Unfinity Categories

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Motivation

Models of homotopy type theory (HoTT)
from
higher-dimensional category theory (HDCT)

Hofmann-Streicher$\rightarrow$Awodey-Warren$\rightarrow \cdots$

Universes are what make dependent type theory tick and for HoTT we'd like (more, and simpler, examples of) *univalent* universes

Motivation

Models of homotopy type theory (HoTT)
from
higher-dimensional category theory (HDCT)

Hofmann-Streicher→Awodey-Warren→ $\cdots$

Universes are what make dependent type theory tick
and for HoTT we'd like (more, and simpler, examples of)
univalent universes
but their study in HDCT seems under-developed

when constructing universes,
size considerations are less of an issue
than "microcosmic" ones

Higher-dimensional category theory

"2-category theory sucks"

A famous logician
(who uses category theory)

So where does that leave ∞-category theory?

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So where does that leave ∞-category theory?

Observation from [AMP, Proc. TYPES 2014]:
one can use nominal techniques to
make dimensionality implicit

Does it help?
The jury is still out

Road map

$$
\begin{tikzcd}
 & \begin{array}{c}\upsilon \\ \text{(unfinity)}\end{array} \arrow[dr, bend left] & \\
\upsilon\text{-fold groupoids} \arrow[dr, bend left] & & \text{nominal sets} \arrow[dd] \\
 & \text{(universes?)} & \\
\upsilon\text{-fold categories} \arrow[uu] & & \text{cubical sets} \arrow[ll, bend left] \arrow[dr, bend right] & \\
 & & & \text{(Kan filling)}
\end{tikzcd}
$$

Unfinity, υ

(André Joyal was probably joking when he coined the word, but I quite like it.)

Unfinity, υ

A [constructive] set υ is unfinite if it has

decidable non-equality (#)

$$\forall \mathtt{x}, \mathtt{y} \in \upsilon.\ \mathtt{x} \,\#\, \mathtt{y} \lor \mathtt{x} = \mathtt{y}$$
$$\forall \mathtt{x}, \mathtt{y} \in \upsilon.\ \neg(\mathtt{x} \,\#\, \mathtt{y} \land \mathtt{x} = \mathtt{y})$$

and it is finitely inexhaustible

$$\forall\, \overline{\mathtt{x}} \in \mathrm{Fin}\, \upsilon, \exists\, \mathtt{x} \in \upsilon.\ \mathtt{x} \notin \overline{\mathtt{x}}$$

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and it is finitely inexhaustible

$$\forall\, \overline{\mathtt{x}} \in \mathrm{Fin}\,\upsilon, \exists\, \mathtt{x} \in \upsilon.\ \mathtt{x} \notin \overline{\mathtt{x}}$$

least collection of subsets of υ containing $\emptyset$ and closed under taking union with singletons

$$\frac{}{\mathtt{x} \notin \emptyset} \qquad \frac{\mathtt{x} \notin \overline{\mathtt{y}} \qquad \mathtt{x} \mathbin{\#} \mathtt{y}}{\mathtt{x} \notin \overline{\mathtt{y}} \cup \{\mathtt{y}\}}$$

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A [constructive] set υ is unfinite if it has

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$$\forall x, y \in \upsilon.\ x \mathbin{\#} y \lor x = y$$
$$\forall x, y \in \upsilon.\ \neg(x \mathbin{\#} y \land x = y)$$

and it is finitely inexhaustible

$$\forall \overline{x} \in \mathrm{Fin}\,\upsilon,\ \exists\, x \in \upsilon.\ x \notin \overline{x}$$

"*mere*" existence

Unfinity, υ

A [constructive] set υ is unfinite if it has

decidable non-equality ($\#$)

$$\forall \mathtt{x}, \mathtt{y} \in \upsilon.\ \mathtt{x} \# \mathtt{y} \vee \mathtt{x} = \mathtt{y}$$
$$\forall \mathtt{x}, \mathtt{y} \in \upsilon.\ \neg(\mathtt{x} \# \mathtt{y} \wedge \mathtt{x} = \mathtt{y})$$

and it is finitely inexhaustible

$$\forall\, \bar{\mathtt{x}} \in \mathrm{Fin}\, \upsilon, \exists\, \mathtt{x} \in \upsilon.\ \mathtt{x} \notin \bar{\mathtt{x}}$$

From now on υ is a fixed, unfinite set (e.g. $\mathbb{N}$)

whose elements are called *dimensions*

and are written $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$, ...

