Constructing Initial Algebras Using Inflationary Iteration

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An old theorem of Adámek constructs initial algebras for sufficiently cocontinuous endofunctors via transfinite iteration over ordinals in classical set theory. We prove a new version that works in constructive logic, using "inflationary" iteration over a notion of *size* that abstracts from limit ordinals just their transitive, directed and well-founded properties. Borrowing from Taylor's constructive treatment of ordinals, we show that sizes exist with upper bounds for any given signature of indexes. From this it follows that there is a rich class of endofunctors to which the new theorem applies, provided one admits a weak form of choice (WISC) due to Streicher, Moerdijk, van den Berg and Palmgren, and which is known to hold in the internal constructive logic of many kinds of topos.

1   Introduction

Initial algebras for endofunctors are a simple category-theoretic concept that has proved very useful in logic and computer science. Recall that an *initial algebra* $(\mu F, \iota)$ for an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a category $\mathscr{C}$ is a morphism $\iota : F(\mu F) \to \mu F$ in $\mathscr{C}$ with the property that for any morphism $a : F(A) \to A$, there is a unique $\hat{a} : \mu F \to A$ that is an F-algebra morphism, that is, satisfies $\hat{a} \circ \iota = a \circ F(\hat{a})$. In functional programming, $\hat{a}$ is sometimes called the *catamorphism* associated with the algebra (A, a) [23]. By varying the choice of $\mathscr{C}$ and F, such initial algebras give semantics for various kinds of inductive (or dually, coinductive) structures and, via their catamorphisms, associated (co)recursion schemes. We refer the reader to the draft book by Adámek, Milius, and Moss [6] for an account of this within classical logic.

Here we make a contribution to the existence of initial algebras within *constructive* logics. Our reason for seeking a constructive treatment is not philosophical, nor motivated by the computational insights that a constructive approach can bring, important though both those thing are. Rather, we are interested in the semantics of dependent type theories with inductive constructions, such as types that are inductive [22], inductive-recursive [11], inductive-inductive [15], quotient (inductive-)inductive [9, 19] and more generally higher-inductive [37]. Toposes are often used when constructing models of such type theories and sometimes the easiest way of doing so is to use their "internal logic" [18, Part D] to express the constructions; see [27, 21], for example. Although there are different candidates for what is the internal logic of toposes, in general they are not classical. So we are led to ask for what categories $\mathscr{C}$ and functors $F : \mathscr{C} \to \mathscr{C}$ that are describable in such an internal logic it is the case that an initial F-algebra can be constructed.

We pursue this question by developing a constructive version of Adámek's classical theorem about existence of initial algebras via transfinite iteration over ordinals [4] (we discuss a different constructive approach [5] in Section 5). Recall, or see Adámek et al. [6, section 6.1] for example, that if $F : \mathscr{C} \to \mathscr{C}$ is an endofunctor on a category $\mathscr{C}$ with all small colimits (colimits of small chains are enough), then we get

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a large chain in $\mathscr{C}$, $(F^\alpha 0)_{\alpha \in \mathrm{Ord}}$ indexed by the totally ordered class of ordinals Ord, defined by recursion over the ordinals:

$$F^\alpha 0 = \begin{cases} 0 \quad \text{(initial object in $\mathscr{C}$)} & \text{if } \alpha = 0, \text{ the ordinal zero} \\ F(F^\beta 0) & \text{if } \alpha = \beta^+ \text{ is a successor ordinal} \\ \mathrm{colim}_{\beta < \lambda} F^\beta 0 & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases} \tag{1}$$

The links in the chain are $\mathscr{C}$-morphisms $i_\alpha : F^\alpha 0 \to F^{\alpha^+} 0$ also defined by ordinal recursion:

$$i_\alpha = \begin{cases} \text{unique morphism given by initiality of } 0 & \text{if } \alpha = 0, \text{ the ordinal zero} \\ F(i_\beta) & \text{if } \alpha = \beta^+ \text{ is a successor ordinal} \\ \text{induced by the universal property of colimits} & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases} \tag{2}$$

Theorem 1.1 [CLASSICAL] (Adámek [4]). *If i_α is an isomorphism for some $\alpha \in \mathrm{Ord}$, then $(F^\alpha 0, i_\alpha^{-1})$ is an initial algebra for $F : \mathscr{C} \to \mathscr{C}$. So in particular, if F preserves colimits of shape λ for some limit ordinal λ, then (by the definition of "preserves colimits") i_λ is an isomorphism and $F^\lambda 0$ is an initial F-algebra.* ◁

This theorem is labelled [CLASSICAL] because its proof uses classical logic: the properties of ordinal numbers that it relies upon require the Law of Excluded Middle ($\forall p.\ p \vee \neg p$). In Section 3 we show that by replacing the use of ordinals with a weaker notion of "size" and modifying the way F is iterated, one can obtain a constructive version of Adámek's theorem (see Theorem 3.8).

Not only the proof, but also the application of Adámek's theorem can require classical logic: the Axiom of Choice [AC] is often invoked to find a suitably large limit ordinal λ for which a particular functor of interest preserves λ-colimits. Such uses of [AC] are not always necessary. In particular, existence of initial algebras for polynominal functors $F_{A,B}(\_) = \sum_{a \in A} (\_)^{B(a)} : \mathbf{Set} \to \mathbf{Set}$ (where $A \in \mathbf{Set}$ and $B \in \mathbf{Set}^A$) can be proved constructively; see [24, Proposition 3.6]. These initial algebras are the categorical analogue of W-types [1, 16] and we will make use of the fact that they exist in toposes with natural number object in what follows. However, for non-polynomial functors, especially ones whose specification involves both exponentiation by infinite sets and taking quotients by equivalence relations (such as Example 4.14 below), it is not immediately clear that [AC] can be avoided. In fact, we show in Section 4 that a much weaker choice principle than [AC], the "Weakly Initial Sets of Covers" [WISC] axiom [32, 24, 38], is enough to ensure that our constructive version of Adámek's theorem applies to a rich class of endofunctors. [WISC] has been called "constructively acceptable" because it is valid in a wide range of elementary toposes [38]. In particular it holds in presheaf and realizability toposes that have been used to construct models of dependent type theory that mix quotients and inductive constructions, which, as we mentioned above, motivates our pursuit of a constructive treatment of initial algebras.

2 Constructive meta-theory

The results in this paper are presented in the usual informal language of mathematics, but only making use of intuitionistically valid logical principles (and, to obtain the results of Section 4, extended by the WISC axiom). In particular we avoid use of the Law of Excluded Middle, or more generally the Axiom of Choice.

More specifically, our results can be soundly interpreted in any elementary topos with natural number object and universes [33] (satisfying [WISC], for the last part of the paper). Thus when we refer to the

category **Set** of small sets and functions, we mean the generalised elements of some such universe, which we always assume contains the subobject classifier. In fact, in order to interpret quantification over such small sets in a straightforward way, we tacitly assume there is a countable nested sequence of such universes, $\mathbf{Set} = \mathbf{Set}_0 \in \mathbf{Set}_1 \in \cdots$. A suitable version of Martin-Löf's Extensional Type Theory [22] extended with an impredicative universe of propositions can be used as the internal language of such toposes.

