The Winning Ways of Concurrent Games

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***Abstract*—A bicategory of concurrent games, where nondeterministic strategies are formalized as certain maps of event structures, was introduced recently. This paper studies an extension of concurrent games by winning conditions, specifying players' objectives. The introduction of winning conditions raises the question of whether such games are determined, that is, if one of the players has a winning strategy. This paper gives a positive answer to this question when the games are well-founded and satisfy a structural property, race-freedom, which prevents one player from interfering with the moves available to the other. Uncovering the conditions under which concurrent games with winning conditions are determined opens up the possibility of further applications of concurrent games in areas such as logic and verification, where both winning conditions and determinacy are most needed. A concurrent-game semantics for predicate calculus is provided as an illustration.**

I. Introduction

The games that have arisen in mathematical logic [2] have typically been games between two players (we call them Player and Opponent), trying to achieve complementary goals. The goals are given by winning conditions, specifying which sets of plays lead to a win for one player or the other. Games have a long history in logic and philosophy but in the last few decades have become invaluable in computer science as a tool to express and solve complex problems, both in the formal semantics of computational processes and in algorithmic questions. Solutions to a great many problems can be naturally phrased in terms of the existence of a winning strategy for one of the two players.

Not surprisingly, such reductions to games with winning conditions generally rely on the existence of winning strategies for one or other player—on the fact that the games are determined. For traditional games this is usually the case since the winning conditions obtained most often form Borel sets, and as shown in Martin's seminal result [8] Borel games are determined; the problem being represented by the game then has a solution (although the solution might not be computable).

Logic games are usually played on graphs, the nodes of which determine whether it is the turn of Player or Opponent, and of a very sequential nature. This feature makes traditional two player games an unnatural model in some contexts, for instance, when dealing with distributed and concurrent systems. As a result, in the last decade, a number of games models where the two players can interact concurrently have been proposed. In this paper we study the model developed in [11], a notion of concurrent game based on event structures. Event structures are the concurrency analogue of trees; just as transition systems, an "interleaving" model, unfold to trees so do Petri nets, a "concurrent" model, unfold to event structures. In this model, games are represented by *event structures with polarities*, and a strategy on a game A is a (certain) polarity-preserving map of event structures $\sigma : S \to A$. In [11] concurrent games and strategies were shown to form a bicategory, the aim to establish a new, alternative basis for the semantics of programming languages. Albeit general, the games model introduced in [11] was not equipped with the means to express players' objectives, a feature needed to model several algorithmic problems in areas such as logic and verification. In order to overcome this limitation this paper extends the framework of concurrent games, based on event structures, with winning conditions. As concurrent games on event structures encompass traditional approaches of games and generalize them by allowing the players to interact in a highly distributed fashion, we expect this games model to be a fruitful framework, for instance, well adapted to the formal study of concurrent and distributed systems.

Following in the steps of Martin, our first goal is to provide classes of concurrent games that are determined. As we will see, the high level of concurrency present in our framework makes the problem very subtle, even for finite games. The paper contains three main technical contributions: firstly, we extend the results of [11] by giving a very general bicategory of concurrent games with winning conditions and nondeterministic winning strategies. Secondly, for well-founded games (*i.e.* when all configurations are finite), we characterise determined games as those which satisfy a property called race-freedom, which prevents one player from interfering with the moves available to the other. Thirdly, in order to illustrate the use of concurrent games with winning conditions, we show how to give a concurrent-game interpretation of first-order predicate logic consistent with Tarski's semantics. Our interpretation exploits the additional mathematical space surrounding concurrent games and provides techniques to effectively build and deconstruct nondeterministic winning strategies in a compositional manner.

Related work: Concurrent games and determinacy problems have been studied elsewhere, though separately in most cases. Melliès et al. [1], [9], [10] have done extensive work on concurrent games on asynchronous transition systems; however, determinacy issues were not addressed. Concurrent games on graphs [3], [4] have also been studied in order to solve verification problems for open systems; such games are undetermined in the general case and as a consequence stochastic strategies are used. Of the several treatments of winning conditions in game semantics ours is close to Hyland's [6], which it can be seen as generalizing directly. Finally, concurrent games on partial orders were developed in

[5]; in this case a determinacy result is given when restricted to regular winning conditions and very simple game boards.

The paper is structured as follows: Sections II-IV give an introduction to event structures and the bicategory of concurrent games and nondeterministic strategies. Section V contains the extension of the concurrent games model with winning conditions as well as a study of some of its properties. Then, in Sections VI and VII, the proof of determinacy is presented. Finally, in Section VIII, the concurrent-game semantics for predicate calculus is described.

Full proofs may be found in [13], though the determinacy proof in [13] has been improved by the proof sketched here.

II. Event structures

An *event structure* comprises $(E, \mathrm{Con}, \leq)$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}$$
$$X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the *finite* configurations of an event structure E.

Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ is a configuration.

Notation 1. Let E be an event structure. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subset y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \overset{e}{-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \overset{e}{-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{-\!\!\subset} y$ for some y.

A. Maps of event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \rightharpoonup E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$e_1, e_2 \in x \;\&\; f(e_1) = f(e_2) \text{ (with both defined)} \implies e_1 = e_2.$$

The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Partial maps of event structures compose as partial functions, with identity maps given by identity functions. We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. A total map of event structures which preserves causal dependency is called *rigid*.

B. Process operations

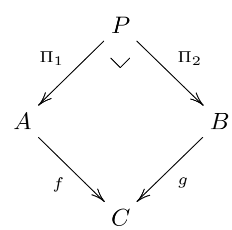
1) Products: The category of event structures with partial maps has *products* $A \times B$ with projections Π_1 to A and Π_2 to B. The effect is to introduce arbitrary synchronisations between events of A and events of B in the manner of process algebra.

2) Restriction: The restriction of an event structure E to a subset of events R, written $E \upharpoonright R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by E.

3) Synchronized compositions and pullbacks: Synchronized compositions play a central role in process algebra, with such seminal work as Milner's CCS and Hoare's CSP. Synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining

$$P =_{\mathrm{def}} A \times B \upharpoonright \{p \in A \times B \mid f\Pi_1(p) = g\Pi_2(p) \text{ with both defined}\}$$

we obtain a pullback square



in the category of event structures. When f and g are total the same construction gives the pullback in the category of event structures with *total* maps.

4) Projection: Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v' \;\&\; v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con} \;\&\; X \subseteq V$.