Nominal sets, 101

A nominal set is a set $\mathcal{C}$ (elements called *cells*, written a, b, c, …)

with a function for swapping dimensions in cells

$$\begin{array}{rcl} \mathcal{C}\times\upsilon\times\upsilon & \to & \mathcal{C}\\ (a,\mathtt{x},\mathtt{y}) & \mapsto & a\left[\begin{smallmatrix}\mathtt{x}\\ \mathtt{y}\end{smallmatrix}\right] \end{array}$$

satisfying

$$\forall a,\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{w}.\ a\left[\begin{smallmatrix}\mathtt{x}\\ \mathtt{x}\end{smallmatrix}\right] = a = a\left[\begin{smallmatrix}\mathtt{x} & \mathtt{x}\\ \mathtt{y} & \mathtt{y}\end{smallmatrix}\right] \quad \wedge \cdots$$

$$\boxed{\left(a\left[\begin{smallmatrix}\mathtt{x}\\ \mathtt{y}\end{smallmatrix}\right]\right)\left[\begin{smallmatrix}\mathtt{x}\\ \mathtt{y}\end{smallmatrix}\right]}$$

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$$\begin{array}{ccc} \mathcal{C} \times \mho \times \mho & \to & \mathcal{C} \\ (a, \mathtt{x}, \mathtt{y}) & \mapsto & a\left[\begin{smallmatrix}\mathtt{x}\\\mathtt{y}\end{smallmatrix}\right] \end{array}$$

satisfying

$$\forall a, \mathtt{x}, \mathtt{y}, \mathtt{z}, \mathtt{w}.\ a\left[\begin{smallmatrix}\mathtt{x}\\\mathtt{x}\end{smallmatrix}\right] = a = a\left[\begin{smallmatrix}\mathtt{x} & \mathtt{x}\\\mathtt{y} & \mathtt{y}\end{smallmatrix}\right] \wedge a\left[\begin{smallmatrix}\mathtt{x} & \mathtt{z}\\\mathtt{y} & \mathtt{w}\end{smallmatrix}\right] = a\left[\begin{matrix}\mathtt{z} & \mathtt{x}\left[\begin{smallmatrix}\mathtt{z}\\\mathtt{w}\end{smallmatrix}\right]\\ \mathtt{w} & \mathtt{y}\left[\begin{smallmatrix}\mathtt{z}\\\mathtt{w}\end{smallmatrix}\right]\end{matrix}\right] \quad \wedge \cdots$$

$\boxed{\text{if } \mathtt{x} = \mathtt{z} \text{ then } \mathtt{w} \text{ else if } \mathtt{x} = \mathtt{w} \text{ then } \mathtt{z} \text{ else } \mathtt{x}}$

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satisfying

$$\forall a, \mathtt{x}, \mathtt{y}, \mathtt{z}, \mathtt{w}.\ a\left[\begin{smallmatrix}\mathtt{x}\\\mathtt{x}\end{smallmatrix}\right] = a = a\left[\begin{smallmatrix}\mathtt{x} & \mathtt{x}\\\mathtt{y} & \mathtt{y}\end{smallmatrix}\right] \ \wedge\ a\left[\begin{smallmatrix}\mathtt{x} & \mathtt{z}\\\mathtt{y} & \mathtt{w}\end{smallmatrix}\right] = a\begin{bmatrix}\mathtt{z} & \mathtt{x}\left[\begin{smallmatrix}\mathtt{z}\\\mathtt{w}\end{smallmatrix}\right]\\ \mathtt{w} & \mathtt{y}\left[\begin{smallmatrix}\mathtt{z}\\\mathtt{w}\end{smallmatrix}\right]\end{bmatrix} \quad \wedge$$

$\forall a, \exists\, \overline{\mathtt{x}}, \forall \mathtt{x}, \mathtt{y} \notin \overline{\mathtt{x}}.\ a\left[\begin{smallmatrix}\mathtt{x}\\\mathtt{y}\end{smallmatrix}\right] = a$ (finite support property)

Nominal sets, 101

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$$\begin{array}{ccc} \mathcal{C} \times \upsilon \times \upsilon & \rightarrow & \mathcal{C} \\ (a, \mathrm{x}, \mathrm{y}) & \mapsto & a[^{\mathrm{x}}_{\mathrm{y}}] \end{array}$$

satisfying…

A morphism of nominal sets is a function $F : \mathcal{C} \rightarrow \mathcal{D}$
which is equivariant: $F(a[^{\mathrm{x}}_{\mathrm{y}}]) = (F\, a)[^{\mathrm{x}}_{\mathrm{y}}]$

