs\ !\ i\#\} \subseteq\# mset\ (take\ i\ xs)$
using *assms*
proof (*induction xs arbitrary*: *i rule*: *list.induct*)

```
  case (Cons x xs i)
  show ?case
  proof (cases i)
    case 0
   thus ?thesis using Cons.prems by (auto simp: filter_mset_is_empty_iff)
  next
    case (Suc i')
    have {#y ∈# mset (x # xs). y < (x # xs) ! i#} ⊆# add_mset x {#y
∈# mset xs. y < xs ! i'#}
      using Suc Cons.prems by (auto)
    also have ... ⊆# add_mset x (mset (take i' xs))
      unfolding mset_subset_eq_add_mset_cancel using Cons.prems Suc
      by (intro Cons.IH) (auto)
    also have ... = mset (take i (x # xs)) by (simp add: Suc)
    finally show ?thesis .
  qed
qed auto

lemma sorted_filter_greater_subset_drop:
  assumes sorted xs and i < length xs
  shows   {#x ∈# mset xs. x > xs ! i#} ⊆# mset (drop (Suc i) xs)
  using assms
proof (induction xs arbitrary: i rule: list.induct)
  case (Cons x xs i)
  show ?case
  proof (cases i)
    case 0
    thus ?thesis by (auto simp: sorted_append filter_mset_is_empty_iff)
  next
    case (Suc i')
    have {#y ∈# mset (x # xs). y > (x # xs) ! i#} ⊆# {#y ∈# mset xs.
y > xs ! i'#}
      using Suc Cons.prems by (auto simp: set_conv_nth)
    also have ... ⊆# mset (drop (Suc i') xs)
      using Cons.prems Suc by (intro Cons.IH) (auto)
    also have ... = mset (drop (Suc i) (x # xs)) by (simp add: Suc)
    finally show ?thesis .
  qed
qed auto
```

49.2 Chopping a list into equally-sized bits

```
fun chop :: nat ⇒ 'a list ⇒ 'a list list where
  chop 0 _  = []
```

| *chop* _ [] = []
| *chop n xs* = *take n xs* # *chop n* (*drop n xs*)

lemmas [*simp del*] = *chop.simps*
lemmas [*simp*] = *chop.simps*(*1*)

This is an alternative induction rule for *chop*, which is often nicer to use.

lemma *chop_induct'* [*case_names trivial reduce*]:
assumes $\bigwedge n\ xs.\ n = 0 \vee xs = [] \Longrightarrow P\ n\ xs$
assumes $\bigwedge n\ xs.\ n > 0 \Longrightarrow xs \neq [] \Longrightarrow P\ n\ (drop\ n\ xs) \Longrightarrow P\ n\ xs$
shows *P n xs*
using *assms*
proof *induction_schema*
show *wf* (*measure* (*length* ∘ *snd*))
by *auto*
qed (*blast* | *simp*)+

lemma *chop_eq_Nil_iff* [*simp*]: *chop n xs* = [] ⟷ *n* = *0* ∨ *xs* = []
by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) *auto*

lemma *chop_Nil* [*simp*]: *chop n* [] = []
by (*cases n*) *auto*

lemma *chop_reduce*: *n* > *0* ⟹ *xs* ≠ [] ⟹ *chop n xs* = *take n xs* # *chop n* (*drop n xs*)
by (*cases n*; *cases xs*) (*auto simp*: *chop.simps*)

lemma *concat_chop* [*simp*]: *n* > *0* ⟹ *concat* (*chop n xs*) = *xs*
by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) *auto*

lemma *chop_elem_not_Nil* [*dest*]: *ys* ∈ *set* (*chop n xs*) ⟹ *ys* ≠ []
by (*induction n xs rule*: *chop.induct*; *subst* (*asm*) *chop.simps*)
(*auto simp*: *eq_commute*[*of* []] *split*: *if_splits*)

lemma *length_chop_part_le*: *ys* ∈ *set* (*chop n xs*) ⟹ *length ys* ≤ *n*
by (*induction n xs rule*: *chop.induct*; *subst* (*asm*) *chop.simps*) (*auto split*: *if_splits*)

lemma *length_chop*:
assumes *n* > *0*
shows *length* (*chop n xs*) = *nat* ⌈*length xs* / *n*⌉
proof −
from ‹*n* > *0*› **have** *real n* ∗ *length* (*chop n xs*) ≥ *length xs*
by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) (*auto simp*:

field_simps)
moreover from ‹*n* > *0*› **have** *real n* ∗ *length* (*chop n xs*) < *length xs* + *n*
by (*induction n xs rule*: *chop.induct*; *subst chop.simps*)
(*auto simp*: *field_simps split*: *nat_diff_split_asm*)+
ultimately have *length* (*chop n xs*) ≥ *length xs* / *n* **and** *length* (*chop n xs*) < *length xs* / *n* + *1*
using *assms* **by** (*auto simp*: *field_simps*)
thus *?thesis* **by** *linarith*
qed

lemma *sum_msets_chop*: *n* > *0* ⟹ (∑ *ys*←*chop n xs*. *mset ys*) = *mset xs*
by (*subst mset_concat* [*symmetric*]) *simp_all*

lemma *UN_sets_chop*: *n* > *0* ⟹ (⋃ *ys*∈*set* (*chop n xs*). *set ys*) = *set xs*
by (*simp only*: *set_concat* [*symmetric*] *concat_chop*)

lemma *chop_append*: *d dvd length xs* ⟹ *chop d* (*xs* @ *ys*) = *chop d xs* @ *chop d ys*
by (*induction d xs rule*: *chop_induct'*) (*auto simp*: *chop_reduce dvd_imp_le*)

lemma *chop_replicate* [*simp*]: *d* > *0* ⟹ *chop d* (*replicate d xs*) = [*replicate d xs*]
by (*subst chop_reduce*) *auto*

lemma *chop_replicate_dvd* [*simp*]:
assumes *d dvd n*
shows *chop d* (*replicate n x*) = *replicate* (*n div d*) (*replicate d x*)
proof (*cases d* = *0*)
case *False*
from *assms* **obtain** *k* **where** *k*: *n* = *d* ∗ *k*
by *blast*
have *chop d* (*replicate* (*d* ∗ *k*) *x*) = *replicate k* (*replicate d x*)
using *False* **by** (*induction k*) (*auto simp*: *replicate_add chop_append*)
thus *?thesis* **using** *False* **by** (*simp add*: *k*)
qed *auto*

lemma *chop_concat*:
assumes ∀ *xs*∈*set xss*. *length xs* = *d* **and** *d* > *0*
shows *chop d* (*concat xss*) = *xss*
using *assms*
proof (*induction xss*)
case (*Cons xs xss*)

have *chop d* (*concat* (*xs* # *xss*)) = *chop d* (*xs* @ *concat xss*)
by *simp*
also have ... = *chop d xs* @ *chop d* (*concat xss*)
using *Cons.prems* **by** (*intro chop_append*) *auto*
also have *chop d xs* = [*xs*]
using *Cons.prems* **by** (*subst chop_reduce*) *auto*
also have *chop d* (*concat xss*) = *xss*
using *Cons.prems* **by** (*intro Cons.IH*) *auto*
finally show *?case* **by** *simp*
qed *auto*

