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Tripos theory

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Introduction. One of the most important constructions in topos theory is that of the category Shv (A) of sheaves on a locale (= complete Heyting algebra) A. Normally, the objects of this category are described as 'presheaves on A satisfying a gluing condition'; but, as Higgs(7) and Fourman and Scott(5) have observed, they may also be regarded as 'sets structured with an A-valued equality predicate' (briefly, 'A-valued sets'). From the latter point of view, it is an inessential feature of the situation that every sheaf has a canonical representation as a 'complete' A-valued set. In this paper, our aim is to investigate those properties which A *must* have for us to be able to construct a topos of A-valued sets: we shall see that there is one important respect, concerning the relationship between the finitary (propositional) structure and the infinitary (quantifier) structure, in which the usual definition of a locale may be relaxed, and we shall give a number of examples (some of which will be explored more fully in a later paper (8)) to show that this relaxation is potentially useful.

To motivate what we are about to do, let us examine the concept of locale in some detail. To take care of the propositional logic of 'A-valued sets', we require first of all that A should be a *Heyting algebra*: that is, a partially ordered set which (considered as a category) is finitely complete and cocomplete (i.e. has all finite meets and joins) and is cartesian closed (i.e. has an implication operator $\rightarrow$ such that $a \leqslant (b \rightarrow c)$ if and only if $(a \wedge b) \leqslant c$). To handle predicate logic, we require A to be complete (equivalently, cocomplete); normally, we express this by saying that we have join and meet maps

$$\bigvee : PA \longrightarrow A, \quad \bigwedge : PA \longrightarrow A$$

where PA is the power-set of A. But we may equivalently express this condition by saying that for any set I we can form sups and infs of I-indexed families of elements of A, and so we have maps

$$\exists_I : A^I \longrightarrow A, \quad \forall_I : A^I \longrightarrow A$$

which are respectively left and right adjoint to the diagonal map $A \longrightarrow A^I$. (This interpretation of quantifiers as adjoints to substitution is due to Lawvere(11)). More generally, for any function $f: I \longrightarrow J$, the map $A^f: A^J \longrightarrow A^I$ (i.e. 'compose with f') has left and right adjoints $\exists_f$, $\forall_f$, computed by taking sups and infs over the fibres of f. Further, the fact that these operations are computed fibrewise tells us that the 'Beck conditions' hold, i.e. for any pullback square

$$\begin{array}{ccc} P & \xrightarrow{k} & I \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ J & \xrightarrow{g} & K \end{array}$$

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we have $A^g \exists_f = \exists_h A^k$ and $A^g \forall_f = \forall_h A^k \colon A^I \longrightarrow A^J$. We have thus arrived at the idea that A is complete as an indexed category over the category of sets (see Bénabou (2), Paré and Schumacher (12), Johnstone ((9), appendix), etc.).

The generalization which we propose to study is the following: we consider a **Set**-indexed category $\mathscr{P}$, for which each category $\mathscr{P}I$ of I-indexed families is actually a pre-ordered set. We think of the elements of $\mathscr{P}I$ as '(nonstandard) predicates on I' and the pre-order as 'entailment'. To take care of the propositional logic, each $\mathscr{P}I$ must be finitely complete and cocomplete and cartesian closed; as indicated above, we can include quantifiers by requiring that $\mathscr{P}$ be complete as an indexed category; and finally, to take care of higher-order logic, we require $\mathscr{P}$ to have a generic (or universal) predicate.

The definition will be explained in more detail in the next section. Since the chief function of a structure $\mathscr{P}$ as above is to codify the internal logic of a topos of $\mathscr{P}$-valued sets, we shall call it a 'topos-representing indexed pre-ordered set'. We hope that our use of the acronym 'tripos' as an abbreviation for this phrase will not be construed as frivolity!

Our use of the language of indexed categories indicates that one might go on to consider preorders indexed over toposes other than **Set** (and perhaps even over more general categories) satisfying (analogues of) the above conditions. (Moreover, it is by now generally recognized that the study of internal locales in toposes is an important technique in topos theory – see (10), for example.) However, in the present paper our feet will remain firmly planted in **Set**; we leave the generalization to other base categories for a subsequent work (13). We shall, though, endeavour to draw attention to the points in our argument where we make use of non-constructive methods (such as the axiom of choice).

This paper is a falsification of history to the extent that it presents the recursive realizability tripos (Example 1·7 below) as one example of a general theory, rather than the spur for developing that theory. The idea of 'building a topos out of recursive realizability' was developed by the first author in 1978, and expounded in a course of lectures (attended by the other two authors) early in 1979. (Details of the particular properties of this topos, which make it of no special interest, will appear in a subsequent paper (8).) The use of indexed preorders to codify the logic of this and related toposes was suggested by the third author, and the general theory of such preorders was developed by all three authors in the summer of 1979.

Independently, W. Powell (14) has arrived at a concept of 'complete Heyting filtered algebra' very similar to our 'triposes'; he too was motivated by the example of recursive realizability, though in his case the objective was to build not a topos but a 'hierarchy of V_α's'. Our notion is also closely related to the 'formal toposes' of M. Coste (4), and less closely to the 'logical categories' of H. Volger (18); in the former case, the main difference is that in a formal topos the equality is already built into the logic, whereas for us it is the *addition* of equality to the logic that brings about the passage from a tripos $\mathscr{P}$ to the (actual) topos of $\mathscr{P}$-valued sets.

1. *Indexed systems of predicates.* We shall be concerned with the notion of a collection P of 'abstract predicates'. At the least, we shall have a notion of 'entailment',

written $\vdash$, between predicates which is reflexive and transitive; thus $(P, \vdash)$ is a preorder. Rather curiously, it is an essential feature of the structure we are building that $\vdash$ is *not* required to be antisymmetric – we shall make this remark more precise in section 4. We regard preorders as categories in the usual way; that is, there is one morphism $p \longrightarrow q$ if and only if $p \vdash q$. Of course a functor between two preorders is just an order-preserving map.

1·1. *Definition.* A *Heyting pre-algebra* is a preorder which has finite limits and colimits, and is cartesian closed. We interpret this as meaning that we are given binary operations $\wedge$ (meet), $\vee$ (join) and $\rightarrow$ (Heyting implication), together with distinguished elements $\top$ (top) and $\bot$ (bottom); of course these are determined only up to isomorphism by their universal properties.

1·2. *Definition.* A *tripos* $\mathscr{P}$ consists of the following:

(i) for each set I, a Heyting pre-algebra $(\mathscr{P}I, \vdash_I)$;

(ii) for each map $f: I \longrightarrow J$, functors $\mathscr{P}f: \mathscr{P}J \longrightarrow \mathscr{P}I$ and $\exists f, \forall f: \mathscr{P}I \longrightarrow \mathscr{P}J$, such that

(*a*) $\exists f$ (respectively $\forall f$) is left (respectively right) adjoint to $\mathscr{P}f$,

(*b*) $\mathscr{P}f$ preserves implication (note that it already preserves meets and joins by virtue of (*a*)),

(*c*) $\mathscr{P}$ (and hence also $\exists$ and $\forall$) is pseudo-functorial: that is, $\mathscr{P}(\mathrm{id}_I) \dashv\vdash \mathrm{id}_{\mathscr{P}I}$, and $\mathscr{P}(gf) \dashv\vdash \mathscr{P}f.\mathscr{P}g$ for maps $f: I \longrightarrow J$, $g: J \longrightarrow K$, and

(*d*) the Beck condition holds for $\forall$ (and hence also for $\exists$, by taking left adjoints): that is, for any pullback square

$$\begin{array}{ccc} P & \xrightarrow{k} & I \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ J & \xrightarrow{g} & K \end{array}$$

we have $\mathscr{P}f.\forall g \dashv\vdash \forall k.\mathscr{P}h$;

(iii) a *generic predicate* $\sigma \in \mathscr{P}\Sigma$, for some set Σ: that is, for any $\phi \in \mathscr{P}I$, there is a map $f: I \longrightarrow \Sigma$ with $\phi \dashv\vdash \mathscr{P}f(\sigma)$. (Note: since there may be many such f, we interpret this condition as meaning that we are given a particular choice of maps

$$\{\ \}_I: \mathscr{P}I \longrightarrow \mathbf{Set}(I, \Sigma) \quad \text{with} \quad \phi \dashv\vdash \mathscr{P}\{\phi\}_I(\sigma)$$

– this is analogous to the requirement that we be given particular choices of left and right adjoints for $\mathscr{P}f$.)

1·3 *Remarks.* (i) The preservation of $\rightarrow$ by $\mathscr{P}f$ (condition (ii)(*b*) above) cannot be deduced from the Beck conditions as in standard categorical logic(17), since $\wedge$ is not given by a pullback. A familiar argument shows that the preservation of $\rightarrow$ is equivalent to either of the conditions

$$\exists f(\mathscr{P}f(\psi) \wedge \phi) \dashv\vdash \psi \wedge \exists f(\phi) \quad \text{or} \quad \forall f(\mathscr{P}f(\psi) \rightarrow \phi) \dashv\vdash \psi \rightarrow \forall f(\phi)$$

where $f: I \longrightarrow J$, $\phi \in \mathscr{P}I$ and $\psi \in \mathscr{P}J$. We shall return to this remark in 1·5 below.

(ii) The generic predicate σ allows us to define, for each set I, a *membership predicate* $\in_I \in \mathscr{P}(I \times \Sigma^I)$. Specifically, let $\in_I$ be $\mathscr{P}(\mathrm{ev}_I)(\sigma)$, where $\mathrm{ev}_I: I \times \Sigma^I \longrightarrow \Sigma$ is the evaluation map. Then $\in_I$ has the following property: for any set J and predicate $\phi \in \mathscr{P}(I \times J)$, there is a map $f: J \longrightarrow \Sigma^I$ with $\mathscr{P}(\mathrm{id}_I \times f)(\in_I) \dashv\vdash \phi$.

We should now give some examples of triposes. However, to cut down the amount of work needed to verify that the structures we describe do satisfy Definition 1·2, we shall find it convenient to investigate which parts of that definition are expressible in terms of the rest. This is the analogue of the definability of second-order logic from $\forall$ and $\rightarrow$ (Scott, Prawitz (15), etc.), and the reader should keep the proof of the latter in mind in what follows – although there is not a word-for-word correspondence.

Suppose then that we are given a structure $\mathscr{P}$ consisting of

(i)′ for each set I, a preorder $(\mathscr{P}I, \vdash_I)$, together with a binary operation $\rightarrow$ on $\mathscr{P}I$ which models purely implicational logic: that is,

(*a*) $\phi \vdash_I \psi \rightarrow \phi$,

(*b*) $\theta \rightarrow (\phi \rightarrow \psi) \vdash_I (\theta \rightarrow \phi) \rightarrow (\theta \rightarrow \psi)$,

(*c*) if $\theta \vdash_I \phi \rightarrow \psi$ and $\theta \vdash_I \phi$, then $\theta \vdash_I \psi$, and

(*d*) if $\phi \vdash_I \psi$ then $\theta \vdash_I \phi \rightarrow \psi$,

for any $\theta, \phi, \psi \in \mathscr{P}I$;

(ii)′ for each map $f: I \longrightarrow J$, functors $\mathscr{P}f: \mathscr{P}J \longrightarrow \mathscr{P}I$ and $\forall f: \mathscr{P}I \longrightarrow \mathscr{P}J$ satisfying the relevant parts of 1·2 (ii) (*a*), (*b*), (*c*) and (*d*), plus

(*e*) $\forall f(\mathscr{P}f(\psi) \rightarrow \phi) \dashv\vdash \psi \rightarrow \forall f(\phi)$ for all $f: I \longrightarrow J, \phi \in \mathscr{P}I$ and $\psi \in \mathscr{P}J$;

(iii)′ a generic predicate, just as in 1·2 (iii). Then we have

1·4 THEOREM. *Suppose given a structure* $\mathscr{P}$ *satisfying conditions* (i)′, (ii)′ *and* (iii)′ *above. Then if we define propositional operations on* $\mathscr{P}I$ *by*

$$\top = \forall\pi(\in_I \rightarrow \in_I),$$
$$\bot = \forall\pi(\in_I),$$
$$\phi \wedge \psi = \forall\pi(\mathscr{P}\pi\phi \rightarrow (\mathscr{P}\pi\psi \rightarrow \in_I)) \rightarrow \in_I),$$
$$\phi \vee \psi = \forall\pi((\mathscr{P}\pi\phi \rightarrow \in_I \wedge \mathscr{P}\pi\psi \rightarrow \in_I) \rightarrow \in_I)$$

(*where* π *is the projection* $I \times \Sigma^I \longrightarrow I$), *and existential quantification along* $f: I \longrightarrow J$ *by*

$$\exists f(\phi) = \forall\pi'(\forall(f \times \mathrm{id})(\mathscr{P}\pi\phi \rightarrow \mathscr{P}(f \times \mathrm{id})(\in_J)) \rightarrow \in_J)$$

(*where* $\pi: I \times \Sigma^J \longrightarrow I$ *and* $\pi': J \times \Sigma^J \longrightarrow J$), *we obtain a tripos.*

Proof. In what follows, we shall frequently appeal to valid purely implicational entailments; we assume that the reader can show that (i)′ (*a*), (*b*), (*c*) and (*d*) generate all such, or else that he is willing to consult some suitable treatment of propositional logic – e.g. (16). We begin with the universal properties of the propositional operations:

(i) Since $\mathscr{P}\pi(\phi) \vdash (\in_I \rightarrow \in_I)$ is valid, we obtain $\phi \vdash \forall\pi(\in_I \rightarrow \in_I) = \top$ by adjointness, so $\top$ is a top element of $\mathscr{P}I$.