5) Prefixes and sums: The prefix of an event structure A, written $\bullet.A$, comprises the event structure in which all the events of A are made to causally depend on an event $\bullet$. The category of event structures has sums given as coproducts; a coproduct $\sum_{i \in I} E_i$ is obtained as the disjoint juxtaposition of an indexed collection of event structures, making events in distinct components inconsistent. In Section VIII we shall use prefixed sums $\sum_{i \in I} \bullet.A_i$ in games for modelling first-order logical quantifiers.

III. Event structures with polarity

Both a game and a strategy in a game are to be represented as an event structure with polarity, which comprises (E, pol) where E is an event structure with a polarity function pol :

$E \to \{+, -\}$ ascribing a polarity + (Player) or − (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

A. Basic operations

1) Dual: The *dual*, $E^{\perp}$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities.

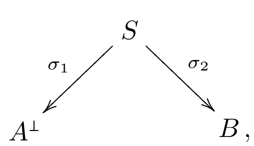
2) Simple parallel composition: This operation juxtaposes two event structures with polarity. Let $(A, \leq_A, \mathrm{Con}_A, pol_A)$ and $(B, \leq_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The events of $A \| B$ are $(\{1\} \times A) \cup (\{2\} \times B)$, their polarities unchanged, with the causal dependency relation given by $(1, a) \leq (1, a')$ iff $a \leq_A a'$ and $(2, b) \leq (2, b')$ iff $b \leq_B b'$; a subset of events C is consistent in $A \| B$ iff $\{a \mid (1, a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2, b) \in C\} \in \mathrm{Con}_B$. The empty event structure with polarity, written $\varnothing$, is the unit w.r.t. $\|$.

IV. Concurrent strategies

A. Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section V). A *pre-strategy* in A is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S. Two pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be essentially the same when they are isomorphic, *i.e.* there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$; then we write $\sigma \cong \tau$.

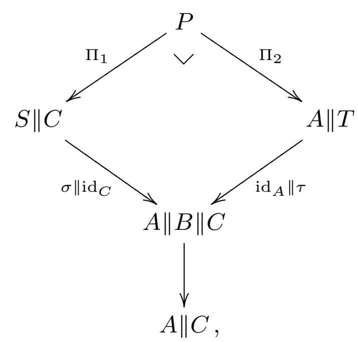
Let A and B be event structures with polarity. Following Joyal [7], a pre-strategy from A to B is a pre-strategy in $A^{\perp} \| B$, so a total map $\sigma : S \to A^{\perp} \| B$. It thus determines a span



of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies from A to B will be isomorphic when they are isomorphic as pre-strategies in $A^{\perp} \| B$, or equivalently are isomorphic as spans. We write $\sigma : A \twoheadrightarrow B$ to express that σ is a pre-strategy from A to B. Note that a pre-strategy σ in a game A coincides with a pre-strategy from the empty game $\sigma : \varnothing \twoheadrightarrow A$.

B. Composing pre-strategies

We can present the composition of pre-strategies via pullbacks.[1] Given two pre-strategies $\sigma : S \to A^{\perp} \| B$ and $\tau : T \to B^{\perp} \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures as shown below



where the map $A \| B \| C \to A \| C$ is undefined on B and acts as identity on A and C. Note there are three kinds of events $p \in P$: *synchronizations* between events of S and T, where $\sigma_2 \Pi_1(p) = \tau_1 \Pi_2(p) \in B$; *asynchronous* occurrences of events in S, where $\sigma_1 \Pi_1(p) \in A$; *asynchronous* occurrences of events in T, where $\tau_2 \Pi_2(p) \in C$. The partial map from P to $A \| C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P \downarrow V$, where V is the set of events of P at which the map $P \to A \| C$ is defined, and a total map $P \downarrow V \to A \| C$. The resulting total map gives us the composition $\tau \odot \sigma : P \downarrow V \to A^{\perp} \| C$ once we reinstate polarities.

C. Concurrent copy-cat

Identities w.r.t. composition are given by copy-cat strategies. Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^{\perp} \| A$. It describes a concurrent strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^{\perp} \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}\mathbb{C}_A$ to comprise the event structure with polarity $A^{\perp} \| A$ together with the extra causal dependencies generated by $\overline{c} \leq_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $pol_{A^{\perp} \| A}(c) = +$. The *copy-cat* pre-strategy $\gamma_A : A \twoheadrightarrow A$ is defined to be the map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^{\perp} \| A$ where γ_A is the identity on the common set of events.

D. Strategies

The main result of [11], presented summarily here, is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures

[1] The construction here gives the same result as that via synchronized composition in [11]— we are grateful to Nathan Bowler for this observation. Notice the analogy with the composition of relations $S \subseteq A \times B$, $T \subseteq B \times C$ which can be defined as $T \circ S = (S \times C \cap A \times T) \downarrow A \times C$, the image of $S \times C \cap A \times T$ under the projection of $A \times B \times C$ to $A \times C$.

an openness to all possible moves of Opponent. Innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game.

Receptivity. A pre-strategy σ is *receptive* iff $\sigma x \overset{a}{\longrightarrow\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}$ & $\sigma(s) = a$.

Innocence. A pre-strategy σ is *innocent* when it is both
+-innocent: if $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$, and
−-innocent: if $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

Theorem 2 (from [11]). *Let $\sigma : A \rightarrow\!\!\!\!+ B$ be pre-strategy. Copy-cat behaves as identity w.r.t. composition, i.e. $\sigma \circ \gamma_A \cong \sigma$ and $\gamma_B \circ \sigma \cong \sigma$, iff σ is receptive and innocent. Copy-cat pre-strategies $\gamma_A : A \rightarrow\!\!\!\!+ A$ are receptive and innocent.*

E. The bicategory of concurrent games and strategies

Theorem 2 motivated the definition of a *strategy* as a pre-strategy which is receptive and innocent. In fact, we obtain a bicategory, $\mathbf{Games}$, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!+ B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

A strategy $\sigma : A \rightarrow\!\!\!\!+ B$ corresponds to a dual strategy $\sigma^\perp : B^\perp \rightarrow\!\!\!\!+ A^\perp$. This duality arises from the correspondence between pre-strategies $\sigma : S \to A^\perp \| B$ and $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$. The dual of copy-cat, $\gamma_A^\perp$, is isomorphic to the copy-cat of the dual, $\gamma_{A^\perp}$, for A an event structure with polarity. The dual of a composition of pre-strategies $(\tau \odot \sigma)^\perp$ is isomorphic to the composition $\sigma^\perp \odot \tau^\perp$.