49.3 Selection

definition *select* :: *nat* ⇒ ('*a* :: *linorder*) *list* ⇒ '*a* **where**
select k xs = *sort xs* ! *k*

lemma *select_0*: *xs* ≠ [] ⟹ *select 0 xs* = *Min* (*set xs*)
by (*simp add*: *hd_sort select_def flip*: *hd_conv_nth*)

lemma *select_mset_cong*: *mset xs* = *mset ys* ⟹ *select k xs* = *select k ys*
using *sort_mset_cong*[*of xs ys*] **unfolding** *select_def* **by** *auto*

lemma *select_in_set* [*intro*,*simp*]:
assumes *k* < *length xs*
shows *select k xs* ∈ *set xs*
proof −
from *assms* **have** *sort xs* ! *k* ∈ *set* (*sort xs*) **by** (*intro nth_mem*) *auto*
also have *set* (*sort xs*) = *set xs* **by** *simp*
finally show *?thesis* **by** (*simp add*: *select_def*)
qed

lemma
assumes *n* < *length xs*
shows *size_less_than_select*: *size* {#*y* ∈# *mset xs*. *y* < *select n xs*#}
≤ *n*
and *size_greater_than_select*: *size* {#*y* ∈# *mset xs*. *y* > *select n xs*#}
< *length xs* − *n*
proof −
have *size* {#*y* ∈# *mset* (*sort xs*). *y* < *select n xs*#} ≤ *size* (*mset* (*take*
n (*sort xs*)))
unfolding *select_def* **using** *assms*
by (*intro size_mset_mono sorted_filter_less_subset_take*) *auto*
thus *size* {#*y* ∈# *mset xs*. *y* < *select n xs*#} ≤ *n*
by *simp*

have *size* {#*y* ∈# *mset* (*sort xs*). *y* > *select n xs*#} ≤ *size* (*mset* (*drop* (*Suc n*) (*sort xs*)))
unfolding *select_def* **using** *assms*
by (*intro size_mset_mono sorted_filter_greater_subset_drop*) *auto*
thus *size* {#*y* ∈# *mset xs*. *y* > *select n xs*#} < *length xs* − *n*
using *assms* **by** *simp*
qed

49.4 The designated median of a list

definition *median* **where** *median xs* = *select* ((*length xs* − *1*) *div 2*) *xs*

lemma *median_in_set* [*intro*, *simp*]:
assumes *xs* ≠ []
shows *median xs* ∈ *set xs*
proof −
from *assms* **have** *length xs* > *0* **by** *auto*
hence (*length xs* − *1*) *div 2* < *length xs* **by** *linarith*
thus *?thesis* **by** (*simp add*: *median_def*)
qed

lemma *size_less_than_median*: *size* {#*y* ∈# *mset xs*. *y* < *median xs*#} ≤ (*length xs* − *1*) *div 2*
proof (*cases xs* = [])
case *False*
hence *length xs* > *0*
by *auto*
hence *less*: (*length xs* − *1*) *div 2* < *length xs*
by *linarith*
show *size* {#*y* ∈# *mset xs*. *y* < *median xs*#} ≤ (*length xs* − *1*) *div 2*
using *size_less_than_select*[*OF less*] **by** (*simp add*: *median_def*)
qed *auto*

lemma *size_greater_than_median*: *size* {#*y* ∈# *mset xs*. *y* > *median xs*#} ≤ *length xs div 2*
proof (*cases xs* = [])
case *False*
hence *length xs* > *0*
by *auto*
hence *less*: (*length xs* − *1*) *div 2* < *length xs*
by *linarith*
have *size* {#*y* ∈# *mset xs*. *y* > *median xs*#} < *length xs* − (*length xs* − *1*) *div 2*
using *size_greater_than_select*[*OF less*] **by** (*simp add*: *median_def*)

also have ... = *length xs div 2 + 1*
using ‹*length xs > 0*› **by** *linarith*
finally show *size {#y ∈# mset xs. y > median xs#} ≤ length xs div 2*
by *simp*
qed *auto*

lemmas *median_props = size_less_than_median size_greater_than_median*

49.5 A recurrence for selection

definition *partition3 :: 'a ⇒ 'a :: linorder list ⇒ 'a list × 'a list × 'a list*
where
partition3 x xs = (filter (λy. y < x) xs, filter (λy. y = x) xs, filter (λy. y > x) xs)

lemma *partition3_code* [*code*]:
partition3 x [] = ([], [], [])
partition3 x (y # ys) =
(case partition3 x ys of (ls, es, gs) ⇒
if y < x then (y # ls, es, gs) else if x = y then (ls, y # es, gs) else (ls, es, y # gs))
by (*auto simp*: *partition3_def*)

lemma *length_partition3*:
assumes *partition3 x xs = (ls, es, gs)*
shows *length xs = length ls + length es + length gs*
using *assms* **by** (*induction xs arbitrary*: *ls es gs*)
(*auto simp*: *partition3_code split*: *if_splits prod.splits*)

lemma *sort_append*:
assumes *∀x∈set xs. ∀y∈set ys. x ≤ y*
shows *sort (xs @ ys) = sort xs @ sort ys*
using *assms* **by** (*intro properties_for_sort*) (*auto simp*: *sorted_append*)

lemma *select_append*:
assumes *∀y∈set ys. ∀z∈set zs. y ≤ z*
shows *k < length ys ⟹ select k (ys @ zs) = select k ys*
and *k ∈ {length ys..<length ys + length zs} ⟹*
select k (ys @ zs) = select (k − length ys) zs
using *assms* **by** (*simp_all add*: *select_def sort_append nth_append*)

lemma *select_append'*:
assumes *∀y∈set ys. ∀z∈set zs. y ≤ z* **and** *k < length ys + length zs*
shows *select k (ys @ zs) = (if k < length ys then select k ys else select*

```
(k − length ys) zs)
  using assms by (auto intro!: select_append)

theorem select_rec_partition:
  assumes k < length xs
  shows select k xs = (
          let (ls, es, gs) = partition3 x xs
          in
            if k < length ls then select k ls
            else if k < length ls + length es then x
            else select (k − length ls − length es) gs
        ) (is _ = ?rhs)
proof −
  define ls es gs where ls = filter (λy. y < x) xs and es = filter (λy. y =
x) xs
                    and gs = filter (λy. y > x) xs
  define nl ne where [simp]: nl = length ls ne = length es
  have mset_eq: mset xs = mset ls + mset es + mset gs
    unfolding ls_def es_def gs_def by (induction xs) auto
  have length_eq: length xs = length ls + length es + length gs
    unfolding ls_def es_def gs_def
    using [[simp_depth_limit = 1]] by (induction xs) auto
  have [simp]: select i es = x if i < length es for i
  proof −
    have select i es ∈ set (sort es) unfolding select_def
      using that by (intro nth_mem) auto
    thus ?thesis
      by (auto simp: es_def)
  qed

  have select k xs = select k (ls @ (es @ gs))
    by (intro select_mset_cong) (simp_all add: mset_eq)
  also have ... = (if k < nl then select k ls else select (k − nl) (es @ gs))
    unfolding nl_ne_def using assms
    by (intro select_append′) (auto simp: ls_def es_def gs_def length_eq)
  also have ... = (if k < nl then select k ls else if k < nl + ne then x
                   else select (k − nl − ne) gs)
  proof (rule if_cong)
    assume ¬k < nl
    have select (k − nl) (es @ gs) =
                (if k − nl < ne then select (k − nl) es else select (k − nl −
ne) gs)
      unfolding nl_ne_def using assms ‹¬k < nl›
      by (intro select_append′) (auto simp: ls_def es_def gs_def length_eq)
```

also have ... = (*if* $k < nl + ne$ *then* x *else select* $(k - nl - ne)$ gs)
using ‹$\neg k < nl$› **by** *auto*
finally show *select* $(k - nl)$ $(es$ @ $gs) = \ldots$ **.**
qed *simp_all*
also have ... = *?rhs*
by (*simp add*: *partition3_def ls_def es_def gs_def*)
finally show *?thesis* **.**
qed

