Cambridge Pure Mathematics Part IV Course Proposal

Decision Methods over Real and Algebraically Closed Fields

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Overview An axiomatic theory T is called *decidable* if there exists an algorithm which, for any conjecture φ in the language of T, can determine in finitely many steps whether or not φ is a theorem of T. Many core axiomatic theories, including the elementary arithmetical theories of $\mathbb{Z}$ and $\mathbb{Q}$, are not decidable. Amazingly, the elementary theories of algebra and geometry over $\mathbb{R}$ and $\mathbb{C}$ *are* decidable. This ten lecture course will study decision methods for elementary algebra and geometry over $\mathbb{R}$ and $\mathbb{C}$. We will examine methods based on Gröbner bases ($\mathbb{C}$), Muchnik sign matrices ($\mathbb{C}$ and $\mathbb{R}$), cylindrical algebraic decomposition ($\mathbb{R}$) and Stengle's Weak Positivstellensatz ($\mathbb{R}$). The course will presume only a basic grounding in commutative algebra, first-order logic and general topology. The machinery developed should be of interest to logicians, algebraists and algebraic geometers.

Lecture Roadmap

1. Introduction to logical decision problems and quantifier elimination; Introduction to the elementary theories of Algebraically Closed Fields of characteristic zero (ACF_0) and Real Closed Fields (RCF)
2. Hilbert's Weak Nullstellensatz (HWN); Gröbner bases - Part I
3. Gröbner bases - Part II; $\exists$ fragment of ACF_0 decidable via Gröbner bases and HWN
4. ACF_0 admits quantifier elimination (via Muchnik) - Part I
5. ACF_0 admits quantifier elimination (via Muchnik) - Part II
6. RCF admits quantifier elimination (via Muchnik)
7. Cylindrical algebraic decomposition - Part I
8. Cylindrical algebraic decomposition - Part II
9. Cylindrical algebraic decomposition - Part III
10. $\exists$ fragment of RCF decidable via Stengle's Weak Positivstellensatz