F. The subcategory of deterministic strategies

Say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = -\ \&\ \exists s \in X.\ s' \le s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Lemma 3. *An event structure with polarity S is deterministic iff for all $s, s' \in S, x \in \mathcal{C}(S)$,*

$$x \overset{s}{\longrightarrow\!\!\subset}\ \&\ x \overset{s'}{\longrightarrow\!\!\subset}\ \&\ pol(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

A copy-cat strategy γ_A can fail to be deterministic. However, γ_A is deterministic iff immediate conflict in A respects polarity, or equivalently that there is no immediate conflict between +ve and −ve events, a condition we call 'race-free.'

Lemma 4. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff for all $x \in \mathcal{C}(A), a, a' \in A$,*

$$x \overset{a}{\longrightarrow\!\!\subset}\ \&\ x \overset{a'}{\longrightarrow\!\!\subset}\ \&\ pol(a) \ne pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\mathbf{Race-free})$$

Lemma 5. *The composition of deterministic strategies is deterministic.*

Lemma 6. *A deterministic strategy $\sigma : S \to A$ is injective on configurations (equivalently, σ is mono in the category of event structures with polarity).*

We obtain a sub-bicategory $\mathbf{DGames}$ of $\mathbf{Games}$ by restricting objects to race-free games and strategies to being deterministic. Via Lemma 6, deterministic strategies in a game correspond to certain subfamilies of configurations of the game. A characterisation of those subfamilies which correspond to deterministic strategies [11] shows them to coincide with the receptive ingenuous strategies of Mimram and Melliès [10]. Via the presentation of deterministic strategies as families $\mathbf{DGames}$ is equivalent to an order-enriched category.

Melliès programme of "asynchronous games" arose from his earlier work with Abramsky where deterministic concurrent strategies were represented essentially by partial closure operators on the domain of configurations of an event structure [1]. For us, a deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = +\ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Because $\mathcal{C}^\infty(S)$ forms a subfamily $\mathcal{C}^\infty(A)$, a deterministic strategy does indeed give rise to a partial closure operator on $\mathcal{C}^\infty(A)$. (Strictly speaking, instead of working with partial closure operators, Abramsky and Melliès worked with closure operators on domains $\mathcal{C}^\infty(A)^\top$, extended with a top element $\top$, with every configuration of A unreachable according to the strategy being sent to $\top$.)

V. Winning strategies

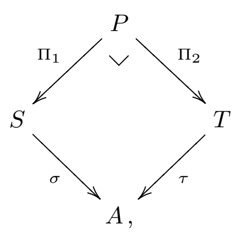
A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $L =_{\text{def}} \mathcal{C}^\infty(A) \smallsetminus W$. Clearly a game with winning conditions is fully defined once we specify either its winning or losing conditions.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is +-maximal if whenever $x \overset{s}{\longrightarrow\!\!\subset}$ then the event s has −ve polarity.

Clearly, we can equivalently say a strategy $\sigma : S \to A$ in G is winning if it always prescribes Player moves to avoid ending up in a losing configuration; a strategy $\sigma : S \to A$ in G is winning if $\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$. Any achievable position $z \in \mathcal{C}^\infty(S)$ of the game can be extended to a +-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy.

Note that for a game A, if winning conditions $W = \mathcal{C}^\infty(A)$, *i.e.* every configuration is winning, then any strategy in A is a winning strategy. Also note that in the special case of a deterministic strategy $\sigma : S \to A$ in G, it is winning iff $\sigma\varphi(x) \in W$ for all $x \in \mathcal{C}^\infty(S)$, where φ is the closure operator $\varphi : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$ determined by σ—see Section IV-F.

We can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose $\sigma : S \to A$ is a strategy in a game with winning conditions (A, W). A counter-strategy is strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!\!+} A$ and τ as a strategy $\tau : A \mathrel{\longrightarrow\!\!\!\!\!+} \varnothing$. Their composition $\tau \odot \sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!\!+} \varnothing$ is not in itself so informative. Rather it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,



to obtain the event structure P resulting from the interaction of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z in $\mathcal{C}^\infty(P)$ images to a configuration $\sigma\Pi_1 z = \tau\Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of the interaction of σ and τ to be

$$\langle\sigma, \tau\rangle =_{\mathrm{def}} \{\sigma\Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P)\}\,.$$

We shall show the strategy σ is winning for Player iff all the results of the interaction $\langle\sigma, \tau\rangle$ lie within W, for any counter-strategy $\tau : T \to A^\perp$ of Opponent.

It will be convenient to have facts about +-maximality in the broader context of the composition of arbitrary strategies.

Lemma 7. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Let P be the pullback of $\sigma \| \mathrm{id}_C$ and $\mathrm{id}_A \| \tau$—see Section IV-B. Then,*

$z \in \mathcal{C}^\infty(P)$ *is +-maximal iff*

$\Pi_1 z \in \mathcal{C}^\infty(S)$ *is +-maximal* & $\Pi_2 z \in \mathcal{C}^\infty(T)$ *is +-maximal.*

Proof sketch. A convention is being adopted. Refer to Section IV-B. A synchronization event in P is regarded as not having a polarity; otherwise, an event of P adopts the polarity of its image in $A^\perp$ or C. A configuration $z \in \mathcal{C}^\infty(P)$ is +-maximal if whenever $z \stackrel{p}{\longrightarrow\!\!\!\subset}$ then p has −ve polarity. If z is not +-maximal, $z \stackrel{p}{\longrightarrow\!\!\!\subset}$ where either p is +ve or a synchronisation. In either case, $\Pi_1 z \stackrel{\Pi_1(p)}{\longrightarrow\!\!\!\subset}$ or $\Pi_2 z \stackrel{\Pi_2(p)}{\longrightarrow\!\!\!\subset}$, ensuring $\Pi_1 z$ or $\Pi_2 z$ is not +-maximal. Conversely, if *e.g.* $\Pi_1 z$ is not +-maximal, $\Pi_1 z \stackrel{s}{\longrightarrow\!\!\!\subset}$ with s +ve. Either $\sigma_1(s) \in A^\perp$ when there is a +ve $p \in P$ with $\Pi_1(p) = s$, associated with the asynchronous occurrence of s, or $\sigma_2(s) \in B$ when by receptivity of τ there is a synchronization $p \in P$ with $\Pi_1(p) = s$ and $z \stackrel{p}{\longrightarrow\!\!\!\subset}$. □

Lemma 8. *Let $\sigma : S \to A$ be a strategy in a game (A, W). The strategy σ is winning for Player iff $\langle\sigma, \tau\rangle \subseteq W$ for all (deterministic) strategies $\tau : T \to A^\perp$.*

Proof. "Only if": Suppose σ is winning, *i.e.* $\sigma x \in W$ for all +-maximal $x \in \mathcal{C}^\infty(S)$. Let $\tau : T \to A^\perp$ be a strategy. As a special case of Lemma 7,

$x \in \mathcal{C}^\infty(P)$ is +-maximal

iff

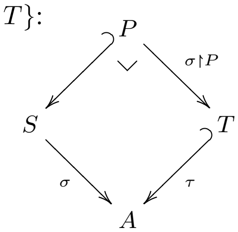
$\Pi_1 x \in \mathcal{C}^\infty(S)$ is +-maximal & $\Pi_2 x \in \mathcal{C}^\infty(T)$ is +-maximal.