49.6 The size of the lists in the recursive calls

We now derive an upper bound for the number of elements of a list that are smaller (resp. bigger) than the median of medians with chopping size 5. To avoid having to do the same proof twice, we do it generically for an operation $\prec$ that we will later instantiate with either $<$ or $>$.

context
fixes $xs :: 'a :: linorder\ list$
fixes M **defines** $M \equiv median\ (map\ median\ (chop\ 5\ xs))$
begin

lemma *size_median_of_medians_aux*:
fixes $R :: 'a :: linorder \Rightarrow 'a \Rightarrow bool$ (**infix** ‹$\prec$›$\ 50$)
assumes R: $R \in \{(<), (>)\}$
shows $size\ \{\#y \in\# mset\ xs.\ y \prec M\#\} \leq nat\ \lceil 0.7 * length\ xs + 3\rceil$
proof −
define n **and** m **where** [*simp*]: $n = length\ xs$ **and** $m = length\ (chop\ 5\ xs)$

We define an abbreviation for the multiset of all the chopped-up groups:

We then split that multiset into those groups whose medians is less than M and the rest.

define $Y\_small\ (‹Y_\prec›)$ **where** $Y_\prec = filter\_mset\ (\lambda ys.\ median\ ys \prec M)\ (mset\ (chop\ 5\ xs))$
define $Y\_big\ (‹Y_\succeq›)$ **where** $Y_\succeq = filter\_mset\ (\lambda ys.\ \neg(median\ ys \prec M))\ (mset\ (chop\ 5\ xs))$
have $m = size\ (mset\ (chop\ 5\ xs))$ **by** (*simp add*: *m_def*)
also have $mset\ (chop\ 5\ xs) = Y_\prec + Y_\succeq$ **unfolding** *Y_small_def Y_big_def*
by (*rule multiset_partition*)
finally have *m_eq*: $m = size\ Y_\prec + size\ Y_\succeq$ **by** *simp*

At most half of the lists have a median that is smaller than the median of medians:

have $size\ Y_\prec = size\ (image\_mset\ median\ Y_\prec)$ **by** *simp*

also have *image_mset median* $Y_\prec$ = {#*y* ∈# *mset* (*map median* (*chop 5 xs*)). *y* ≺ *M*#}
unfolding *Y_small_def* **by** (*subst filter_mset_image_mset* [*symmetric*]) *simp_all*
also have *size* ... ≤ (*length* (*map median* (*chop 5 xs*))) *div 2*
unfolding *M_def* **using** *median_props*[*of map median* (*chop 5 xs*)] *R* **by** *auto*
also have ... = *m div 2* **by** (*simp add*: *m_def*)
finally have *size_Y_small*: *size* $Y_\prec$ ≤ *m div 2* **.**

We estimate the number of elements less than M by grouping them into elements coming from $Y_\prec$ and elements coming from $Y_\succeq$:

have {#*y* ∈# *mset xs*. *y* ≺ *M*#} = {#*y* ∈# (∑ *ys*←*chop 5 xs*. *mset ys*). *y* ≺ *M*#}
by (*subst sum_msets_chop*) *simp_all*
also have ... = (∑ *ys*←*chop 5 xs*. {#*y* ∈# *mset ys*. *y* ≺ *M*#})
by (*subst filter_mset_sum_list*) (*simp add*: *o_def*)
also have ... = (∑ *ys*∈#*mset* (*chop 5 xs*). {#*y* ∈# *mset ys*. *y* ≺ *M*#})
by (*subst sum_mset_sum_list* [*symmetric*]) *simp_all*
also have *mset* (*chop 5 xs*) = $Y_\prec$ + $Y_\succeq$
by (*simp add*: *Y_small_def Y_big_def not_le*)
also have (∑ *ys*∈#.... {#*y* ∈# *mset ys*. *y* ≺ *M*#}) =
(∑ *ys*∈#$Y_\prec$. {#*y* ∈# *mset ys*. *y* ≺ *M*#}) + (∑ *ys*∈#$Y_\succeq$. {#*y* ∈# *mset ys*. *y* ≺ *M*#})
by *simp*

Next, we overapproximate the elements contributed by $Y_\prec$: instead of those elements that are smaller than the median, we take *all* the elements of each group. For the elements contributed by $Y_\succeq$, we overapproximate by taking all those that are less than their median instead of only those that are less than M.

also have ... ⊆# (∑ *ys*∈#$Y_\prec$. *mset ys*) + (∑ *ys*∈#$Y_\succeq$. {#*y* ∈# *mset ys*. *y* ≺ *median ys*#})
using *R*
by (*intro subset_mset.add_mono sum_mset_mset_mono mset_filter_mono*) (*auto simp*: *Y_big_def*)
finally have *size* {# *y* ∈# *mset xs*. *y* ≺ *M*#} ≤ *size* ...
by (*rule size_mset_mono*)
hence *size* {# *y* ∈# *mset xs*. *y* ≺ *M*#} ≤
(∑ *ys*∈#$Y_\prec$. *length ys*) + (∑ *ys*∈#$Y_\succeq$. *size* {#*y* ∈# *mset ys*. *y* ≺ *median ys*#})
by (*simp add*: *size_mset_sum_mset_distrib multiset.map_comp o_def*)

Next, we further overapproximate the first sum by noting that each group has at most size 5.

also have $(\sum ys \in\# Y_{\prec}.\ length\ ys) \leq (\sum ys \in\# Y_{\prec}.\ 5)$
by (*intro sum_mset_mono*) (*auto simp*: *Y_small_def length_chop_part_le*)
also have $\ldots = 5 * size\ Y_{\prec}$ **by** *simp*

Next, we note that each group in $Y_{\succeq}$ can have at most 2 elements that are smaller than its median.

also have $(\sum ys \in\# Y_{\succeq}.\ size\ \{\#y \in\#\ mset\ ys.\ y \prec median\ ys\#\}) \leq$
$(\sum ys \in\# Y_{\succeq}.\ length\ ys\ div\ 2)$
proof (*intro sum_mset_mono*, *goal_cases*)
fix *ys* **assume** $ys \in\# Y_{\succeq}$
hence $ys \neq []$
by (*auto simp*: *Y_big_def*)
thus $size\ \{\#y \in\#\ mset\ ys.\ y \prec median\ ys\#\} \leq length\ ys\ div\ 2$
using *R median_props*[*of ys*] **by** *auto*
qed
also have $\ldots \leq (\sum ys \in\# Y_{\succeq}.\ 2)$
by (*intro sum_mset_mono div_le_mono diff_le_mono*)
(*auto simp*: *Y_big_def dest*: *length_chop_part_le*)
also have $\ldots = 2 * size\ Y_{\succeq}$ **by** *simp*

Simplifying gives us the main result.