(ii) Given any $a: I \longrightarrow \Sigma$, define $b: I \longrightarrow \Sigma^I$ by $b(i)(i') = a(i)$. Then

$$\begin{aligned}\bot = \forall\pi(\in_I) &\vdash \forall\pi\forall(\mathrm{id}_I, b)\mathscr{P}(\mathrm{id}_I, b)(\in_I)\\ &\dashv\vdash \forall(\mathrm{id}_I)\mathscr{P}(\mathrm{ev}_I . (\mathrm{id}_I, b))(\sigma)\\ &\dashv\vdash a(\sigma).\end{aligned}$$

Putting $a = \{\phi\}_I$, we obtain $\bot \vdash \phi$; so $\bot$ is a bottom element of $\mathscr{P}I$.

(iii) With a and b as above, for any ϕ and ψ in $\mathscr{P}I$ we have

$$\begin{aligned}\phi \wedge \psi &= \forall\pi((\mathscr{P}\pi\phi \rightarrow (\mathscr{P}\pi\psi \rightarrow \in_I)) \rightarrow \in_I)\\ &\vdash \forall(\pi . (\mathrm{id}_I, b))\mathscr{P}(\mathrm{id}_I, b)((\mathscr{P}\pi\phi \rightarrow (\mathscr{P}\pi\psi \rightarrow \in_I)) \rightarrow \in_I).\end{aligned}$$

So by functoriality of $\mathscr{P}$ and $\forall$, and the fact that $\mathscr{P}$ preserves $\to$, we have

$$\phi \wedge \psi \vdash (\phi \to (\psi \to \mathscr{P}a(\sigma)) \to \mathscr{P}a(\sigma). \tag{1}$$

Putting $a = \{\phi\}_I$ and $a = \{\psi\}_I$ in turn, and applying propositional logic, we obtain

$$\phi \wedge \psi \vdash_I \phi \quad \text{and} \quad \phi \wedge \psi \vdash_I \psi.$$

Conversely suppose $\theta \vdash_I \phi$ and $\theta \vdash_I \psi$; then $\mathscr{P}\pi\theta \vdash \mathscr{P}\pi\phi$ and $\mathscr{P}\pi\theta \vdash \mathscr{P}\pi\psi$, so by propositional logic

$$\mathscr{P}\pi\theta \vdash (\mathscr{P}\pi\phi \to (\mathscr{P}\pi\psi \to \in_I)) \to \in_I,$$

and thus $\theta \vdash_I \phi \wedge \psi$. So $\phi \wedge \psi$ is the meet of ϕ and ψ in $\mathscr{P}I$.

(iv) Next we show that $(-) \wedge \phi$ is left adjoint to $\phi \to (-)$. Putting $\phi \to \psi$ for ψ and $\{\psi\}_I$ for a in (1) above, we obtain

$$\begin{aligned} \phi \wedge (\phi \to \psi) &\vdash_I (\phi \to ((\phi \to \psi) \to \psi)) \to \psi \\ &\vdash_I \psi \end{aligned}$$

by propositional logic. Conversely,

$$\mathscr{P}\pi\phi \vdash \mathscr{P}\pi\psi \to ((\mathscr{P}\pi\phi \to (\mathscr{P}\pi\psi \to \in_I)) \to \in_I)$$

is always valid, so, by adjointness and condition (*e*) of (ii)′, we have

$$\phi \vdash_I \psi \to (\phi \wedge \psi).$$

(v) Finally, we consider joins in $\mathscr{P}I$. We always have

$$\mathscr{P}\pi\phi \vdash (\mathscr{P}\pi\phi \to \in_I) \to \in_I,$$

so

$$\mathscr{P}\pi\phi \vdash ((\mathscr{P}\pi\phi \to \in_I) \wedge (\mathscr{P}\pi\psi \to \in_I)) \to \in_I$$

and hence $\phi \vdash_I \phi \vee \psi$; similarly $\psi \vdash_I \phi \vee \psi$. Now suppose $\phi \vdash_I \theta$ and $\psi \vdash_I \theta$. With $a = \{\theta\}_I$ and b defined as before, we have

$$\begin{aligned} \phi \vee \psi &\vdash_I \phi\, \forall\pi\forall(\mathrm{id}_I, b)\ \mathscr{P}(\mathrm{id}_I, b)((\mathscr{P}\pi\phi \to \in_I \wedge \mathscr{P}\pi\psi \to \in_I) \to \in_I) \\ &\vdash_I (\phi \to \theta \wedge \psi \to \theta) \to \theta \\ &\vdash_I \theta \end{aligned}$$

by propositional logic.

It remains to show that $\exists f$ is left adjoint to $\mathscr{P}f$. Suppose $\phi \vdash_I \mathscr{P}f(\psi)$; then by functoriality and propositional logic we obtain

$$\begin{aligned} \exists f(\phi) &\vdash_J \forall\pi'(\forall(f \times \mathrm{id})\,(\mathscr{P}\pi\mathscr{P}f\psi \to \mathscr{P}(f \times \mathrm{id}) \in_J) \to \in_J) \\ &\vdash_J \forall\pi'(\forall(f \times \mathrm{id})\,\mathscr{P}(f \times \mathrm{id})\,(\mathscr{P}\pi'\psi \to \in_J) \to \in_J) \\ &\vdash_J \forall\pi'((\mathscr{P}\pi'\psi \to \in_J) \to \in_J). \end{aligned}$$

Then with $a = \{\psi\}_J : J \longrightarrow \Sigma$ and $b : J \longrightarrow \Sigma^J$ defined as before,

$$\begin{aligned} \exists f(\phi) &\vdash_J \forall\pi'\,\forall(\mathrm{id}, b)\,\mathscr{P}(\mathrm{id}, b)((\mathscr{P}\pi'\psi \to \in_J) \to \in_J) \\ &\vdash_J (\psi \to \psi) \to \psi \\ &\vdash_J \psi. \end{aligned}$$

Conversely if $\exists f(\phi) \vdash_J \psi$, then

$$\begin{aligned}\phi &\vdash_I \forall\pi\mathscr{P}\pi(\phi)\\ &\vdash_I \forall\pi((\mathscr{P}\pi\phi \to \mathscr{P}(f\times \mathrm{id})\in_J)\to\mathscr{P}(f\times\mathrm{id})\in_J)\\ &\vdash_I \forall\pi(\mathscr{P}(f\times\mathrm{id})\,\forall(f\times\mathrm{id})\,(\mathscr{P}\pi\phi\to\mathscr{P}(f\times\mathrm{id})\in_J)\to\mathscr{P}(f\times\mathrm{id})\in_J)\\ &\vdash_I \mathscr{P}f\forall\pi'(f\times\mathrm{id})\,(\mathscr{P}\pi\phi\to\mathscr{P}(f\times\mathrm{id})\in_J)\to\in_J)\\ &\vdash_I \mathscr{P}f(\psi),\end{aligned}$$

using the Beck condition for the pullback square

$$\begin{array}{ccc} I\times\Sigma^J & \xrightarrow{f\times\mathrm{id}} & J\times\Sigma^J\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'}\\ I & \xrightarrow{f} & J\end{array}$$

and the fact that $\mathscr{P}(f\times\mathrm{id})$ preserves $\to$.

1·5 *Remark.* In the proof of Theorem 1·4, we used condition (e) of (ii)′ (on commuting $\forall$ past $\to$) to establish the cartesian closedness of $(\mathscr{P}I, \vdash_I)$. We hope that some appreciation of its significance will emerge from the following points:

(i) in the presence of the adjunctions $\exists f \dashv \mathscr{P}f$ and $(-)\wedge\phi \dashv \phi\to(-)$, there is an equivalence between the conditions

$$\phi\wedge\exists f(\psi)\vdash\exists f(\mathscr{P}f(\phi)\wedge\psi) \quad\text{and}\quad \mathscr{P}f(\phi)\to\mathscr{P}f(\psi)\vdash\mathscr{P}f(\phi\to\psi);$$

(ii) the condition $\exists f(\mathscr{P}f(\phi)\wedge\psi)\vdash\phi\wedge\exists f(\psi)$ is an immediate consequence of $\exists f\dashv\mathscr{P}f$;

(iii) the condition $\mathscr{P}f(\phi\to\psi)\vdash\mathscr{P}f(\phi)\to\mathscr{P}f(\psi)$, usually thought of as a consequence of the adjunction $(-)\wedge\phi\dashv\phi\to(-)$ and the left exactness of $\mathscr{P}f$, is (in the absence of $\wedge$ but the presence of $\mathscr{P}f\dashv\forall f$ and monotonicity of $\to$ in the second variable) equivalent to

$$\phi\to\forall f(\psi)\vdash\forall f(\mathscr{P}f(\phi)\to\psi);$$

(iv) the condition

$$\forall f(\mathscr{P}f(\phi)\to\psi)\vdash\phi\to\forall f(\psi)$$

follows from $\mathscr{P}f\dashv\forall f$ and $(-)\wedge\phi\dashv\phi\to(-)$.

Thus by (iii) the operative part of condition (ii)′ (e) must be (as indeed it was in the proof of 1·4) the condition (iv) above. With the higher-order structure, it is sufficient to recapture cartesian closedness. Then the other basic laws relating quantifiers and connectives follow by (i) and (ii).

We now turn to some examples of triposes, starting with the kind that were sketched in the Introduction.

1·6. *Localic examples.* (i) Given a locale A, we define the *canonical tripos* of A by setting $\mathscr{P}I = A^I$, with $\vdash_I$ simply the pointwise partial order on functions $I\longrightarrow A$. The propositional connectives are similarly defined pointwise; the maps $\mathscr{P}f$ are induced by composition with f, and $\forall f\colon\mathscr{P}I\longrightarrow\mathscr{P}J$ sends $\phi\in A^I$ to the map

$$j\longmapsto\bigwedge\nolimits_A\{\phi(i)\,|\,f(i)=j\}$$

(with a similar definition for $\exists f$). The generic predicate is of course $\mathrm{id}_A \in \mathscr{P}A = A^A$. As we indicated in the Introduction, it is easy to verify that the conditions of Definition 1·2 are satisfied.

(ii) Given a filter Φ on a locale A, we can modify the above example if we set $\mathscr{P}I = A^I$ as before, but

$$\phi \vdash_I \psi \quad \text{iff} \quad \bigwedge_A \{\phi(i) \to \psi(i) \mid i \in I\} \in \Phi,$$

the remaining structure being defined as in (i). Because Φ is a filter, it is not hard to see that $\vdash_I$ is a preorder on A^I, and the conditions of 1·2 are again easy to verify. More generally, if Φ is a filter on any tripos $\mathscr{P}$ (i.e. a filter on the Heyting pre-algebra $\mathscr{P}1$), redefining the preorder on each $\mathscr{P}I$ as above yields a new tripos $\mathscr{P}_\Phi$.

(iii) To emphasize the importance of condition (iii) (existence of a generic predicate) in Definition 1·2, we give an example of an indexed preorder which is not a tripos. Let A be an infinite Heyting algebra (not necessarily complete), and for any set I define $\mathscr{P}I$ to be the set of all maps $I \longrightarrow A$ with finite image. It is easy to verify that if we define the substitution maps $\mathscr{P}f$, logical connectives and quantifiers as in (i) we obtain an indexed preorder satisfying (i) and (ii) of Definition 1·2. However, any generic predicate $\sigma \in \mathscr{P}\Sigma$ would have to be a *surjection* $\Sigma \longrightarrow A$, so no such predicate can exist. For a particular A, there may be possibilities intermediate between this one and taking $\mathscr{P}I = A^I$; for example if A is the set of rationals in $[0, 1]$ with its usual total order, we can take $\mathscr{P}I$ to be the set of functions $f: I \longrightarrow A$ such that all accumulation points of $\{f(i) \mid i \in I\}$ in $[0, 1]$ are rational. (The reader may find it of interest to carry through the construction of the category $\mathscr{P}$-**Set** described in §2 for this particular $\mathscr{P}$; the result is a logos in the sense of (6), but not a topos.)

1·7 *Realizability examples.* A *partial applicative structure* consists of a set A together with a partial binary operation (denoted $(a, b) \longmapsto a(b)$) on A, such that there are elements $e, k, s \in A$ with

$$e(a) \asymp a,$$

$$k(a)(b) \asymp a,$$

and

$$s(a)(b)(c) \asymp a(c)(b(c))$$

for all $a, b, c \in A$, where (following Freyd (6)) '$\asymp$' denotes 'one side is defined iff the other is and then they are equal' (i.e. it is equality for partial elements in the sense of (5)). Such a structure is of course intended to model an untyped theory of (partial) functional application.

For such a structure A, we define a binary operation $\to$ on the power-set PA of A by

$$p \to q = \{a \in A \mid \text{for all } b \in p,\ a(b) \text{ is defined and } a(b) \in q\}.$$

(The idea of treating the realizability interpretation of implication in this model-theoretic way is due to Scott. If we think of subsets of A as 'propositions', then elements of A are 'proofs' of these propositions, with $a \in p$ read as 'a proves p'.)

