gw 5/94 ①

Introduction to Sequentiality

References:

P-L. Curien "Categorical Combinators, Sequential Algorithms & Functional Prog." Birkhauser '93.

P-L Curien "Sequentiality & full abstraction" – previous handout.

T. Ehrhard "Hypercoherences" MSCS, Dec. 93.

A. Bucciarelli's Ph.D thesis '93.

$\mathbb{N}$ flat cpo of integers

A continuous function

$$f : \mathbb{N}^m \longrightarrow \mathbb{N} \qquad (m \geq 1)$$

is <u>sequential</u> (Vuillemin)

iff $\forall \vec{x} \in \vec{\mathbb{N}}.$

$$\exists i. \ \forall \vec{y} \sqsupseteq \vec{x}. \quad f(\vec{y}) \not\sqsupseteq f(\vec{x})$$

$$\Rightarrow \quad y_i \underset{\neq}{\sqsupseteq} x_i$$

A continuous function

$$f : \mathbb{N}^m \longrightarrow \mathbb{N}^q \qquad (m \geq 1)$$

is sequential (Vuillemin) iff each composition $\pi_j \circ f$, $j = 1, \ldots, q$, is sequential as above.

The Gustav - function (Berry)

a stable but non-sequential function

$$G : \mathbb{B}^3 \longrightarrow \mathbb{O}$$

minimum s.t.

$$G(t, f, \bot) = \top$$

$$G(\bot, t, f) = \top$$

$$G(f, \bot, t) = \top$$

[One reason why restricting to stable functions does not give full abstraction]

A <u>sequential structure</u> consists of

$$S = (C, E, i, \lhd)$$

C – cells

E – events, initial event $i \in E$,

$\lhd$ – accessibility relation

$\lhd \subseteq C \times E \cup E \times C$

$\lhd^*$ a p.o.

(Cell / Event) <u>occurrences</u> consist of finite non-empty alternating sequences $e_0 c_1 e_1 c_2 e_2 \cdots$ for which $e_0 = i$ & $\cdots e_k \lhd c_{k+1} \lhd e_{k+1} \cdots$.

A <u>configuration</u> of S consists of x a subset of occurrences s.t.

nonempty: $i \in x$

prefix-closed: $sd \in x \Rightarrow s \in x$, $d \in C \cup E$

event-determined: $sc \in x$ & $c \in C \Rightarrow \exists e \in E.\ sce \in x$

consistent: $se_1, se_2 \in x$ & $e_1, e_2 \in E \Rightarrow e_1 = e_2$

Write $\Gamma(S)$ for set of configurations.

Alternatively, could define configurations as "partial strategies"

$x \subseteq$ Occurrences

nonempty: $\iota \in x$

prefix closed: $sd \in x \Rightarrow s \in x$

cell-closed: $s \in x$ & sc a cell-occurrence $\Rightarrow sc \in x$

consistent: $se_1, se_2 \in x$ & $e_1, e_2 \in E \Rightarrow e_1 = e_2$.

Notation

For a seql. structure S,
Write

C^*, E^*, D^* for cell, event & all

occurrences.

Write $\leq$ for the order of extension on

occurrences. D^* forms a tree, ~~and~~ root

i, and has meet $\wedge$.

For $x, y \in \Gamma(S)$, $c \in C^*$

$x \prec_c y$ means $x \cup \{ce\} = y$ &
$x \subsetneq y$, for some $e \in E$.

$x \subset_c y$ means $x \prec_c z \subseteq y$, for some $z \in \Gamma(S)$

Prop. A config. x of S is determined by its event occurrences $x \wedge E^*$.

$(\Gamma(S), \subseteq)$ is a filiform concrete domain.
(cf. Curien's book).

Let S, S' be seql. structures.

A continuous function

$$f : (\Gamma(S), \subseteq) \to (\Gamma(S'), \subseteq)$$

is <u>sequential</u> (Kahn - Plotkin)

iff $\forall x \in \Gamma(S)$.

$\forall c' \in C'^{*}$.

if $\exists y \supseteq x.\ f(y) \supset_{c'} f(x)$

then

$\exists c \in C^{*}$.

$\forall y \supseteq x.\ f(y) \supset_{c'} f(x)$

$\Rightarrow y \supset_c x$

From here there are two routes to sequentiality at higher types

1. Berry & Curien's sequential algorithms [intensional model]

2. Bucciarelli & Ehrhard's strongly stable functions w.r.t. coherences (& their representation via "hypercoherences") [extensional model]

We first look at 1.