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$$\begin{array}{rcl} \mathcal{C} \times \upsilon \times \upsilon & \to & \mathcal{C} \\ (a, \mathrm{x}, \mathrm{y}) & \mapsto & a\left[{\mathrm{x} \atop \mathrm{y}}\right] \end{array}$$

satisfying…

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The category $\boxed{\mathcal{N}om}$ of nominal sets & equivariant functions is so nice there are whole books about it:

AMP, *Nominal Sets* (CUP 2013)
Mikołaj Bojańczyk, *Slightly Infinite Sets* (2019, on-line)

(but not written from a constructive point of view and not oriented towards HDCT)

Name abstraction

Given $\mathcal{C} \in \mathcal{N}om$, the nominal set $\boxed{\wp\, \mathcal{C}}$ of paths in $\mathcal{C}$ consists of equivalence classes $\langle \mathtt{x} \to a \rangle$ of pairs $(\mathtt{x}, a) \in \mathbb{v} \times \mathcal{C}$ for the equivalence relating $(\mathtt{x}, a)$ and $(\mathtt{y}, b)$ when

$$\exists\, \mathtt{z} \mathbin{\#} \mathtt{x}, \mathtt{y}, a, b.\ a[{}^{\mathtt{z}}_{\mathtt{x}}] = b[{}^{\mathtt{z}}_{\mathtt{y}}]$$

Freshness relation: $\boxed{\mathtt{x} \mathbin{\#} a}$ if $\mathtt{x} \notin \overline{\mathtt{x}}$, for some support $\overline{\mathtt{x}}$ of a
where $\overline{\mathtt{x}} \in \mathrm{Fin}\, \mathbb{v}$ supports $a \in \mathcal{C}$ if $\forall \mathtt{x}, \mathtt{y} \notin \overline{\mathtt{x}},\ a[{}^{\mathtt{x}}_{\mathtt{y}}] = a$
(read "$\mathtt{x} \mathbin{\#} a$" as "cell a is degenerate in dimension $\mathtt{x}$")

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$$\exists\, \mathtt{z} \mathbin{\#} \mathtt{x}, \mathtt{y}, a, b.\ a[{}^{\mathtt{z}}_{\mathtt{x}}] = b[{}^{\mathtt{z}}_{\mathtt{y}}]$$

Swapping action in $\wp\,\mathcal{C}$ is well-defined by:
$\langle \mathtt{x} \to a \rangle [{}^{\mathtt{y}}_{\mathtt{z}}] \triangleq \langle \mathtt{x}[{}^{\mathtt{y}}_{\mathtt{z}}] \to a[{}^{\mathtt{y}}_{\mathtt{z}}] \rangle$

Hence the freshness relation for $\wp\,\mathcal{C}$ is:
$\mathtt{y} \mathbin{\#} \langle \mathtt{x} \to a \rangle \;\Leftrightarrow\; \mathtt{y} = \mathtt{x} \vee \mathtt{y} \mathbin{\#} a$

Paths can be "concreted" at a fresh dimension:
$\langle \mathtt{x} \to a \rangle \mathbin{@} \mathtt{y} \triangleq a[{}^{\mathtt{x}}_{\mathtt{y}}]$ well-defined when $\mathtt{y} \mathbin{\#} \langle \mathtt{x} \to a \rangle$

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Hence the freshness relation for $\wp\,\mathcal{C}$ is:
$\mathtt{y} \mathrel{\#} \langle \mathtt{x} \to a \rangle \;\Leftrightarrow\; \mathtt{y} = \mathtt{x} \lor \mathtt{y} \mathrel{\#} a$

Paths can be "concreted" at a fresh dimension:
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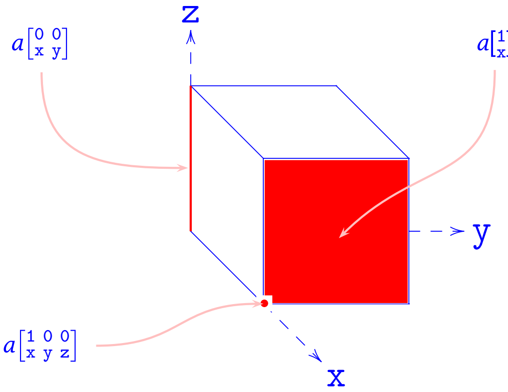
But as yet, paths don't start or end anywhere.