also have $5 * size\ Y_{\prec} + 2 * size\ Y_{\succeq} = 2 * m + 3 * size\ Y_{\prec}$
by (*simp add*: *m_eq*)
also have $\ldots \leq 3.5 * m$
using ‹$size\ Y_{\prec} \leq m\ div\ 2$› **by** *linarith*
also have $\ldots = 3.5 * \lceil n\ /\ 5 \rceil$
by (*simp add*: *m_def length_chop*)
also have $\ldots \leq 0.7 * n + 3.5$
by *linarith*
finally have $size\ \{\#y \in\#\ mset\ xs.\ y \prec M\#\} \leq 0.7 * n + 3.5$
by *simp*
thus $size\ \{\#y \in\#\ mset\ xs.\ y \prec M\#\} \leq nat\ \lceil 0.7 * n + 3 \rceil$
by *linarith*
qed

lemma *size_less_than_median_of_medians*:
$size\ \{\#y \in\#\ mset\ xs.\ y < M\#\} \leq nat\ \lceil 0.7 * length\ xs + 3 \rceil$
using *size_median_of_medians_aux*[*of* $(<)$] **by** *simp*

lemma *size_greater_than_median_of_medians*:
$size\ \{\#y \in\#\ mset\ xs.\ y > M\#\} \leq nat\ \lceil 0.7 * length\ xs + 3 \rceil$
using *size_median_of_medians_aux*[*of* $(>)$] **by** *simp*

end

49.7 Efficient algorithm

We handle the base cases and computing the median for the chopped-up sublists of size 5 using the naive selection algorithm where we sort the list using insertion sort.

definition *slow_select* **where**
slow_select k xs = insort xs ! k

definition *slow_median* **where**
slow_median xs = slow_select ((length xs − 1) div 2) xs

lemma *slow_select_correct*: *slow_select k xs = select k xs*
by (*simp add*: *slow_select_def select_def insort_correct*)

lemma *slow_median_correct*: *slow_median xs = median xs*
by (*simp add*: *median_def slow_median_def slow_select_correct*)

The definition of the selection algorithm is complicated somewhat by the fact that its termination is contingent on its correctness: if the first recursive call were to return an element for x that is e.g. smaller than all list elements, the algorithm would not terminate.

Therefore, we first prove partial correctness, then termination, and then combine the two to obtain total correctness.

function *mom_select* **where**
mom_select k xs = (
let n = length xs
in if n ≤ 20 then
slow_select k xs
else
let M = mom_select (((n + 4) div 5 − 1) div 2) (map slow_median (chop 5 xs));
(ls, es, gs) = partition3 M xs;
nl = length ls
in
if k < nl then mom_select k ls
else let ne = length es in if k < nl + ne then M
else mom_select (k − nl − ne) gs
)
by *auto*

If *mom_select* terminates, it agrees with *select*:

lemma *mom_select_correct_aux*:
assumes *mom_select_dom (k, xs)* **and** *k < length xs*
shows *mom_select k xs = select k xs*

```
  using assms
proof (induction rule: mom_select.pinduct)
  case (1 k xs)
  show mom_select k xs = select k xs
  proof (cases length xs ≤ 20)
    case True
    thus mom_select k xs = select k xs using 1.prems 1.hyps
    by (subst mom_select.psimps) (auto simp: select_def slow_select_correct)
  next
    case False
    define x where
     x = mom_select (((length xs + 4) div 5 − 1) div 2) (map slow_median
(chop 5 xs))
    define ls es gs where ls = filter (λy. y < x) xs and es = filter (λy. y
= x) xs
                      and gs = filter (λy. y > x) xs
    define nl ne where nl = length ls and ne = length es
    note defs = nl_def ne_def x_def ls_def es_def gs_def
    have tw: (ls, es, gs) = partition3 x xs
      unfolding partition3_def defs One_nat_def ..
    have length_eq: length xs = nl + ne + length gs
      unfolding nl_def ne_def ls_def es_def gs_def
      using [[simp_depth_limit = 1]] by (induction xs) auto
    note IH = 1.IH(2)[OF refl False x_def tw refl refl refl]
             1.IH(3)[OF refl False x_def tw refl refl refl _ refl]

    have mom_select k xs = (if k < nl then mom_select k ls else if k < nl
+ ne then x
                                else mom_select (k − nl − ne) gs) using 1.hyps
False
      by (subst mom_select.psimps) (simp_all add: partition3_def flip: defs
One_nat_def)
    also have ... = (if k < nl then select k ls else if k < nl + ne then x
                      else select (k − nl − ne) gs)
     using IH length_eq 1.prems by (simp add: ls_def es_def gs_def nl_def
ne_def)
      try0
    also have ... = select k xs using ‹k < length xs›
      by (subst (3) select_rec_partition[of _ _ x]) (simp_all add: nl_def
ne_def flip: tw)
    finally show mom_select k xs = select k xs .
  qed
qed
```

mom_select indeed terminates for all inputs:

lemma *mom_select_termination*: *All mom_select_dom*
proof (*relation measure* (*length* ∘ *snd*); (*safe*)*?*)
fix *k* :: *nat* **and** *xs* :: *'a list*
assume ¬ *length xs* ≤ *20*
thus ((((*length xs* + *4*) *div 5* − *1*) *div 2*, *map slow_median* (*chop 5 xs*)), *k*, *xs*)
∈ *measure* (*length* ∘ *snd*)
by (*auto simp*: *length_chop nat_less_iff ceiling_less_iff*)
next
fix *k* :: *nat* **and** *xs ls es gs* :: *'a list*
define *x* **where** *x* = *mom_select* ((((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
assume *A*: ¬ *length xs* ≤ *20*
(*ls*, *es*, *gs*) = *partition3 x xs*
mom_select_dom ((((*length xs* + *4*) *div 5* − *1*) *div 2*,
map slow_median (*chop 5 xs*))
have *less*: ((*length xs* + *4*) *div 5* − *1*) *div 2* < *nat* ⌈*length xs* / *5*⌉
using *A*(*1*) **by** *linarith*

For termination, it suffices to prove that *x* is in the list.

have *x* = *select* ((((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
using *less* **unfolding** *x_def* **by** (*intro mom_select_correct_aux A*) (*auto simp*: *length_chop*)
also have ... ∈ *set* (*map slow_median* (*chop 5 xs*))
using *less* **by** (*intro select_in_set*) (*simp_all add*: *length_chop*)
also have ... ⊆ *set xs*
unfolding *set_map*
proof *safe*
fix *ys* **assume** *ys*: *ys* ∈ *set* (*chop 5 xs*)
hence *median ys* ∈ *set ys*
by *auto*
also have *set ys* ⊆ ⋃(*set* ‘ *set* (*chop 5 xs*))
using *ys* **by** *blast*
also have ... = *set xs*
by (*rule UN_sets_chop*) *simp_all*
finally show *slow_median ys* ∈ *set xs*
by (*simp add*: *slow_median_correct*)
qed
finally have *x* ∈ *set xs* **.**
thus ((*k*, *ls*), *k*, *xs*) ∈ *measure* (*length* ∘ *snd*)
and ((*k* − *length ls* − *length es*, *gs*), *k*, *xs*) ∈ *measure* (*length* ∘ *snd*)
using *A*(*1*,*2*) **by** (*auto simp*: *partition3_def intro*!: *length_filter_less*[*of*

x])
qed

termination *mom_select* **by** (*rule mom_select_termination*)

lemmas [*simp del*] = *mom_select.simps*

lemma *mom_select_correct*: $k < length\ xs \Longrightarrow mom\_select\ k\ xs = select\ k\ xs$
  using *mom_select_correct_aux* **and** *mom_select_termination* **by** *blast*