In fact the use of impredicative quantification is not necessary: we have developed a formalisation of the results of this paper using the Agda proof assistant [8], which can provide a dependent type theory with a predicative universe of (proof irrelevant) propositions and convenient mechanisms (such as pattern-matching) for using inductively defined types. We then have to postulate as axioms some things which are derivable in the logic of toposes, namely axioms for propositional extensionality, quotient sets and unique choice (and [WISC], when we need it). Our Agda development is available at [28].

3   Size-indexed inflationary iteration

Throughout this section we fix a large, locally small category[1] $\mathscr{C}$ and an endofunctor $F : \mathscr{C} \to \mathscr{C}$. We will consider sequences of objects in $\mathscr{C}$ built up by iterating F while taking certain colimits. For simplicity we assume that $\mathscr{C}$ is cocomplete, that is, has colimits of all small diagrams.[2]

From a constructive point of view, the problem with the sequence (1) is that it makes use of ordinals, which rely on the Law of Excluded Middle [LEM] for their good properties; in particular, the definition in (1) is by cases according to whether an ordinal is zero, or a successor, *or not*. In the case that $\mathscr{C}$ is a complete partially ordered set (with joins denoted by $\bigvee$), Abel and Pientka [3, section 4.5] point out that one can avoid this case distinction, while still achieving within constructive logic the same result in the (co)limit, by instead taking the approach of Sprenger and Dam [31] and using what they term an *inflationary iteration*:

$$\mu_i F = \bigvee_{j<i} F(\mu_j F) \tag{3}$$

We only need i to range over the elements of a set equipped with a binary relation $<$ that is well-founded for this definition to make sense. Here we generalise from complete posets to cocomplete categories, replacing joins by colimits. Definition 3.2 sums up what we need of the indexes i and the relation $<$ between them in order to ensure that the inflationary sequence can be defined and yields an initial algebra for F if it becomes stationary up to isomorphism.

Definition 3.1. Recall that a *semi-category* is like a category, but lacks identity morphisms. A semi-category is *thin* if there is at most one morphism between any pair of objects. Thus a small thin semi-category is the same thing as a set κ (the set of objects) equipped with a transitive relation $\_ < \_ \subseteq \kappa \times \kappa$ (the existence-of-a-morphism relation). Given such a $(\kappa, <)$, a *diagram* $D : \kappa \to \mathscr{C}$ in a category $\mathscr{C}$ is by definition a semi-functor from κ to $\mathscr{C}$: thus D maps each $i \in \kappa$ to a $\mathscr{C}$-object D_i, each pair (j, i) with $j < i$ to a $\mathscr{C}$-morphism $D_{j,i} : D_j \to D_i$, and these morphisms satisfy $D_{j,i} \circ D_{k,j} = D_{k,i}$ for all $k < j < i$ in κ.   ◁

Definition 3.2. A *size* is a small thin semi-category $(\kappa, <)$ that is
- *directed*: every finite subset of κ has an upper bound with respect to $<$; specifically, we assume we are given a distinguished element $0^s \in \kappa$ and a binary operation $\_ \sqcup^s \_ : \kappa \times \kappa \to \kappa$ satisfying $\forall i, j \in \kappa. \ i < i \sqcup^s j \ \wedge \ j < i \sqcup^s j$

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[1]The collection of objects is in **Set**$_1$ and the collection of morphisms between any pair of objects is in **Set**.

[2]This means that we are given a function assigning a choice of colimit for each small diagram, since we work in a constructive setting and in particular have to avoid the use of the Axiom of Choice.

- *well-founded*: for all $K \subseteq \kappa$, if $\forall i \in \kappa.(\forall j < i.\, j \in K) \Rightarrow i \in K$, then $K = \kappa$.   ◁

Note that the directedness property in particular gives a successor operation $\uparrow^s : \kappa \to \kappa$ on the elements of a size, defined by $\uparrow^s i \triangleq i \sqcup^s i$ and satisfying $\forall i \in \kappa.\, i < \uparrow^s i$. (We do not need a successor that also preserves $<$, although the sizes constructed in the next section have one that does so.)

Example 3.3. In the next section we will define a rich class of sizes derived from algebraic signatures (see Proposition 4.2). For now, we note that the natural numbers $\mathbb{N}$ with their usual strict order is a size.[3] In classical logic, an ordinal is a size iff its usual strict total order is directed, which happens iff it is a limit ordinal.   ◁

Remark 3.4. Since we are working constructively, the well-foundedness property of a size is stated in a suitably positive form; classically, it is equivalent to the non-existence of infinite descending chains for $<$. Well-foundedness of $<$ allows one to define size-indexed families by *well-founded recursion* [36, section 6.3]: given a size κ and a κ-indexed family of sets $(A_i)_{i \in \kappa}$, from each family of functions $(f_i : (\prod_{j<i} A_j) \to A_i)_{i \in \kappa}$ we get a family of elements $(a_i \in A_i)_{i \in \kappa}$, uniquely defined by the requirement $\forall i \in \kappa.\, a_i = f_i((a_j)_{j<i})$.   ◁

Given a size κ, for each element $i \in \kappa$ we get a small thin semi-category[4] $\downarrow(i)$ whose vertices are the elements $j \in \kappa$ with $j < i$ and whose morphisms are the instances of the $<$ relation. Thus a diagram $D : \downarrow(i) \to \mathscr{C}$ maps each $j < i$ to a $\mathscr{C}$-object D_j and each pair (k,j) with $k < j < i$ to a $\mathscr{C}$-morphism $D_{k,j} : D_k \to D_j$, satisfying $D_{k,j} \circ D_{l,k} = D_{l,j}$ for all $l < k < j < i$. We write

$$(\mathrm{inc}^D_j : D_j \to \mathrm{colim}_{j<i} D_j)_{j<i} \tag{4}$$

for the colimit of this diagram (recall that we are assuming $\mathscr{C}$ is cocomplete). Thus for all $k < j < i$ it is the case that $\mathrm{inc}^D_k = \mathrm{inc}^D_j \circ D_{k,j}$; and given any cocone in $\mathscr{C}$

$$(f_j : D_j \to X)_{j<i} \qquad \forall k < j < i.\, f_k = f_j \circ D_{k,j}$$

there is a unique $\mathscr{C}$-morphism $\hat{f} : \mathrm{colim}_{j<i} D_j \to X$ satisfying $\forall j < i.\, \hat{f} \circ \mathrm{inc}^D_j = f_j$.

Since $<$ is transitive, if $j < i$ in κ, then $\downarrow(j)$ is a sub-semi-category of $\downarrow(i)$ and each diagram $D : \downarrow(i) \to \mathscr{C}$ restricts to a diagram $D|_j : \downarrow(j) \to \mathscr{C}$. We write

$$c^D_{j,i} : \mathrm{colim}_{k<j} D_k \to \mathrm{colim}_{k<i} D_k \tag{5}$$

for the unique $\mathscr{C}$-morphism satisfying $\forall k < j < i.\, c^D_{j,i} \circ \mathrm{inc}^{D|_j}_k = \mathrm{inc}^D_k$.

