Nominal Cubical
model of type theory

Part 2

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These slides are at www.cl.cam.ac.uk/~amp12/talks

Plan

- Motivation: the univalence axiom [HoTT]
- Overview of the Cohen-Coquand-Huber-Mörtberg presheaf model of univalent type theory [CCHM,OP]
- Toposes of $\mathbb{M}$-sets
- CCHM cubical sets as finitely supported $\mathbb{M}$-sets [Pit]
- Path objects
- Cofibrant propositions and fibrant families
- A univalent universe [CCHM]

De Morgan sets

Recall:

The category **Dms** of **De Morgan sets** is the full subcategory of $\mathbf{Set}^{\mathbb{M}}$ consisting of those $\mathbb{M}$-sets Γ such that every $x \in \Gamma$ possesses a finite support.

$\mathbb{M}$ = finitary endomorphisms of the free De Morgan algebra $\mathbb{I}$ on countably many generators $\mathfrak{I} \subseteq \mathbb{I}$

Every $d \in \mathbb{I}$ can be put in disjunctive normal form as a finite join of finite meets of finite subsets of $\mathfrak{I} \cup \{1 - i \mid i \in \mathfrak{I}\}$.

Elements of $\mathbb{M}$ are finite substitutions $(d_1/i_1) \circ \cdots \circ (d_n/i_n)$ for some distinct $i_1, \ldots, i_n \in \mathfrak{I}$ and some $d_1, \ldots, d_n \in \mathbb{I}$.

An $\mathbb{M}$-set Γ is in **Dms** if for each $x \in \Gamma$, there is some $I \subseteq_{\mathrm{fin}} \mathfrak{I}$ with $i \notin I \;\Rightarrow\; (d/i) \cdot x = x$ (any $d \in \mathbb{I}$).

The interval $\mathbb{I}$ in $\mathbf{Dms}$

$\mathbb{I} \equiv$ countably infinitely generated free De Morgan algebra (generators $\mathfrak{I} \subseteq \mathbb{I}$).
$\mathbb{M}$ acts on $\mathbb{I}$ via function application.

$\mathbb{I} \in \mathbf{Dms}$, because each $d \in \mathbb{I}$ is supported by the finite set of directions occurring in its normal form.

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Since each endomorphism $m \in \mathbb{M}$ preserves the De Morgan algebra structure of $\mathbb{I}$ we get morphisms in $\mathbf{Dms}$

$$\mathbf{0}, \mathbf{1} : 1 \to \mathbb{I} \qquad \vee, \wedge : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \qquad \mathbf{1} - (\_) : \mathbb{I} \to \mathbb{I}$$

making $\mathbb{I}$ an internal De Morgan algebra in the topos $\mathbf{Dms}$.

$\mathbf{0}, \mathbf{1}$ give source and target of $\mathbb{I}$-paths

$\mathbf{1} - (\_)$ gives $\mathbb{I}$-path reversal

$\vee, \wedge$ give a “connection” structure, e.g. used to prove that singleton types w.r.t. $\mathbb{I}$-paths are contractible

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making $\mathbb{I}$ an internal De Morgan algebra in the topos $\mathbf{Dms}$.

In the internal logic of the topos $\mathbf{Dms}$, $\mathbb{I}$ does not look much like the classical interval $[\mathbf{0}, \mathbf{1}]$, e.g. it is not totally ordered,

but it is (logically) connected ($\mathbf{2}^{\mathbb{I}} \cong \mathbf{2}$).

Paths

Given $\Gamma \in \mathbf{Dms}$, the object of $\mathbb{I}$-paths in Γ is just the exponential $\Gamma^{\mathbb{I}}$.

General exponentials Γ^{Δ} of (finitely supported) M-sets have a somewhat complicated description (compared with G-sets).

But when $\Delta = \mathbb{I}$, there is a simple characterisation of $\Gamma^{\mathbb{I}}$ in terms of the nominal sets notions of

name abstraction

and

freshness

...

Freshness

Given $\Gamma \in \mathbf{Dms}$

we say direction $i \in \mathfrak{I}$ is **fresh** for $x \in \Gamma$
and write $i \mathbin{\#} x$
if $(0/i) \cdot x = x$

in which case $(d/i) \cdot x = x$ for any $d \in \mathbb{I}$.