49.8 Running time analysis

time_fun *partition3* **equations** *partition3_code*

lemma *T_partition3*: *T_partition3 x xs* = *length xs* + *1*
  by (*induction x xs rule*: *T_partition3.induct*) *auto*

time_definition *slow_select*

lemmas *T_slow_select_def* [*simp del*] = *T_slow_select.simps*

time_fun *slow_median*

lemma *T_slow_select_le*:
  assumes $k < length\ xs$
  shows $T\_slow\_select\ k\ xs \leq length\ xs\ \hat{}\ 2 + 3 * length\ xs + 1$
proof −
  have *T_slow_select k xs* = *T_insort xs* + *T_nth* (*Sorting.insort xs*) *k*
    unfolding *T_slow_select_def* **..**
  also have $T\_insort\ xs \leq (length\ xs + 1)\ \hat{}\ 2$
    by (*rule T_insort_length*)
  also have *T_nth* (*Sorting.insort xs*) *k* = *k* + *1*
    using *assms* **by** (*subst T_nth*) (*auto simp*: *length_insort*)
  also have $k + 1 \leq length\ xs$
    using *assms* **by** *linarith*
  also have $(length\ xs + 1)\ \hat{}\ 2 + length\ xs = length\ xs\ \hat{}\ 2 + 3 * length\ xs + 1$
    by (*simp add*: *algebra_simps power2_eq_square*)
  finally show *?thesis* **by** − *simp_all*
qed

lemma *T_slow_median_le*:

assumes $xs \neq []$
shows $T\_slow\_median\ xs \leq length\ xs\ \widehat{}\ 2 + 4 * length\ xs + 2$
proof −
have $T\_slow\_median\ xs = length\ xs + T\_slow\_select\ ((length\ xs - 1)\ div\ 2)\ xs + 1$
by (*simp add*: *T_length*)
also from *assms* **have** $length\ xs > 0$
by *simp*
hence $(length\ xs - 1)\ div\ 2 < length\ xs$
by *linarith*
hence $T\_slow\_select\ ((length\ xs - 1)\ div\ 2)\ xs \leq length\ xs\ \widehat{}\ 2 + 3 * length\ xs + 1$
by (*intro T_slow_select_le*) *auto*
also have $length\ xs + \ldots + 1 = length\ xs\ \widehat{}\ 2 + 4 * length\ xs + 2$
by (*simp add*: *algebra_simps*)
finally show *?thesis* **by** − *simp_all*
qed

time_fun *chop*

lemmas [*simp del*] = *T_chop.simps*

lemma *T_chop_Nil* [*simp*]: $T\_chop\ d\ [] = 1$
by (*cases d*) (*auto simp*: *T_chop.simps*)

lemma *T_chop_0* [*simp*]: $T\_chop\ 0\ xs = 1$
by (*auto simp*: *T_chop.simps*)

lemma *T_chop_reduce*:
$n > 0 \Longrightarrow xs \neq [] \Longrightarrow T\_chop\ n\ xs = T\_take\ n\ xs + T\_drop\ n\ xs + T\_chop\ n\ (drop\ n\ xs) + 1$
by (*cases n*; *cases xs*) (*auto simp*: *T_chop.simps*)

lemma *T_chop_le*: $T\_chop\ d\ xs \leq 5 * length\ xs + 1$
by (*induction d xs rule*: *T_chop.induct*) (*auto simp*: *T_chop_reduce T_take T_drop*)

time_fun *mom_select*

lemmas [*simp del*] = *T_mom_select.simps*

lemma *T_mom_select_simps*:
$length\ xs \leq 20 \Longrightarrow T\_mom\_select\ k\ xs = T\_slow\_select\ k\ xs + T\_length$

```
xs + 1
  length xs > 20 ⟹ T_mom_select k xs = (
      let xss = chop 5 xs;
          ms = map slow_median xss;
          idx = (((length xs + 4) div 5 − 1) div 2);
          x = mom_select idx ms;
          (ls, es, gs) = partition3 x xs;
          nl = length ls;
          ne = length es
      in
        (if k < nl then T_mom_select k ls
         else T_length es + (if k < nl + ne then 0 else T_mom_select (k
− nl − ne) gs)) +
         T_mom_select idx ms + T_chop 5 xs + T_map T_slow_median
xss +
         T_partition3 x xs + T_length ls + T_length xs + 1
    )
  by (subst T_mom_select.simps; simp add: Let_def case_prod_unfold)+

function T'_mom_select :: nat ⇒ nat where
  T'_mom_select n =
    (if n ≤ 20 then
       483
     else
       T'_mom_select (nat ⌈0.2*n⌉) + T'_mom_select (nat ⌈0.7*n+3⌉)
+ 19 * n + 54)
  by force+
termination by (relation measure id; simp; linarith)

lemmas [simp del] = T'_mom_select.simps


lemma T'_mom_select_ge: T'_mom_select n ≥ 483
 by (induction n rule: T'_mom_select.induct; subst T'_mom_select.simps)
auto

lemma T'_mom_select_mono:
  m ≤ n ⟹ T'_mom_select m ≤ T'_mom_select n
proof (induction n arbitrary: m rule: less_induct)
  case (less n m)
  show ?case
  proof (cases m ≤ 20)
    case True
    hence T'_mom_select m = 483
```

```
    by (subst T'_mom_select.simps) auto
  also have ... ≤ T'_mom_select n
    by (rule T'_mom_select_ge)
  finally show ?thesis .
 next
  case False
  hence T'_mom_select m =
          T'_mom_select (nat ⌈0.2*m⌉) + T'_mom_select (nat ⌈0.7*m
+ 3⌉) + 19 * m + 54
    by (subst T'_mom_select.simps) auto
  also have ... ≤ T'_mom_select (nat ⌈0.2*n⌉) + T'_mom_select (nat
⌈0.7*n + 3⌉) + 19 * n + 54
    using ‹m ≤ n› and False by (intro add_mono less.IH; linarith)
  also have ... = T'_mom_select n
    using ‹m ≤ n› and False by (subst T'_mom_select.simps) auto
  finally show ?thesis .
 qed
qed

lemma T_mom_select_le_aux:
 assumes k < length xs
 shows   T_mom_select k xs ≤ T'_mom_select (length xs)
 using assms
proof (induction k xs rule: T_mom_select.induct)
 case (1 k xs)
 define n where [simp]: n = length xs
 define x where
   x = mom_select (((n + 4) div 5 - 1) div 2) (map slow_median (chop
5 xs))
 define ls es gs where ls = filter (λy. y < x) xs and es = filter (λy. y =
x) xs
                   and gs = filter (λy. y > x) xs
 define nl ne where nl = length ls and ne = length es
 note defs = nl_def ne_def x_def ls_def es_def gs_def
 have tw: (ls, es, gs) = partition3 x xs
   unfolding partition3_def defs One_nat_def ..
 note IH = 1.IH(1)[OF n_def]
           1.IH(2)[OF n_def _ x_def tw refl refl nl_def]
           1.IH(3)[OF n_def _ x_def tw refl refl nl_def _ ne_def]

 show ?case
 proof (cases length xs ≤ 20)
   case True — base case
   hence T_mom_select k xs ≤ (length xs)^2 + 4 * length xs + 3
```

```
    using T_slow_select_le[of k xs] ‹k < length xs›
    by (subst T_mom_select_simps(1)) (auto simp: T_length)
  also have ... ≤ 20^2 + 4 * 20 + 3
    using True by (intro add_mono power_mono) auto
  also have ... = 483
    by simp
  also have ... = T'_mom_select (length xs)
    using True by (simp add: T'_mom_select.simps)
  finally show ?thesis by simp
next
  case False — recursive case
  have ((n + 4) div 5 - 1) div 2 < nat ⌈n / 5⌉
    using False unfolding n_def by linarith
  hence x = select (((n + 4) div 5 - 1) div 2) (map slow_median (chop
5 xs))
    unfolding x_def n_def by (intro mom_select_correct) (auto simp:
length_chop)
  also have ((n + 4) div 5 - 1) div 2 = (nat ⌈n / 5⌉ - 1) div 2
    by linarith
  also have select ... (map slow_median (chop 5 xs)) = median (map
slow_median (chop 5 xs))
    by (auto simp: median_def length_chop)
  finally have x_eq: x = median (map slow_median (chop 5 xs)) .
```