Definition 3.5. Let κ be a size. Given an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a cocomplete category $\mathscr{C}$, a diagram $D : \kappa \to \mathscr{C}$ is an *inflationary iteration of F over κ* if for all $i \in \kappa$

$$D_i = \mathrm{colim}_{j<i} F(D_j) \ \wedge\ \forall j < i.\, D_{j,i} = c^{F \circ D}_{j,i} \tag{◁}$$

Lemma 3.6. Given an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a cocomplete category $\mathscr{C}$, for each size κ an inflationary iteration of F over κ exists (and is unique).   ◁

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[3] $\mathbb{N}$ will be the smallest size once one has developed a comparison relation between sizes. To do that one probably has to restrict to sizes that are *extensional*, that is, satisfy $\forall i, j \in \kappa.\, \{k \in \kappa \mid k < i\} = \{k \in \kappa \mid k < j\} \Rightarrow i = j$. However, we have no need of that property for the results in this paper.

[4] Well-foundedness is preserved, but directedness is not, so $\downarrow(i)$ is not necessarily a size.

Proof. Given $i \in \kappa$, say that a diagram $D : \downarrow(i) \to \mathscr{C}$ is an inflationary iteration of F *up to i* if for all $j < i$, $D_j = \text{colim}_{k<j} F(D_k)$ and $\forall k < j. D_{k,j} = c_{k,j}^{F \circ D}$. Note that given such a diagram, for any $j < i$ we have that $D|_j : \downarrow(j) \to \mathscr{C}$ is an inflationary iteration of F up to j. Using well-founded induction for $<$, one can prove that

$$\forall i \in \kappa, \text{ any two inflationary iterations of } F \text{ up to } i \text{ are equal} \tag{6}$$

Then one can use well-founded recursion for $<$ (Remark 3.4) to define for each $i \in \kappa$ an inflationary iteration of F up to i, $D^{(i)} : \downarrow(i) \to \mathscr{C}$. If $j < i$, then $D^{(j)}$ and $D^{(i)}|_j$ are both inflationary iterations of F up to j and so are equal by (6). From this it follows that

$$(D_i \triangleq \text{colim}_{j<i} F(D_j^{(i)}))_{i \in \kappa} \wedge (D_{j,i} \triangleq c_{j,i}^{F \circ D^{(i)}})_{j,i \in \kappa | j < i}$$

defines an inflationary iteration of F. (Furthermore, since any such restricts to an up-to-i inflationary iteration, uniqueness follows from (6).)   □

Remark 3.7. We record some simple properties of inflationary iteration that we need in the proof of the theorem below. Let $D : \kappa \to \mathscr{C}$ be the inflationary iteration of $F : \mathscr{C} \to \mathscr{C}$ over κ. Note that for all $j < i$ in κ, the components of the colimit cocone $\text{inc}_j^{F \circ D|_i} : F(D_j) \to \text{colim}_{j<i} F(D_j)$ are morphisms $\iota_{j,i} : F(D_j) \to D_i$ satisfying

$$\forall k < j < i. D_{j,i} \circ \iota_{k,j} = \iota_{k,i} = \iota_{j,i} \circ F(D_{k,j}) \tag{7}$$

The first equation follows from the fact that $D_{j,i} = c_{j,i}^{F \circ D}$ and the second from the definition of $\iota_{j,i}$ as a component of a cocone. Since that cocone is colimiting, one also has for all $i \in \kappa$ and all $\mathscr{C}$-morphisms $f, g : D_i \to X$ that

$$(\forall j < i. f \circ \iota_{j,i} = g \circ \iota_{j,i}) \Rightarrow f = g \tag{8}$$

◁

The proof of Lemma 3.6 only used the transitive and well-founded properties of the relation $<$ on a size κ, whereas the following theorem needs its directedness property as well.

Theorem 3.8 (Initial algebras via inflationary iteration). Suppose $\mathscr{C}$ is a cocomplete category, $F : \mathscr{C} \to \mathscr{C}$ is an endofunctor and there is a size κ such that F preserves colimits of diagrams $\kappa \to \mathscr{C}$. Then F has an initial algebra whose underlying $\mathscr{C}$-object is the colimit $\mu F = \text{colim}_{i \in \kappa} \mu_i F$ of the inflationary iteration $\mu_i F$ (Definition 3.5) of F over κ.   ◁

Proof. By Lemma 3.6 there is an inflationary iteration of $F : \mathscr{C} \to \mathscr{C}$ over κ; call it $D : \kappa \to \mathscr{C}$ and define $\mu F \triangleq \text{colim}_{i \in \kappa} D_i$. For each $i \in \kappa$, as in Definition 3.2 we have $\uparrow^s i \in \kappa$ with $i < \uparrow^s i$ and hence a $\mathscr{C}$-morphism

$$\iota_i \triangleq \left(F(D_i) \xrightarrow{\iota_{i,\uparrow^s i}} D_{\uparrow^s i} \xrightarrow{\text{inc}_{\uparrow^s i}^D} \text{colim}_{i \in \kappa} D_i = \mu F \right)$$

By (7), $(\iota_i)_{i \in \kappa}$ is a cocone under the diagram $F \circ D : \kappa \to \mathscr{C}$ and so induces $\hat{\iota} : \text{colim}_{i \in \kappa} F(D_i) \to \mu F$. Then since F preserves the colimit of D, we get a morphism

$$\iota \triangleq \left(F(\mu F) = F(\text{colim}_{i \in \kappa} D_i) \cong \text{colim}_{i \in \kappa} F(D_i) \xrightarrow{\hat{\iota}} \mu F \right) \tag{9}$$

Therefore μF has the structure of an F-algebra. To see that it is initial, suppose we are given $a : F(A) \to A$. We have to show that there is a unique F-algebra morphism $(\mu F, \iota) \to (A, a)$.

If $h : \mu F \to A$ is such an algebra morphism, that is $h \circ \iota = a \circ F(h)$, then by definition of ι in (9) it follows that the associated cocone $(h_i \triangleq h \circ \mathrm{inc}_i^D : D_i \to A)_{i \in \kappa}$ satisfies $h_{\uparrow^s i} \circ \iota_{i, \uparrow^s i} = a \circ F(h_i)$. From this, using the directedness property of sizes, we get

$$\forall i \in \kappa. \forall j < i. \ h_i \circ \iota_{j,i} = a \circ F(h_i \circ D_{j,i}) \tag{10}$$

So if h and h' are both F-algebra morphisms $(\mu F, \iota) \to (A, a)$, one can prove by well-founded induction for $<$, using (8) and (10), that $\forall i \in \kappa. \ h \circ \mathrm{inc}_i^D = h' \circ \mathrm{inc}_i^D$ and hence that $h = h'$.

So it just remains to prove that there is such an h. It suffices to construct a cocone $(h_i : D_i \to A)_{i \in \kappa}$ satisfying (10) and then take h to be the morphism given by the universal property of the colimit; for then we have $\forall i \in \kappa. \ h_{\uparrow^s i} \circ \iota_{i, \uparrow^s i} = a \circ F(h_i)$ and hence $h \circ \iota = a \circ F(h)$, as required.