Path objects in $\mathbf{Dms}$

Given $\Gamma \in \mathbf{Dms}$, equivalence relation $\sim$ on $\mathbb{I} \times \Gamma$:
$(i, x) \sim (i', x')$ holds iff

$$(\exists j \in \mathbb{I})\ j \mathbin{\#} (i, x, i', x') \ \wedge \ (j/i) \cdot x = (j/i') \cdot x'$$

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Path object $\mathcal{P}\,\Gamma \in \mathbf{Dms}$ is $(\mathbb{J} \times \Gamma)/\!\sim$
$\sim$-equiv class of (i, x) written $\boxed{\langle i \to x \rangle}$
$\mathbb{M}$-action: $m \cdot \langle i \to x \rangle \equiv \langle j \to m \cdot (j/i) \cdot x \rangle$
for some/any $j \# (m, i, x)$

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Theorem. $\mathcal{P}\,\Gamma \cong \Gamma^{\mathbb{I}}$

Application $@ : \mathcal{P}\,\Gamma \times \mathbb{I} \to \Gamma$ satisfies $\langle i \to x \rangle \,@\, d = (d/i) \cdot x$.

Currying of $\gamma \in \mathbf{Dms}(\Delta \times \mathbb{I}, \Gamma)$ is $\mathbf{cur}\,\gamma \in \mathbf{Dms}(\Delta, \mathcal{P}\,\Gamma)$ where

$$\mathbf{cur}\,\gamma\, y \equiv \langle i \to \gamma(y, i) \rangle \quad \text{for some/any } i \text{ with } i \,\#\, y$$

Operations on paths

Source/target: $\boxed{\partial_0, \partial_1 \in \mathbf{Dms}(\mathcal{P}\,\Gamma, \Gamma)}$

$$\partial_0\langle i \to x\rangle = (0/i)\cdot x \qquad \partial_1\langle i \to x\rangle = (1/i)\cdot x$$

Degenerate paths: $\boxed{\iota \in \mathbf{Dms}(\Gamma, \mathcal{P}\,\Gamma)}$

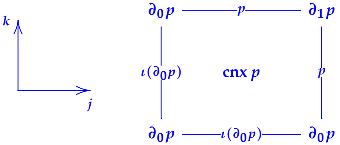
$\iota\, x \equiv \langle i \to x\rangle$ for some/any $i \,\#\, x$

Reversal: $\boxed{\mathbf{rev} : \mathbf{Dms}(\mathcal{P}\,\Gamma, \mathcal{P}\,\Gamma)}$

$$\mathbf{rev}\langle i \to x\rangle = \langle i \to ((1 - i)/i)\cdot x\rangle$$

Connection: $\boxed{\mathbf{cnx} : \mathbf{Dms}(\mathcal{P}\,\Gamma, \mathcal{P}(\mathcal{P}\,\Gamma))}$

$\mathbf{cnx}\langle i \to x\rangle = \langle j \to \langle k \to ((j \wedge k)/i)\cdot x\rangle\rangle$ (some/any $j, k \,\#\, (i, x)$)



CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Recall:

Objects $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ are sets equipped with an $\mathbb{M}$-action

Morphisms $\gamma \in \mathbf{Set}^{\mathbb{M}}(\Delta, \Gamma)$ are functions preserving the $\mathbb{M}$-action

Families $A \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$ are families of sets $(A\,x \in \mathbf{Set} \mid x \in \Gamma)$ equipped with a dependently-typed $\mathbb{M}$-action

Elements $\alpha \in \mathbf{Set}^{\mathbb{M}}(\Gamma \vdash A)$ are dependent functions $\alpha \in \prod_{x \in \Gamma} A\,x$ preserving the $\mathbb{M}$-action

$$m \cdot (\alpha\,x) = \alpha(m \cdot x)$$

$\mathbf{Dms}$ as a CwF

Same as the CwF for $\mathbf{Set}^{\mathbb{M}}$ except that for a De Morgan set $\Gamma \in \mathbf{Dms}$ the families in $\mathbf{Dms}(\Gamma)$ are all the families $A = (A\,x \mid x \in \Gamma) \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$ with a (dependent) finite support property:

> for every $x \in \Gamma$ and $a \in A\,x$ there is a finite subset $I \subseteq_{\mathbf{fin}} \mathfrak{I}$ that supports x in Γ and such that for all $i \in \mathfrak{I}$, if $i \notin I$ then
>
> $$(\mathbf{0}/i) \cdot a = a \in A\,x = A((\mathbf{0}/i) \cdot x)$$