Now for any set I let $\mathscr{P}I$ be PA^I, with $\vdash_I$ defined by $\phi \vdash_I \psi$ if and only if there is $a \in A$ with $a \in (\phi(i) \to \psi(i))$ for all $i \in I$. Given $f: I \longrightarrow J$, we define $\mathscr{P}f$ to be composition with f, whilst $\forall f$ sends $\phi \in PA^I$ to the map

$$j \longmapsto \bigcap \{[\![f(i) = j]\!] \to \phi(i) \mid i \in I\}$$

where $[\![f(i) = j]\!] = \{a \in A \mid f(i) = j\}$; i.e. it is A if $f(i) = j$, and $\varnothing$ otherwise. The generic predicate σ is $\mathrm{id}_{PA} \in \mathscr{P}(PA)$; we claim that the above definitions make $\mathscr{P}$ into a tripos.

To prove this, we shall make use of Theorem 1·4. Note first that for any $\phi \in \mathscr{P}I$ we have $e \in \phi(i) \to \phi(i)$ for all i, so $\vdash_I$ is reflexive. Similarly, if

$$a \in \theta(i) \to \phi(i) \quad \text{and} \quad b \in \phi(i) \to \psi(i) \quad \text{for all } i,$$

then $s(k(a))(b) \in \theta(i) \to \psi(i)$ for all i; so $\vdash_I$ is transitive.

For any index set I, the elements k and s yield 'proofs' of the propositional tautologies (i)′ (*a*) and (*b*) of 1·4. To verify (*c*), suppose

$$a \in \theta(i) \to (\phi(i) \to \psi(i)) \quad \text{and} \quad b \in \theta(i)\,\phi(i)$$

for all i; then $s(a)(b) \in \theta(i) \to \psi(i)$ for all i. Similarly if $a \in \phi(i) \to \psi(i)$ for all i, then $k(a) \in \theta(i) \to (\phi(i) \to \psi(i))$ for all i, and so (*d*) holds.

It is clear that $\mathscr{P}f$ is functorial and $\mathscr{P}$ is itself a functor. For the rest of the conditions in 1·4 (ii)′, we make use of the translation of the λ-calculus into combinatory logic. Thus for example, to show that $\forall f$ is functorial, take $a \in \phi(i) \to \psi(i)$ for all i and note that, in λ-notation,

$$\lambda xy\,.\,a(x(y)) \in (\forall f\phi)(j) \to (\forall f\psi)(j)$$

for all $j \in J$. But we may take $s(k(a))$ for $\lambda xy\,.\,a(x(y))$; so $\forall f(\phi) \vdash_J \forall f(\psi)$. The other conditions are similar. Finally, condition (iii)′ of 1·4 is trivial from the definition of $\mathscr{P}f$; so we have a tripos, as claimed.

1·8. *Remarks.* (i) When we defined universal quantification for the realizability tripos, the reader might have expected to see the formula

$$(\forall f\phi)(j) = \bigcap \{\phi(i) \mid f(i) = j\},$$

rather than the more complicated one which we gave; but unfortunately the above definition fails to be right adjoint to $\mathscr{P}f$ if f is not surjective. Nevertheless, we shall see in Proposition 1·12 below that there is a map $\bigwedge : P(PA) \longrightarrow PA$ with the property that $\forall f(\phi)$ is isomorphic to the map $j \longmapsto \bigwedge \{\phi(i) \mid f(i) = j\}$, for all f and ϕ.

(ii) The most familiar example of a partial applicative structure is the set $\mathbb{N}$ of natural numbers with the partial application $n(m)$ = value of the nth partial recursive function at m (if defined). This gives rise to the *recursive realizability tripos*, which was the motivating example for the whole development of this paper. However, there are other realizability triposes of interest; for example, the various models of the (untyped) λ-calculus give rise to them.

We conclude this section with a number of results which extend the definability theorem (1·4), in that they tell us that 'up to equivalence' a tripos may always be taken to satisfy some of the conditions of Definition 1·2 in a 'stricter' sense than we originally envisaged. Note first that, in all the examples described above, $\mathscr{P}I$ is actually Σ^I, where Σ is the index set of a generic predicate for I, and $\mathscr{P}f$ is composition with f. We shall call such a tripos *canonically presented.*

1·9 Proposition. *Any tripos is equivalent to a canonically presented one.* (Note: by 'is equivalent to', we really mean 'represents the same topos as'. How a tripos represents a topos will be made precise in §2; for the moment, we may take 'equivalent' to mean 'equivalent as a pseudofunctor from **Set** to the 2-category of preorders'.)

Proof. Given a tripos $\mathscr{P}$, with generic predicate $\sigma \in \mathscr{P}\Sigma$, define a functor $\bar{\mathscr{P}}$ from sets to preorders by $\bar{\mathscr{P}}I = (\Sigma^I, \vdash_I)$, where $g \vdash_I h$ in Σ^I if and only if $\mathscr{P}g(\sigma) \vdash_I \mathscr{P}h(\sigma)$ in $\mathscr{P}I$, with $\bar{\mathscr{P}}f = \Sigma^f: \Sigma^J \longrightarrow \Sigma^I$ for a map $f: I \longrightarrow J$. The definition of a generic predicate now tells us that the maps $\phi \longrightarrow \{\phi\}_I$ and $g \longrightarrow \mathscr{P}g(\sigma)$ set up an equivalence between $\mathscr{P}I$ and Σ^I, which is pseudo-natural in I. So we may use it to transport the rest of the structure in 1·2 from $\mathscr{P}$ to $\bar{\mathscr{P}}$; for example ,we may define implication in $\bar{\mathscr{P}}I$ by

$$g \rightarrow h = \{\mathscr{P}g(\sigma) \rightarrow \mathscr{P}h(\sigma)\}_I.$$

So $\bar{\mathscr{P}}$ is a canonically presented tripos equivalent to $\mathscr{P}$.

1·10 *Remark.* From the point of view of indexed categories, Proposition 1·9 says that we may assume that the (pseudo)functor $\mathscr{P}$ satisfies a 'descent condition' on objects; i.e. its composite with the forgetful functor from preorders to sets is a sheaf for the canonical topology on **Set**. However, we cannot require $\mathscr{P}$ to satisfy a similar condition on morphisms; indeed, as we shall see in §4, it is precisely the difference between the 'uniform' preorder $\vdash_I$ on $\mathscr{P}I = \mathscr{P}(1)^I$ and the pointwise preorder $(\vdash_1)^I$ which gives rise to the difference between locales and triposes in general. This is (at least initially) rather surprising, since we are accustomed to consider indexed categories which are locally internal (i.e. 'have small homs' in the terminology of (12)), and therefore automatically satisfy a descent condition on morphisms; whereas descent conditions on objects do not normally play an important role in indexed category theory.

1·11 PROPOSITION. *Let $\mathscr{P}$ be a canonically presented tripos. Then we may choose the propositional operations on each $\mathscr{P}I$ so that they are induced pointwise by the operations on $\mathscr{P}1$.*

Proof. Let $m: \Sigma \times \Sigma \longrightarrow \Sigma$ be $\pi_1 \wedge \pi_2 \in \mathscr{P}(\Sigma \times \Sigma)$, where $\pi_1, \pi_2: \Sigma \times \Sigma \longrightarrow \Sigma$ are the product projections. Then for any $f, g \in \mathscr{P}I = \Sigma^I$, we have

$$\begin{aligned} m.(f,g) &= \mathscr{P}(f,g)(\pi_1 \wedge \pi_2) \\ &\dashv\vdash \mathscr{P}(f,g)(\pi_1) \wedge \mathscr{P}(f,g)(\pi_2) \\ &= f \wedge g, \end{aligned}$$

so we may redefine the meet operation in $\mathscr{P}I$ to be composition with m. Then for any $i \in I$ we have $(f \wedge g)(i) = m(fi, gi) = fi \wedge gi$. Similarly for the other propositional operations.

Thus the propositional structure of $\mathscr{P}$ is determined by that of Σ. Furthermore, the entailment relations $\vdash_I$ are determined by that on Σ, in the following sense. Let $D = \{p \in \Sigma \mid \top_1 \vdash_1 p\}$ be the set of propositions isomorphic to the top element of Σ. Then given $f, g \in \Sigma^I$, we have

$$\begin{aligned} f \vdash_I g \quad &\text{iff} \quad \mathscr{P}I(\top_1) = \top_I \vdash_I f \rightarrow g \\ &\text{iff} \quad \top_1 \vdash_1 \forall I(f \rightarrow g) \\ &\text{iff} \quad \forall I(f \rightarrow g) \in D \end{aligned}$$

where I denotes the unique map $I \longrightarrow 1$.

To complete this transference of structure on $\mathscr{P}$ to structure on Σ, we need to show that the quantifiers can be computed fibre-wise (i.e. so that the Beck conditions hold up to equality, and not just up to isomorphism). The next proposition says that we can do this; but it differs from the previous two results in that it makes use of the axiom of choice in the base category **Set**.

1·12 Proposition. *Let $\mathscr{P}$ be a canonically presented tripos. Then the quantifiers in $\mathscr{P}$ may be chosen so that we have*

$$(\forall f\phi)(j) = \bigwedge\{\phi(i)\,|\,f(i) = j\} \quad \textit{and} \quad (\exists f\phi)(j) = \bigvee\{\phi(i)\,|\,f(i) = j\}$$

for some maps $\bigwedge, \bigvee: P\Sigma \longrightarrow \Sigma$.

Proof. Let $\in_\Sigma \subseteq \Sigma \times P\Sigma$ be the (standard) membership relation on Σ (not the predicate defined in 1·3 (ii)), and let $e: \in_\Sigma \longrightarrow \Sigma$, $n: \in_\Sigma \longrightarrow P\Sigma$ be the restrictions to $\in_\Sigma$ of the two projection maps. We define $\bigwedge, \bigvee: P\Sigma \longrightarrow \Sigma$ to be $\forall n(e)$ and $\exists n(e)$ respectively.

Given $f: I \longrightarrow J$ and $\phi: I \longrightarrow \Sigma$, let $K = \{(\phi i, fi)\,|\,i \in I\} \subseteq \Sigma \times J$. Then we have a pullback square

$$\begin{array}{ccc} K & \xrightarrow{r} & \in_\Sigma \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle n} \\ J & \xrightarrow{s} & P\Sigma \end{array}$$

where $s(j) = \{\phi i\,|\,fi = j\}$, $r(p,j) = (p, s(j))$ and $f'(p,j) = j$. So by the Beck conditions

$$\bigwedge . s = \mathscr{P}s(\forall n(e)) \dashv\vdash_J \forall f'(\mathscr{P}r(e)) = \forall f'(\phi')$$

and

$$\bigvee . s = \mathscr{P}s(\exists n(e)) \dashv\vdash_J \exists f'(\mathscr{P}r(e)) = \exists f'(\phi')$$

where ϕ' is the composite $er: K \longrightarrow \Sigma$. So it remains to show that $\forall f'(\phi') \dashv\vdash_J \forall f(\phi)$ and $\exists f'(\phi') \dashv\vdash_J \exists f(\phi)$. But if $q: I \longrightarrow K$ is the map $i \longmapsto (\phi i, fi)$, then q is surjective; so by (AC) it has a right inverse, and hence $\forall q . \mathscr{P}q \dashv\vdash \mathrm{id}_{\mathscr{P}K} \dashv\vdash \exists q . \mathscr{P}q$. Thus

$$\forall f'(\phi') \dashv\vdash_J \forall f'(\forall q(\mathscr{P}q(\phi'))) \dashv\vdash_J \forall(f'q)(\phi' q) = \forall f(\phi)$$

and similarly $\exists f'(\phi') \dashv\vdash_J \exists f(\phi)$.

Taken together, Propositions 1·9, 1·11 and 1·12 assert that (if we accept the axiom of choice in **Set**) our notion of tripos is effectively no more general than Powell's 'complete Heyting filtered algebras' (14), or the very similar 'models of second order propositional logic' used by the first author in a preliminary draft of (8). Nevertheless, we believe that the conceptual advantages of the indexed-category approach will quickly become apparent when we embark on the construction of the topos of $\mathscr{P}$-valued sets in the next section.

2. *The topos of $\mathscr{P}$-sets.* In this section, our aim is to construct from an arbitrary tripos $\mathscr{P}$ a topos of '$\mathscr{P}$-valued sets' in a way which generalizes the constructions of Higgs (7) and Fourman and Scott (5). Before proceeding to the particular notion of a $\mathscr{P}$-valued set, however, it will be convenient to make a few remarks about the general concept of a *$\mathscr{P}$-relational structure*.

Given a many-sorted purely relational first-order language $\mathscr{L}$ without equality, a

$\mathscr{P}$-interpretation of $\mathscr{L}$ assigns to each sort (type) $\mathbf{X}$ a set X, and to each relation symbol $\mathbf{R}$ of sort $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ an element R of $\mathscr{P}(\Pi_{i=1}^n X_i)$. Given such an interpretation we may proceed to define, for each formula ϕ of $\mathscr{L}$ and each string of variables

$$\vec{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

containing the free variables of ϕ, an interpretation $[\![\phi(\vec{\mathbf{x}})]\!] \in \mathscr{P}(\Pi X_i)$ (where $\mathbf{X}_i$ is the sort of the variable $\mathbf{x}_i$), by an obvious induction on the structure of ϕ. In this interpretation, we have $[\![\mathbf{R}(\vec{\mathbf{x}})]\!] = \mathscr{P}\pi(R)$ where π is the projection map from ΠX_i to the product of the (interpretations of) sorts actually occurring in $\mathbf{R}$; the propositional connectives are interpreted by the Heyting pre-algebra operations in $\mathscr{P}(\Pi X_i)$, and the quantifiers $\forall \mathbf{y}$, $\exists \mathbf{y}$ by $\forall \pi$, $\exists \pi$, where $\pi: Y \times \Pi X_i \longrightarrow \Pi X_i$ is the projection. (Note: it would be perfectly possible to extend this interpretation to a language having function-symbols as well as relation-symbols; but we shall not need to avail ourselves of this possibility, except for a moment in the proof of Proposition 2·12.)