Letting x be maximal in $\mathcal{C}^\infty(P)$ it is certainly +-maximal, whence $\Pi_1 x$ is +-maximal in $\mathcal{C}^\infty(S)$. It follows that $\sigma\Pi_1 x \in W$ as σ is winning. Hence $\langle\sigma, \tau\rangle \subseteq W$.

"If": Assume $\langle\sigma, \tau\rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$. Suppose x is +-maximal in $\mathcal{C}^\infty(S)$. Define T to be the event structure given as the restriction

$$T =_{\mathrm{def}} A^\perp \upharpoonright \sigma x \cup \{a \in A^\perp \mid pol_{A^\perp} = -\}\,.$$

The pre-strategy $\tau : T \to A^\perp$ defined to be the inclusion map $T \hookrightarrow A^\perp$ can be checked to be receptive and innocent, so a strategy. (In fact, τ is a *deterministic* strategy as all its +ve events lie within the configuration σx.) One way to describe a pullback of τ along σ is as the "inverse image" $P =_{\mathrm{def}} S \upharpoonright \{s \in S \mid \sigma(s) \in T\}$:



From the definition of T and P we see $x \in \mathcal{C}^\infty(P)$; and moreover that x is maximal in $\mathcal{C}^\infty(P)$ as x is +-maximal in $\mathcal{C}^\infty(S)$. Hence $\sigma x \in \langle\sigma, \tau\rangle$ ensuring $\sigma x \in W$, as required. The proof is unaffected if we restrict to *deterministic* counter-strategies $\tau : T \to A^\perp$. □

Corollary 9. *There are the following four equivalent ways to say that a strategy $\sigma : S \to A$ is winning in (A, W)—we write L for the losing configurations $\mathcal{C}^\infty(A) \smallsetminus W$:*

1) *$\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to reach a winning configuration, no matter what the activity or inactivity of Opponent;*
2) *$\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent;*

3) $\langle\sigma,\tau\rangle \subseteq W$ *for all strategies* $\tau : T \to A^\perp$, *i.e. all plays against counter-strategies of the Opponent result in a win for Player;*
4) $\langle\sigma,\tau\rangle \subseteq W$ *for all deterministic strategies* $\tau : T \to A^\perp$, *i.e. all plays against deterministic counter-strategies of the Opponent result in a win for Player.*

Not all games with winning conditions have winning strategies. Consider the game A consisting of one Player move $\oplus$ and one Opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as its winning conditions. This game has no winning strategy; any strategy $\sigma : S \to A$, being receptive, will have an event $s \in S$ with $\sigma(s) = \ominus$, and so the losing $\{s\}$ as a +-maximal configuration.

A. Operations

1) Dual: There is an obvious *dual* of a game with winning conditions $G = (A, W_G)$:

$$G^\perp =_{\rm def} (A^\perp, \mathcal{C}^\infty(A) \smallsetminus W_G)\,,$$

reversing the role of Player and Opponent, and consequently that of winning and losing conditions.

2) Parallel composition: The parallel composition of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is

$$G \parr H =_{\rm def} (A \| B,\ W_G \| \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \| W_H)$$

where $X \| Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \&\ y \in Y\}$ when X and Y are subsets of configurations. In other words, for $x \in \mathcal{C}^\infty(A \| B)$,

$$x \in W_{G \parr H} \quad \text{iff} \quad x_1 \in W_G \text{ or } x_2 \in W_H\,,$$

where $x_1 = \{a \mid (1,a) \in x\}$ and $x_2 = \{b \mid (2,b) \in x\}$. To win in $G \parr H$ is to win in either game. Its losing conditions are $L_A \| L_B$—to lose is to lose in both games G and H. The unit of $\parr$ is $(\emptyset, \emptyset)$.

3) Tensor: Defining $G \otimes H =_{\rm def} (G^\perp \parr H^\perp)^\perp$ we obtain a game where to win is to win in both games G and H—so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\rm def} (A \| B,\ W_A \| W_B)\,.$$

The unit of $\otimes$ is $(\emptyset, \{\emptyset\})$.

4) Function space: With $G \multimap H =_{\rm def} G^\perp \parr H$ a win in $G \multimap H$ is a win in H conditional on a win in G.

Proposition 10. *Let* $G = (A, W_G)$ *and* $H = (B, W_H)$ *be games with winning conditions. Write* $W_{G \multimap H}$ *for the winning conditions of* $G \multimap H$, *so* $G \multimap H = (A^\perp \| B, W_{G \multimap H})$. *For* $x \in \mathcal{C}^\infty(A^\perp \| B)$,

$$x \in W_{G \multimap H} \quad \textit{iff} \quad x_1 \in W_G \implies x_2 \in W_H\,.$$

B. The bicategory of winning strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from G, a game with winning conditions, to another H is a (winning) strategy in $G \multimap H = G^\perp \parr H$. We compose strategies as before. We first show that the composition of winning strategies is winning.

Lemma 11. *Let* σ *be a winning strategy in* $G \multimap H$ *and* τ *be a winning strategy in* $H \multimap K$. *Their composition* $\tau \odot \sigma$ *is a winning strategy in* $G \multimap K$.