The cost of computing the medians of all the subgroups:

```
  define T_ms where T_ms = T_map T_slow_median (chop 5 xs)
  have T_ms ≤ 10 * n + 48
  proof -
    have T_ms = (∑ ys←chop 5 xs. T_slow_median ys) + length (chop
5 xs) + 1
      by (simp add: T_ms_def T_map)
    also have (∑ ys←chop 5 xs. T_slow_median ys) ≤ (∑ ys←chop 5
xs. 47)
    proof (intro sum_list_mono)
      fix ys assume ys ∈ set (chop 5 xs)
      hence length ys ≤ 5 ys ≠ []
        using length_chop_part_le[of ys 5 xs] by auto
      from ‹ys ≠ []› have T_slow_median ys ≤ (length ys) ^ 2 + 4 *
length ys + 2
        by (rule T_slow_median_le)
      also have ... ≤ 5 ^ 2 + 4 * 5 + 2
        using ‹length ys ≤ 5› by (intro add_mono power_mono) auto
      finally show T_slow_median ys ≤ 47 by simp
    qed
```

also have $(\sum ys \leftarrow chop\ 5\ xs.\ 47) + length\ (chop\ 5\ xs) + 1 =$
$48 * nat\ \lceil real\ n\ /\ 5 \rceil + 1$
by (*simp add*: *map_replicate_const length_chop*)
also have $\ldots \leq 10 * n + 48$
by *linarith*
finally show $T\_ms \leq 10 * n + 48$ **by** *simp*
qed

The cost of the first recursive call (to compute the median of medians):

define *T_rec1* **where**
T_rec1 = *T_mom_select* (((*n* + 4) *div* 5 − 1) *div* 2) (*map slow_median* (*chop* 5 *xs*))
from *False* **have** ((*length xs* + 4) *div* 5 − *Suc* 0) *div* 2 < *nat* ⌈*real* (*length xs*) / 5⌉
by *linarith*
hence *T_rec1* ≤ *T'_mom_select* (*length* (*map slow_median* (*chop* 5 *xs*)))
using *False* **unfolding** *T_rec1_def* **by** (*intro IH*(1)) (*auto simp*: *length_chop*)
hence *T_rec1* ≤ *T'_mom_select* (*nat* ⌈0.2 * *n*⌉)
by (*simp add*: *length_chop*)

The cost of the second recursive call (to compute the final result):

define *T_rec2* **where** *T_rec2* = (*if k* < *nl then T_mom_select k ls*
else if k < *nl* + *ne then* 0
else T_mom_select (*k* − *nl* − *ne*) *gs*)
consider *k* < *nl* | *k* ∈ {*nl*..<*nl*+*ne*} | *k* ≥ *nl*+*ne*
by *force*
hence *T_rec2* ≤ *T'_mom_select* (*nat* ⌈0.7 * *n* + 3⌉)
proof *cases*
assume *k* < *nl*
hence *T_rec2* = *T_mom_select k ls*
by (*simp add*: *T_rec2_def*)
also have $\ldots \leq$ *T'_mom_select* (*length ls*)
by (*rule IH*(2)) (*use* ‹*k* < *nl*› *False* **in** ‹*auto simp*: *defs*›)
also have *length ls* ≤ *nat* ⌈0.7 * *n* + 3⌉
unfolding *ls_def* **using** *size_less_than_median_of_medians*[*of xs*]
by (*auto simp*: *length_filter_conv_size_filter_mset slow_median_correct*[*abs_def*] *x_eq*)
hence *T'_mom_select* (*length ls*) ≤ *T'_mom_select* (*nat* ⌈0.7 * *n* + 3⌉)
by (*rule T'_mom_select_mono*)
finally show *?thesis* **.**
next

```
    assume k ∈ {nl..<nl + ne}
    hence T_rec2 = 0
      by (simp add: T_rec2_def)
    thus ?thesis
      using T'_mom_select_ge[of nat ⌈0.7 * n + 3⌉] by simp
  next
    assume k ≥ nl + ne
    hence T_rec2 = T_mom_select (k − nl − ne) gs
      by (simp add: T_rec2_def)
    also have ... ≤ T'_mom_select (length gs)
    proof (rule IH(3))
      show ¬n ≤ 20
        using False by auto
      show ¬ k < nl ¬k < nl + ne
        using ‹k ≥ nl + ne› by (auto simp: nl_def ne_def)
      have length xs = nl + ne + length gs
        unfolding defs by (rule length_partition3) (simp_all add: parti-
tion3_def)
      thus k − nl − ne < length gs
       using ‹k ≥ nl + ne› ‹k < length xs› by (auto simp: nl_def ne_def)
    qed
    also have length gs ≤ nat ⌈0.7 * n + 3⌉
     unfolding gs_def using size_greater_than_median_of_medians[of
xs]
     by (auto simp: length_filter_conv_size_filter_mset slow_median_correct[abs_def]
x_eq)
      hence T'_mom_select (length gs) ≤ T'_mom_select (nat ⌈0.7 * n
+ 3⌉)
      by (rule T'_mom_select_mono)
    finally show ?thesis .
  qed
```

Now for the final inequality chain:

```
  have T_mom_select k xs =
         (if k < nl then T_mom_select k ls
          else T_length es +
            (if k < nl + ne then 0 else T_mom_select (k − nl − ne) gs))
+
          T_mom_select (((n + 4) div 5 − 1) div 2) (map slow_median
(chop 5 xs)) +
        T_chop 5 xs + T_map T_slow_median (chop 5 xs) + T_partition3
x xs +
          T_length ls + T_length xs + 1 using False
    by (subst T_mom_select_simps;
```

```
        unfold Let_def n_def [symmetric] x_def [symmetric] nl_def
[symmetric]
        ne_def [symmetric] prod.case tw [symmetric]) simp_all
    also have ... ≤ T_rec2 + T_rec1 + T_ms + 2 * n + nl + ne +
T_chop 5 xs + 5 using False
      by (auto simp add: T_rec1_def T_rec2_def T_partition3
                         T_length T_ms_def nl_def ne_def)
    also have nl ≤ n by (simp add: nl_def ls_def)
    also have ne ≤ n by (simp add: ne_def es_def)
    also note ‹T_ms ≤ 10 * n + 48›
    also have T_chop 5 xs ≤ 5 * n + 1
      using T_chop_le[of 5 xs] by simp
    also note ‹T_rec1 ≤ T'_mom_select (nat ⌈0.2*n⌉)›
    also note ‹T_rec2 ≤ T'_mom_select (nat ⌈0.7*n + 3⌉)›
    finally have T_mom_select k xs ≤
                 T'_mom_select (nat ⌈0.7*n + 3⌉) + T'_mom_select (nat
⌈0.2*n⌉) + 19 * n + 54
      by simp
    also have ... = T'_mom_select n
      using False by (subst T'_mom_select.simps) auto
    finally show ?thesis by simp
  qed
qed
```