We shall be working with the version of intuitionistic predicate logic where each step in a deduction is labelled by a string of (free) variables denoting the types of elements which are assumed to have been picked at that step; so the fundamental (syntactic) entailment-notion appears as

$$\Gamma \vdash^{\vec{\mathbf{x}}} \phi$$

where Γ is a finite collection of formulae, ϕ a formula, and $\vec{\mathbf{x}}$ a string containing all the free variables of the formulae in $\Gamma \cup \{\phi\}$. A detailed description of this entailment-notion will be found in (3).

Given a particular $\mathscr{P}$-interpretation, we also have a semantic entailment-notion

$$\Gamma \vDash^{\vec{\mathbf{x}}}_{\mathscr{P}} \phi$$

whose interpretation is the statement that

$$\bigwedge_{\gamma \in \Gamma} [\![\gamma(\vec{\mathbf{x}})]\!] \vdash [\![\phi(\vec{\mathbf{x}})]\!] \quad \text{in} \quad \mathscr{P}(\Pi X_i).$$

These two entailment-notions are linked by

2·1 LEMMA (Soundness Lemma). *For any tripos $\mathscr{P}$ and any $\mathscr{P}$-interpretation of a language $\mathscr{L}$ as above, if $\Gamma \vdash^{\vec{\mathbf{x}}} \phi$ holds in intuitionistic predicate logic, then $\Gamma \vDash^{\vec{\mathbf{x}}} \phi$ holds in the interpretation.*

The proof of the Soundness Lemma is a standard induction over the definition of $\vdash^{\vec{\mathbf{x}}}$, and we shall not give the details. Nevertheless, we shall use the Soundness Lemma repeatedly in what follows: in order to establish that $\phi(\vec{\mathbf{x}})$ is valid (i.e. $\varnothing \vDash^{\vec{\mathbf{x}}} \phi$) in some $\mathscr{P}$-interpretation, we shall simply appeal to the fact that $\phi(\vec{\mathbf{x}})$ is deducible intuitionistically from formulae which we already know to be valid. (We assume that the reader is capable of carrying out straightforward deductions in intuitionistic predicate logic.)

2·2 *Definition.* Let $\mathscr{P}$ be a tripos. A *$\mathscr{P}$-valued set* (briefly, *$\mathscr{P}$-set*) is a $\mathscr{P}$-relational structure $(X, =)$ where $=$ is a binary relation such that

$$\vDash_{\mathscr{P}} \forall \mathbf{x}, \mathbf{x}' \quad (\mathbf{x} = \mathbf{x}' \to \mathbf{x}' = \mathbf{x})$$

and

$$\vDash_{\mathscr{P}} \forall \mathbf{x}, \mathbf{x}', \mathbf{x}'' \quad (\mathbf{x} = \mathbf{x}' \wedge \mathbf{x}' = \mathbf{x}'' \to \mathbf{x} = \mathbf{x}'').$$

Thus in the sense of $\mathscr{P}$-logic, equality is symmetric and transitive. We do not require it to be reflexive; instead, we *define* the interpretation of an 'existence' or 'membership' predicate E_X by

$$[\![E_X(\mathbf{x})]\!] = [\![\mathbf{x} = \mathbf{x}]\!].$$

(We shall frequently write '$\mathbf{x} \in X$' in place of '$E_X(\mathbf{x})$'.) This predicate clearly satisfies

$$\models_{\mathscr{P}} \forall \mathbf{x} \quad (\mathbf{x} = \mathbf{x} \leftrightarrow E_X(\mathbf{x})). \qquad (*)$$

For certain purposes (particularly when we come to consider subobjects in the category of $\mathscr{P}$-sets) it will be convenient to regard E_X as a second primitive predicate, and $(*)$ as an additional axiom; clearly, this does not essentially change our notion of $\mathscr{P}$-valued set. We should also mention here that we shall frequently abuse notation by using the same letter for a $\mathscr{P}$-set and its underlying set, although at some points (where we have to consider more than one $\mathscr{P}$-set structure on the same set) we shall have to be careful of the distinction.

2·3 *Definition.* A *relation* on a $\mathscr{P}$-set $(X, =)$ is an element R of $\mathscr{P}X$ which respects the equality predicate, i.e.

$$\models_{\mathscr{P}} \forall \mathbf{x}, \mathbf{x}' \quad (\mathbf{x} = \mathbf{x}' \wedge R(\mathbf{x}) \to R(\mathbf{x}')).$$

We say R is *strict* if in addition it respects the existence predicate, i.e.

$$\models_{\mathscr{P}} \forall \mathbf{x} \quad (R(\mathbf{x}) \to E_X(\mathbf{x})).$$

Clearly if $(X_1, =_1), (X_2, =_2), \ldots, (X_n, =_n)$ are $\mathscr{P}$-sets, we may make the product $\Pi_{i=1}^n X_i$ into a $\mathscr{P}$-set if we define equality by

$$[\![\vec{\mathbf{x}} = \vec{\mathbf{y}}]\!] = [\![\mathbf{x}_1 =_1 \mathbf{y}_1 \wedge \ldots \wedge \mathbf{x}_n =_n \mathbf{y}_n]\!].$$

Thus the above definitions for unary relations may immediately be extended to n-ary relations.

2·4 *Definition.* If $(X, =)$ and $(Y, =)$ are $\mathscr{P}$-sets, we say a relation F on $X \times Y$ is *functional* if it is strict, single-valued and total, where 'F is single-valued' means

$$\models_{\mathscr{P}} \forall \mathbf{x}, \mathbf{y}, \mathbf{y}' \ (F(\mathbf{x}, \mathbf{y}) \wedge F(\mathbf{x}, \mathbf{y}') \to \mathbf{y} = \mathbf{y}')$$

and 'F is total' means

$$\models_{\mathscr{P}} \forall \mathbf{x} \quad (\mathbf{x} \in X \to \exists \mathbf{y}(F(\mathbf{x}, \mathbf{y}))).$$

We say two functional relations F and G are *equivalent* if

$$\models_{\mathscr{P}} \forall \mathbf{x}, \mathbf{y} \quad (F(\mathbf{x}, \mathbf{y}) \leftrightarrow G(\mathbf{x}, \mathbf{y})).$$

(Note: since G is strict and single-valued and F is total, it would in fact be sufficient to have $\models_{\mathscr{P}} \forall \mathbf{x}, \mathbf{y} \, (F(\mathbf{x}, \mathbf{y}) \to G(\mathbf{x}, \mathbf{y}))$.) It is an immediate application of the Soundness Lemma that this does define an equivalence relation on the set of functional relations. Finally, we define a *morphism of $\mathscr{P}$-sets* $f\colon (X, =) \to (Y, =)$ to be an equivalence class of functional relations. We adopt the convention that if a lower-case letter such as f denotes a morphism of $\mathscr{P}$-sets, the corresponding capital letter denotes (the name of) a functional relation representing it.

2·5 LEMMA. *$\mathscr{P}$-sets and their morphisms form a category $\mathscr{P}$-*Set.

Proof. First we have to define identity morphisms and composition. The identity morphism $(X, =) \longrightarrow (X, =)$ is simply the equivalence class of the functional relation

$[\![x = x']\!] \in \mathscr{P}(X \times X)$. Given morphisms $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, we form their composite as the equivalence class of the functional relation

$$[\![\exists y(F(x,y) \wedge G(y,z))]\!] \in \mathscr{P}(X \times Z).$$

It is a straightforward application of the soundness lemma to show that this relation is functional from X to Z, and that its equivalence class does not depend on the choice of the representatives F and G.

The requirement that relations be extensional (i.e. that they respect equality) tells us at once that identity morphisms, as defined above, are identities for composition; and the associative law for composition follows from the triviality

$$\models_{\mathscr{P}} \forall x, w(\exists z(\exists y(F(x,y) \wedge G(y,z)) \wedge H(z,w)) \leftrightarrow \exists y(F(x,y) \wedge \exists z(G(y,z) \wedge H(z,w)))$$

We now embark on the proof that $\mathscr{P}$-**Set** is a topos. The argument should have a familiar appearance to anyone who has read (5); the only technical complications are mild ones arising from the fact that we do not have any canonical way of choosing representative functional relations for morphisms of $\mathscr{P}$-sets.

2·6 LEMMA. *The category* $\mathscr{P}$-**Set** *has finite limits.*

Proof. First we construct a terminal object. Let 1 be any one-element set, and make it into a $\mathscr{P}$-set by defining $=$ to be the top element of $\mathscr{P}(1 \times 1)$. For any $\mathscr{P}$-set $(X, =)$, if we regard the existence predicate E_X as an element of $\mathscr{P}(X \times 1)$, it is functional from X to 1; and it is clearly the unique such relation up to equivalence.

The product $(X, =) \times (Y, =)$ of $\mathscr{P}$-sets has already been defined after 2·3. The product projections $p: X \times Y \longrightarrow X$ and $q: X \times Y \longrightarrow Y$ are represented by the relations

$$[\![x = x' \wedge y \in Y]\!] \in \mathscr{P}(X \times Y \times X) \quad \text{and} \quad [\![x \in X \wedge y = y']\!] \in \mathscr{P}(X \times Y \times Y)$$

respectively. Given the morphisms $f: Z \longrightarrow X$ and $g: Z \longrightarrow Y$, then the pairing $(f,g): Z \longrightarrow X \times Y$ is represented by

$$[\![F(z,x) \wedge G(z,y)]\!] \in \mathscr{P}(Z \times X \times Y).$$

As usual, it is a straightforward application of soundness to show that the above relations are functional, that $p(f,g) = f$ and $q(f,g) = g$, and that (f,g) is uniquely determined by its composites with p and q.

To complete the proof of the Lemma, we have to define equalizers. Let $f, g: X \rightrightarrows Y$ be a parallel pair of morphisms in $\mathscr{P}$-**Set**. We define a morphism $h: E \longrightarrow X$ as follows: the $\mathscr{P}$-set E has the same underlying set as X, with membership predicate

$$[\![x \in E]\!] = [\![\exists y(F(x,y) \wedge G(x,y))]\!]$$

and equality $\qquad [\![x =_E x']\!] = [\![x =_X x' \wedge x \in E \wedge x' \in E]\!].$

(Here we have used the freedom to treat membership as a separate primitive; since F and G are strict relations, it is clear that

$$x \in E \models_{\mathscr{P}} x \in X,$$

and hence that the axiom $(*)$ is satisfied.) The morphism h is represented by H, where

$$[\![H(x,x')]\!] = [\![x \in E \wedge x =_X x']\!];$$

from the definition of E, it is easy to verify that $fh = gh$. Conversely if $k: Z \longrightarrow X$ satisfies $fk = gk$, then

$$\models^{x,y,z}_{\mathscr{P}} K(\mathbf{z},\mathbf{x}) \wedge F(\mathbf{x},\mathbf{y}) \rightarrow G(\mathbf{x},\mathbf{y}),$$

whence (since F is total) we obtain

$$\models^{x,z}_{\mathscr{P}} K(\mathbf{z},\mathbf{x}) \rightarrow \exists \mathbf{y}(F(\mathbf{x},\mathbf{y}) \wedge G(\mathbf{x},\mathbf{y})).$$

So K is still strict (and therefore functional) as a relation on $Z \times E$; that is, it defines a morphism $\tilde{k}: Z \longrightarrow E$, which is clearly the unique factorization of k through h.

2·7 *Remark.* In the construction of equalizers in the proof of 2·6, it is clear that the object E depends on the representatives F and G, and not just on the morphisms f and g. So unless we assume the axiom of choice in our meta-logic, we cannot assert that $\mathscr{P}$-**Set** has equalizers in the 'constructive' sense that there is a function assigning a choice of equalizers to each parallel pair of morphisms.

From the proof of Lemma 2·6, it is clear that the construction of finite limits in $\mathscr{P}$-**Set** is obtained by 'writing down the formulae which define limits in **Set**' and interpreting them in the $\mathscr{P}$-logic rather than in standard logic. This observation has a converse: to recognize that a given finite diagram in $\mathscr{P}$-**Set** is a limit, it suffices to check the validity of the appropriate formulae in the $\mathscr{P}$-logic. As an example, we state

2·8 LEMMA. *A commutative square*

$$\begin{array}{ccc} P & \xrightarrow{k} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

in $\mathscr{P}$-**Set** *is a pullback if and only if we have*

$$\models_{\mathscr{P}} \forall \mathbf{x},\mathbf{y},\mathbf{z} \quad (F(\mathbf{x},\mathbf{z}) \wedge G(\mathbf{y},\mathbf{z}) \rightarrow \exists \mathbf{p}(H(\mathbf{p},\mathbf{y}) \wedge K(\mathbf{p},\mathbf{x})))$$

and $$\models_{\mathscr{P}} \forall \mathbf{p},\mathbf{p}',\mathbf{x},\mathbf{y} \quad (H(\mathbf{p},\mathbf{y}) \wedge H(\mathbf{p}',\mathbf{y}) \wedge K(\mathbf{p},\mathbf{x}) \wedge K(\mathbf{p}',\mathbf{x}) \rightarrow \mathbf{p} = \mathbf{p}')$$

for some (*equivalently, for any*) *choice of representatives* F, G, H, K *for* f, g, h, k.