Proof. Suppose $x \in \mathcal{C}^\infty(T \odot S)$ is +-maximal. The event structure $T \odot S$ is obtained as the projection of the pullback P to the set of 'visible' events V. Hence the down-closure $[x]$ in P forms a configuration $[x] \in \mathcal{C}^\infty(P)$. By Zorn's Lemma we can extend $[x]$ to a maximal configuration $z \supseteq [x]$ in $\mathcal{C}^\infty(P)$ with the property that all events of $z \smallsetminus [x]$ are synchronizations. Then, z will be +-maximal in $\mathcal{C}^\infty(P)$ with

$$\sigma_1 \Pi_1 z = \sigma_1 \Pi_1 [x] \quad \& \quad \tau_2 \Pi_2 z = \tau_2 \Pi_2 [x]\,. \tag{1}$$

By Lemma 7,

$$\Pi_1 z \text{ is +-maximal in } S \quad \& \quad \Pi_2 z \text{ is +-maximal in } T\,.$$

As σ and τ are winning,

$$\sigma \Pi_1 z \in W_{G \multimap H} \quad \& \quad \tau \Pi_2 z \in W_{H \multimap K}\,.$$

Now $\sigma \Pi_1 z \in W_{G \multimap H}$ expresses that

$$\sigma_1 \Pi_1 z \in W_G \implies \sigma_2 \Pi_1 z \in W_H \tag{2}$$

and $\tau \Pi_2 z \in W_{H \multimap K}$ that

$$\tau_1 \Pi_2 z \in W_H \implies \tau_2 \Pi_2 z \in W_K\,, \tag{3}$$

by Proposition 10. But $\sigma_2 \Pi_1 z = \tau_1 \Pi_2 z$, so (2) and (3) yield

$$\sigma_1 \Pi_1 z \in W_G \implies \tau_2 \Pi_2 z \in W_K\,.$$

By (1),

$$\sigma_1 \Pi_1 [x] \in W_G \implies \tau_2 \Pi_2 [x] \in W_K\,,$$

i.e. from the definition of $\tau \odot \sigma$,

$$(\tau \odot \sigma)_1\, x \in W_G \implies (\tau \odot \sigma)_2\, x \in W_K$$

in the span of the composition $\tau \odot \sigma$. Hence $x \in W_{G \multimap K}$, as required to show $\tau \odot \sigma$ is winning. □

For a general game with winning conditions (A, W) the copy-cat strategy need not be winning, as shown in the following example.

Example 12. Let A consist of two events, one +ve event $\oplus$ and one −ve event $\ominus$, inconsistent with each other. Take as winning conditions the set $\{\{\oplus\}\}$. The event structure $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{ccc} A^\perp & \ominus \rightarrow \oplus & A \\ & \oplus \leftarrow \ominus & \end{array}$$

To see $\mathbb{C}\mathbb{C}_A$ is not winning consider the configuration x consisting of the two −ve events in $\mathbb{C}\mathbb{C}_A$. Then x is +-maximal as any +ve event is inconsistent with x. However,

$x_1 \in W$ while $x_2 \notin W$, failing the winning condition of $(A, W) \multimap (A, W)$.

Each event structure with polarity A possesses a 'Scott order' on its configurations $\mathcal{C}^\infty(A)$:

$$x' \sqsubseteq x \;\text{ iff }\; x' \supseteq^- x \cap x' \subseteq^+ x\,.$$

Above we use the special inclusions

$$\begin{aligned} &x \subseteq^- y \text{ iff } x \subseteq y \;\&\; pol_A(y \setminus x) \subseteq \{-\}\,, \text{ and} \\ &x \subseteq^+ y \text{ iff } x \subseteq y \;\&\; pol_A(y \setminus x) \subseteq \{+\} \end{aligned}$$

for $x, y \in \mathcal{C}^\infty(A)$. The 'Scott-order' is indeed a partial order, in which there are two ways to increase in the order: adjoin more 'output' in the form of +ve events, or use less 'input' in the form of −ve events.

A necessary and sufficient condition for copy-cat to be winning w.r.t. a game (A, W):

$$\begin{aligned} &\text{if } x' \sqsubseteq x \;\&\; x' \text{ is } +\text{-maximal} \;\&\; x \text{ is } -\text{-maximal}, \\ &\text{then } \;\; x \in W \implies x' \in W, \text{ for all } x, x' \in \mathcal{C}^\infty(A)\,. \end{aligned} \qquad \textbf{(Cwins)}$$

Lemma 13. *Let (A, W) be a game with winning conditions. The copy-cat strategy $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ is winning iff (A, W) satisfies* **(Cwins)**.

Race-freedom, seen earlier in Lemma 4, is a robust condition sufficient to ensure that copy-cat is a winning strategy for all choices of winning conditions.

Proposition 14. *Let A be an event structure with polarity. Copy-cat is a winning strategy for all games (A, W) with winning conditions W iff A is race-free.*

We can now refine the bicategory of strategies $\mathbf{Games}$ to the bicategory $\mathbf{WGames}$ with objects games with winning conditions $G, H, \cdots$ satisfying $(\mathbf{Cwins})$ and arrows winning strategies $G \mathrel{+\!\!\!\twoheadrightarrow} H$; 2-cells, their vertical and horizontal composition is as before. Its restriction to deterministic strategies yields a bicategory $\mathbf{WDGames}$ equivalent to a simpler order-enriched category.

VI. On determined games

In this section, we define and make some observations on determinacy on concurrent games. In particular we show that games that are not *race-free* (see Lemma 4) are not necessarily determined, and that race-free games need not have a deterministic winning strategy.

A game with winning conditions G is said to be *determined* when either Player or Opponent has a winning strategy, *i.e.* either there is a winning strategy in G or in $G^\perp$. Not all games are determined. Neither the game G consisting of one Player move $\oplus$ and one Opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as winning conditions, nor the game $G^\perp$ have a winning strategy. Note that G is not *race-free* (see Lemma 4), so it is reasonable to assume race-freedom in a characterisation of determinacy. We are now going to prove a first direction of this equivalence: that whenever an event structure with polarity A is not race-free, there is a set W of winning configurations such that (A, W) is undetermined. This uses the following notion of reachability:

Notation 15. Let $\sigma : S \to A$ be a strategy. We say $y \in \mathcal{C}^\infty(A)$ is *σ-reachable* iff $y = \sigma x$ for some $x \in \mathcal{C}^\infty(S)$. Let $y' \subseteq y$ in $\mathcal{C}^\infty(A)$. Say y' is *$-$-maximal in y* iff $y \overset{-}{\longrightarrow\!\!\!\subset} y''$ implies $y'' \notin y$. Similarly, say y' is *$+$-maximal in y* iff $y \overset{+}{\longrightarrow\!\!\!\subset} y''$ implies $y'' \notin y$.