Cubical sets

We take cubical sets to be nominal sets $\mathcal{C}$ equipped with some extra structure, namely face maps $a \in \mathcal{C} \mapsto a[{}^{i}_{\mathtt{x}}] \in \mathcal{C}$, for $i = 0, 1$ and $\mathtt{x} \in \mathbb{v}$



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$$\begin{array}{rcl} \mathrm{d}^i : \wp\,\mathcal{C} & \to & \mathcal{C} \\ \langle \mathrm{x} \to a \rangle & \mapsto & a[^i_{\mathrm{x}}] \end{array} \quad (i \in \{0,1\})$$

morphisms in $\mathcal{N}om$ satisfying

(equivariance $a[^i_{\mathrm{x}}\ ^{\mathrm{y}}_{\mathrm{z}}] = a\left[^{\mathrm{y}}_{\mathrm{z}}\ ^{\quad i}_{\mathrm{x}[^{\mathrm{y}}_{\mathrm{z}}]}\right]$ – implied)

(binding $\mathrm{x} \# a[^i_{\mathrm{x}}]$ – implied)

degeneracy $\mathrm{x} \# a \Rightarrow a[^i_{\mathrm{x}}] = a$

independence $\mathrm{x} \# \mathrm{y} \Rightarrow a\left[^i_{\mathrm{x}}\ ^j_{\mathrm{y}}\right] = a\left[^j_{\mathrm{y}}\ ^i_{\mathrm{x}}\right]$

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cf. nominal restriction sets

Cubical sets

$\boxed{Cub}$ = category of nominal sets equipped with face maps + equivariant functions preserving face maps

Theorem. Cub is equivalent to the presheaf category $[\mathbb{C}, Set]$, where $\mathbb{C}$ is the category whose objects are finite ordinals and whose morphisms are given by:

$$\begin{aligned}\mathbb{C}(m,n) = \{f \in Set(m+2, n+2) \mid & f\,0 = 0 \;\wedge\; f\,1 = 1 \wedge \\ & \forall i,j > 1.\, f\,i = f\,j > 1 \;\Rightarrow\; i = j\}\end{aligned}$$

Proof: see Theorem 2.13 of [AMP, Proc. TYPES 2014] (after a theorem of Sam Staton about nominal restriction sets)

(There are similar theorems for other varieties of cubical sets and also for simplicial sets [Eric Faber *Homogeneous models and their toposes of supported sets*, PhD 2020].)

Type Theory in *Cub*

Bezem, Coquand & Huber, Proc. TYPES 2013

show for families in $[\mathbb{C}, \mathcal{S}et]$ satisfying a

uniform Kan filling condition

that $\wp$ does model (propositional) identity types

and

there is a universe classifying such families
which is univalent.

Constructively valid, but the details are complicated.

Type Theory in *Cub*

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uniform Kan filling condition

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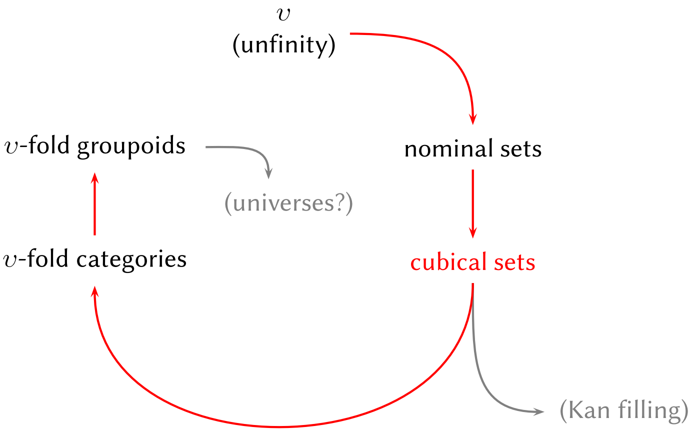
and

there is a universe classifying such families which is univalent.

Constructively valid, but the details are complicated.

Is it possible to replace Kan filling by (weak) higher-dimensional category/groupoid structure and still get univalent universes?

Road map



υ
(unfinity)
υ-fold groupoids
nominal sets
(universes?)
υ-fold categories
cubical sets
(Kan filling)

Dimension-labelled path composition

Given $\mathcal{C} \in \mathcal{C}ub$,

a *path composition* should be a morphism in $\mathcal{C}ub$

$$\_ \circ \_ : \{(p, q) \in \wp\,\mathcal{C} \times \wp\,\mathcal{C} \mid \mathrm{d}^0 p = \mathrm{d}^1 q\} \to \wp\,\mathcal{C} \qquad \text{(compose paths)}$$

satisfying… [*to be determined*].