Proof. It we construct the pullback $X \times_Z Y$ in the usual way from products and equalizers, it is an easy application of soundness to show that it satisfies the above conditions. Conversely, if the conditions are satisfied, then the induced morphism $P \longrightarrow X \times_Z Y$ may be represented by a relation which is functional in either direction, and hence this morphism is an isomorphism.

2·9 COROLLARY. *A morphism* $f: X \longrightarrow Y$ *in* Y *in* $\mathscr{P}$-**Set** *is a monomorphism if and only if*

$$\models_{\mathscr{P}} \forall \mathbf{x},\mathbf{x}',\mathbf{y} \quad (F(\mathbf{x},\mathbf{y}) \wedge F(\mathbf{x}',\mathbf{y}) \rightarrow \mathbf{x} = \mathbf{x}')$$

for some (*equivalently, for any*) *representative* F *for* f.

Proof. Use Lemma 2·8, and the fact that f is mono if and only if

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback.

In the construction of equalizers in 2·6, we used the fact that, given a strict unary relation A on a $\mathscr{P}$-set $(X, =_X)$, we may define a monomorphism

$$m: \|A\| \longrightarrow X$$

by defining $\|A\|$ to be the $\mathscr{P}$-set with underlying set X, $[\![\mathbf{x} \in \|A\|\,]\!] = [\![A(\mathbf{x})]\!]$ and $[\![\mathbf{x} =_A \mathbf{x}']\!] = [\![\mathbf{x} =_X \mathbf{x}' \wedge \mathbf{x} \in \|A\| \wedge \mathbf{x}' \in \|A\|\,]\!]$, and m to be the morphism represented by M, where $[\![M(\mathbf{a}, \mathbf{x})]\!] = [\![\mathbf{a} \in \|A\| \wedge \mathbf{a} =_X \mathbf{x}]\!]$. We shall call a monomorphism which arises in this way *canonical*.

2·10 Lemma. *Any subobject of a $\mathscr{P}$-set X can be represented by a canonical monomorphism.*

Proof. Given a mono $f\colon Y \longrightarrow X$, consider the strict relation A defined by

$$[\![A(\mathbf{x})]\!] = [\![\exists \mathbf{y}(F(\mathbf{y}, \mathbf{x}))]\!]$$

where F is a representative for f. It is clear from the definition and Corollary 2·9 that F defines a relation on $Y \times \|A\|$ which is functional in either direction, i.e. there is an isomorphism $\bar{f}\colon Y \longrightarrow \|A\|$ such that $m\bar{f} = f$.

The name 'canonical' is however slightly misleading, since a subobject of X is not *uniquely* representable by a canonical monomorphism. If A and B are relations such that

$$\models_{\mathscr{P}} \forall \mathbf{x} \quad (A(\mathbf{x}) \leftrightarrow B(\mathbf{x})),$$

then $\|A\|$ and $\|B\|$ are isomorphic over X, but not necessarily equal.

2·11 Lemma. *Let $f\colon Y \longrightarrow X$ be a morphism of $\mathscr{P}$-sets, and A a strict relation on X. Then there is a pullback diagram*

$$\begin{array}{ccc} \|f^{-1}A\| & \longrightarrow & \|A\| \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

*in $\mathscr{P}$-*Set*, where the vertical arrows are canonical monos and $f^{-1}A$ is defined by*

$$[\![f^{-1}A(\mathbf{y})]\!] = [\![\exists \mathbf{x}(F(\mathbf{y}, \mathbf{x}) \wedge A(\mathbf{x}))]\!]$$

for some representative F for f.

Proof. Since F is a strict relation, it is clear that $f^{-1}A$ is strict. The top arrow in the diagram is just the restriction of f, i.e. it is represented by

$$[\![F(\mathbf{y}, \mathbf{x}) \wedge A(\mathbf{x})]\!] \in \mathscr{P}(Y \times X).$$

It is then clear that the diagram commutes; the fact that it is a pullback follows easily from Lemma 2·8.

2·12 Proposition. *The category $\mathscr{P}$-*Set* has power-objects; i.e., for any object X, the functor*

$$\mathrm{Sub}\,(X \times -)\colon (\mathscr{P}\text{-}\mathbf{Set})^{\mathrm{op}} \to \mathbf{Set}$$

is representable.

Proof. Let $(X, =)$ be a $\mathscr{P}$-set. We define its power-object PX to have underlying set Σ^X (where Σ is the index set of the generic predicate for $\mathscr{P}$), with

$$[\![\mathbf{R} \in PX]\!] = [\![\forall \mathbf{x}, \mathbf{x}'(\in_X(\mathbf{x}, \mathbf{R}) \wedge \mathbf{x} = \mathbf{x}' \to \in_X(\mathbf{x}', \mathbf{R})) \wedge \forall \mathbf{x}(\in_X(\mathbf{x}, \mathbf{R}) \to E_X(\mathbf{x}))]\!]$$

and
$$[\![\mathbf{R} =_{PX} \mathbf{S}]\!] = [\![\mathbf{R} \in PX \wedge \mathbf{S} \in PX \wedge \forall \mathbf{x}(\in_X(\mathbf{x}, \mathbf{R}) \leftrightarrow \in_X(\mathbf{x}, \mathbf{S}))]\!]$$

where $\in_X \in \mathscr{P}(X \times \Sigma^X)$ is the membership predicate defined in 1·3(ii) and $\mathbf{R}$, $\mathbf{S}$ are variables of type Σ^X. (Henceforth we shall write '$\mathbf{x} \in_X \mathbf{R}$' in place of '$\in_X(\mathbf{x}, \mathbf{R})$'.)

We have a strict relation E on $X \times PX$ defined by

$$[\![E(\mathbf{x}, \mathbf{R})]\!] = [\![\mathbf{R} \in PX \wedge \mathbf{x} \in_X \mathbf{R}]\!]$$

and hence a subobject $\|E\| \longrightarrow X \times PX$. By the Yoneda Lemma, this induces a natural transformation

$$\mathscr{P}\text{-}\mathbf{Set}(-, PX) \longrightarrow \mathrm{Sub}(X \times -)$$

which by Lemma 2·11 sends a morphism $f\colon Y \longrightarrow PX$ to the subobject represented by $\|(\mathrm{id}_X \times f)^{-1}E\| \longrightarrow X \times Y$. We have to show that this transformation is a bijection.

Suppose given a subobject of $X \times Y$, which (by Lemma 2·10) we may assume to be canonically represented by a strict relation A on $X \times Y$. By Remark 1·3(ii), there is a function $g\colon Y \longrightarrow \Sigma^X$ in **Set** such that $\mathscr{P}(\mathrm{id}_X \times g)(\in_X) \dashv\vdash A$ in $\mathscr{P}(X \times Y)$. Define $F \in \mathscr{P}(Y \times \Sigma^X)$ by

$$[\![F(\mathbf{y}, \mathbf{R})]\!] = [\![\mathbf{y} \in Y \wedge g(\mathbf{y}) =_{PX} \mathbf{R}]\!]$$

(note that we are here extending our language by using g as a function-symbol). Since A is a strict relation, it is easy to see that

$$\mathbf{y} \in Y \models^{\mathbf{y}}_{\mathscr{P}} g(\mathbf{y}) \in PX,$$

from which it follows directly that F is a functional relation, and so defines a morphism $f\colon Y \longrightarrow PX$ in $\mathscr{P}$-**Set**. It is straightforward to check that f is independent of the choice of representatives A and g, so that we have a function

$$\mathrm{Sub}(X \times Y) \longrightarrow \mathscr{P}\text{-}\mathbf{Set}(Y, PX).$$

Now the isomorphism $\mathscr{P}(\mathrm{id} \times g)(\in_X) \dashv\vdash A$ in $\mathscr{P}(X \times Y)$ can be interpreted as the statement

$$\models^{\mathbf{x}, \mathbf{y}}_{\mathscr{P}} A(\mathbf{x}, \mathbf{y}) \leftrightarrow \exists \mathbf{R}(g(\mathbf{y}) = \mathbf{R} \wedge \mathbf{x} \in_X \mathbf{R}),$$

whence we obtain

$$\models^{\mathbf{x}, \mathbf{y}}_{\mathscr{P}} A(\mathbf{x}, \mathbf{y}) \leftrightarrow \exists \mathbf{R}(F(\mathbf{y}, \mathbf{R}) \wedge E(\mathbf{x}, \mathbf{R}))$$

since A is a strict relation; thus we have $\|A\| = \|(\mathrm{id} \times f)^{-1}E\|$ as a subobject of $X \times Y$. A similar argument shows that the composite

$$\mathscr{P}\text{-}\mathbf{Set}(Y, PX) \longrightarrow \mathrm{Sub}(X \times Y) \longrightarrow \mathscr{P}\text{-}\mathbf{Set}(Y, PX)$$

is the identity; so the natural transformation defined above is an isomorphism.

Combining Proposition 2·12 with Lemma 2·6, we at once obtain

2·13 THEOREM. *For any tripos $\mathscr{P}$, the category $\mathscr{P}$-*Set* is a topos.*

2·14 *Remark.* In particular, 2·13 tells us that $\mathscr{P}$-**Set** has finite colimits, exponentials and a subobject classifier. It is convenient to describe explicit constructions for some of these (though we shall omit the proofs).

(i) Finite coproducts: The initial object of $\mathscr{P}$-**Set** is of course the empty set, with its unique (up to isomorphism) equality structure. Given $\mathscr{P}$-sets $(X, =_X)$ and $(Y, =_Y)$, their coproduct has underlying set $X \amalg Y$, with equality given by

$$\exists(i \times i)(=_X) \vee \exists(j \times j)(=_Y) \in \mathscr{P}((X \amalg Y) \times (X \amalg Y))$$

where $i: X \longrightarrow X \amalg Y$ and $j: Y \longrightarrow X \amalg Y$ are the coproduct inclusions.

(ii) Exponentials: The exponential $(Y, =)^{(X, =)}$ has underlying set $\Sigma^{X \times Y}$, with

$$[\![\mathbf{F} \in Y^X]\!] = [\![\text{'}\mathbf{F} \text{ is a functional relation'}]\!]$$

and $$[\![\mathbf{F} = \mathbf{G}]\!] = [\![\mathbf{F} \in Y^X \wedge \mathbf{G} \in Y^X \wedge \forall \mathbf{x}, \mathbf{y}\, ((\mathbf{x}, \mathbf{y}) \in \mathbf{F} \leftrightarrow (\mathbf{x}, \mathbf{y}) \in \mathbf{G})]\!].$$

(iii) Subobject classifier: The subobject classifier Ω has Σ as underlying set, with $[\![\mathbf{p} = \mathbf{q}]\!] = [\![\mathbf{p} \leftrightarrow \mathbf{q}]\!]$.

2·15 *Examples.* To conclude this section, we shall revisit some of the examples of triposes which we described in § 1, and point out some particular features of the toposes they generate.

(i) Firstly, if $\mathscr{P}$ is the canonical tripos of a locale A (1·6 (i)), then the notion of $\mathscr{P}$-set is easily seen to be equivalent to that of an A-valued set as defined in (7); so $\mathscr{P}$-**Set** is equivalent to the Grothendieck topos Shv (A) of (canonical) sheaves on A.

(ii) Next, suppose $\mathscr{P}$ is induced by a filter Φ on a locale A. For each $\mathscr{P}$-set $(X, =_X)$, we can find $a \in \Phi$ (depending on X) such that

$$a \leqslant [\![\text{'}=_X \text{ is symmetric and transitive'}]\!]$$

in $\mathscr{P}(1) = A$; then it is clear that we may regard X as an A_a-valued set, where A_a is the open sublocale of A determined by a. Similarly, any morphism of $\mathscr{P}$-sets may be regarded as a morphism of A_a-valued sets for some $a \in \Phi$; so we conclude that $\mathscr{P}$-**Set** is the filtered colimit of the toposes Shv (A_a), $a \in \Phi$ (the transition maps

$$\text{Shv}(A_a) \longrightarrow \text{Shv}(A_b)$$

being the logical functors induced by the open inclusions $A_b \longrightarrow A_a$ when $b \leqslant a$). Equivalently, $\mathscr{P}$-**Set** may be described as the *filterpower* Shv $(A)_\Phi$ constructed in ((9), § 9·3). (A similar argument shows that, if $\mathscr{P}_\Phi$ is the tripos obtained from a filter Φ on an arbitrary tripos $\mathscr{P}$, then $\mathscr{P}_\Phi$-**Set** $\simeq (\mathscr{P}\text{-}\mathbf{Set})_{\hat{\Phi}}$, where $\hat{\Phi}$ denotes the filter of those subobjects of 1 in $\mathscr{P}$-**Set** which may be canonically represented by predicates in Φ.)

Note in particular that in this example the lattice of subobjects of 1 in $\mathscr{P}$-**Set** may be identified with the Heyting algebra quotient A/Φ, which is not always complete; for example, if A is the open-set locale of a space X and Φ the filter of open neighbourhoods of a point x, then A/Φ is the lattice of 'germs of open sets' at x. Thus toposes of this form are not always Grothendieck toposes; indeed it can be shown (1) that the topos Shv $(A)_\Phi$ is Grothendieck if and only if the filter Φ is principal.