Lemma 16. *Let (A, W) be a game with winning conditions. Let $y \in \mathcal{C}(A)$. Suppose*

$$\begin{aligned} &\forall y' \in \mathcal{C}(A). \\ &y' \subseteq y \;\&\; y' \textit{ is } -\textit{-maximal in } y \;\&\; \textit{not } +\textit{-maximal in } y \\ &\;\implies \\ &\{y'' \in \mathcal{C}(A) \mid y' \subseteq^+ y'' \;\&\; (y'' \setminus y') \cap y = \varnothing\} \cap W = \varnothing\,. \end{aligned}$$

Then y is σ-reachable in all winning strategies σ.

Lemma 17. *If A, an event structure with polarity, is not race-free, then there are winning conditions W for which the game (A, W) is not determined.*

Proof sketch. If A is not race-free there is $y \in \mathcal{C}(A)$ such that $y \overset{a}{\longrightarrow\!\!\!\subset} y_1$ and $y \overset{a'}{\longrightarrow\!\!\!\subset} y_2$ and $pol(a) = -$ & $pol(a') = +$ and $y \cup \{a, a'\} \notin \mathcal{C}(A)$. Let W be defined by the following rules:

(i) for y'' with $y_1 \subseteq^+ y''$, assign $y'' \notin W$;
(ii) for y'' with $y_2 \subseteq^- y''$, assign $y'' \in W$;
(iii) for y'' with $y' \subseteq^+ y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' $-$-maximal and not $+$-maximal in y, assign $y'' \notin W$;
(iv) for y'' with $y' \subseteq^- y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' $+$-maximal and not $-$-maximal in y, assign $y'' \in W$;
(v) assign arbitrarily in all other cases.

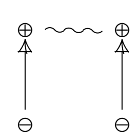
The assignment is well-defined and complete. Moreover, y is reachable for any winning strategy, either for Player or Opponent. W.r.t. any winning strategy for Player y_1 must be reachable, by receptivity, but by construction this entails there is +-maximal configuration of the strategy whose image in A is losing. Similarly, Opponent has no winning strategy. □

It is tempting to believe that a nondeterministic winning strategy always has a winning deterministic sub-strategy. However, this is not so, as the following example shows.

Example 18. A winning strategy need not have a winning deterministic sub-strategy. Consider the game (A, W) where A consists of two −ve events $1, 2$ and one +ve event 3 all consistent with each other and

$$W = \{\varnothing, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Let S be the event structure



and $\sigma : S \to A$ the only possible total map of event structures with polarity. Then σ is a winning strategy for A. However,

A has no *deterministic* winning strategy: as we have seen in Section IV-F, any deterministic strategy on A yields a (partial) closure operator φ on $\mathcal{C}^\infty(A)$. Moreover, this closure operator is necessarily *stable*, *i.e.* $\varphi(x_1 \cap x_2) = \varphi(x_1) \cap \varphi(x_2)$ for x_1, x_2 within its domain of definition. If φ comes from a winning strategy, we must have $\varphi(\{1\}) = \{1,3\}$ and $\varphi(\{2\}) = \{2,3\}$, and therefore $\varphi(\varnothing) = \{3\}$. But $\{3\}$ is a +-maximal losing configuration, so the deterministic strategy φ cannot come from any winning strategy.

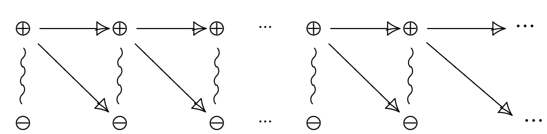
Therefore, σ is a winning strategy for which there is no deterministic sub-strategy. □

The above example shows that determinacy does not hold if we restrict to deterministic strategies. Note that some of the previous approaches to concurrent games [1], [10] were restricted to deterministic strategies, hence by the example above could not enjoy determinacy. In our setting, the ability to handle nondeterminism permits a determinacy result.

The following example shows that for non-well-founded games, race-freedom is not sufficient to ensure determinacy. It also shows that the existence of a winning receptive pre-strategy does not imply that there is a winning strategy.

Example 19. Consider the infinite game A comprising the event structure with polarity

where Player wins iff
(i) Player plays all ⊕ moves and Opponent does nothing, or
(ii) Player plays finitely many ⊕ moves and Opponent plays.
In this case there is a winning *pre-strategy* for Player. Informally, this is to continue playing moves until Opponent moves, then stop. Formally, it is described by the event structure with polarity S



with pre-strategy the unique total map to A. The pre-strategy is receptive and winning in the sense that its +-maximal configurations image to winning configurations in A. It follows that there is no winning strategy for Opponent. (If σ is a winning receptive pre-strategy then $\langle \sigma, \tau \rangle$ will be a subset of winning configurations, exactly as in the proof of Lemma 8, so must result in a loss for τ, which cannot be winning.) Nor is there a winning strategy for Player. Suppose $\sigma : S \to A$ were. For σ to win against the empty strategy there must be $x \in S$ such that σx comprises all +ve events of A. But now, using receptivity and −-innocence, there must be $s \in S$ such that $\sigma(s) = \ominus$ with $x \cup \{s\} \in \mathcal{C}^\infty(S)$ losing and +-maximal—a contradiction. □

VII. Determinacy for well-founded games

Definition 20. *A game A is* well-founded *if every configuration in $\mathcal{C}^\infty(A)$ is finite.*

It is shown that any well-founded, race-free concurrent game is determined.

Definition 21. *Let A be an event structure with polarity. Let $W \subseteq \mathcal{C}^\infty(A)$. Let $y \in \mathcal{C}^\infty(A)$. Define A/y to be the event structure with polarity comprising events*

$$\{a \in A \setminus y \mid y \cup [a]_A \in \mathcal{C}^\infty(A)\},$$

also called A/y, with consistency relation

$$X \in \mathrm{Con}_{A/y} \text{ iff } X \subseteq_{\mathrm{fin}} A/y \;\&\; y \cup [X]_A \in \mathcal{C}^\infty(A),$$

and causal dependency the restriction of that on A. Define $W/y \subseteq \mathcal{C}^\infty(A/y)$ by

$$z \in W/y \text{ iff } z \in \mathcal{C}^\infty(A/y) \;\&\; y \cup z \in W.$$

Finally, define $(A, W)/y =_{\mathrm{def}} (A/y, W/y)$.