49.9 Akra–Bazzi Light

```
lemma akra_bazzi_light_aux1:
  fixes a b :: real and n n0 :: nat
  assumes ab: a > 0 a < 1 n > n0
  assumes n0 ≥ (max 0 b + 1) / (1 - a)
  shows nat ⌈a*n+b⌉ < n
proof -
  have a * real n + max 0 b ≥ 0
    using ab by simp
  hence real (nat ⌈a*n+b⌉) ≤ a * n + max 0 b + 1
    by linarith
  also {
    have n0 ≥ (max 0 b + 1) / (1 - a)
      by fact
    also have ... < real n
      using assms by simp
    finally have a * real n + max 0 b + 1 < real n
      using ab by (simp add: field_simps)
  }
```

```
  finally show nat ⌈a*n+b⌉ < n
    using ‹n > n0› by linarith
qed

lemma akra_bazzi_light_aux2:
  fixes f :: nat ⇒ real
  fixes n0 :: nat and a b c d :: real and C1 C2 C1 C2 :: real
  assumes bounds: a > 0 c > 0 a + c < 1 C1 ≥ 0
  assumes rec: ∀ n>n0. f n = f (nat ⌈a*n+b⌉) + f (nat ⌈c*n+d⌉) + C1 *
n + C2
  assumes ineqs: n0 > (max 0 b + max 0 d + 2) / (1 − a − c)
                 C3 ≥ C1 / (1 − a − c)
                 C3 ≥ (C1 * n0 + C2 + C4) / ((1 − a − c) * n0 − max 0 b
− max 0 d − 2)
                 ∀ n≤n0. f n ≤ C4
  shows   f n ≤ C3 * n + C4
proof (induction n rule: less_induct)
  case (less n)
  have 0 ≤ C1 / (1 − a − c)
    using bounds by auto
  also have ... ≤ C3
    by fact
  finally have C3 ≥ 0 .

  show ?case
  proof (cases n > n0)
    case False
    hence f n ≤ C4
      using ineqs(4) by auto
    also have ... ≤ C3 * real n + C4
      using bounds ‹C3 ≥ 0› by auto
    finally show ?thesis .
  next
    case True
    have nonneg: a * n ≥ 0 c * n ≥ 0
      using bounds by simp_all

    have (max 0 b + 1) / (1 − a) ≤ (max 0 b + max 0 d + 2) / (1 − a
− c)
      using bounds by (intro frac_le) auto
    hence n0 ≥ (max 0 b + 1) / (1 − a)
      using ineqs(1) by linarith
    hence rec_less1: nat ⌈a*n+b⌉ < n
     using bounds ‹n > n0› by (intro akra_bazzi_light_aux1[of _ n0]) auto
```

```
    have (max 0 d + 1) / (1 − c) ≤ (max 0 b + max 0 d + 2) / (1 − a
− c)
      using bounds by (intro frac_le) auto
    hence n₀ ≥ (max 0 d + 1) / (1 − c)
      using ineqs(1) by linarith
    hence rec_less2: nat ⌈c∗n+d⌉ < n
     using bounds ‹n > n₀› by (intro akra_bazzi_light_aux1[of _ n₀]) auto

    have f n = f (nat ⌈a∗n+b⌉) + f (nat ⌈c∗n+d⌉) + C₁ ∗ n + C₂
      using ‹n > n₀› by (subst rec) auto
    also have ... ≤ (C₃ ∗ nat ⌈a∗n+b⌉ + C₄) + (C₃ ∗ nat ⌈c∗n+d⌉ +
C₄) + C₁ ∗ n + C₂
      using rec_less1 rec_less2 by (intro add_mono less.IH) auto
   also have ... ≤ (C₃ ∗ (a∗n+max 0 b+1) + C₄) + (C₃ ∗ (c∗n+max 0
d+1) + C₄) + C₁ ∗ n + C₂
     using bounds ‹C₃ ≥ 0› nonneg by (intro add_mono mult_left_mono
order.refl; linarith)
    also have ... = C₃ ∗ n + ((C₃ ∗ (max 0 b + max 0 d + 2) + 2 ∗
C₄ + C₂) −
                                (C₃ ∗ (1 − a − c) − C₁) ∗ n)
      by (simp add: algebra_simps)
    also have ... ≤ C₃ ∗ n + ((C₃ ∗ (max 0 b + max 0 d + 2) + 2 ∗
C₄ + C₂) −
                                (C₃ ∗ (1 − a − c) − C₁) ∗ n₀)
      using ‹n > n₀› ineqs(2) bounds
     by (intro add_mono diff_mono order.refl mult_left_mono) (auto simp:
field_simps)
    also have (C₃ ∗ (max 0 b + max 0 d + 2) + 2 ∗ C₄ + C₂) − (C₃ ∗
(1 − a − c) − C₁) ∗ n₀ ≤ C₄
      using ineqs bounds by (simp add: field_simps)
    finally show f n ≤ C₃ ∗ real n + C₄
      by (simp add: mult_right_mono)
  qed
qed

lemma akra_bazzi_light:
  fixes f :: nat ⇒ real
  fixes n₀ :: nat and a b c d C₁ C₂ :: real
  assumes bounds: a > 0 c > 0 a + c < 1 C₁ ≥ 0
  assumes rec: ∀ n>n₀. f n = f (nat ⌈a∗n+b⌉) + f (nat ⌈c∗n+d⌉) + C₁ ∗
n + C₂
  shows ∃ C₃ C₄. ∀ n. f n ≤ C₃ ∗ real n + C₄
proof −
```

```
  define n0' where n0' = max n0 (nat ⌈(max 0 b + max 0 d + 2) / (1 −
a − c) + 1⌉)
  define C4 where C4 = Max (f ‘ {..n0'})
  define C3 where C3 = max (C1 / (1 − a − c))
                       ((C1 * n0' + C2 + C4) / ((1 − a − c) * n0' − max 0
b − max 0 d − 2))

  have f n ≤ C3 * n + C4 for n
  proof (rule akra_bazzi_light_aux2[OF bounds _])
    show ∀ n>n0'. f n = f (nat ⌈a*n+b⌉) + f (nat ⌈c*n+d⌉) + C1 * n +
C2
      using rec by (auto simp: n0'_def)
  next
    show C3 ≥ C1 / (1 − a − c)
     and C3 ≥ (C1 * n0' + C2 + C4) / ((1 − a − c) * n0' − max 0 b −
max 0 d − 2)
      by (simp_all add: C3_def)
  next
    have (max 0 b + max 0 d + 2) / (1 − a − c) < nat ⌈(max 0 b + max
0 d + 2) / (1 − a − c) + 1⌉
      by linarith
    also have ... ≤ n0'
      by (simp add: n0'_def)
    finally show (max 0 b + max 0 d + 2) / (1 − a − c) < real n0' .
  next
    show ∀ n≤n0'. f n ≤ C4
      by (auto simp: C4_def)
  qed
  thus ?thesis by blast
qed

lemma akra_bazzi_light_nat:
  fixes f :: nat ⇒ nat
  fixes n0 :: nat and a b c d :: real and C1 C2 :: nat
  assumes bounds: a > 0 c > 0 a + c < 1 C1 ≥ 0
  assumes rec: ∀ n>n0. f n = f (nat ⌈a*n+b⌉) + f (nat ⌈c*n+d⌉) + C1 *
n + C2
  shows ∃ C3 C4. ∀ n. f n ≤ C3 * n + C4
proof −
  have ∃ C3 C4. ∀ n. real (f n) ≤ C3 * real n + C4
    using assms by (intro akra_bazzi_light[of a c C1 n0 f b d C2]) auto
  then obtain C3 C4 where le: ∀ n. real (f n) ≤ C3 * real n + C4
    by blast
  have f n ≤ nat ⌈C3⌉ * n + nat ⌈C4⌉ for n
```

```
  proof −
    have real (f n) ≤ C3 ∗ real n + C4
      using le by blast
    also have ... ≤ real (nat ⌈C3⌉) ∗ real n + real (nat ⌈C4⌉)
      by (intro add_mono mult_right_mono; linarith)
    also have ... = real (nat ⌈C3⌉ ∗ n + nat ⌈C4⌉)
      by simp
    finally show ?thesis by linarith
  qed
  thus ?thesis by blast
qed

lemma T'_mom_select_le': ∃ C1 C2. ∀ n. T'_mom_select n ≤ C1 ∗ n +
C2
proof (rule akra_bazzi_light_nat)
  show ∀ n>20. T'_mom_select n = T'_mom_select (nat ⌈0.2 ∗ n + 0⌉)
+
                T'_mom_select (nat ⌈0.7 ∗ n + 3⌉) + 19 ∗ n + 54
    using T'_mom_select.simps by auto
qed auto

end
```