Dependently-typed paths

For each family $A \in \mathbf{Dms}(\Gamma)$, dependently-typed choice gives:

$$\left(\textstyle\sum_{x \in \Gamma} A\,x\right)^{\mathbb{I}} \;\cong\; \textstyle\sum_{f \in \Gamma^{\mathbb{I}}} \prod_{d \in \mathbb{I}} A(f\,d)$$

$$\left(\textstyle\sum_{x \in \Gamma} A\,x\right)^{\mathbb{I}} \xrightarrow{\mathbf{fst}^{\mathbb{I}}} \Gamma^{\mathbb{I}} \xleftarrow{\mathbf{fst}} \textstyle\sum_{f \in \Gamma^{\mathbb{I}}} \prod_{d \in \mathbb{I}} A(f\,d)$$

Dependently-typed paths

For each family $A \in \mathbf{Dms}(\Gamma)$, dependently-typed choice gives:

$$\mathcal{P}(\Gamma.A) \;\cong\; \textstyle\sum_{p \in \mathcal{P}\,\Gamma} \prod_{d \in \mathbb{I}} A(p \mathbin{@} d)$$

$$\mathcal{P}(\Gamma.A) \xrightarrow{\mathcal{P}\,\mathbf{fst}} \mathcal{P}\,\Gamma \xleftarrow{\mathbf{fst}} \textstyle\sum_{p \in \mathcal{P}\,\Gamma} \prod_{d \in \mathbb{I}} A(p \mathbin{@} d)$$

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$$\mathcal{P}(\Gamma.A) \xrightarrow{\mathcal{P}\,\mathbf{fst}} \mathcal{P}\,\Gamma \xleftarrow{\mathbf{fst}} \mathcal{P}\,\Gamma.\mathcal{P}\,A$$

$\mathcal{P}\,A \in \mathbf{Dms}(\mathcal{P}\,\Gamma)$ is the family of dependently-typed paths over paths in Γ.

For each $p \in \mathcal{P}\,\Gamma$, $(\mathcal{P}\,A)(p)$ consists of $\sim$-equiv. classes $\langle i \to a \rangle$ where $i \in \mathbb{I}$ and $a \in A(p \mathbin{@} i)$

$$(i, a) \sim (i', a') \equiv (\exists j \mathbin{\#} p, i, a, i', a')\ (j/i) \cdot a = (j/i') \cdot a' \in A(p \mathbin{@} j)$$

Dependently-typed paths

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$\mathcal{P}\,A \in \mathbf{Dms}(\mathcal{P}\,\Gamma)$ is the family of dependently-typed paths over paths in Γ.

From this we get families $\mathbf{path}_A \in \mathbf{Dms}(\Gamma.A.(A \circ \mathbf{fst}))$

$$\mathbf{path}_A((x, a_0), a_1) \equiv \{p \in (\mathcal{P}\,A)(\iota\, x) \mid \partial_0 p = a_0 \wedge \partial_1 p = a_1\}$$

Do these give identification types in the CwF $\mathbf{Dms}$?

Dependently-typed paths

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Do these give identification types in the CwF $\mathbf{Dms}$?

Not quite. . .

Coquand's axioms for propositional identification types

$A : \mathcal{U}, a, a' : A \vdash a = a' : \mathcal{U}$

$$A : \mathcal{U}, a : A \vdash \mathbf{refl} : a = a$$

$$A : \mathcal{U}, a, a' : A, p : a = a' \vdash \mathbf{contr} : (a, \mathbf{refl}) = (a', p)$$

$$A : \mathcal{U}, B : A \to \mathcal{U}, a, a' : A \vdash \mathbf{subst} : (a = a') \to B\,a \to B\,a'$$

$$A : \mathcal{U}, B : A \to \mathcal{U}, a : A, B; B\,a \vdash \mathbf{scomp} : \mathbf{subst}\,\mathbf{refl}\,b = b$$

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Lumsdaine [unpublished]: given **subst** without **scomp**, can always find a new **subst′** with a **scomp′**

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$$\sout{A : \mathcal{U}, B : A \to \mathcal{U}, a : A, B; B\,a \vdash \mathbf{scomp} : \mathbf{subst}\,\mathbf{refl}\,b = b}$$

Can use $\iota \in \mathbf{Dms}(\Gamma, \mathcal{P}\,\Gamma)$ and $\mathbf{cnx} : \mathbf{Dms}(\mathcal{P}\,\Gamma, \mathcal{P}(\mathcal{P}\,\Gamma))$ to get $\mathbf{refl}$ and $\mathbf{contr}$ for $\mathbb{I}$-paths in $\mathbf{Dms}$.