(iii) Finally, we consider the recursive realizability tripos of 1·8 (ii). Our concern here is to make explicit the behaviour of coproducts in $\mathscr{P}$-**Set**. It is easily verified that the disjunction in $\mathscr{P}1 = P\mathbb{N}$ may be defined by

$$A \vee B = \{\langle 0, a\rangle \mid a \in A\} \cup \{\langle 1, b\rangle \mid b \in B\}$$

where $\langle -, - \rangle$ is a suitable pairing function $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$. Thus $(X, =) \amalg (Y, =)$ has underlying set $X \amalg Y$, with equality

$$\begin{aligned} [\![x = x']\!] &= \{\langle 0, a\rangle \mid a \in [\![x = x']\!]\} \quad \text{if} \quad x, x' \in X \\ &= \{\langle 1, a\rangle \mid a \in [\![x = x']\!]\} \quad \text{if} \quad x, x' \in Y \\ &= \varnothing \quad \text{otherwise.} \end{aligned}$$

It is now obvious how we may extend this definition to the disjoint union of a countable family $(X_n \mid n \in \mathbb{N})$ of $\mathscr{P}$-sets, and straightforward to verify that it does yield the coproduct of the X_n in $\mathscr{P}$-**Set**. In particular, a countable copower of 1 exists in $\mathscr{P}$-**Set**; up to isomorphism, it may be described as the set $\mathbb{N}$ with equality

$$\begin{aligned} [\![m = n]\!] &= \{n\} \quad \text{if} \quad m = n \\ &= \varnothing \quad \text{otherwise.} \end{aligned}$$

It is further straightforward to verify that this object (with the obvious definitions for the zero and successor maps) is a natural number object in $\mathscr{P}$-**Set**. (The details will appear in (8).)

On the other hand, if $(X_\alpha \mid \alpha \in A)$ is a family of $\mathscr{P}$-sets having a coproduct X in $\mathscr{P}$-**Set** and x_α, x_β are elements of X_α, X_β with $\alpha \neq \beta$ and $[\![x_\alpha \in X_\alpha]\!] \neq \varnothing \neq [\![x_\beta \in X_\beta]\!]$, then it is easy to show that there must exist elements y_α, y_β of X with $J_\alpha(x_\alpha, y_\alpha) \neq \varnothing \neq J_\beta(x_\beta, y_\beta)$ (where J_α is a representative for the coproduct inclusion $j_\alpha: X_\alpha \longrightarrow X$), and that for any such pair of elements $[\![y_\alpha \in X]\!]$ and $[\![y_\beta \in X]\!]$ must be disjoint. (This is because we may define a family of maps $f_\alpha: X_\alpha \longrightarrow 1 \amalg 1$ which 'send' x_α and x_β to elements of $1 \amalg 1$ with disjoint extents.) It follows that the number of *inhabited* $\mathscr{P}$-sets in the family $(X_\alpha \mid \alpha \in A)$ must be countable; loosely, we can say that $\mathscr{P}$-**Set** has no uncountable coproducts. In particular, $\mathscr{P}$-**Set** is not a Grothendieck topos.

3. *Geometric morphisms.* It is well known that continuous maps between locales correspond precisely to geometric morphisms between their associated sheaf toposes. In this section we generalize this result to triposes. To be able to construct direct image functors, we must first consider a process analogous to the construction of the sheaf generated by a locale-valued set ((5), 4·17).

Let $\mathscr{P}$ be a tripos and $(X, =)$ a $\mathscr{P}$-**Set**. We define a predicate S ('is a singleton') in $\mathscr{P}(\Sigma^X)$ by

$$[\![S(\mathbf{R})]\!] = [\![\exists \mathbf{x}\,(\mathbf{x} \in X \wedge \forall \mathbf{x}'(\mathbf{x}' \in_X \mathbf{R} \leftrightarrow \mathbf{x}' = \mathbf{x}))]\!].$$

This is easily seen to be a strict relation on the power-object PX of X (as defined in the proof of 2·12), so it determines a canonical monomorphism which we write as

$$SX \longrightarrow PX.$$

Up to isomorphism, this is just the singleton map $\{\,\}: X \longrightarrow PX$ in the topos $\mathscr{P}$-**Set**; more explicitly, we have

3·1 LEMMA. *The predicate in* $\mathscr{P}(X \times \Sigma^X)$ *given by*

$$[\![\mathbf{x} \in_X \mathbf{R} \wedge S(\mathbf{R})]\!]$$

represents an isomorphism $X \longrightarrow SX$ *in* $\mathscr{P}$-**Set**.

Proof. A routine application of the Soundness Lemma (2·1).

3·2 *Definition.* Given $\mathscr{P}$-**Sets** $(X, =)$ and $(Y, =)$, we say that $F \in \mathscr{P}(X \times Y)$ is a *partial functional relation* from X to Y if it is a strict single-valued relation (in the sense of Definitions 2·3 and 2·4) on $X \times Y$. We say that $(Y, =)$ is *weakly complete* if, given any partial functional relation F from X to Y, there is a function $f\colon X \longrightarrow Y$ in **Set** such that

$$[\![\exists \mathbf{y}\,(F(\mathbf{x},\mathbf{y}))]\!] \dashv\vdash [\![F(\mathbf{x},f(\mathbf{x}))]\!],$$

in $\mathscr{P}X$.

When $\mathscr{P}$ is the canonical tripos of a locale A, the A-**Set** SX has a natural A-sheaf structure which makes it into the sheaf generated by the A-**Set** X ((5), theorem 4·18). The notion of weak completeness represents as much of this additional structure as we are able to salvage in our more general context:

3·3 Proposition. *For any $\mathscr{P}$-set X, SX is weakly complete. In particular, any $\mathscr{P}$-set is isomorphic to a weakly complete one.*

Proof. The second assertion follows from the first by Lemma 3·1. To prove the first, let $F \in \mathscr{P}(Y \times \Sigma^X)$ be a partial functional relation from Y to SX, and define

$$G \in \mathscr{P}(X \times Y)$$

by

$$[\![G(\mathbf{x},\mathbf{y})]\!] = [\![\exists \mathbf{R}\,(\mathbf{x} \in_X \mathbf{R} \wedge F(\mathbf{y},\mathbf{R}))]\!].$$

Then by Remark 1·3 (ii) we have a map $f\colon Y \to \Sigma^X$ with

$$[\![\mathbf{x} \in_X f(\mathbf{y})]\!] \dashv\vdash [\![G(\mathbf{x},\mathbf{y})]\!]$$

in $\mathscr{P}(X \times Y)$. Now we certainly have

$$F(\mathbf{y},f(\mathbf{y})) \models^{\mathbf{y}}_{\mathscr{P}} \exists \mathbf{R}(F(\mathbf{y},\mathbf{R}));$$

but, since F is a partial functional relation, it is clear that

$$\models^{\mathbf{y},\mathbf{R}}_{\mathscr{P}} F(\mathbf{y},\mathbf{R}) \to \mathbf{R} \in SX \wedge f(\mathbf{y}) \in SX \wedge \exists \mathbf{x}\,(\mathbf{x} \in_X \mathbf{R} \wedge \mathbf{x} \in_X f(\mathbf{y})),$$

i.e.

$$\models^{\mathbf{y},\mathbf{R}}_{\mathscr{P}} F(\mathbf{y},\mathbf{R}) \to \mathbf{R} =_{SX} f(\mathbf{y}),$$

so that

$$\exists \mathbf{R}(F(\mathbf{y},\mathbf{R})) \models F(\mathbf{y},f(\mathbf{y})).$$

3.4. *Definition.* Let $\mathscr{P}$ and $\mathscr{R}$ be triposes. A *geometric morphism* $f\colon \mathscr{P} \longrightarrow \mathscr{R}$ is given by a pair of **Set**-indexed functors

$$f_*\colon \mathscr{P} \longrightarrow \mathscr{R}, \quad f^*\colon \mathscr{R} \longrightarrow \mathscr{P}$$

such that, for each set I, the functor $(f^*)_I$ is left adjoint to $(f_*)_I$ and preserves finite limits (= meets).

Given such a morphism f, we wish to 'extend' it to a geometric morphism of toposes $\bar{f}\colon \mathscr{P}$-**Set** $\longrightarrow \mathscr{R}$-**Set**. Now since f_* preserves substitution (i.e. commutes with the functors $\mathscr{R}f$, up to isomorphism), on taking left adjoints we see that f^* preserves $\exists$, as well as $\vdash$, $\top$ and $\wedge$. That is, it preserves all the logical structure we needed to construct the category $\mathscr{R}$-**Set** and show it had finite limits. It follows at once that we can define a left exact functor $\bar{f}^*: \mathscr{R}$-**Set** $\longrightarrow \mathscr{P}$-**Set** by setting $\bar{f}^*(Y, =) = (Y, f^*(=))$ for an $\mathscr{R}$-set $(Y, =)$, and letting $\bar{f}^*(g)$ (for a morphism $g: (Y, =) \longrightarrow (Y', =)$, represented by $G \in \mathscr{R}(Y \times Y')$, say) be the morphism represented by $f^*G \in \mathscr{P}(Y \times Y')$.

The definition of $\bar{f}_*$ is less straightforward, since f_* does not in general preserve $\exists$; so, if we apply it to a total functional relation between $\mathscr{P}$-sets, the result will be only a partial functional relation of $\mathscr{P}$-sets. However, the existence of weak completions, which we demonstrated in Proposition 3·3, enables us to get round this difficulty.

3·5 PROPOSITION. *Let $f: \mathscr{P} \to \mathscr{R}$ be a geometric morphism of triposes. Then there is a geometric morphism $\bar{f}: \mathscr{P}$-**Set** $\to \mathscr{R}$-**Set**, whose inverse image is the functor $\bar{f}^*$ defined above.*

Proof. Given a $\mathscr{P}$-set $(X, =)$, we define $\bar{f}_*(X, =)$ to be $(\Sigma^X, f_*(=_{SX}))$. Consider the predicate $E \in \mathscr{P}(\Sigma^X \times X)$ given by

$$[\![E(\mathbf{R}, \mathbf{x})]\!] = f^*f_*[\![\mathbf{R} \in SX]\!] \wedge [\![\mathbf{x} \in_X \mathbf{R}]\!].$$

Since $f^*f_* \vdash \mathrm{id}$, it follows easily from Lemma 3·1 that E represents a morphism

$$\epsilon_X: \bar{f}^*\bar{f}_*(X, =) \longrightarrow (X, =)$$

in $\mathscr{P}$-**Set**. We shall show that ϵ_X is universal among maps $\bar{f}^* Y \longrightarrow X$, so that $\bar{f}_*$ may be made into a functor right adjoint to $\bar{f}^*$.

Given a morphism $g: \bar{f}^* Y \longrightarrow X$, we first compose it with the isomorphism of 3·1 to obtain a morphism g', which we may represent by a predicate $G' \in \mathscr{P}(Y \times \Sigma^X)$. Define $\bar{G} \in \mathscr{R}(Y \times \Sigma^X)$ by

$$[\![\bar{G}(\mathbf{y}, \mathbf{R})]\!] = f_*[\![G'(\mathbf{y}, \mathbf{R})]\!] \wedge [\![\mathbf{y} \in \mathbf{Y}]\!].$$

It follows easily that $\bar{G}$ is a strict single-valued relation from Y to $\bar{f}_* X$. To see that it is total, we apply Proposition 3·3 to obtain a map $\gamma: Y \longrightarrow \Sigma^X$ in **Set** with

$$[\![\exists \mathbf{R}(G'(\mathbf{y}, \mathbf{R}))]\!] \dashv\vdash [\![G'(\mathbf{y}, \gamma(\mathbf{y}))]\!]$$

in $\mathscr{P}Y$. Then since G' is total we have

$$\begin{aligned}
[\![\mathbf{y} \in Y]\!] \vdash f_*f^*[\![\mathbf{y} \in Y]\!] \vdash f_* &[\![\exists \mathbf{R}(G'(\mathbf{y}, \mathbf{R}))]\!] \\
&\dashv\vdash f_*[\![G'(\mathbf{y}, \gamma(\mathbf{y}))]\!] \\
&\dashv\vdash [\![(f_* G')(\mathbf{y}, \gamma(\mathbf{y}))]\!] \\
&\vdash [\![\exists \mathbf{R}((f_* G')(\mathbf{y}, \mathbf{R}))]\!].
\end{aligned}$$

So $\bar{G}$ is total, and represents a morphism $\bar{g}: Y \longrightarrow \bar{f}_* X$ in $\mathscr{R}$-**Set**; it is clear that $\bar{g}$ depends only on g and not on G'.

Now the composite $\epsilon_X . \bar{f}^*(\bar{g}): \bar{f}^* Y \longrightarrow X$ is represented by

$$[\![\exists \mathbf{R}(f^*\bar{G}(\mathbf{y}, \mathbf{R}) \wedge E(\mathbf{R}, \mathbf{x}))]\!],$$

i.e. by
$$[\![\exists \mathbf{R}(f^*f_*(G'(\mathbf{y}, \mathbf{R}) \wedge \mathbf{R} \in SX) \wedge \mathbf{x} \in_X \mathbf{R})]\!],$$

which entails
$$[\![\exists \mathbf{R}(G'(\mathbf{y}, \mathbf{R}) \wedge \mathbf{R} \in SX \wedge \mathbf{x} \in_X \mathbf{R})]\!]$$

in $\mathscr{P}(Y \times X)$. But the latter represents the composite

$$\bar{f}^* Y \xrightarrow{g'} SX \cong X$$

which is our original morphism g. By the remark which follows the definition of equivalence between functional relations in Definition 2·4, it follows that we have

$$g = \epsilon_X . \bar{f}^*(\bar{g}),$$

i.e. $\bar{g}$ is a morphism in the comma category $(\bar{f}^* \downarrow X)$ from (Y, g) to $(\bar{f}_* X, \epsilon_X)$.