For each $i \in \kappa$, say that a morphism $h' : D_i \to A$ is an *up-to-i algebra morphism* if $\forall j < i. \ h' \circ \iota_{j,i} = a \circ F(h' \circ D_{j,i})$ (*cf.* (10)). Given such a morphism, then for any $j < i$, $h' \circ D_{j,i} : D_j \to A$ is an up-to-j algebra morphism. From this it follows by well-founded induction for $<$ that any two up-to-i algebra morphisms are equal. A well-founded recursion for $<$ allows one to construct an up-to-i algebra morphism $h_i : D_i \to A$ for each $i \in \kappa$; and the uniqueness of up-to algebra morphisms implies that $h_j = h_i \circ D_{j,i}$ when $j < i$. Thus $(h_i)_{i \in \kappa}$ is the required cocone satisfying (10). □

Corollary 3.9. With the same assumptions on $\mathscr{C}$, F and κ as in Theorem 3.8, then free F-algebras exist, that is, the forgetful functor from the category of F-algebras to $\mathscr{C}$ has a left adjoint. ◁

Proof. The free F-algebra on an object $X \in \mathscr{C}$ is the same thing as an initial algebra for the endofunctor $F(\_) + X$. So by the theorem, it suffices to check that $F(\_) + X$ preserves colimits of diagrams $\kappa \to \mathscr{C}$. It does so because F does by assumption and because κ is directed (*cf.* Proposition 4.7(4) below). □

4 Initial algebras for sized endofunctors

In classical set theory with the Axiom of Choice, given a set of operation symbols $A \in \mathbf{Set}$ with associated arities $B \in \mathbf{Set}^A$, the associated polynomial endofunctor $X \mapsto \sum_{a \in A} X^{B(a)}$ on $\mathbf{Set}$ preserves λ-colimits when the ordinal λ is large enough; specifically it does so if for all $a \in A$, λ has upper bounds (with respect to the strict total order given by membership) for all $B(a)$-indexed families of ordinals less than λ. We will see that this notion of "large enough" is also the right one for sizes in our constructive setting.

Definition 4.1. A *signature* (also known as a *container* [1, 16]) is specified by a set $A \in \mathbf{Set}$ and an A-indexed family of sets $B \in \mathbf{Set}^A$. We write $\mathbf{Sig} \in \mathbf{Set}_1$ for the large set of all such signatures. Given $\Sigma = (A, B) \in \mathbf{Sig}$, we say that a size $(\kappa, <)$ is Σ-*filtered* if for all $a \in A$ and every function $f : B(a) \to \kappa$, there exists $i \in \kappa$ with $\forall x \in B(a). \ f(x) < i$. ◁

We can deduce the existence of Σ-filtered sizes by abstracting from the constructive analysis of Conway's surreal numbers by Shulman [30], which in turn is inspired by Taylor's constructive notion of "plump" ordinal [35]. For each $\Sigma = (A, B) \in \mathbf{Sig}$, let W_Σ be the initial algebra for the associated polynominal endofunctor $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$, $F_{A,B}(X) = \sum_{a \in A} X^{B(a)}$. Thus W_Σ is an example of a W-type [26, Chapter 15]. The function $\Sigma \mapsto W_\Sigma$ exists in our constructive setting, because W-types can be constructed in elementary toposes with natural number objects [24, Proposition 3.6]; one can take the elements of W_Σ to be well-founded trees representing the algebraic terms inductively generated by the signature Σ. Each such term t is uniquely of the form $\sup_a f$ where $\sup_a$ is the $B(a)$-arity operation symbol named by $a \in A$ and,

inductively, $f = (t_x)_{x \in B(a)}$ is a $B(a)$-tuple of well-founded algebraic terms over Σ. The *plump* ordering on W_Σ is given by the least relations $\_ < \_ \subseteq W_\Sigma \times W_\Sigma$ and $\_ \le \_ \subseteq W_\Sigma \times W_\Sigma$ satisfying for all $a \in A$, $f : B(a) \to W_\Sigma$ and $t \in W_\Sigma$

$$(\forall x \in B(a). \, f(x) < t) \Rightarrow \sup_a f \le t \quad \text{and} \quad (\exists x \in B(a). \, t \le f(x)) \Rightarrow t < \sup_a f \tag{11}$$

As noted in [14, Example 5.4], $<$ is transitive and well-founded, and $\le$ is a preorder (reflexive and transitive). In particular, since $\le$ is reflexive, from (11) we deduce that $\forall x \in B(a). \, f(x) < \sup_a f$, in other words for each arity set $B(a)$ in the signature, any function $f : B(a) \to W_\Sigma$ is bounded above in the $<$ relation by $\sup_a f$. This allows us to construct Σ-filtered sizes:

Proposition 4.2. There is a Σ-filtered size $(\kappa_\Sigma, <)$ for every signature Σ. $\triangleleft$

Proof. Given a signature $\Sigma = (A, B)$, we extend it to a signature (A', B') by adding fresh nullary and binary operation symbols. Thus $A' \triangleq A \uplus \{n, b\}$ and $B' \in \mathbf{Set}^{A'}$ satisfies $B'(a) \triangleq B(a)$ for $a \in A$, $B'(n) \triangleq \emptyset$ and $B'(b) \triangleq \{0, 1\}$. Let set κ_Σ be the W-type $W_{(A', B')}$ and let $<$ be the plump order given by (11). As noted above, $<$ is transitive and well-founded and has upper bounds for any arity-indexed family and hence in particular it is Σ-filtered. It just remains to see that it is directed (Definition 3.2). Since A' contains the nullary operation symbol n, κ_Σ contains $0^s \triangleq \sup_n \emptyset$; and given $i, j \in \kappa_\Sigma$, letting $f : B'(b) = \{0, 1\} \to \kappa_\Sigma$ map 0 to i and 1 to j, then $i \sqcup^s j \triangleq \sup_b f$ is an upper bound for i and j with respect to $<$. $\square$

Definition 4.3. Given a signature $\Sigma \in \mathbf{Sig}$, a functor $F : \mathscr{C} \to \mathscr{D}$ between cocomplete categories is Σ-*sized* if it preserves colimits of all diagrams $\kappa \to \mathscr{C}$ for any Σ-filtered size κ. A functor is *sized* it there exists a signature Σ for which it is Σ-sized. $\triangleleft$

Theorem 4.4 (Sized endofunctors have initial algebras). Assuming $\mathscr{C}$ is a cocomplete category, if $F : \mathscr{C} \to \mathscr{C}$ is sized, then there exists an initial algebra for F. More precisely, there is a function assigning to each signature Σ and each Σ-sized endofunctor F an initial algebra for F. $\triangleleft$

Proof. If F is Σ-sized for some $\Sigma \in \mathbf{Sig}$, then F preserves colimits of diagrams for the Σ-filtered size κ_Σ constructed in the proof of Proposition 4.2. Hence by Theorem 3.8, it has an initial algebra, given by taking the colimit of its inflationary iteration. $\square$

To apply this theorem one needs a rich collection of sized functors. The rest of the section is devoted to exploring closure properties of sized functors. To do so we use the following operation on signatures:

Definition 4.5. Suppose $\Sigma_c = (A_c, B_c)$ is a family of signatures indexed by the elements c of some set C. Then the *signature sum* $\bigoplus_{c \in C} \Sigma_c$ is the signature (A, B) where $A \triangleq \sum_{c \in C} A_c = \{(c, a) \mid c \in C \wedge a \in A_c\}$ and $B \in \mathbf{Set}^A$ maps each (c, a) to the set $B_c(a)$. As a special case when $I = \{0, 1\}$, we have the *binary sum* $\Sigma_0 \oplus \Sigma_1$. There is also an *empty signature* $0 = (\emptyset, \emptyset)$ which acts as a unit for $\oplus$ up to isomorphism (for a suitable notion of signature morphism). $\triangleleft$

Remark 4.6. Note that if a size is $(\bigoplus_{c \in C} \Sigma_c)$-filtered, it is also Σ_c-filtered for each $c \in C$. Conversely, given a single signature Σ, if a size is Σ-filtered, it is also $(\bigoplus_{c \in C} \Sigma)$-filtered. $\triangleleft$

Proposition 4.7. Suppose that $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ are cocomplete categories.