To also get $\mathbf{subst}$, we restrict attention to families equipped with a suitable Kan-style fibration structure...

(Maybe there are other ways to get $\mathbf{subst}$?)

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Topos structure of $\mathbf{Set}^{\mathbb{M}}$

Recall:

Subobjects of $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ correspond to subsets of $U\,\Gamma \in \mathbf{Set}$ that are closed under the $\mathbb{M}$-action.

Subobject classifier:

$$\Omega \equiv \{\varphi \subseteq \mathbb{M} \mid (\forall m, m')\; m \in \varphi \Rightarrow m' \circ m \in \varphi\}$$
$$m \cdot \varphi \equiv \{m' \in \mathbb{M} \mid m' \circ m \in \varphi\}$$

Recall that in any topos,
each object Γ has a partial map classifier,
viz. a mono $\eta : \Gamma \rightarrowtail \tilde{\Gamma}$ with the universal property that

Recall that in any topos,
each object $\mathbf{\Gamma}$ has a partial map classifier,
viz. a mono $\boldsymbol{\eta} : \mathbf{\Gamma} \rightarrowtail \tilde{\mathbf{\Gamma}}$ with the universal property that

for any partial morphism

$$\begin{array}{ccc} \cdot & \xrightarrow{\gamma} & \mathbf{\Gamma} \\ \downarrow & & \\ \mathbf{\Delta} & & \end{array}$$

there is a unique morphism $\boldsymbol{\delta} : \mathbf{\Delta} \to \tilde{\mathbf{\Gamma}}$ making the following square a pullback:

$$\begin{array}{ccc} \cdot & \xrightarrow{\gamma} & \mathbf{\Gamma} \\ \downarrow & \lrcorner & \downarrow \eta \\ \mathbf{\Delta} & \xrightarrow[\delta]{} & \tilde{\mathbf{\Gamma}} \end{array}$$

Topos structure of $\mathbf{Set}^{\mathbb{M}}$

Partial map classifier for $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ is

$$\tilde{\Gamma} \equiv \{(\varphi, f) \in \textstyle\sum_{\varphi \in \Omega} \Gamma^{\varphi} \mid (\forall m, m')\; m \in \varphi \Rightarrow m' \cdot (f\,m) = f(m' \circ m)\}$$

with $\mathbb{M}$-action

$$m \cdot (\varphi, f) \equiv (m \cdot \varphi\,,\, \lambda m' \to f(m' \circ m))$$

and $\eta : \Gamma \rightarrowtail \tilde{\Gamma}$ given by

$$\eta\, x \equiv (\mathbb{M}, \lambda m \to m \cdot x)$$

Topos structure of $\mathbf{Dms}$

Limits are created by the forgetful functor $U : \mathbf{Dms} \hookrightarrow \mathbf{Set}^{\mathbb{M}^{\mathrm{op}}} \to \mathbf{Set}$.

Exponential of $\Gamma, \Delta \in \mathbf{Dms}$ is coreflection $(\Delta^{\Gamma})_{\mathbf{fs}}$ of exponential in $\mathbf{Set}^{\mathbb{M}^{\mathrm{op}}}$.

Subobjects of $\Gamma \in \mathbf{Dms}$ correspond to subsets of $U\,\Gamma \in \mathbf{Set}$ that are closed under the $\mathbb{M}$-action.

Subobject classifier is $(\Omega)_{\mathbf{fs}}$.

Partial map classifier of $\Gamma \in \mathbf{Dms}$ is $(\tilde{\Gamma})_{\mathbf{fs}}$.

Cofibrant propositions

In $\mathbf{Dms}$, the subobject $\mathbb{F} \rightarrowtail \mathbf{\Omega_{fs}}$ of **cofibrant propositions** consists of those $\varphi \in \mathbf{\Omega_{fs}}$ satisfying

$$(\forall m \in \mathbb{M})\ m \in \varphi \vee m \notin \varphi$$
$$(\forall s \in \mathbb{M}_s)(\forall m \in \mathbb{M})\ s \circ m \in \varphi \Rightarrow m \in \varphi$$

where $\mathbb{M}_s \subseteq \mathbb{M}$ consists of those $s \in \mathbb{M}$ satisfying $(\forall i \in \mathbf{dom}(s))\ s(i) \notin \{0,1\}$ ("strict" substitution)

E.g. the top element $\mathbb{M} \in \mathbf{\Omega_{fs}}$ is in $\mathbb{F}$.