Proposition 22. *Let A be an event structure with polarity and $y \in \mathcal{C}^\infty(A)$. Then,*

$$z \in \mathcal{C}^\infty(A/y) \text{ iff } z \subseteq A/y \;\&\; y \cup z \in \mathcal{C}^\infty(A).$$

Definition 23. *The* value $v(x)$ *of $x \in \mathcal{C}^\infty(A)$ is defined as + is Player has a winning strategy on A/x, − if Opponent has a winning strategy on A/x, 0 otherwise.*

Lemma 24. *Suppose A is race-free. If $x \in \mathcal{C}^\infty(A)$ such that $x \stackrel{a}{\longrightarrow\!\!\subset}$ with $pol(a) = +$ and $v(x \cup \{a\}) = +$, then $v(x) = +$.*

Proof sketch. Given a winning strategy $\sigma : S \to A/(x \cup \{a\})$, we build a new strategy $\mathrm{ext}_a\sigma : S' \to A/x$ by adding a new minimal Player event s in S, mapped to a by $\mathrm{ext}_a\sigma$. Here, the fact that A is race-free is used in a crucial way to prove that $\mathrm{ext}_a\sigma$ is receptive. It is winning because all +-maximal configurations of S' contain s, therefore they are in bijection with +-maximal configurations of S and map to winning configurations of A/x. □

The case of negative extensions requires to introduce the following lemma, proved by taking an adequate quotient.

Lemma 25. *If $\sigma : S \to A$ is innocent and weakly receptive,* i.e. *for all $x \in \mathcal{C}(S)$ such that $\sigma x \stackrel{a}{\longrightarrow\!\!\subset}$ with $pol(a) = -$ there is at least one $s \in S$ such that $x \stackrel{s}{\longrightarrow\!\!\subset}$ and $\sigma(s) = a$, then there is a strategy $\sigma' : S' \to A$ and a rigid map of event structures with polarity $h : S \to S'$ surjective on configurations such that $\sigma = \sigma' \circ h$.*

Lemma 26. *If $x \in \mathcal{C}(A)$ is such that $x \in W_A$ and that for all $e \in A$ such that $pol(e) = -$ and $x \stackrel{e}{\longrightarrow\!\!\subset}$ we have $v(x \cup \{e\}) = +$, then $v(x) = +$.*

Proof sketch. Take the family $(e_i)_{i \in I}$ of negative extensions of x, *i.e.* events in A such that $pol(e_i) = -$ and $x \stackrel{e_i}{\longrightarrow\!\!\subset}$. For each $i \in I$, $v(x \cup \{e_i\}) = +$ so there exists a winning strategy $\sigma_i : S_i \to A/(x \cup \{e_i\})$. The proof relies on the construction of a strategy $\mathrm{case}_{i \in I}\sigma_i : S' \to A/x$, which picks $i \in I$ nondeterministically and plays according to σ_i.

For each $i \in I$, we first define the event structure with polarity $\ominus_i \ltimes S_i$ as the maximum prefixing of a new event $\ominus_i$ with an event structure with polarity S_i allowed by innocence, *i.e.* $\ominus_i \leq s$ iff $pol(s) = +$ or $pol(s) = -$ and $e_i \leq_A \sigma_i(s)$. Then we define $S = \Sigma_{i \in I} \ominus_i \ltimes S_i$, with all events in the i-th copy conflicting with all events in the j-th copy if $i \neq j$. We define $\sigma : S \to A/x$ by:

$$\begin{aligned}\sigma(\ominus_i) &= e_i \\ \sigma((i,s)) &= \sigma_i(s)\end{aligned}$$

One can show that it is a winning innocent pre-strategy. It is not necessarily receptive since many negative minimal events in $\sum_{i\in I} \ominus_i \ltimes S_i$ may be mapped by σ to the same negative minimal event in A/x. However it is weakly receptive, hence by Lemma 26 we get a strategy $\sigma' : S' \to A$ and a map $h : S \to S'$ surjective on configurations. Then, $\text{case}_{i\in I}\sigma_i = \sigma'$ is winning. Indeed if $y' \in \mathcal{C}(S')$ is +-maximal there must be $y \in \mathcal{C}(S)$ such that $h(y) = y'$. Moreover, y is +-maximal as well since h preserves polarity, thus $\sigma'(y') = \sigma(y) \in W_{A/x}$ and $\text{case}_{i\in I}\sigma_i$ is a winning strategy, so $v(x) = +$. □

Theorem 27. *Let A be a well-founded game. Then A is race-free iff (A, W) is determined for all winning conditions W.*

Proof. We have already proved in Lemma 17 that if (A, W) is determined for all winning conditions W, then A is race-free. Suppose that A is race-free, and let $W \subseteq \mathcal{C}(A)$ be arbitrary winning conditions on A. Let $x \in \mathcal{C}(A)$ be maximal such that $v(x) = 0$. If there exists $a \in A$ such that $pol(a) = +$ and $x \xrightarrow{a}\subset$ and $v(x \cup \{a\}) = +$, then $v(x) = +$ by Lemma 24, a contradiction. By the same argument on $A^\perp/x$ if there is $a \in A$ such that $pol(a) = -$ and $x \xrightarrow{a}\subset$ and $v(x \cup \{a\}) = -$, then $v(x) = -$ by Lemma 24. If $x \in W_A$, then let $(e_i)_{i\in I}$ be the family of negative extensions of x. By the reasoning above, for all $i \in I$ we have $v(x \cup \{e_i\}) = +$, therefore $v(x) = +$ by Lemma 26, a contradiction. Similarly, if $x \notin W_A$, then $x \in W_{A^\perp}$ and an application of Lemma 26 on $A^\perp$ shows that $v(x) = -$, a contradiction. Therefore, there is no such maximal x. Since A is well-founded, this implies that all configurations of A have non-zero value, so A is determined. □

VIII. EXAMPLE

We now apply the tools developed in the previous sections of this paper to give an interpretation of first-order predicate logic. Although similar in spirit to the usual games interpretation of first-order logic, our construction differs technically by exploiting the extra space allowed by concurrency. In particular only quantifiers add new events, logical connectives are modelled in a completely concurrent way by variants of the parallel composition operation.

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\ \phi \mid \forall x.\ \phi$$

where R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots, x_k$ over variables.

A model M for the predicate calculus comprises a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. Following Tarski we can then define by structural induction the truth of a formula of predicate logic w.r.t. an assignment of values in V_M to the variables of the formula. We write $\rho \vDash_M \phi$ iff formula ϕ is true in M w.r.t. environment ρ; we take an environment to be a function from variables to values.

W.r.t. a model M and an environment ρ, we can denote a formula ϕ by $[\![\phi]\!]_M\rho$, a concurrent game with winning conditions, so that $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M\rho$ has a winning strategy.