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satisfying… [*to be determined*]. From this we get, for each $\mathtt{x} \in \upsilon$

$$\_ \circ_{\mathtt{x}} \_ : \{(a, b) \in \mathcal{C} \times \mathcal{C} \mid a[{}^{0}_{\mathtt{x}}] = b[{}^{1}_{\mathtt{x}}]\} \to \mathcal{C} \quad \text{(compose cells in dimension x)}$$

by defining $a \circ_{\mathtt{x}} b = (\langle \mathtt{x} \to a \rangle \circ \langle \mathtt{x} \to b \rangle)\ @\ \mathtt{x}$

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by defining $a \circ_{\mathtt{x}} b = (\langle \mathtt{x} \to a \rangle \circ \langle \mathtt{x} \to b \rangle) \mathbin{@} \mathtt{x}$

Conversely, given $\_ \circ_{\mathtt{x}} \_$ satisfying… [*to be determined*], we can recover

$$p \circ q = \mathrm{fresh}\,\mathtt{x}\ \mathrm{in}\ \langle \mathtt{x} \to (p \mathbin{@} \mathtt{x}) \circ_{\mathtt{x}} (q \mathbin{@} \mathtt{x}) \rangle$$

$\boxed{\langle \mathtt{x} \to (p \mathbin{@} \mathtt{x}) \circ_{\mathtt{x}} (q \mathbin{@} \mathtt{x}) \rangle \text{ for some/any } \mathtt{x} \mathbin{\#} (p, q)}$

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satisfying... [*to be determined*]. From this we get, for each $\mathtt{x} \in \upsilon$

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Recall the uni-sorted definition of “category” in which objects are identified with their identity morphisms:

Sets and functions $C \underset{t}{\overset{s}{\rightrightarrows}} C \xleftarrow{\circ} \{(a, b) \mid s(a) = t(b)\}$ with

$$s(s(a)) = s(a) = t(s(a))$$
$$s(t(a)) = t(a) = t(t(a))$$
$$s(a \circ b) = s(b)$$
$$t(a \circ b) = t(a)$$
$$a \circ s(a) = a = t(a) \circ a$$
$$a \circ (b \circ c) = (a \circ b) \circ c$$

Strict υ-fold categories

are given by

$\mathcal{C} \in \mathcal{C}ub$ with for each $\mathrm{x} \in \upsilon$ an operation

$$\{(a, b) \in \mathcal{C} \times \mathcal{C} \mid a[{}^{0}_{\mathrm{x}}] = b[{}^{1}_{\mathrm{x}}]\} \to \mathcal{C}$$
$$(a, b) \mapsto a \circ_{\mathrm{x}} b$$

which is equivariant, respects face maps and satisfies

$$(a \circ_{\mathrm{x}} b)[{}^{0}_{\mathrm{x}}] = b[{}^{0}_{\mathrm{x}}] \quad (a \circ_{\mathrm{x}} b)[{}^{1}_{\mathrm{x}}] = a[{}^{1}_{\mathrm{x}}]$$
$$a \circ_{\mathrm{x}} a[{}^{0}_{\mathrm{x}}] = a = a[{}^{1}_{\mathrm{x}}] \circ_{\mathrm{x}} a$$
$$(a \circ_{x} b) \circ_{x} c = a \circ_{x} (b \circ_{\mathrm{x}} c)$$
$$\boxed{\mathrm{x} \# \mathrm{y} \Rightarrow (a \circ_{\mathrm{x}} b) \circ_{\mathrm{y}} (c \circ_{\mathrm{x}} d) = (a \circ_{\mathrm{y}} c) \circ_{\mathrm{x}} (b \circ_{\mathrm{y}} d)}$$

(interchange law)

Strict υ-fold categories

are given by

$\mathcal{C} \in \mathit{Cub}$ with for each $\mathtt{x} \in \upsilon$ an operation

$$\{(a, b) \in \mathcal{C} \times \mathcal{C} \mid a[{}^{0}_{\mathtt{x}}] = b[{}^{1}_{\mathtt{x}}]\} \to \mathcal{C}$$
$$(a, b) \mapsto a \circ_{\mathtt{x}} b$$

which is equivariant, respects face maps and satisfies...
A functor of strict υ-fold categories is a morphism in Cub that preserves composition.