We say that a subobject $\mathbf{\Delta} \rightarrowtail \mathbf{\Gamma}$ in $\mathbf{Dms}$ is cofibrant if its classifier $\mathbf{\Gamma} \to \mathbf{\Omega_{fs}}$ factors through $\mathbb{F} \rightarrowtail \mathbf{\Omega_{fs}}$.

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where $\mathbb{M}_s \subseteq \mathbb{M}$ consists of those $s \in \mathbb{M}$ satisfying $(\forall i \in \mathbf{dom}(s))\; s(i) \notin \{0,1\}$ ("strict" substitution)

Every $m \in \mathbb{M}$ factors as $m = s \circ f$ where s is strict and f is a face substitution: $\mathbb{M}_f \equiv \{f \in \mathbb{M} \mid (\forall i \in \mathbf{dom}(f))\; f(i) \in \{0,1\}\}$

Lemma. $\varphi \in \mathbb{F}$ iff there are finitely many face substitutions $f_1, \ldots, f_n$ such that

$$m \in \varphi \;\Leftrightarrow\; \bigvee_{k=1,\ldots,n} \bigwedge_{i \in \mathbf{dom}(f_k)} m(i) = f_k(i)$$

Cofibrant partial elements

Given $\Gamma \in \mathbf{Dms}$ and $A \in \mathbf{Dms}(\Gamma)$, we can consider the partial map classifier for $\mathbf{fst} : \Gamma.A \to \Gamma$ in $\mathbf{Dms}/\Gamma$ restricted to partial maps whose domains of definition are cofibrant subobjects.

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$$\Box\, A \in \mathbf{Dms}(\mathbf{\Gamma})$$

$$\Box\, A\, x \equiv \{(\varphi, \alpha) \in \textstyle\sum_{\varphi \in \mathbb{F}} \prod_{m \in \varphi} A(m \cdot x) \mid \\ (\forall m, m')\; m \in \varphi \Rightarrow m' \cdot (\alpha\, m) = \alpha(m' \circ m)\}_{\mathbf{fs}}$$

$$m \cdot (\varphi, \alpha) \equiv (m \cdot \varphi, \lambda m' \to \alpha(m' \circ m))$$

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$$m \cdot (\varphi, \alpha) \equiv (m \cdot \varphi, \lambda m' \to \alpha(m' \circ m))$$

Every $a \in A\, x$ gives $(\mathbb{M}, \lambda m \to m \cdot a) \in \Box A\, x$.

We say a cofibrant partial element $(\varphi, \alpha) \in \Box A\, x$
extends to a total element $a \in A\, x$
and write $\boxed{(\varphi, \alpha) \nearrow a}$
if $(\forall m \in \varphi)\; \alpha\, m = m \cdot a \in A(m \cdot x)$

CCHM fibrations

Given $\Gamma \in \mathbf{Dms}$

a **fibration structure** for a family $A \in \mathbf{Dms}(\Gamma)$ is a function $\mathbf{comp}_A$ mapping every

- $p \in \mathcal{P}\,\Gamma$ (path in Γ)
- $(\varphi, \pi) \in \square(\mathcal{P}\,A)\,p$ (cofibrant partial path over p)
- $a_0 \in A(\partial_0 p)$ extending $(\varphi, \partial_0 \pi)$

to $\mathbf{comp}_A(p, \varphi, \pi, a_0) \in A(\partial_1 p)$ extending $(\varphi, \partial_1 \pi)$.
Furthermore, $\mathbf{comp}_A$ must respect the $\mathbb{M}$-action.

(A remarkably simple definition – honestly! In particular it implies a Kan-style path-lifting property.)

CCHM fibrations

There is a family $\boxed{\mathbf{Fib}\,A \in \mathbf{Dms}(\mathcal{P}\,\Gamma)}$ whose elements are fibration structures for $A \in \mathbf{Dms}(\Gamma)$.