50 Time Functions in Locales — An Example

```
theory Time_Locale_Example
imports
  HOL−Library.Time_Functions
  HOL−Library.AList
  Map_Specs
begin
```

If you want to reason about the time complexity of functions in a locale, you need to parameterize the locale with time functions for all functions that are utilized. More precisely, if you are in a locale parameterized by some function *f* and you define a function *g* that uses *f*, and you want to define *T_g*, it will depend on *T_f*, which you have to make an additional parameter of the locale. Only then will *time_fun g* work. Below we show a realistic example.

50.1 Basic Time Functions

```
time_fun AList.update
```

time_fun needs uncurried defining equations

lemma
map_of_simps': *map_of* [] (*x*::'*a*) = (*None* :: '*b option*)
map_of ((*a*::'*a*,*b*::'*b*)#*ps*) *x* = (*if x*=*a then Some b else map_of ps x*)
by *auto*

time_fun *map_of* **equations** *map_of_simps'*

lemma *T_map_ub*: *T_map_of ps a* ≤ *length ps* + *1*
by(*induction ps*) *auto*

lemma *T_update_ub*: *T_update a b ps* ≤ *length ps* + *1*
by(*induction ps*) *auto*

lemma *length_AList_update_ub*: *length* (*AList.update a b ps*) ≤ *length ps* + *1*
by(*induction ps*) *auto*

50.2 Locale

Counting the elements in a list by means of a map that associates elements with their multiplicities in the list, like a 'histogram'. The locale is parameterized with the map ADT and the timing functions for *lookup* and *update*.

locale *Count_List* = *Map* **where** *update* = *update* **for** *update* :: '*a* ⇒ *nat* ⇒ '*m* ⇒ '*m* +
fixes *T_lookup* :: '*m* ⇒ '*a* ⇒ *nat*
and *T_update* :: '*a* ⇒ *nat* ⇒ '*m* ⇒ *nat*
begin

definition *lookup_nat* :: '*m* ⇒ '*a* ⇒ *nat* **where**
lookup_nat m x = (*case lookup m x of None* ⇒ *0* | *Some n* ⇒ *n*)

time_definition *lookup_nat*

fun *count* :: '*m* ⇒ '*a list* ⇒ '*m* **where**
count m [] = *m* |
count m (*x*#*xs*) = *count* (*update x* (*lookup_nat m x* + *1*) *m*) *xs*

time_fun *count*

end

50.3 Interpretation

Interpretation of *Count_List* with association lists as maps.

lemma *map_of_AList_update*: *map_of* (*AList.update a b ps*) = ((*map_of ps*)(*a* ↦ *b*))
by(*induction ps*) *auto*

lemma *map_of_AList_delete*: *map_of* (*AList.delete a ps*) = (*map_of ps*)(*a* := *None*)
by(*induction ps*) *auto*

global_interpretation *CL*: *Count_List*
where *empty* = [] **and** *lookup* = *map_of*
and *update* = *AList.update* **and** *delete* = *AList.delete* **and** *invar* = λ_. *True*
and *T_lookup* = *T_map_of* **and** *T_update* = *T_update*
defines *CL_count* = *CL.count* **and** *CL_T_count* = *CL.T_count*
proof (*standard*, *goal_cases*)
  case *1*
  show *?case* **by** (*rule ext*) *simp*
next
  case (*2 m a b*)
  show *?case* **by** (*rule map_of_AList_update*)
next
  case (*3 m a*)
  show *?case* **by** (*rule map_of_AList_delete*)
next
  case *4*
  show *?case* **by**(*rule TrueI*)
next
  case (*5 m a b*)
  show *?case* **by**(*rule TrueI*)
next
  case (*6 m a*)
  show *?case* **by**(*rule TrueI*)
qed

50.4 Complexity Proof

lemma *CL.T_count ps xs* ≤ *2* ∗ *length xs* ∗ (*length xs* + *length ps* + *1*) + *1*
proof(*induction xs arbitrary*: *ps*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons a xs*)
  let *?lps'* = *length ps* + *1*

let *?na′ = CL.lookup_nat ps a + 1*
let *?ps′ = AList.update a ?na′ ps*
have *CL_T_count ps (a # xs) =*
T_map_of ps a + T_update a ?na′ ps + CL_T_count (AList.update a ?na′ ps) xs + 1
by *simp*
also have *... ≤ 2 * ?lps′ + CL_T_count ?ps′ xs + 1*
using *T_map_ub T_update_ub add_mono* **by** (*fastforce simp*: *mult_2*)
also have *... ≤ 2 * ?lps′ + 2 * length xs * (length xs + length ?ps′ + 1) + 1 + 1*
using *Cons.IH* **by** (*metis* (*no_types*, *lifting*) *add.assoc add_mono_thms_linordered_semiring*(*3*) *nat_add_left_cancel_le*)
also have *... ≤ 2 * ?lps′ + 2 * length xs * (length xs + ?lps′ + 1) + 1 + 1*
using *length_AList_update_ub*
by (*metis add_mono_thms_linordered_semiring*(*2*) *add_right_mono mult_le_mono2*)
also have *... ≤ 2 * length (a # xs) * (length (a # xs) + length ps + 1) + 1*
by (*auto simp*: *algebra_simps*)
finally show *?case* **.**
qed

end

51 Bibliographic Notes

Red-black trees The insert function follows Okasaki [15]. The delete function in theory *RBT_Set* follows Kahrs [11, 12], an alternative delete function is given in theory *RBT_Set2*.

2-3 trees Equational definitions were given by Hoffmann and O'Donnell [9] (only insertion) and Reade [19]. Our formalisation is based on the teaching material by Turbak [22] and the article by Hinze [8].

1-2 brother trees They were invented by Ottmann and Six [16, 17]. The functional version is due to Hinze [7].

AA trees They were invented by Arne Anderson [3]. Our formalisation follows Ragde [18] but fixes a number of mistakes.

Splay trees They were invented by Sleator and Tarjan [21]. Our formalisation follows Schoenmakers [20].

Join-based BSTs They were invented by Adams [1, 2] and analyzed by Blelloch *et al.* [4].

Leftist heaps They were invented by Crane [6]. A first functional implementation is due to Núñez *et al.* [14].

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