1. Any cocontinuous functor $\mathscr{C} \to \mathscr{D}$ is sized.

2. Identity functors are sized. If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are sized, so is their composition $G \circ F : \mathscr{C} \to \mathscr{E}$.

3. The terminal functor $\mathscr{C} \to 1$ and the projection functors $\pi_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $\pi_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ are sized; if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{E}$ are sized, then so is $\langle F, G \rangle : \mathscr{C} \to \mathscr{D} \times \mathscr{E}$.

4. For any $X \in \mathscr{C}$ the constant functor $1 \to \mathscr{C}$ with value X is sized.

$\triangleleft$

Proof. For part 1, if $F : \mathscr{C} \to \mathscr{D}$ is cocontinuous, then it is Σ-sized for any Σ and in particular for the empty signature.

The first sentence of part 2 follows from part 1. If F is Σ-sized and G is Σ'-sized, then F and G both preserve colimits over any $\Sigma \oplus \Sigma'$-filtered size, because such a size is also Σ- and Σ'-filtered. The composition $G \circ F$ preserves such a colimit because F and G do. Therefore $G \circ F$ is $\Sigma \oplus \Sigma'$-sized.

For part 3 we use the fact that colimits in a product category are computed componentwise. Thus the terminal and projection functors are sized by part 1; and if $F : \mathscr{C} \to \mathscr{D}$ is Σ-sized and $G : \mathscr{C} \to \mathscr{E}$ is Σ'-sized, then $\langle F, G \rangle$ is $\Sigma \oplus \Sigma'$-sized.

For part 4, note that each size κ is directed and hence in particular is a connected semi-category; therefore $\mathrm{colim}_{i \in \kappa} X$ is canonically isomorphic to X. So the constant functor with value X is Σ-sized for any Σ and in particular for the empty signature. $\square$

We can deduce further preservation properties involving infinitary operations on sized functors by assuming a weak form of choice, which following `https://ncatlab.org/nlab/show/WISC` we call the [WISC] axiom. It was introduced in type theory by Streicher [32] under the name TTCA$_f$ ("Type Theoretic Collection Axiom") and independently in constructive set theory by van den Berg and Moerdijk [38] under the name "Axiom of Multiple Choice"; see also Levy [20, Section 5.1].

Axiom 4.8 [WISC]. A (possibly large) *cover* of a set $X \in \mathbf{Set}$ is a surjective function $f : Y \twoheadrightarrow X$ with $Y \in \mathbf{Set}_1$. An indexed family[5] $(E_c)_{c \in C} \in \mathbf{Sig}$ is a *wisc* for $X \in \mathbf{Set}$ if for any cover $f : Y \twoheadrightarrow X$, there exist $c \in C$ and $g : E_c \to Y$ such that $f \circ g$ is surjective. The [WISC] axiom[6] states that for every $X \in \mathbf{Set}$ there exists a family $(E_c)_{c \in C} \in \mathbf{Sig}$ that is a wisc for it. $\triangleleft$

"Wisc" stands for "weakly initial set of covers" and the terminology is justified by the fact that if in $\mathbf{Set}$ the family $(E_c)_{c \in C}$ is a wisc for X, then the family of covers of X whose domains are of the form E_c for some $c \in C$ is weakly initial among all the (possibly large) covers of X: for every $Y \in \mathbf{Set}_1$ and $f : Y \twoheadrightarrow X$, there is some cover $e : E_c \twoheadrightarrow X$ in the family that factors as $e = f \circ g$ for some $g : E_c \to Y$.

Classically, [WISC] is implied by the Axiom of Choice [AC], since the latter implies that every surjection has a right inverse and hence the family whose single member is X is a wisc for X. From the results of van den Berg and Moerdijk [38] (and as noted by Streicher [32]), if any elementary topos $\mathscr{E}$ satisfies [WISC], then so do toposes of (pre)sheaves and realizability toposes built from $\mathscr{E}$; it is in this sense that the axiom is constructively acceptable. In particular, starting from the category of sets in classical set theory with [AC], [WISC] holds in the kinds of topos that have been used to model type theory with various kinds of higher inductive types, whose semantics motivates the work presented here. (However, it does not hold in all toposes [29].)

Lemma 4.9 [WISC]. Suppose [WISC] holds and that $\mathscr{C}$ and $\mathscr{D}$ are cocomplete categories. If $(F_x : \mathscr{C} \to \mathscr{D})_{x \in X}$ is a family of sized functors indexed by a set $X \in \mathbf{Set}$, then there exists a signature $\Sigma \in \mathbf{Sig}$ such that F_x is Σ-sized for all $x \in X$. $\triangleleft$

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[5]We will refer to elements of $\mathbf{Sig}$ as *families* rather than *signatures* when we are not thinking of them as collections of operation symbols of set-valued arity.

[6]For simplicity and following [32], we have given the axiom just for a pair of universes, $(\mathbf{Set}_0, \mathbf{Set}_1)$; more generally one can ask for the property to hold for any pair $(\mathbf{Set}_m, \mathbf{Set}_n)$.

Proof. Consider the large set $S \triangleq \sum_{x \in X} \{\Sigma' \in \mathbf{Sig} \mid F_x$ is a Σ'-sized functor$\}$ in $\mathbf{Set}_1$. By assumption on F, the first projection $\pi_1 : S \to X$ is a large[7] cover of X. By [WISC] there is some surjection $e : X' \twoheadrightarrow X$ in $\mathbf{Set}$ and a function $\Sigma' : X' \to \mathbf{Sig}$ so that for all $x' \in X'$, the functor $F_{e(x')}$ is $\Sigma'_{x'}$-sized; and since e is surjective this implies that each F_x is $\Sigma'_{x'}$-sized for some $x' \in X'$. Consider the signature $\Sigma \triangleq \bigoplus_{x' \in X'} \Sigma'_{x'}$ from Definition 4.5. By Remark 4.6, each F_x is Σ-sized. $\square$