Presumably the category of strict υ-fold categories is equivalent to the category of strict, edge-symmetric ω-fold categories, but I have not checked this

(and maybe one gets strict ω-categories by adding connection structure to υ-fold categories [Brown, Higgins, Mosa])

Strict υ-fold categories

Is there a universe of υ-fold categories?

υ-fold category of paths

If $\mathcal{C}$ is an υ-fold category,

then $\wp\,\mathcal{C}$ inherits not only cubical structure from $\mathcal{C}$

$$P[{}^{i}_{\mathrm{x}}] \triangleq \text{fresh}\ \mathrm{y}\ \text{in}\ \langle \mathrm{y} \to (P\ @\ \mathrm{y})[{}^{i}_{\mathrm{x}}] \rangle$$

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but also path composition structure

$$P \circ_{\mathrm{x}} Q \triangleq \text{fresh}\,\mathrm{y}\ \text{in}\ \langle \mathrm{y} \to (P \mathbin{\texttt{@}} \mathrm{y}) \circ_{\mathrm{x}} (Q \mathbin{\texttt{@}} \mathrm{y}) \rangle$$

making $\wp\,\mathcal{C}$ an υ-fold category.

Universes

Every strict υ-fold category $\mathcal{C}$ yields a category object in $\mathcal{Cub}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{rfl}" description] & \wp\,\mathcal{C} \arrow[l, bend right, "\mathrm{d}^1"'] \arrow[l, bend left, "\mathrm{d}^0"] & \wp\,\mathcal{C} \times_{\mathcal{C}} \wp\,\mathcal{C} \arrow[l, "\circ"']
\end{tikzcd}$$

degenerate path

$\mathrm{rfl}\, a \triangleq \mathrm{fresh}\; \mathrm{x}\; \mathrm{in}\; \langle \mathrm{x} \to a \rangle$

Universes

Every strict υ-fold category $\mathcal{C}$ yields a category object in $\mathcal{C}ub$

$$\begin{tikzcd} \mathcal{C} & \wp\,\mathcal{C} \arrow[l, shift right=2, "d^1"'] \arrow[l, shift left=2, "d^0"] & \wp\,\mathcal{C} \times_{\mathcal{C}} \wp\,\mathcal{C} \arrow[l, "\circ"'] \arrow[from=1-1, to=1-2, "\mathrm{rfl}" description] \end{tikzcd}$$

and hence, taking dimension-zero cells and paths, an underlying (small) category $\boxed{|\mathcal{C}|}$.

Is there a strict υ-fold category $\mathcal{U}$ in $\mathcal{CUB}$ (large cubical sets) with $|\mathcal{U}|$ equivalent to the category of all small υ-fold categories and functors?

Note that, being equivalent to a presheaf topos, $\mathcal{C}ub$ has Hofmann-Streicher style universes, but that only deals with the size problem, not the "microcosm" problem.

Models of Type Theory?

To model type theory (rather than directed type theory)

need to consider υ-fold *groupoids*

Models of Type Theory?

To model type theory (rather than directed type theory)

need to consider υ-fold *groupoids*

Asking for *strict inverses* w.r.t path composition is probably too restrictive if we want to get univalent universes (?)

Recall:

Michael Warren, *The Strict ω-Groupoid Interpretation of Type Theory*, CRM Lecture Notes 53(2011)

(doesn't treat universes)

Models of Type Theory?

To model type theory (rather than directed type theory) need to consider υ-fold *groupoids*

In view of

> Norihiro Yamada, *Game Semantics of Homotopy Type Theory*
> (HOTTEST seminar, 11 Feb 2021)

perhaps one should consider υ-fold categories in which all cells are equivalences (*bi-invertible*, say).

Is there a universe of such υ-fold groupoids?

Road map

$$
\begin{tikzcd}
& \begin{array}{c}\upsilon \\ \text{(unfinity)}\end{array} \arrow[dr, bend left] & & \\
\upsilon\text{-fold groupoids} \arrow[d, bend left] & & \text{nominal sets} \arrow[dd] & \\
& \text{(universes?)} & & \\
\upsilon\text{-fold categories} \arrow[uu] & & \text{cubical sets} \arrow[ll, bend left] \arrow[dr, bend right] & \\
& & & \text{(Kan filling)}
\end{tikzcd}
$$

The end of the road for now

(these slides are here: www.cl.cam.ac.uk/users/amp12/talks/)