Theorem. [CCHM] There are (re-indexing stable) functions

$$\mathbf{Fib}(A) \to \mathbf{Fib}(\mathbf{path}_A)$$
$$\mathbf{Fib}(A) \to \mathbf{Fib}(B) \to \mathbf{Fib}(\Sigma\, A\, B)$$
$$\mathbf{Fib}(A) \to \mathbf{Fib}(B) \to \mathbf{Fib}(\Pi\, A\, B)$$
$$\mathbf{Fib}(A) \to \mathbf{Fib}(B) \to \mathbf{Fib}(W\, A\, B)$$

etc.

Proof. [OP] Clearer to work in the internal language of $\mathbf{Dms}$, since doing shows that one just needs some simple properties of $\mathbb{I}$ and $\mathbb{F}$

$\mathbf{0} \neq \mathbf{1}$, $\mathbf{2}^{\mathbb{I}} \cong \mathbf{2}$, "connection algebra" ($\wedge$, $\vee$),

$(\forall i \in \mathbb{I})\ (i = \mathbf{0}) \in \mathbb{F} \wedge (i = \mathbf{1}) \in \mathbb{F}$, $(\forall p, q \in \mathbb{F})\ p \vee q \in \mathbb{F}$.

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etc.

Get a new CwF $\mathcal{F}$ over $\mathbf{Dms}$ with
$\mathcal{F}(\Gamma) \equiv \sum_{A \in \mathbf{Dms}(\Gamma)} \mathbf{Dms}(\mathcal{P}\,\Gamma \vdash \mathbf{Fib}\,A)$ and
$\mathcal{F}(\Gamma \vdash (A, \mathbf{comp}_A)) \equiv \mathbf{Dms}(\Gamma \vdash A)$ with identification types,
$\mathbf{\Sigma}$-, $\mathbf{\Pi}$-, W-types,... But what about universes?

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Small sets

Let $\mathbf{set} \in \mathbf{Set}$ be a fixed Grothendieck universe.

$$\mathbb{N} \in \mathbf{set}$$
$$x \in y \in \mathbf{set} \;\Rightarrow\; x \in \mathbf{set}$$
$$x, y \in \mathbf{set} \;\Rightarrow\; \{x, y\} \in \mathbf{set}$$
$$x \in \mathbf{set} \;\Rightarrow\; \{y \mid y \subseteq x\} \in \mathbf{set}$$
$$x \in \mathbf{set} \;\wedge\; f \in \mathbf{set}^x \;\Rightarrow\; \bigcup_{y \in x} f\, y \in \mathbf{set}$$

(More generally, can assume there is a countable sequence $\mathbf{set}_0 \in \mathbf{set}_1 \in \mathbf{set}_2 \in \cdots \in \mathbf{Set}$ of Grothendieck universes.)

Small sets

Let $\mathbf{set} \in \mathbf{Set}$ be a fixed Grothendieck universe.

Say that $x \in \mathbf{Set}$ is small if $x \in \mathbf{set}$
(and large otherwise).

We assume that the set $\mathfrak{J}$ is small;
and hence so are $\mathbb{I}$ and $\mathbb{M}$.

Hofmann-Streicher universe in $\mathbf{Dms}$

$\mathcal{S} \in \mathbf{Dms}$ consists of functions $A \in \mathbf{set}^{\mathbb{M}}$ that come equipped with a dependently-typed $\mathbb{M}$-action

$$m, m' \in \mathbb{M}, a \in A\,m \mapsto m' \cdot a \in A(m' \circ m)$$

$$m'' \cdot (m' \cdot a) = (m'' \circ m') \cdot a \in A(m'' \circ m' \circ m)$$
$$\mathbf{id} \cdot a = a \in A\,m$$

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Action of $m \in \mathbb{M}$ on $A \in \mathcal{S}$ is

$$m \cdot A \equiv \lambda m' \to A(m' \circ m)$$

Furthermore, we require $\mathcal{S}$ to be finitely supported w.r.t. this action and that each $a \in A\,m$ is finitely supported, i.e. there is some $I \subseteq_{\mathrm{fin}} \mathcal{I}$ supporting A, containing $\mathbf{dir}(m)$ and satisfying that for all $i \notin I$

$$a = (0/i) \cdot a \in A((0/i) \circ m)$$

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$$= A(m \circ (\mathbf{0}/i)) = ((\mathbf{0}/i) \cdot A)\, m = A\, m$$

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There is a family $\boxed{\mathcal{E}\ell \in \mathbf{Dms}(\mathcal{S})}$ mapping each $A \in \mathcal{S}$ to

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which weakly classifies small families in $\mathbf{Dms}$:

Theorem. For all $\Gamma \in \mathbf{Dms}$, if a family $A \in \mathbf{Dms}(\Gamma)$ satisfies $(\forall x \in \Gamma)\; A\,x \in \mathbf{set}$, then there is a morphism $\ulcorner A \urcorner \in \mathbf{Dms}(\Gamma, \mathcal{S})$ with $A = \mathcal{E}\ell \circ \ulcorner A \urcorner$.