The denotation as a game is defined by structural induction:

$$[\![R(x_1,\cdots,x_k)]\!]_M\rho = \begin{cases}(\varnothing, \{\varnothing\}) & \text{if } \rho \vDash_M R(x_1,\cdots,x_k)\,, \\ (\varnothing, \varnothing) & \text{otherwise.}\end{cases}$$
$$[\![\phi \wedge \psi]\!]_M\rho = [\![\phi]\!]_M\rho \otimes [\![\psi]\!]_M\rho$$
$$[\![\phi \vee \psi]\!]_M\rho = [\![\phi]\!]_M\rho \parr [\![\psi]\!]_M\rho$$
$$[\![\neg\phi]\!]_M\rho = ([\![\phi]\!]_M\rho)^\perp$$
$$[\![\exists x.\ \phi]\!]_M\rho = \bigoplus_{v\in V_M} [\![\phi]\!]_M\rho[v/x]$$
$$[\![\forall x.\ \phi]\!]_M\rho = \bigominus_{v\in V_M} [\![\phi]\!]_M\rho[v/x]\,.$$

We use $\rho[v/x]$ to mean the environment ρ updated to assign value v to variable x. The game $(\varnothing, \{\varnothing\})$, the unit w.r.t. $\otimes$, is the game used to denote true and the game $(\varnothing, \{\varnothing\})$, the unit w.r.t. $\parr$, to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of $\otimes$ and $\parr$ on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe.

The game $\bigoplus_{v\in V}(A_v, W_v)$ has underlying event structure with polarity the sum (=coproduct) $\sum_{v\in V} \oplus.A_v$ where the winning conditions of a component are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. In $\sum_{v\in V} \oplus.A_v$ a configuration is winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of $\bigoplus_{v\in V} G_v$ is not winning—Player must make a move in order to win. The game $\bigominus_{v\in V} G_v$ is defined dually, as $(\bigoplus_{v\in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move. Writing $G_v = (A_v, W_v)$, the underlying event structure of $\bigominus_{v\in V} G_v$ is the sum $\sum_{v\in V} \ominus.A_v$ with a configuration winning iff it is empty or the image under the injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that:

Proposition 28. *For any formula ϕ the game $[\![\phi]\!]_M\rho$ is well-founded and race-free, so a determined game by the result of the last section.*

The following facts are useful for building strategies.

Proposition 29.

(i) *If $\sigma : S \to A$ is a strategy in A and $\tau : T \to B$ is a strategy in B, then $\sigma\|\tau : S\|T \to A\|B$ is a strategy in $A\|B$.*

(ii) *If $\sigma : S \to T$ is a strategy in T and $\tau : T \to B$ is a strategy in B, then their composition as maps of event structures with polarity $\tau\sigma : S \to B$ is a strategy in B.*

There are 'projection' strategies from a tensor product of games to its components:

Proposition 30. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be race-free games with winning conditions. The map of event structures with polarity*

$$\mathrm{id}_{A^\perp} \| \gamma_B : A^\perp \| \mathrm{C\!\!C}_B \to A^\perp \| B^\perp \| B$$

is a winning strategy $p_H : G \otimes H \twoheadrightarrow H$. The map of event structures with polarity

$$\mathrm{id}_{B^\perp} \| \gamma_A : B^\perp \| \mathrm{C\!\!C}_A \to B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$$

is a winning strategy $p_G : G \otimes H \twoheadrightarrow G$.

The following lemma is used to build and deconstruct strategies in prefixed sums of games. The lemma concerns the more basic prefixed sums of event structures. These are built as coproducts $\sum_{i\in I} \bullet.B_i$ of event structures $\bullet.B_i$ in which an event $\bullet$ is prefixed to B_i, making all the events in B_i causally depend on $\bullet$.

Lemma 31. *Suppose $f : A \to \sum_{i\in I} \bullet.B_i$ is a total map of event structures, with codomain a prefixed sum. Then, A is isomorphic to a prefixed sum, $A \cong \sum_{j\in J} \bullet.A_j$, and there is a function $r : J \to I$ and total maps of event structures $f_j : A_j \to B_{r(j)}$ for which the following diagram commutes.*

$$\begin{array}{ccc} \sum_{j\in J} \bullet.A_j & \cong & A \\ {\scriptstyle [\bullet.f_j]_{j\in J}} \downarrow & \swarrow {\scriptstyle f} & \\ \sum_{i\in I} \bullet.B_i & & \end{array}$$

With the help of Propositions 29 and 30 and Lemma 31 we can build and deconstruct strategies to establish the next lemma, and the main theorem of this section. Theorem 33 follows by a straightforward structural induction using Lemma 32.

Lemma 32. *Let G, H, G_v, where $v \in V$, be race-free games with winning conditions. Then,*

(i) *$G \otimes H$ has a winning strategy iff G has a winning strategy and H has a winning strategy.*

(ii) *$\bigoplus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for some $v \in V$.*

(iii) *$\bigominus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for all $v \in V$.*

If in addition G and H are determined,

(iv) *$G \parr H$ has a winning strategy iff G has a winning strategy or H has a winning strategy.*

Theorem 33. *For all formulae ϕ and environments ρ, $\rho \models_M \phi$ iff the game $[\![\phi]\!]_M \rho$ has a winning strategy.*

IX. Conclusion and Further Work

For games one of the most fundamental mathematical questions is that of determinacy. This paper shows that to give a positive answer—even for well-founded (race-free) games—one has to consider nondeterministic (winning) strategies. In particular nondeterministic strategies are needed to faithfully represent parallel disjunctive behaviour, one of the reasons why our concurrent interpretation of predicate calculus is possible. Nondeterministic winning strategies are indeed computationally more powerful than deterministic ones.

In contrast, it may come as a surprise that if a strategy is not winning, then it can always be beaten by a deterministic counter-strategy. This fact is relevant from the point of view of verification since a deterministic strategy on a game corresponds to a subfamily of configurations of the game: whenever a game A is finite, the process of effectively checking whether a strategy $\sigma : S \to A$ is winning can be performed by inspecting the results of playing σ against all possible deterministic counter-strategies $\tau : T \to A^\perp$, and these are bounded within subfamilies of configurations of $A^\perp$. Hence a basic decidability theorem for finite games, which is needed to solve verification problems, follows from the results here.

There are several ways of extending the work on concurrent games with winning conditions: *stochastic behaviour*, perhaps with the use of probabilistic event structures [12], in order to be able to define profiles of mixed strategies and Nash equilibria; *imperfect information* as the key concept for reasoning, more faithfully, about real-life distributed systems; and determinacy results for games with *infinite behaviour* so that more complex winning conditions can be handled, *e.g.* Büchi or parity conditions. These extensions are within the focus of our current research on games.

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