Theorem 4.10 [WISC] **(Colimits of sized functors).** Suppose that [WISC] holds, $\mathscr{C}$ and $\mathscr{D}$ are cocomplete categories, $\mathbf{C}$ is a small category and that $F : \mathbf{C} \times \mathscr{C} \to \mathscr{D}$ is a functor. If for some signature $\Sigma \in \mathbf{Sig}$ the functor $F(c, \_)$ is Σ-sized for each $c \in \mathbf{C}$, then $\mathrm{colim}_{c \in \mathbf{C}} F(c, \_) : \mathscr{C} \to \mathscr{D}$ is also Σ-sized. More generally, if each $F(c, \_)$ is sized, then so is $\mathrm{colim}_{c \in \mathbf{C}} F(c, \_)$. $\triangleleft$

Proof. If $F(c, \_) : \mathscr{C} \to \mathscr{D}$ is Σ-sized for all $c \in \mathbf{C}$ and κ is a Σ-filtered size, then each $F(c, \_)$ preserves colimits of all diagrams $\kappa \to \mathscr{C}$. Thus given such a diagram $D : \kappa \to \mathscr{C}$, we have a canonical isomorphism $F(c, \mathrm{colim}_{i \in \kappa} D_i) \cong \mathrm{colim}_{i \in \kappa} F(c, D_i)$, natural in c. Taking the colimit over $c \in \mathbf{C}$ and writing $F' \triangleq \mathrm{colim}_{c \in \mathbf{C}} F(c, \_)$, we have $F'(\mathrm{colim}_{i \in \kappa} D_i) = \mathrm{colim}_{c \in \mathbf{C}} F(c, \mathrm{colim}_{i \in \kappa} D_i) \cong \mathrm{colim}_{c \in \mathbf{C}} \mathrm{colim}_{i \in \kappa} F(c, D_i)$. Since colimits commute with each other, it follows that the canonical morphism $F'(\mathrm{colim}_{i \in \kappa} D_i) \to \mathrm{colim}_{i \in \kappa} F'(D_i)$ is an isomorphism. Therefore F' is Σ-sized. The last sentence of the theorem follows by Lemma 4.9. $\square$

Corollary 4.11 [WISC]. Suppose [WISC] holds and that $\mathscr{C}$ and $\mathscr{D}$ are cocomplete categories. If $F : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ is sized, then there is a function $X \mapsto \mu Y.F(X,Y)$ assigning to each $X \in \mathscr{C}$ an initial algebra $\mu Y.F(X,Y)$ for the functor $F(X, \_) : \mathscr{D} \to \mathscr{D}$. The induced functor $\mu Y.F(\_,Y) : \mathscr{C} \to \mathscr{D}$ is sized. $\triangleleft$

Proof. Suppose $F : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ is Σ-sized. It follows from Proposition 4.7 and Remark 4.6 that for each $X \in \mathscr{C}$, the functor $F_X \triangleq F(X, \_) : \mathscr{D} \to \mathscr{D}$ is Σ-sized. Therefore by Theorems 4.4 and 3.8, the function $X \mapsto \mu Y.F(X,Y) \triangleq \mathrm{colim}_{i \in \kappa_\Sigma} \mu_i F_X$ is the required function mapping each $X \in \mathscr{C}$ to an initial algebra for $F(X, \_)$. Since each $\mu_i F_X$ is $\mathrm{colim}_{j<i} F(X, \mu_j F_X)$, it follows by well-founded induction on $i \in \kappa_\Sigma$ that each $\mu_i F_X$ is Σ-sized, using Theorem 4.10 (taking $\mathbf{C}$ to be the category generated by the thin semi-category $\downarrow(i)$). Then by Theorem 4.10 again (taking $\mathbf{C}$ to be the category generated by κ) we have that $\mu Y.F(\_,Y) = \mathrm{colim}_{i \in \kappa_\Sigma} \mu_i F_\_$ is Σ-sized. $\square$

Although Proposition 4.7, Theorem 4.10, and Corollary 4.11 show that there is quite a rich collection of sized functors, what is lacking so far is any closure under taking limits, assuming the target category has them; in other words the dual of Theorem 4.10. We consider this for the case $\mathscr{D} = \mathbf{Set}$, leaving consideration of more general complete and cocomplete categories for future work. First note that if $F, G : \mathscr{C} \to \mathbf{Set}$ are sized functors, the equalizer of any parallel pair $F \rightrightarrows G$ of natural transformations is also a sized functor (it is $(\Sigma \oplus \Sigma')$-sized if F is Σ-sized and G is Σ'-sized). This is because each size κ is directed and so taking κ-colimits in $\mathbf{Set}$ commutes with finite limits and hence in particular with equalizers. So to get closure of sized functors under all small limits it suffices to consider small products. For this we need to use a "double cover" signature of a set (the wiscs W and W' in the proof of Theorem 4.13 below), inspired by the use that Swan [34] makes of the indexed form of the WISC Axiom; see also [14]. So we will need wiscs for indexed families of sets; but their existence follows from [WISC]:

Lemma 4.12 [WISC]. Assuming [WISC] holds, then for every family of sets $(X_i)_{i \in I} \in \mathbf{Sig}$ there exists a family $(E_c)_{c \in C} \in \mathbf{Sig}$ that is a wisc for each set X_i. $\triangleleft$

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[7]This proof, as well as that for Lemma 4.12, illustrates the need for a wisc property that quantifies over large covers of small sets.

Proof. Consider $S \triangleq \sum_{i \in I} \{W \in \mathbf{Sig} \mid W \text{ is a wisc for } X_i\} \in \mathbf{Set}_1$. By [WISC], the first projection $\pi_1 : S \to I$ is a large cover of I. Since there is a wisc for I, it follows that there is some surjection $e : J \twoheadrightarrow I$ in **Set** and a function $W : J \to \mathbf{Sig}$ so that for all $j \in J$, W_j is a wisc for $X_{e(j)}$. Consider the signature sum $W \triangleq \bigoplus_{j \in J} W_j \in \mathbf{Sig}$ as in Definition 4.5. Thus writing (C, E) for W and (C_j, E_j) for each W_j, we have $C \triangleq \sum_{j \in J} C_j \in \mathbf{Set}$ and $E \in \mathbf{Set}^C$ is the function mapping each $(j, c) \in \sum_{j \in J} C_j$ to $E_j(c)$. Then we claim that $W \in \mathbf{Sig}$ is a wisc for each set X_i. For, given any cover $f : Y \twoheadrightarrow X_i$, since $e : J \twoheadrightarrow I$ is a surjection, there exists $j \in J$ with $e(j) = i$; then since $W_j = (C_j, E_j)$ is a wisc for $X_{e(j)} = X_i$, there exists $c \in C_j$ and $g : E_j(c) \to Y$ such that $f \circ g$ is surjective. So there exists $(j, c) \in C$ and $g : E(j, c) = E_j(c) \to Y$ such that $f \circ g$ is surjective. Therefore $W = (C, E)$ does indeed have the wisc property for X_i. $\qquad \square$

Theorem 4.13 [WISC] **(Products of set-valued sized functors are sized).** Suppose that $\mathscr{C}$ is a cocomplete category. Assuming [WISC] holds, if $(F_x : \mathscr{C} \to \mathbf{Set})_{x \in X}$ is a family of sized functors indexed by some set $X \in \mathbf{Set}$, then the functor $\prod_{x \in X} F_x : \mathscr{C} \to \mathbf{Set}$ given by taking products in **Set** is also sized. $\qquad \triangleleft$

Proof. By Lemma 4.9, there exists a signature Σ so that each functor F_x is Σ-sized. However, we need a bigger signature than Σ in order to prove that $\prod_{x \in X} F_x$ is sized. Using [WISC], let $W = (E_c)_{c \in C}$ be a wisc for X. Then using Lemma 4.12, let $W' = (E'_{c'})_{c' \in C'}$ be a wisc for the sets in the family $(\ker p)_{c \in C, p : E_c \twoheadrightarrow X}$, where

$$\ker p \triangleq \{(d_1, d_2) \in E_c \times E_c \mid p(d_1) = p(d_2)\} \tag{12}$$

We claim that the functor $F' \triangleq \prod_{x \in X} F_x$ is Σ'-sized when $\Sigma' = \Sigma \oplus W \oplus W'$ (using the signature sum from Definition 4.5).