Composition structure

There is an operation $\boxed{\mathbf{Comp} : \mathbf{Dms}(\Gamma) \to \mathbf{Dms}(\Gamma)}$ which has the property that for each family $A \in \mathbf{Dms}(\Gamma)$ there is a bijection

$$\mathbf{Dms}(\mathcal{P}\,\Gamma \vdash \mathbf{Fib}\,A) \cong \mathbf{Dms}(\Gamma \vdash \mathbf{Comp}\,A)$$

naturally in Γ and A.

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Sattler [unpublished]: $\mathbf{Comp}$ can be constructed from $\mathbf{Fib}$ just using the fact that $\mathbb{I}$ is an atomic object...

“Copaths”

The interval $\mathbb{I} \in \mathbf{Dms}$ is **atomic** in Lawvere's sense, i.e. $(\_)^{\mathbb{I}}$ has a right adjoint $(\_)^{1/\mathbb{I}} : \mathbf{Dms} \to \mathbf{Dms}$

$$\frac{\Gamma^{\mathbb{I}} \to \Delta}{\Gamma \to \Delta^{1/\mathbb{I}}}$$

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Explicit description of $\Gamma^{1/\mathbb{I}}$:

underlying set consists of those functions $f : \mathbb{M} \to \mathfrak{I} \to \Gamma$ satisfying

$$i \# m' \;\Rightarrow\; m' \cdot (f\, m\, i) = f\,(m' \circ m)\, i$$

for which there is some $I \subseteq_{\mathrm{fin}} \mathfrak{I}$ supporting f w.r.t. the action

$$(m \cdot f)\, m'\, i \equiv f\,(m' \circ m)\, i$$

and satisfying

$$i' \# I, m \;\Rightarrow\; f\, m\, i = f\,((i'/i) \circ m)\, i'$$

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- transpose of $\gamma \in \mathbf{Dms}(\mathcal{P}\,\Delta, \Gamma)$ is $\overline{\gamma} \in \mathbf{Dms}(\Delta, \Gamma^{1/\mathbb{I}})$ where

$$\overline{\gamma}\, y\, m\, i \equiv \gamma\langle i \to m \cdot y\rangle \qquad (y \in \Delta, m \in \mathbb{M}, i \in \mathfrak{I})$$

- counit of the adjunction $\mathcal{P} \dashv (\_)^{1/\mathbb{I}}$ at Γ, is $\varepsilon_\Gamma \in \mathbf{Dms}(\mathcal{P}(\Gamma^{1/\mathbb{I}}), \Gamma)$ where

$$\varepsilon_\Gamma\langle i \to f\rangle \equiv f\,\mathbf{id}\,i \qquad (i \in \mathfrak{I}, f \in \Gamma^{1/\mathbb{I}})$$

(and the unit is $\eta_\Gamma \in \mathbf{Dms}(\Gamma, (\mathcal{P}\,\Gamma)^{1/\mathbb{I}})$ where

$$\eta_\Gamma\, x\, m\, i \equiv \langle i \to m \cdot x\rangle \qquad (x \in \Gamma, m \in \mathbb{M}, i \in \mathfrak{I}))$$

There is a dependently-typed version of $(\_)^{1/\mathbb{I}}$: given any family $A \in \mathbf{Dms}(\Gamma)$, there is $\boxed{A^{1/\mathbb{I}} \in \mathbf{Dms}(\Gamma^{1/\mathbb{I}})}$ and an isomorphism

$$\begin{array}{ccc} (\Gamma.A)^{1/\mathbb{I}} & \cong & \Gamma^{1/\mathbb{I}}.A^{1/\mathbb{I}} \\ \mathbf{fst}^{1/\mathbb{I}} \searrow & & \swarrow \mathbf{fst} \\ & \Gamma^{1/\mathbb{I}} & \end{array}$$

(For each $f \in \Gamma^{1/\mathbb{I}}$, the set $A^{1/\mathbb{I}}\, f$ consists of dependent functions $g \in \prod_{m \in \mathbb{M}} \prod_{i \in \mathbb{J}} A(f\, m\, i)$ satisfying... [definition omitted].)