It remains to show that $\bar{g}$ is the unique such map. Suppose given a morphism $h: Y \longrightarrow \bar{f}_* X$ (represented by $H \in \mathscr{R}(Y \times \Sigma^X)$, say) such that $g = \epsilon_X . \bar{f}^*(h)$. Then g' is the composite of $\epsilon_X . \bar{f}^*(h)$ with the isomorphism $X \longrightarrow SX$ of 3·1, i.e.

$$[\![G'(\mathbf{y}, \mathbf{R})]\!] \dashv\vdash [\![\exists \mathbf{R}'(f^*H(\mathbf{y}, \mathbf{R}') \wedge f^*f_*(\mathbf{R} \in SX) \wedge \mathbf{R}' =_{SX} \mathbf{R})]\!]$$

in $\mathscr{P}(Y \times \Sigma^X)$. Thus

$$f^*[\![H(\mathbf{y}, \mathbf{R}) \wedge f_*(\mathbf{R} \in SX)]\!] \vdash [\![G'(\mathbf{y}, \mathbf{R})]\!],$$

and hence (since H is a strict relation)

$$\begin{aligned} [\![H(\mathbf{y}, \mathbf{R})]\!] &\vdash [\![H(\mathbf{y}, \mathbf{R}) \wedge f_*(\mathbf{R} \in SX) \wedge \mathbf{y} \in Y]\!] \\ &\vdash [\![f_* G'(\mathbf{y}, \mathbf{R}) \wedge \mathbf{y} \in Y]\!] \\ &= [\![\bar{G}(\mathbf{y}, \mathbf{R})]\!]. \end{aligned}$$

So by the same remark as before, we have $h = \bar{g}$, as required.

3·6 *Remark*. In the proof of 3·5, we did not need to specify the effect of the functor $\bar{f}_*$ on morphisms. In fact it may be defined as follows: given $g: X \longrightarrow X'$ in $\mathscr{P}$-**Set**, represent the composite

$$SX \cong X \xrightarrow{g} X' \cong SX'$$

by a functional relation $G \in \mathscr{P}(\Sigma^X \times \Sigma^{X'})$; then $\bar{f}_*(g)$ is represented by

$$f_* G \in \mathscr{R}(\Sigma^X \times \Sigma^{X'}).$$

More generally, it can be shown that any left exact indexed functor $t: \mathscr{P} \to \mathscr{R}$ induces in this way a left exact functor $\bar{t}: \mathscr{P}$-**Set** $\longrightarrow \mathscr{R}$-**Set**.

When $\mathscr{P}$ and $\mathscr{R}$ are the canonical triposes of locales A and B, we know that (up to isomorphism) *every* geometric morphism $\mathscr{P}$-**Set** $\longrightarrow \mathscr{R}$-**Set** is induced as in 3·5 by a continuous map $A \longrightarrow B$. The proof of this rests on two facts: first, that in this case $\mathscr{P}$-**Set** and $\mathscr{R}$-**Set** are defined over **Set** by geometric morphisms, and secondly that every geometric morphism between toposes defined over **Set** is a morphism over **Set**. We have seen in 2·15 that, for a general tripos $\mathscr{P}$, the topos $\mathscr{P}$-**Set** need not be defined over **Set**. Nevertheless, we do have a functor Δ: **Set** $\longrightarrow \mathscr{P}$-**Set** which is analogous to the 'constant sheaf' functor in the localic case, and, if we restrict our attention to geometric morphisms $\mathscr{P}$-**Set** $\longrightarrow \mathscr{R}$-**Set** which respect these 'constant' functors, then we can give a converse result to Proposition 3·5.

3·7 *Definition*. We define a functor Δ: **Set** $\longrightarrow \mathscr{P}$-**Set** as follows: For a set X, ΔX is the $\mathscr{P}$-set $(X, \exists\Delta_X(\top_X))$, where $\Delta_X: X \longrightarrow X \times X$ is the diagonal map. For a function $f: X \longrightarrow Y$, $\Delta f: \Delta X \longrightarrow \Delta Y$ is represented by $\exists(\mathrm{id}_X, f)(\top_X) \in \mathscr{P}(X \times Y)$. It is straightforward to verify that this does define a functor.

3·8 LEMMA. *The functor* Δ *is left exact.*

Proof. It is clear that Δ preserves the terminal object. To see that it preserves pullbacks, use the fact that if

$$\begin{array}{ccc} P & \xrightarrow{k} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pullback in **Set**, then

$$\exists f(\top_X) \wedge \exists g(\top_Y) \dashv\vdash \exists(fk)(\top_P)$$

in $\mathscr{P}Z$.

3·9 PROPOSITION. *Let* $f\colon \mathscr{P} \to \mathscr{R}$ *be a geometric morphism of triposes. Then*

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\Delta'} & \mathscr{R}\text{-}\mathbf{Set} \\ & {\scriptstyle \Delta}\searrow & \downarrow{\scriptstyle \bar{f}^*} \\ & & \mathscr{P}\text{-}\mathbf{Set} \end{array}$$

commutes up to natural isomorphism.

Proof. Since f^* preserves $\top$ and $\exists$, it is clear that

$$f^*(=_{\Delta' X}) \dashv\vdash =_{\Delta X}$$

in $\mathscr{P}(X \times X)$, from which the result follows easily.

We can now state the converse of Proposition 3·5. We shall omit most of the details of the proof, since they are generally straightforward and we shall in any case not need to use the result.

3·10. PROPOSITION. *Let* $\mathscr{P}$ *and* $\mathscr{R}$ *be triposes, and* $g\colon \mathscr{P}\text{-}\mathbf{Set} \longrightarrow \mathscr{R}\text{-}\mathbf{Set}$ *a geometric morphism such that*

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\Delta'} & \mathscr{R}\text{-}\mathbf{Set} \\ & {\scriptstyle \Delta}\searrow & \downarrow{\scriptstyle g^*} \\ & & \mathscr{P}\text{-}\mathbf{Set} \end{array}$$

commutes up to isomorphism. Then there is a geometric morphism of triposes $f\colon \mathscr{P} \longrightarrow \mathscr{R}$ *such that* g *is isomorphic to* $\bar{f}$.

Proof. It is easy to see that every predicate $R \in \mathscr{P}X$ is a strict relation on the $\mathscr{P}$-set ΔX. Thus the assignment

$$R \mapsto (\|R\| \rightarrowtail \Delta X)$$

sets up an equivalence between the preorders $\mathscr{P}X$ and $\mathrm{Sub}_{\mathscr{P}\text{-}\mathbf{Set}}(\Delta X)$; and using Lemma 2·11 we may easily show that this equivalence is natural in X – i.e. it is an equivalence of **Set**-indexed categories. So for an element S of $\mathscr{R}X$, we may define f^*S (up to isomorphism) by requiring that

$$(\|f^*S\| \rightarrowtail \Delta X) \cong (g^*\|S\| \rightarrowtail g^*\Delta' X \cong \Delta X)$$

in Sub (ΔX). Similarly, we define f_* up to isomorphism by requiring that

$$\begin{array}{ccc} \|f_* R\| & \rightarrowtail & \Delta' X \\ \downarrow & & \downarrow \eta \\ g_*\|R\| & \rightarrowtail & g_*\,\Delta X \cong g_*\, g^*\Delta' X \end{array}$$

be a pullback in $\mathscr{R}$-**Set**, where η is the unit of $g^* \dashv g_*$.

4. *Relations between a tripos and its topos.* In this section we present some results which relate properties of a tripos $\mathscr{P}$ to properties of the topos $\mathscr{P}$-**Set** which it generates. As a very special case of Proposition 3·5, we now know that triposes which are equivalent as **Set**-indexed categories generate equivalent toposes, so we shall lose no generality by assuming (as we do henceforth) that $\mathscr{P}$ is canonically presented as in 1·9, and that it has pointwise propositional operations and fibre-wise quantification as in 1·11 and 1·12.

In addition to the functor Δ: **Set** $\longrightarrow$ $\mathscr{P}$-**Set** defined in the last section, we shall also need to consider the global section functor Γ: $\mathscr{P}$-**Set** $\longrightarrow$ **Set**. From our construction of weak completions in 3·3 (or directly from the definition of a morphism $1 \longrightarrow X$ in $\mathscr{P}$-**Set**), it is easy to see that we have

$$\Gamma(X, =) = \{f \in \Sigma^X \mid \top \vdash [\![f \in SX]\!]\}/\sim$$

where $\sim$ is the equivalence relation defined by

$$f \sim g \quad \text{iff} \quad \top \vdash [\![f =_{SX} g]\!].$$

We now introduce some notation. Let $A = \Sigma/\dashv\vdash$ be the Heyting algebra obtained from the Heyting pre-algebra $\Sigma = \mathscr{P}1$ by factoring out the intersection of the preorder and its opposite, and let $q: \Sigma \longrightarrow A$ be the quotient map. We shall write $a: A \longrightarrow \Sigma$ for the element $\forall q(\mathrm{id}_\Sigma)$ of $\mathscr{P}A = \Sigma^A$, and similarly $e: A \longrightarrow \Sigma$ for $\exists q(\mathrm{id}_\Sigma)$. We shall say that $\mathscr{P}$ has *standard universal quantification* (briefly, '$\mathscr{P}$ is $\forall$-standard') if the composite qa is the identity on A; since quantification is computed fibre-wise, this is equivalent to saying that for every isomorphism class $E \subseteq \Sigma$ we have $\bigwedge E \in E$, where $\bigwedge$ is the operation defined in 1·12. Similarly, we say $\mathscr{P}$ is $\exists$-standard if $qe = \mathrm{id}_A$.

4·1 Theorem. *The following conditions on a tripos $\mathscr{P}$ are equivalent:*

(i) *$\mathscr{P}$ is $\forall$-standard.*

(ii) *$\forall D(\iota) \in D$, where $D = \{P \in \Sigma \mid \top \vdash P\}$ and $\iota: D \longrightarrow \Sigma$ is the inclusion.*

(iii) *For every set I, the uniform preorder $\vdash_I$ on $\mathscr{P}I = \Sigma^I$ coincides with the pointwise preorder.*

(iv) *The Heyting algebra A is complete, and $q: \Sigma \longrightarrow A$ induces an equivalence between $\mathscr{P}$ and the canonical tripos of A.*

(v) *The functors Γ and Δ define a geometric morphism $\mathscr{P}$-**Set** $\to$ **Set**.*

(vi) *Δ is an inverse image functor.*

(vii) *Δ preserves all small coproducts.*

Proof. (i) $\Rightarrow$ (ii) is trivial, since D is one of the fibres of q.

(ii) $\Rightarrow$ (iii): Clearly uniform preorder is always contained in pointwise preorder, since the maps $P(i)\colon \mathscr{P}I \longrightarrow \mathscr{P}1$ induced by elements $i\colon 1 \longrightarrow I$ are order-preserving. So let f, g be two elements of $\mathscr{P}I = \Sigma^I$ such that $f(i) \vdash g(i)$ for each i. Then

$$f(i) \to g(i) = (f \to g)\,(i) \in D$$

for each i, so we can write $f \to g = \mathscr{P}h(\iota)$ for some $h\colon I \longrightarrow D$. But then we have

$$\forall D(\iota) \vdash \forall D \forall h\, \mathscr{P}h(\iota) \dashv\vdash \forall I(f \to g),$$

so $\forall I(f \to g) \in D$. But by the remarks after Proposition 1·11 this is equivalent to $f \vdash_I g$.

(iii) $\Rightarrow$ (i) is trivial, since if the uniform order on Σ^E is pointwise (where E is one of the fibres of q), then any member of E is a uniform lower bound for the inclusion map $E \to \Sigma$.

(iii) $\Rightarrow$ (iv): Given (iii) (and hence (i)), the maps q and a form an equivalence of pre-ordered sets between Σ and A. Moreover, since Σ^I is ordered pointwise, it is clear that $q^I\colon \Sigma^I \to A^I$ is an equivalence for any I. In particular, we deduce that A is complete as a **Set**-indexed category, and hence as a lattice.

(iv) $\Rightarrow$ (v): The equivalence between $\mathscr{P}$ and (the canonical tripos of) A induces an equivalence between $\mathscr{P}$-**Set** and A-**Set**, which commutes up to isomorphism with the functors Γ and Δ. But, for the topos A-**Set**, it is well known that Γ and Δ form a geometric morphism.

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) is trivial.

(vii) $\Rightarrow$ (ii): The element $\iota \in \Sigma^D$ defines a canonical subobject $\|\iota\|$ of ΔD in $\mathscr{P}$-**Set**. Now for any element d of D, it follows easily from Lemma 2·11 that the pullback of $\|\iota\|$ along $\Delta(d)\colon \Delta 1 \longrightarrow \Delta D$ is the canonical subobject $\|d\|$, which is isomorphic to 1 by the definition of D. But if ΔD is a D-indexed copower of 1, then the family of all maps $\Delta(d)$ is epimorphic, and so the inclusion $\|\iota\| \longrightarrow \Delta D$ must be an isomorphism. That is, we have

$$\models_{\mathscr{P}} \forall \mathbf{d}\,(\top \to \iota(\mathbf{d}))$$

or equivalently $\forall D(\iota) \in D$.