If $D : \kappa \to \mathscr{C}$ is a diagram on a Σ'-filtered size κ, then by Remark 4.6, each F_x is Σ'-sized and so we have a canonical isomorphism $\mathrm{colim}_{i \in \kappa} F_x(D_i) \cong F_x(\mathrm{colim}_{i \in \kappa} D_i)$. Taking the product over $x \in X$, we get $\prod_{x \in X} \mathrm{colim}_{i \in \kappa} F_x(D_i) \cong \prod_{x \in X} F_x(\mathrm{colim}_{i \in \kappa} D_i) = (\prod_{x \in X} F_x)(\mathrm{colim}_{i \in \kappa} D_i)$. So it just remains to show that the canonical function

$$\mathrm{can}_{F,D} : \mathrm{colim}_{i \in \kappa} \left((\prod_{x \in X} F_x) D_i \right) = \mathrm{colim}_{i \in \kappa} \prod_{x \in X} F_x(D_i) \to \prod_{x \in X} \mathrm{colim}_{i \in \kappa} F_x(D_i) \tag{13}$$

is an isomorphism, that is, both an injection and a surjection. The summand W in Σ' ensures that κ has upper bounds for E_c-indexed families for any $c \in C$; and the W' summand ensures the same for $E'_{c'}$-indexed families, for any $c' \in C'$. The first kind of upper bound, together with the wisc property of W, comes into play in proving that $\mathrm{can}_{F,D}$ is injective; and both kinds of upper bounds and the wisc property of W and W' come into play in proving that $\mathrm{can}_{F,D}$ is surjective.

To prove that $\mathrm{can}_{F,D}$ is injective and surjective we use the fact that the colimit in **Set** of a directed diagram $D : \kappa \to \mathbf{Set}$ can be described explicitly as the quotient $(\sum_{i \in \kappa} D_i)/\approx$ where the equivalence relation $\approx$ identifies $(i, d), (i', d') \in \sum_{i \in \kappa} D_i$ if there is some $j \in \kappa$ with $i < j$, $i' < j$ and $D_{i,j}(d) = D_{i',j}(d')$. We will write $[i, d]_\approx$ for the $\approx$-equivalence class of $(i, d) \in \sum_{i \in \kappa} D_i$. Then the function in equation (13) satisfies for all $i \in \kappa$ and $f \in \prod_{x \in X} F_x(D_i)$

$$\mathrm{can}_{F,D}[i, f]_\approx = \lambda x \in X. \, [i, f(x)]_\approx$$

To see that $\mathrm{can}_{F,D}$ is injective, suppose we also have $i' \in \kappa$ and $f' \in \prod_{x \in X} F_x(D_i)$ satisfying $\forall x \in X. \, [i, f(x)]_\approx = [i', f'(x)]_\approx$; we wish to prove that $(i, f) \approx (i', f')$. By definition of $\approx$ we have $\forall x \in X. \exists j \in \kappa. \, i < j \wedge i' < j \wedge F_x(D_{i,j})(f x) = F_x(D_{i',j})(f'x)$. Since $W = (E_c)_{c \in C}$ is a wisc for X, there exist $c \in C$, a surjection $p : E_c \twoheadrightarrow X$ and a function $q : E_c \to \kappa$ so that

$$\forall z \in E_c. \, i < q(z) \wedge i' < q(z) \wedge F_x(D_{i,q(z)})(f(pz)) = F_x(D_{i',q(z)})(f'(pz)) \tag{14}$$

Since W is a summand in Σ' and κ is a Σ'-filtered size, there is an $<$-upper bound $j \in \kappa$ for $q : E_c \to \kappa$; and since κ is directed, we can assume $i < j$ and $i' < j$. So from (14) and surjectiviy of p we deduce that $\forall x \in X.\ F_x(D_{i,j})(fx) = F_x(D_{i',j})(f'x)$, which implies $(i, f) \approx (i', f')$. Therefore the function $\mathrm{can}_{F,D}$ in (13) is indeed injective.

To see that $\mathrm{can}_{F,D}$ is also surjective, suppose we have $g \in \prod_{x \in X} \mathrm{colim}_{i \in \kappa} F_x(D_i)$. Since

$$\forall x \in X.\exists (i,d) \in \textstyle\sum_{i \in \kappa} F_x(D_i).\ g(x) = [i,d]_\approx$$

and W is a wisc for X, there exists some $c \in C$, $p : E_c \twoheadrightarrow X$ and $\langle q_1, q_2 \rangle \in \prod_{z \in E_c} \sum_{i \in \kappa} F_{p(z)}(D_i)$ so that $\forall z \in E_c.\ g(pz) = [q_1(z), q_2(z)]_\approx$. Then since W is a summand in Σ' and κ is a Σ'-filtered size, there is an $<$-upper bound $j \in \kappa$ for $q_1 : E_c \to \kappa$. So we have

$$\forall z \in E_c.\ g(pz) = [j, q'(z)]_\approx \tag{15}$$

where $q' \in \prod_{z \in E_c} F_{p(z)}(D_j)$ is $q'(z) \triangleq F_{p(z)}(D_{q_1(z),j})(q_2(z))$. It follows that the relation $\Phi \subseteq \sum_{x \in X} F_x(D_j)$ given by $\Phi(x,d) \triangleq \exists z \in E_c.\ x = p(z) \wedge d = q'(z)$ is total (because p is surjective); and were it also single-valued, it would determine a function $f \in \prod_{x \in X} F_x(D_j)$ which by virtue of (15) would satisfy $\mathrm{can}_{F,D}[j,f] = g$. However, we need to increase j to get this single-valued property. Recall that W' is a wisc for the kernel (12) of $p : E_c \twoheadrightarrow X$. If $(z_1, z_2) \in \ker p$, then by (15) $[j, q'(z_1)]_\approx = g(pz_1) = g(pz_2) = [j, q'(z_2)]_\approx$. Therefore we have

$$\forall (z_1, z_2) \in \ker p.\exists k \in \kappa.\ j < k \ \wedge\ F_{p(z_1)}(D_{q_1(z_1),k})(q_2(z_1)) = F_{p(z_2)}(D_{q_1(z_2),k})(q_2(z_2))$$