Affine function space

S_0, S_1 seq^l. structures

$S_0 \multimap S_1 =$

$$(E_0 \times C_1,\ E_0 \times E_1 \cup C_0 \times C_1,\ (i_0, i_1),\ \lhd)$$

where

$$(d_0, d_1) \lhd (d_0', d_1') \iff (d_0 \lhd_0 d_0' \ \&\ d_1 = d_1') \text{ or } (d_0 = d_0' \ \&\ d_1 \lhd_1 d_1').$$

Morphisms are to be configurations of the affine fn. space. In order to compose them we use the following characterization.

Let $\alpha \in \Gamma(S_0 \multimap S_1)$.

Then $\alpha \cap E^*$ has two kinds of event occurrences:

$$s(e_0, e_1) \mapsto (s_0 e_0,\ s_1 e_1) \in E_0^* \times E_1^* \qquad\qquad s(c_0, c_1) \mapsto (s_0 c_0,\ s_1 c_1) \in C_0^* \times C_1^*.$$

Configs. α are in 1-1 correspondence with

$$\bar{\alpha} \subseteq E_0^* \times E_1^* \cup C_0^* \times C_1^*$$

s.t.

(1) $(i_0, i_1) \in \alpha$

(2) (a) $(d_0, d_1) \in \bar{\alpha}$ & $d_0 \geq c_0 \in C_0^* \Rightarrow \exists!\, c_1 \in C_1^*.$ $c_1 \leq d_1$ & $(c_0, c_1) \in \bar{\alpha}$

(b) $(d_0, d_1) \in \bar{\alpha}$ & $d_1 \geq e_1 \in E_1^* \Rightarrow \exists!\, e_0 \in E_0^*.$ $e_0 \leq d_0$ & $(e_0, e_1) \in \bar{\alpha}$

(3) If $(d_0, d_1),\ (d_0', d_1') \in \bar{\alpha}$ then

(a) $d_0 \leq d_0'$ & $d_0 \in E_0^* \Rightarrow d_1 \wedge d_1' \in E_1^*$, and

(b) $d_1 \leq d_1'$ & $d_1 \in C_1^* \Rightarrow d_0 \wedge d_0' \in C_0^*$.

Composition in the category is given by relational composition on $\bar{\alpha}$'s. Identities correspond to identity relations on occurrences.

Tensor

$$S_0 \otimes S_1 = (C_0 \times E_1 \cup E_0 \times C_1 ,\ E_0 \times E_1 ,\ (i_0, i_1),\ \vartriangleleft)$$

where $(d_0, d_1) \vartriangleleft (d_0', d_1') \Leftrightarrow (d_0 \vartriangleleft_0 d_0' \ \& \ d_1 = d_1')$ or
$(d_0 = d_0' \ \& \ d_1 \vartriangleleft_1 d_1')$.

Have :

$$S_0 \otimes S_1 \multimap S_2 \cong S_0 \multimap (S_1 \multimap S_2)$$

Product $\times$ disjoint juxtaposition.

Claim $\quad !S_0 \otimes !S_1 \cong !(S_0 \times S_1)$

for def. of exponential below.

Exponential

$!S = (C_!, E_!, \{i\}, \lhd_!)$ where

$C_! = \{xc \mid x \prec_c\}$

$E_! = \Gamma(S)^\circ$ (finite configs. of S)

$\lhd_!$ is least relu. s.t.:

$xc \lhd_! y$ iff $x \prec_c y$

$x \lhd_! xc$

Berry & Curien's sequential algorithms corr. to configs of:

$!S_0 \multimap S_1 = (C, E, i, \lhd)$ where

$C = \Gamma(S_0)^\circ \times C_1$,

$E = \Gamma(S_0)^\circ \times E_1 \cup \{(x_0 c_0, c_1) \mid x \prec_{c_0} \ \&\ c_1 \in C_1\}$,

$i = (\{i_0\}, i_1)$,

$x_0 e_1 \lhd x_0 c_1$ if $e_1 \lhd_1 c_1$, $x_0 c_1 \lhd x_0 e_1$ if $c_1 \lhd_1 e_1$

$x_0 c_1 \lhd x_0 c_0, c_1$ if $x_0 \prec_{c_0}$, $x_0 c_0, c_1 \lhd y_0 c_1$ if $x_0 \prec_{c_0} y_0$.