4·2 COROLLARY. *Let $\mathscr{P}$ be a (canonically presented) tripos, and suppose the preorder on $\Sigma = \mathscr{P}1$ is a partial order. Then Σ is a locale, and $\mathscr{P}$ is its canonical tripos.*

Proof. Since $\vdash$ is a partial order on Σ, D is the singleton subset $\{\top\}$, and so $\forall D$ is an isomorphism. Thus condition (ii) of Theorem 4·1 is trivially satisfied.

We have seen that $\mathscr{P}$-**Set** forms a Grothendieck topos in the 'obvious' way (i.e. with Δ as the inverse image of the geometric morphism $\mathscr{P}$-**Set** $\longrightarrow$ **Set**) if and only if $\mathscr{P}$ is $\forall$-standard. It is natural to ask whether $\mathscr{P}$-**Set** can be a Grothendieck topos in some 'non-obvious' way. Unfortunately, we have not been able to answer this question (though, in the particular case of filterpowers, the answer is known to be no (1)); it is not hard to see that $\mathscr{P}$-**Set** has small hom-sets and a set of generators, so the relevant question is cocompleteness. However, the example of the recursive realizability topos (2.15 (iii)) shows that $\mathscr{P}$-**Set** may have some infinite coproducts even when Δ does not preserve them.

It is possible slightly to refine the equivalence between (iii) and (vii) in Theorem 4·1. Let us call $\mathscr{P}$ *κ-standard* (where κ is a regular cardinal) if the uniform preorder $\vdash_I$ on Σ^I agrees with the pointwise preorder for every set I of cardinality less than κ. (Note that the tripos induced by a filter Φ on a locale A is always ω-standard; it is κ-standard if and only if Φ is κ-complete.)

4.3. Proposition. *A tripos $\mathscr{P}$ is κ-standard if and only if the functor* Δ: **Set** $\longrightarrow$ $\mathscr{P}$-**Set** *preserves coproducts of cardinality less than κ.*

Proof. Suppose $\mathscr{P}$ is κ-standard. Let $(X_i \mid i \in I)$ be a family of sets, where I has cardinality less than κ; write X for the disjoint union of the X_i, and $h: X \to I$ for the map having the X_i as fibres. We shall show directly that ΔX is the coproduct of the ΔX_i in $\mathscr{P}$-**Set**. Suppose given a family of morphisms $f_i: \Delta X_i \longrightarrow (Y, =)$ in $\mathscr{P}$-**Set** (represented by functional relations $F_i: X_i \times Y \longrightarrow \Sigma$, say). We may combine the F_i into a map $F: X \times Y \longrightarrow \Sigma$; in proving that F is functional from ΔX to $(Y, =)$, we use the principle that if something is 'uniformly valid' on each X_i then it is 'uniformly valid' on X. For example, to show that F is total, we have to verify that the quantifier $\forall X$ sends

$$[\![\top_X \to \exists \mathbf{y}\,(\mathbf{y} \in Y \wedge F(\mathbf{x}, \mathbf{y}))]\!] \in \mathscr{P}X$$

to an element of D. But we may factor $\forall X$ as the composite

$$\mathscr{P}X \xrightarrow{\forall h} \mathscr{P}I \xrightarrow{\forall I} \mathscr{P}1,$$

and, since the F_i are total, the (fibre-wise) quantifier $\forall h$ sends this element to an I-indexed family of elements of D. Then since the uniform order on $\mathscr{P}I$ is pointwise, applying $\forall I$ yields an element of D. It is now easy to see that the morphism

$$f: \Delta X \longrightarrow (Y, =)$$

represented by F satisfies $f . \Delta(\nu_i) = f_i$ for each i (where ν_i is the ith coproduct inclusion); the fact that these equations determine f uniquely (i.e. that the family of maps $\Delta(\nu_i)$ is epimorphic) is proved by an argument similar to that already given.

Conversely, suppose Δ preserves coproducts of cardinality less than κ. To show that $\mathscr{P}$ is κ-standard, it suffices by the argument in the proof of 4·1 (ii) $\Rightarrow$ (iii) to show that if $h: I \to D$ is any map with card $I \leqslant \kappa$, then $\forall I(\iota h) \in D$. But we can prove this by considering the subobject $\|\iota h\|$ of ΔI and arguing as in the last part of the proof of 4·1.

Closely related to the notion of κ-standardness is the question of whether Δ preserves the natural number object.

4·4 Proposition. *$\Delta\mathbb{N}$ is a natural number object in $\mathscr{P}$-**Set** if and only if Δ preserves finite colimits. Moreover, these conditions hold if $\mathscr{P}$ is ω_1-standard, and they imply that $\mathscr{P}$ is ω-standard.*

Proof. If Δ preserves finite colimits, then it preserves the natural number object by ((9), proposition 6·16). Conversely, suppose $\Delta\mathbb{N}$ is a natural number object; then for each natural number k the object $\Delta\{0, 1, \ldots, k-1\}$ is a finite cardinal in $\mathscr{P}$-**Set** and hence a k-fold copower of 1; so $\mathscr{P}$ is ω-standard by the argument of Proposition 4·3. To see that Δ also preserves coequalizers, note first that any functor defined on **Set** preserves coequalizers of equivalence relations, since these can be given the structure of split coequalizers; but if Δ preserves $\mathbb{N}$ then it preserves the construction of the equivalence relation generated by a parallel pair of maps, since it preserves finite limits by

Lemma 3·8. Finally, if $\mathscr{P}$ is ω_1-standard then $\Delta\mathbb{N}$ is a countable copower of 1, from which it is easy to verify directly that it is a natural number object.

When $\mathscr{P}$ is the tripos induced by a filter Φ on a locale A, then we may factor Δ as the composite of the inverse image functor $\mathbf{Set} \longrightarrow A\text{-}\mathbf{Set}$ and the logical functor

$$A\text{-}\mathbf{Set} \longrightarrow (A\text{-}\mathbf{Set})_\Phi;$$

so it preserves the natural number object. So ω_1-standardness is not a necessary condition for $\mathscr{P}$ to satisfy the conditions of Proposition 4·4. On the other hand, we do not know any example to show that these conditions are not implied by ω-standardness (though it is not true in general that $\Delta: \mathbf{Set} \longrightarrow \mathscr{P}\text{-}\mathbf{Set}$ preserves coequalizers). Likewise, we do not know any example of a $\mathscr{P}$ for which $\mathscr{P}$-**Set** fails to have a natural number object altogether.

Even when $\mathscr{P}$ is not $\forall$-standard, it may still happen that existential quantification in $\mathscr{P}$ is standard, i.e. that $qe = \mathrm{id}_A$. This is the case, for example, in the realizability tripos of a partial applicative system (Example 1·7), since there are only two isomorphism classes in Σ, one of them being the singleton $\{\varnothing\}$.

4·5 Lemma. *Suppose $\mathscr{P}$ is $\exists$-standard. Then the Heyting algebra $A = \Sigma/\dashv\vdash$ is complete, and the maps $q^I: \mathscr{P}I \rightarrow A^I$ and $e^I: A^I \longrightarrow \mathscr{P}I$ define a geometric morphism from the canonical tripos of A to $\mathscr{P}$.*

Proof. First we show that q^I is left adjoint to e^I. Let $f \in \Sigma^I$ and $g \in A^I$; then if $f \vdash_I eg$, we have $f(i) \vdash eg(i)$ for all i, and hence $qf(i) \leqslant qeg(i) = g(i)$ for all i. Since the order on A^I is by definition pointwise, we thus have $qf \leqslant g$ in A^I. Conversely if $qf \leqslant g$, then we clearly have $qf(i) \leqslant g(i)$ and hence $f(i) \vdash eg(i)$ for each i; we have to show that this entailment is uniform in i. Define $h = f \vee eg$ in $\mathscr{P}I$; then we clearly have $f \vdash_I h$, and $h(i) \dashv\vdash eg(i)$ for each i, so that $qh = g$. Now using the definition of e we have

$$h = \mathscr{P}h(\mathrm{id}_\Sigma) \vdash_I \mathscr{P}h(\mathscr{P}q(\exists q(\mathrm{id}_\Sigma))) = \mathscr{P}(qh)(e) = \mathscr{P}g(e) = eg$$

and so by transitivity $f \vdash_I eg$.

It is clear that q^I preserves finite meets, so it only remains to show that A is complete. Let $f: I \longrightarrow J$ be any map, $g \in A^I$, $h \in A^J$. Then $hf \leqslant g$ if and only if $qehf \leqslant g$, which in turn is equivalent to $ehf \vdash_I eg$, and hence to $eh \vdash_J \forall f(eg)$ and to

$$h = qeh \leqslant q(\forall f(eg)).$$

So we conclude that the composite $q^J.\forall f.e^I: A^I \longrightarrow A^J$ acts as universal quantification for the **Set**-indexed poset A; it is straightforward to verify that the Beck condition is satisfied, and so A is complete.

4·6 Corollary. *Let $\mathscr{P}$ be an $\exists$-standard tripos, with $A = \mathscr{P}1/\dashv\vdash$. Then there is an inclusion of toposes e:* Shv $(A) \longrightarrow \mathscr{P}$-**Set**.

Proof. The geometric morphism of triposes constructed in Lemma 4·5 induces a geometric morphism Shv $(A) \simeq A$-**Set** $\longrightarrow \mathscr{P}$-**Set** (Proposition 3·5). Since $qe = \mathrm{id}_A$, the counit of the adjunction $(q^I \dashv e^I)$ is an isomorphism; it is easy to see from the proof of 3·5 that this condition is inherited by the adjunction $(\bar{q} \dashv \bar{e})$, and so this geometric morphism is an inclusion.

4·7 *Remark.* If $\mathscr{P}$ is the realizability tripos of a partial applicative system, then the locale A has only two elements and so Shv(A) is equivalent to **Set**. If we follow the prescriptions of 3·5 for constructing the direct and inverse image functors of the morphism $\bar{e}$ in this case, we find that they are (isomorphic to) the functors Δ and Γ respectively. So in this case the functor Δ is *right* adjoint to Γ, and not left adjoint as in the localic case.

We conclude the paper with a partial converse to Corollary 4·6.

4·8 Proposition. *Let $\mathscr{P}$ be a tripos, and suppose there is a diagram*

$$\begin{array}{ccc} & \mathscr{E} & \\ {\scriptstyle v}\swarrow & & \searrow{\scriptstyle u} \\ \mathbf{Set} & & \mathscr{P}\text{-}\mathbf{Set} \end{array}$$

of toposes and geometric morphisms such that $u^*\Delta$ is naturally isomorphic to v^* and u^* is conservative on subobjects of 1. *Then $\mathscr{P}$ is $\exists$-standard.*

(In passing, we note that the above conditions are satisfied by the diagram

$$\begin{array}{ccc} & \mathrm{Shv}(A) & \\ \swarrow & & \searrow{\scriptstyle \bar{e}} \\ \mathbf{Set} & & \mathscr{P}\text{-}\mathbf{Set} \end{array}$$

whenever $\mathscr{P}$ is $\exists$-standard, by Proposition 3·9 and the fact that the lattice of subobjects of 1 in $\mathscr{P}$-**Set** is isomorphic to A.)

Proof. Let E be an isomorphism class of elements of Σ, and consider the canonical subobject $\|\iota\|$ of ΔE in $\mathscr{P}$-**Set**, where $\iota\colon E \longrightarrow \Sigma$ is the inclusion map. Clearly, the image of the composite

$$\|\iota\| \longrightarrow \Delta E \longrightarrow \Delta 1$$

is the canonical subobject $\|\exists E(\iota)\|$. Now for any element e of E, the pullback of $\|\iota\|$ along $\Delta(e)\colon \Delta 1 \longrightarrow \Delta E$ is the subobject $\|e\|$; i.e. it is the subobject $\bar{E}$ which corresponds to the element E of A. Now u^* preserves pullbacks, and it sends ΔE to $v^*(E)$, which is an E-indexed copower of 1 in $\mathscr{E}$ ((9), proposition 4·41). So the subobject $u^*\|\iota\|$ is determined by its pullbacks along the maps $v^*(e)$, and must therefore be isomorphic to $v^*(E) \times u^*(\bar{E})$. In particular, the image of $u^*\|\iota\| \longrightarrow 1$ in $\mathscr{E}$ is isomorphic to $u^*(\bar{E})$; that is, we have $u^*\|\exists E(\iota)\| \simeq u^*(\bar{E})$ as subobjects of 1. But u^* is conservative on subobjects of 1, so $\|\exists E(\iota)\| \simeq \bar{E}$ in $\mathscr{P}$-**Set**, i.e. $\exists E(\iota) \in E$.

Note that the hypothesis 'u^* is conservative on subobjects of 1' in Proposition 4·8 does not imply that u is a surjection – indeed, in our canonical example of a diagram satisfying the hypotheses of 4·8, it is a nontrivial inclusion – since $\mathscr{P}$-**Set** is not generated by subobjects of 1. One would like to be able to replace this hypothesis with something less demanding; but some such 'nontriviality' condition is clearly necessary, since otherwise we could take $\mathscr{E}$ to be the degenerate topos.

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