So since $W' = (E_{c'})_{c' \in C'}$ is a wisc for $\ker p$ and κ has $E_{c'}$-indexed upper bounds for any $c' \in C'$ and is directed, it follows that there exists $c' \in C'$, $\langle p_1, p_2 \rangle : E_{c'} \twoheadrightarrow \ker p$ and $k \in \kappa$ with $j < k$ and

$$\forall z'' \in E_{c'}.F_{p(p_1 z'')}(D_{q_1(p_1 z''),k})(q_2(p_1 z'')) = F_{p(p_2 z'')}(D_{q_1(p_2 z''),k})(q_2(p_2 z'')) \tag{16}$$

Now if we let $q'' \in \prod_{z \in E_c} F_{p(z)}(D_k)$ be $q''(z) \triangleq F_{p(z)}(D_{q_1(z),k})(q_2(z))$, then from (15) we have

$$\forall z \in E_c.\ g(pz) = [k, q''(z)]_\approx \tag{17}$$

Let the relation $\Phi' \subseteq \sum_{x \in X} F_x(D_k)$ be $\Phi'(x,d) \triangleq \exists z \in E_c.\ x = p(z) \wedge d = q''(z)$. It is total because p is surjective; but it is also single-valued because if $\Phi'(x,d) \wedge \Phi'(x,d')$, then $d = q''(z_1) \wedge d' = q''(z_2)$ for some $(z_1, z_2) \in \ker p$, so that there exists $z'' \in E_{c'}$ with $p_1(z'') = z_1 \wedge p_2(z'') = z_2$ and hence $d = q''(z_1) = q''(z_2) = d'$ by (16). Therefore Φ' is the graph of a function $f \in \prod_{x \in X} F_x(D_k)$; and by virtue of (17) we have $\forall x \in X.\ g(x) = [k, f(x)]_\approx$, so that $g = \mathrm{can}_{F,D}[k,f]$. Thus $\mathrm{can}_{F,D}$ is indeed surjective. $\square$

Example 4.14. The *symmetric containers* of Gylterud [17] generalize ordinary signatures by replacing the set of operation symbols by a groupoid $\mathbf{A}$ and the arity function by a functor $B : \mathbf{A} \to \mathbf{Set}$. The associated endofunctor $S_{\mathbf{A},B} : \mathbf{Set} \to \mathbf{Set}$ maps each set $X \in \mathbf{Set}$ to the colimit

$$S_{\mathbf{A},B}(X) \triangleq \mathrm{colim}_{a \in \mathbf{A}} X^{B(a)} \tag{18}$$

Applying Theorem 4.4, Proposition 4.7 and Theorem 4.13 we have that any topos with universes satisfying [WISC] has initial algebras for symmetric containers.

In fact these initial algebras are special cases of QW-types [14]: they can be seen as sets of terms quotiented by the symmetries given by the groupoid structure on the arguments of an operation symbol. So their existence in toposes with [WISC] follows from the results of that paper. However, the construction here in terms of a colimit of an inflationary iteration gives a simpler description than for the general case of a QW-type. $\triangleleft$

5   Related and future work

The results in this paper make use of the constructive techniques introduced by the authors and Fiore in our prior paper [14]: the use of sizes given by "plump" well-founded orders on W-types and the use of a WISC axiom to see that certain functors preserve colimits of that shape. That paper constructs a large class of quotient-inductive types, called *QWI-types*, which by definition are initial among algebras for indexed containers [1] satisfying a given system of equations. Although the construction proceeds by forming a size-indexed family of objects in the case $\mathscr{C}$ is $\mathbf{Set}^I$ (with $I \in \mathbf{Set}$) and taking its colimit, it does not appear to be a direct corollary of Theorem 3.8. Conversely, the results here do not follow from the ones in [14], since for one thing here we consider general cocomplete categories $\mathscr{C}$, rather than just products of $\mathbf{Set}$. In this respect we are closer to the approach of Fiore and Hur [12] and it would be interesting to see whether our techniques can be extended to give constructive proofs of existence of free algebras for the very general notion of equational system on a category that is introduced in that paper. This may involve investigating the extent to which our approach allows a constructive treatment of some of the classical theory of locally presentable and accessible categories [7], which is future work.

   The inflationary iteration indexed by a notion of size that we have introduced in the paper generalises from complete posets to cocomplete categories aspects of Abel and Pientka's work [2, 3]. These papers develop a theory of *sized types* and its semantics. Abel has added a version of this to the type theory provided by the Agda proof assistant [8]. Unfortunately recent versions of Agda contain features that allow one to use sized types to prove a logical contradiction. The problem is that, in contrast to the notion of size used here, the one by Abel et al. [2, 3] features a generic size ∞ at which sized-indexed sequences become stationary. Currently in Agda (version 2.6.2) one both has $\infty < \infty$ and can prove that $<$ is well-founded, leading to a contradiction. For us, the intuitive and important aspect of "size" is that there is well-founded ordering, thus permitting definitions by well-founded recursion on a set of sizes. Then having a single size ∞ at which all sequences become stationary is semantically problematic. So we avoid having an explicit stationary size ∞, at the expense of having to take a colimit to obtain an initial algebra, instead of just instantiating an inflationary iteration at ∞.

   We hope Agda's sized types will get fixed, since they are useful in practice; they are most often used (together with copatterns) to demonstrate that recursively defined functions on a coinductively defined record type are well-defined (that is, are "productive") [3]. Here, while avoiding sized types, we can still dualise Theorem 3.8. Applying it to the opposite category $\mathscr{C}^{\mathrm{op}}$, we have that if $\mathscr{C}$ is complete and $F : \mathscr{C} \to \mathscr{C}$ preserves limits of diagrams $\kappa \to \mathscr{C}$ for some size κ, then *F has a final coalgebra νF given by the limit of a deflationary iteration* $(\nu_i F = \lim_{j<i} F(\nu_j F))_{i \in \kappa}$. We have yet to investigate whether this is useful, that is, how rich the class of such endofunctors is in a constructive setting.

   Adámek, Milius and Moss [5] take a different approach to constructive initial algebra theorems than the one here, avoiding iteration of the endofunctor. They consider categories $\mathscr{C}$ with colimits of diagrams of monomorphisms (from some well-behaved class) and endofunctors $F : \mathscr{C} \to \mathscr{C}$ that preserve those monomorphisms. Using the intuitionistically valid fixed point theorem of Pataraia (see [10, Theorem 3.2]), they prove that such an F has an initial algebra iff it has a prefixed point (an algebra whose structure morphism is a monomorphism). Preserving monomorphisms seems less of a condition on a functor than the one we need for Theorem 3.8, that is, preserving colimits of some size κ (although the two conditions are independent). However, as we saw in Theorem 4.10, our class of sized endofunctors is closed under taking coequalizers, so that we get initial algebras for constructs involving quotients, such as Example 4.14, whereas endofunctors preserving monomorphisms are not in general closed under taking coequalizers. Another difference to [5] is that it uses impredicative principles (the proof of Pataraia's fixed point theorem uses impredicative quantification), whereas our Agda development [28] shows that

our initial algebra theorem (Theorem 3.8) is valid in a predicative constructive logic.

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