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$\mathbf{Comp}\, A \in \mathbf{Dms}(\Gamma)$ is defined to be the re-indexing

$$(\mathbf{Fib}\, A)^{1/\mathbb{I}} \circ \eta_\Gamma$$

of $(\mathbf{Fib}\, A)^{1/\mathbb{I}} \in \mathbf{Dms}((\mathcal{P}\,\Gamma)^{1/\mathbb{I}})$ along the counit $\eta_\Gamma \in \mathbf{Dms}(\Gamma, (\mathcal{P}\,\Gamma)^{1/\mathbb{I}})$ of the adjunction $\mathcal{P} \dashv (\_)^{1/\mathbb{I}}$.

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So we have a pullback square in $\mathbf{Dms}$:

$$\begin{array}{ccccc}
\Gamma.\,\mathbf{Comp}\, A & \longrightarrow & (\mathcal{P}\,\Gamma)^{1/\mathbb{I}}.(\mathbf{Fib}\, A)^{1/\mathbb{I}} & \cong & (\mathcal{P}\,\Gamma.\mathbf{Fib}\, A)^{1/\mathbb{I}} \\
\downarrow \mathbf{fst} & \lrcorner & \downarrow \mathbf{fst} & & \downarrow \mathbf{fst}^{1/\mathbb{I}} \\
\Gamma & \xrightarrow[\eta_\Gamma]{} & (\mathcal{P}\,\Gamma)^{1/\mathbb{I}} & = & (\mathcal{P}\,\Gamma)^{1/\mathbb{I}}
\end{array}$$

Hence, sections of $\begin{array}{c}\Gamma.\,\mathbf{Comp}\, A \\ \downarrow \mathbf{fst} \\ \Gamma\end{array}$ correspond to sections of $\begin{array}{c}\mathcal{P}\,\Gamma.\mathbf{Fib}\, A \\ \downarrow \mathbf{fst} \\ \mathcal{P}\,\Gamma\end{array}$

There is a dependently-typed version of $(\_)^{1/\mathbb{I}}$: given any family $A \in \mathbf{Dms}(\Gamma)$, there is $A^{1/\mathbb{I}} \in \mathbf{Dms}(\Gamma^{1/\mathbb{I}})$

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\end{array}$$

Hence, elements of $\mathbf{Dms}(\Gamma \vdash \mathbf{Comp}\, A)$ correspond to elements of $\mathbf{Dms}(\mathcal{P}\,\Gamma \vdash \mathbf{Fib}\, A)$, i.e. fibration structures for A.

CwF of fibrations

Now we can (re)define $\mathfrak{F}$ to be the CwF over $\mathbf{Dms}$ with

– $\mathfrak{F}(\Gamma) \equiv \sum_{A \in \mathbf{Dms}(\Gamma)} \mathbf{Dms}(\Gamma \vdash \mathbf{Comp}\, A)$

– $\mathfrak{F}(\Gamma \vdash (A, \alpha)) \equiv \mathbf{Dms}(\Gamma \vdash A)$

Theorem. [CCHM] $\mathfrak{F}$ is a model of UTT.

CwF of fibrations

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Theorem. [CCHM] $\mathcal{F}$ is a model of UTT.

The univalent universe in $\mathcal{F}$ has underlying De Morgan set $\mathcal{U} \equiv \mathcal{S}.\,\mathbf{Comp}\,\mathcal{E}\ell$. There's a family in $\mathcal{F}(\mathcal{U})$ that weakly classifies small families in $\mathcal{F}$ and this is univalent (and $\mathcal{U}$ is itself a fibration over $\mathbf{1}$).

(Proof, via "glueing", uses closure of $\mathbb{F}$ under $\mathbb{I}$-indexed $\forall$, and a construction that allows one to strictify some isomorphisms into equalities in the ambient set theory.)

In conclusion

- I spent 4 hrs and still didn't manage to give you a convincingly detailed proof that the CCHM model is univalent :-(
- A proof entirely in a language of type theory would be better – to do that it seems one needs a modality to express global nature of the universe construction.
- Can the nominal/$\mathbb{M}$-sets approach usefully be applied to (a constructive version of) the simplicial model of UTT?
- Do non-truncated models of UTT have to be this complicated? (and can we avoid Kan-filling in some way?)